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Stability analysis and soliton solutions for unstable nonlinear Schrödinger equation via two potential methods

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Abstract

This paper presents the analytical techniques to investigate the unstable nonlinear Schrödinger equation (NLSE). The exact solutions to the unstable NLSE which are found based on the generalized extended trial equation scheme and the improved Bernoulli sub-ODE scheme (IBSOS) with three cases, by utilizing Maple software. A system of nonlinear algebra differential equations is obtained, afterwards, this system by help of Maple is solved. The discovered solutions include hyperbolic function, trigonometric function, exponential, and rational solutions. Plenty of such types nonlinear equations arising in basic fabric of communications network technology and nonlinear optics which are investigated via mentioned methods. It offers theoretical application value for the study of complex wave dynamics in various scientific domains, such as plasma physics, and nonlinear optics. Firstly, the wave transform converts the considered model into a system of ordinary differential equations. Then, novel exact solitary wave solutions are developed as periodic, dark, combined hyperbolic, and rational functions. Specific parameter values help demonstrate the dynamic nature of the constructed solutions through their implementation. In addition, the stability of generated solitary wave solution through the Hamiltonian technique is investigated.

Keywords. Solitary wave solutions, Extended trial equation method, Improved Bernoulli sub-ODE method, Unstable nonlinear Schrödinger equation, Stability analysis.
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1. Introduction

Nonlinear wave equations are indispensable tools in the study of complex physical systems, as they model the interplay of various forces that shape wave dynamics [20, 44, 47]. These equations provide a mathematical framework to understand phenomena such as the propagation of light in optical fibers, the motion of water waves in shallow or deep fluids, and the intricate behaviors of plasma in high-energy astrophysical and laboratory environments [57]. Unlike linear equations, which are limited to describing simple waveforms, nonlinear equations account for interactions between waves, leading to phenomena such as wave steepening, dispersion, and energy localization [10, 30, 50]. These nonlinear effects are essential in capturing the richness of wave behavior observed in nature and technology [28, 62]. Chaos, fractal and solitons constitute the main body of nonlinear science. With the gradual development of solitons, solitons theory has attracted the interest of many mathematicians and physicists. For example, Korteweg-deVries (KdV) equation, which has been characterized mainly by the study of the integrability of the equation [30].

One of the most intriguing outcomes of nonlinear wave equations is the formation of solitary waves or solitons [34, 51]. Solitons are self-reinforcing waveforms that maintain their shape and energy while traveling over long distances [12, 61]. This stability arises from a delicate balance between nonlinearity, which tends to steepen the wave, and dispersion, which tends to spread it out. Solitons are of particular importance because of their practical applications in fields such as optical communication, where they enable high-capacity data transmission, and in hydrodynamics, where they

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describe wave patterns in oceans and channels. Their robustness also makes them valuable in understanding energy transport in nonlinear media and exploring fundamental wave dynamics [40, 55].

The study of the nonlinear Schrödinger equation has been an area of significant interest due to its ability to support diverse solitary wave solutions [54]. Bright solitons, characterized by localized peaks of intensity, arise when nonlinearity balances anomalous dispersion [16]. These solitons are critical in applications like optical fiber communications, where their stability ensures minimal signal degradation [29]. Conversely, dark solitons, which manifest as localized intensity dips on a continuous wave background, emerge under conditions of normal dispersion. Dark solitons have been observed in both optics and hydrodynamics, providing insights into wave dynamics in various media. Singular solitons represent another intriguing class of solutions [5]. These solitons, distinguished by singularities in their profiles, exhibit unique mathematical and physical properties. Hybrid solutions, such as bright-dark solitons, where bright and dark components coexist, add to the diversity of wave phenomena described by the nonlinear Schrödinger equation [35]. These solutions highlight the richness of nonlinear dynamics and offer potential for novel applications in areas like signal processing and energy localization.

The nonlinear Schrödinger equation (NLSE) [18, 19, 22, 48, 56] is considered as

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} \pm 2\gamma|\psi|^2\psi = 0, \quad 0 < x < L, \quad t > 0, \quad i = \sqrt{-1}, \tag{1.1}$$

where γ is an arbitrary real constant and the above equation plays a attractive role in the plenty of areas nonlinear sciences such as biology, the nonlinear optics, quantum mechanics and plasma physics. The unstable nonlinear Schrödinger equation (UNLSE) is a type of NLSE with space and time replaced [44, 57]. The dimensionless form of the unstable NLSE is giving by [26, 53]:

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2\lambda|\psi|^2\psi - 2\gamma\psi = 0, \quad 0 < x < L, \quad t > 0, \quad i = \sqrt{-1}, \tag{1.2}$$

where λ and γ are the arbitrary real constants. Various complex nonlinear aspects appearing in various fields of nonlinear sciences and engineering, including; optical fiber, plasmas physics and hydrodynamics among others are modeled in forms of nonlinear Schrödinger equations (NLSEs) in which were given by [31, 36–39].

Nonlinear partial differential equations (NLPDEs) play an indispensable role in simulating complex phenomena and control in various scientific fields, including models in nonlinear optics, fluid dynamics, quantum mechanics, plasma physics, and other domains [14, 15]. Over the past decades, significant progress has been made in developing effective methods for constructing solitary wave solutions of NLPDEs. Contemporary approaches, including the sine-Gordon expansion method, systematically generate localized solutions (e.g., dark solitons, breathers) by leveraging hyperbolic function bases [43]. The generalized exponential differential function method has also been proposed to capture multi-peakon and hybrid wave structures through exponential-trigonometric combinations [46]. These advancements have not only deepened our understanding of the underlying physical phenomena but also provided powerful tools for analyzing complex nonlinear systems [1, 17]. Numerical methods are essential in the study of solitary wave solutions, as they not only provide approximate solutions to complex equations but also help verify the accuracy of theoretical models [2]. Such characteristics play a crucial role in offering profound understandings of the complexity inherent in diverse scientific domains [8, 60]. In the past few years, researchers have made significant progress in demonstrating the well-known integrable models. Such as theoretical analysis for miscellaneous soliton waves [52], the dynamics of solitary waves [45], the fractional generalized CBS-BK equation arising in fluid mechanics [33], analytical approach for polar magnetooptics [58], the shallow water wave equation with variable coefficients [13], the cubic B-splines method [3], the modulation instability analysis for coupled fractional Lakshmanan-Porsezian-Daniel equation [4], sixth-order Benney-Luke equation [25], the extended homoclinic breather wave solutions [49], the spatial symmetric nonlinear dispersive wave model [63], the conservation law, stability analysis and degenerate lump solutions [59], three-dimensional nonlinear Volterra integral equations with 3D-Legendre polynomials [32], the reduced differential transform method [41] and the fifth-order integrable equations [24]. In [23], authors investigated the effects of Toxicode and Google Blockly on computational aspects. Asgarov et al presented the application of unsupervised machine learning technique [11]. Mammadzada examined three main task distribution models in Human-Robot Collaborative systems [27]. Mutashar et al [42] studied the simulation algorithm to the nonlinear fractional biological population model. A mixed problem with time derivative was considered in [6].

The perturbed higher-order modified Gerdjikov-Ivanov equation was considered the focus of the improved modified extended tanh-function method for producing brilliant and unique solitons such as dark and bright solitons [21]. A (2+1)-dimensional generalized Korteweg-de Vries problem was investigated by using the generalized Arnous technique and the Riccati equation approach [9]. Optical solitons for generalized perturbed nonlinear Schrödinger model in the presence of dual-power law nonlinear medium were studied in [7].
These applications highlight the dual role of NLPDEs as both mathematical descriptors of natural laws and theoretical cornerstones for technological advancements, necessitating systematic investigations into their exact solutions and dynamical behaviors. We offer two analytical methods to solve a well-known equation to obtain the exact solutions. New solutions not only enrich the theoretical understanding of nonlinear systems but also offer potential for practical applications. However, the complexity of the NLS equation often necessitates advanced solution techniques to uncover previously unreported wave forms. The provided search results indicate that the unstable nonlinear Schrödinger equation (UNLS) is a subject of ongoing research. Studies focus on finding exact solutions, particularly for modified versions of the equation, and understanding solitary wave patterns. Research also explores applications related to modulated wave train instabilities. The investigations often involve finding solutions and analyzing the behavior of these equations under specific conditions. In this work, we tackle this challenge by utilizing two analytical schemes: the generalized extended trial equation scheme and the improved Bernoulli sub-ODE method. These methods are chosen for their robustness and efficiency in solving nonlinear differential equations.

This paper is organized as follows. In next Section, we describe the main steps of the extended trial equation scheme (ETES) and its application. The improved Bernoulli sub-ODE scheme (IBSOS) is offered and then its application to solve (1.2) are given in section 3. The stability analysis is investigated in Section 4. Finally, some concluding remarks are drawn in section 5.

2. Extended trial equation eechnique

In the following section, the ETE technique Starting with the transformation as:
Step 1. Consider the special nonlinear PDE as

$$\mathcal{F}_1(\psi, \psi_x, \psi_t, \psi_{xx}, \psi_{tt}, \ldots) = 0. \tag{2.1}$$

Utilizing the wave transformation

$$\psi(x,t) = u(\eta), \qquad \eta = k\left(x - \lambda t\right), \tag{2.2}$$

where $\lambda \neq 0$ and $k \neq 0$ and inserting (2.2) into Eq. (2.1), we can get the following equation

$$\mathcal{F}_2(\psi, k\psi', -k\lambda\psi', k^2\psi'', k^2\lambda^2\psi'', \ldots) = 0. \tag{2.3}$$

Step 2. Consider the following trial equation:

$$u(\eta) = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{2.4}$$

where

$$(\Gamma')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \ldots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \ldots + \zeta_1 \Gamma + \zeta_0}. \tag{2.5}$$

Employing the Eqs. (2.4) and (2.5), one obtains

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right)^2, \tag{2.6}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2} \right). \tag{2.7}$$

After some computations, we get the following polynomial $\Lambda(\Gamma)$:

$$\Lambda(\Gamma) = \varrho_s \Gamma^s + ... + \varrho_1 \Gamma + \varrho_0 = 0. \tag{2.8}$$

The balance principle on (2.8) is determined to get a relation between θ, ϵ and δ.
Step 3. We solve the following algebraic equations:

$$\varrho_i = 0, \quad i = 1, 2, ..., s. \tag{2.9}$$

Step 4. The final step is to solve the following integral:

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Omega(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma, \tag{2.10}$$

where η_0 is free.

2.1. **Application of ETEM to UNLSE.** This familiar form captures the essential dynamics of nonlinear wave propagation without the additional higher-order effects, making it a foundational model for studying solitary waves in simpler contexts. This generalized form of the NLS equation provides a versatile framework for analyzing the rich dynamics of nonlinear wave systems, accounting for both fundamental and higher-order effects. Its comprehensive nature allows researchers to explore a wide range of physical phenomena and identify novel wave structures in diverse applications. To discover Eq. (1.2), the travelling wave transformation is used as

$$\psi(x,t) = u(\xi) e^{i\phi(x,t)}, \quad \xi = kx + \omega t, \tag{2.11}$$

where

$$\phi = px + \nu t, \tag{2.12}$$

where $\phi(x,t)$ offers the phase component, β is the frequency of solitons, ν shows the wave number, θ represents the phase constant. Applying Eq. (2.11) into Eq. (1.2) converts the nonlinear Schrödinger equation into two parts.
From the imaginary component of the equation, we can write as follows

$$\omega = -2pk, \tag{2.13}$$

and also the real part sees the following

$$k^2 u'' - \left(p^2 + \nu + 2\gamma\right) u + 2\lambda u^3 = 0. \tag{2.14}$$

By balancing the terms of u^3 and u'' in Eq. (2.14), we can see the following relationship between parameters δ, θ, and ϵ:

$$2\delta = \theta - \epsilon - 2. \tag{2.15}$$

For various values of δ, θ, and ϵ, we get:
Merge I: $\boldsymbol{\delta = 1, \theta = 4}$, **and** $\boldsymbol{\epsilon = 0}$.
By using **Merge I** for Eqs. (2.4) and (2.5), one receives

$$u(\eta) = \tau_0 + \tau_1 \Gamma, \tag{2.16}$$

$$(u'(\eta))^2 = \frac{\tau_1^2(\xi_4 \Gamma^4 + \xi_3 \Gamma^3 + \xi_2 \Gamma^2 + \xi_1 \Gamma + \xi_0)}{\zeta_0}, \quad \xi_4 \neq 0, \quad \zeta_0 \neq 0. \tag{2.17}$$

When the previous system is solved, several results are available:

- **Result 1**:

$$\begin{aligned} &k = k, \quad \nu = \nu, \quad p = p, \quad \zeta_0 = -\tfrac{k^2 \xi_4}{\lambda \tau_1^2}, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad \xi_0 = \xi_0, \\ &\xi_1 = -\tfrac{2\tau_0 \xi_4 (-2\lambda \tau_0^2 + p^2 + 2\gamma + \nu)}{(\tau_1^3 \lambda)}, \end{aligned} \tag{2.18}$$

$$\xi_2 = -\frac{\xi_4(-6\lambda\tau_0^2 + p^2 + 2\gamma + \nu)}{\lambda\tau_1^2}, \quad \xi_3 = \frac{4\tau_0\xi_4}{\tau_1}, \quad \xi_4 = \xi_4. \tag{2.19}$$

Putting these findings into Eqs. (2.5) and (2.10), we can write

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{-\frac{k^2}{\lambda\tau_1^2}}d\Gamma}{\sqrt{\Gamma^4 + \frac{4\tau_0}{\tau_1}\Gamma^3 - \frac{(-6\lambda\tau_0^2+p^2+2\gamma+\nu)}{\lambda\tau_1^2}\Gamma^2 - \frac{2\tau_0(-2\lambda\tau_0^2+p^2+2\gamma+\nu)}{(\tau_1^3\lambda)}\Gamma + \frac{\xi_0}{\xi_4}}}. \tag{2.20}$$

Integrating Eq. (2.20), the following results are reached as:

$$\pm(\eta - \eta_0) = -\frac{\Pi}{\Gamma - \alpha_1}, \tag{2.21}$$

$$\pm(\eta - \eta_0) = \frac{2\Pi}{\alpha_1 - \alpha_2}\sqrt{\frac{\Gamma - \alpha_2}{\Gamma - \alpha_1}}, \quad \alpha_2 > \alpha_1, \tag{2.22}$$

$$\pm(\eta - \eta_0) = \frac{\Pi}{\alpha_1 - \alpha_2}\ln\left|\frac{\Gamma - \alpha_1}{\Gamma - \alpha_2}\right|, \tag{2.23}$$

$$\pm(\eta - \eta_0) = \frac{\Pi}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}}\ln\left|\frac{\sqrt{(\alpha_1 - \alpha_3)(\Gamma - \alpha_2)} - \sqrt{(\alpha_1 - \alpha_2)(\Gamma - \alpha_3)}}{\sqrt{(\alpha_1 - \alpha_3)(\Gamma - \alpha_2)} + \sqrt{(\alpha_1 - \alpha_2)(\Gamma - \alpha_3)}}\right|, \tag{2.24}$$

$$\pm(\eta - \eta_0) = \frac{2\Pi}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}F(\varphi, l), \quad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \tag{2.25}$$

where

$$\Pi = \sqrt{-\frac{k^2}{\lambda\tau_1^2}}, \qquad F(\varphi, l) = \int_0^{\varphi}\frac{d\psi}{\sqrt{1 - l^2\sin^2\psi}}, \tag{2.26}$$

and

$$\varphi = \arcsin\sqrt{\frac{(\alpha_2 - \alpha_4)(\Gamma - \alpha_1)}{(\alpha_1 - \alpha_4)(\Gamma - \alpha_2)}}, \qquad l^2 = \frac{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}. \tag{2.27}$$

In the above relations suppose $\alpha_1, \alpha_2, \alpha_3$ and α_4 be the following roots as:

$$\Gamma^4 + \frac{4\tau_0}{\tau_1}\Gamma^3 - \frac{(-6\lambda\tau_0^2 + p^2 + 2\gamma + \nu)}{\lambda\tau_1^2}\Gamma^2 - \frac{2\tau_0(-2\lambda\tau_0^2 + p^2 + 2\gamma + \nu)}{(\tau_1^3\lambda)}\Gamma + \frac{\xi_0}{\xi_4} = 0. \tag{2.28}$$

By solving the above equation, we can get roots of fourth order equations as:

$$\begin{aligned}
\alpha_1 &= 1/2\,\frac{-2\,\lambda\,\tau_0\tau_1 + \sqrt{2\,\tau_1{}^2\lambda\,(p^2+2\,\gamma+\nu) + 2\,\sqrt{(-2\,\lambda\,\tau_0{}^2+2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)(-2\,\lambda\,\tau_0{}^2-2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)}\lambda\,\tau_1{}^2}}{\lambda\,\tau_1{}^2},\\
\alpha_2 &= 1/2\,\frac{-2\,\lambda\,\tau_0\tau_1 - \sqrt{2\,\tau_1{}^2\lambda\,(p^2+2\,\gamma+\nu) + 2\,\sqrt{(-2\,\lambda\,\tau_0{}^2+2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)(-2\,\lambda\,\tau_0{}^2-2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)}\lambda\,\tau_1{}^2}}{\lambda\,\tau_1{}^2},\\
\alpha_3 &= 1/2\,\frac{-2\,\lambda\,\tau_0\tau_1 + \sqrt{2\,\tau_1{}^2\lambda\,(p^2+2\,\gamma+\nu) - 2\,\sqrt{(-2\,\lambda\,\tau_0{}^2+2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)(-2\,\lambda\,\tau_0{}^2-2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)}\lambda\,\tau_1{}^2}}{\lambda\,\tau_1{}^2},\\
\alpha_4 &= 1/2\,\frac{-2\,\lambda\,\tau_0\tau_1 - \sqrt{2\,\tau_1{}^2\lambda\,(p^2+2\,\gamma+\nu) - 2\,\sqrt{(-2\,\lambda\,\tau_0{}^2+2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)(-2\,\lambda\,\tau_0{}^2-2\,\lambda\,\tau_1{}^2+p^2+2\,\gamma+\nu)}\lambda\,\tau_1{}^2}}{\lambda\,\tau_1{}^2}.
\end{aligned} \tag{2.29}$$

Inserting the solutions (2.21)-(2.26) into (2.4), we obtain the following traveling wave solutions for Eq. (1.2):

$$u_1(x,t) = e^{i(px+\nu t)}\left\{\tau_0 + \tau_1\alpha_1 - \frac{\Pi}{kx - 2pkt - \eta_0}\right\}, \tag{2.30}$$

$$u_2(x,t) = e^{i(px+\nu t)}\left\{\tau_0 + \tau_1\alpha_1 + \frac{4(\alpha_2-\alpha_1)\tau_1\Pi^2}{4\Pi^2 - (\alpha_1-\alpha_2)^2(kx - 2pkt - \eta_0)^2}\right\}, \tag{2.31}$$

$$u_3(x,t) = e^{i(px+\nu t)}\left\{\tau_0 + \tau_1\alpha_2 \pm \frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{\Pi}\left(kx - 2pkt - \eta_0\right)\right] - 1}\right\}, \tag{2.32}$$

$$u_4(x,t) = e^{i(px+\nu t)}\left\{\tau_0 + \tau_1\alpha_1 - \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\Theta_1\right]}\right\},$$
$$\Theta_1 = \frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{\Pi}\left(kx - 2pkt - \eta_0\right), \tag{2.33}$$

$$u_5(x,t) = e^{i(px+\nu t)}\left\{\tau_0 + \tau_1\alpha_2 + \frac{2(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\Theta_2\right]}\right\},$$
$$\Theta_2 = \mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\Pi}\left(kx - 2pkt - \eta_0\right), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}. \tag{2.34}$$

If we take $\tau_0 = -\tau_1\alpha_1$ and $\eta_0 = 0$, then the solutions (2.30) and (2.33) change to rational function solutions as

$$u_6(x,t) = e^{i(px+\nu t)}\left\{-\frac{\Pi}{kx - 2pkt}\right\}, \tag{2.35}$$

$$u_7(x,t) = e^{i(px+\nu t)}\left\{\frac{4(\alpha_2-\alpha_1)\tau_1\Pi^2}{4\Pi^2 - (\alpha_1-\alpha_2)^2(kx - 2pkt)^2}\right\}. \tag{2.36}$$

If we take $\tau_0 = -\tau_1\alpha_1$ and $\eta_0 = 0$, then the solution (2.33) change to hyperbolic function solution

$$u_8(x,t) = e^{i(px+\nu t)}\left\{-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{\Pi}\left(kx - 2pkt\right)\right]}\right\}. \tag{2.37}$$

If we take $\tau_0 = -\tau_1\alpha_2$ and $\eta_0 = 0$, then the solutions (2.32) and (2.34) change to kink-singular and Jacobi elliptic function solution, respectively as

$$u_9(x,t) = e^{i(px+\nu t)}\left\{\tau_0 + \tau_1\alpha_2 \pm \frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{\Pi}\left(kx - 2pkt\right)\right] - 1}\right\}, \tag{2.38}$$

$$u_{10}(x,t) = e^{i(px+\nu t)}\left\{\frac{2(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\Pi}\left(kx - 2pkt\right), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}\right\}. \tag{2.39}$$

If the modulus $l \to 1$, then the solution (2.39) change to the solitary wave solution as

$$u_{11}(x,t) = e^{i(px+\nu t)}\left\{\frac{2(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\tanh^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{2\Pi}\left(kx - 2pkt\right)\right]}\right\}, \tag{2.40}$$

where $\alpha_3 = \alpha_4$. If the modulus $l \to 0$, then the solution (2.39) change to the periodic wave solution as

$$u_{12}(x,t) = e^{i(px+\nu t)} \left\{ \frac{2(\alpha_1-\alpha_3)(\alpha_4-\alpha_3)\tau_1}{\alpha_4-\alpha_3+(\alpha_1-\alpha_4)\sin^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{2\Pi}(kx-2pkt)\right]} \right\}, \tag{2.41}$$

where $\alpha_2 = \alpha_3$.

Merge II: $\delta = 1, \theta = 5$, and $\epsilon = 1$.

Consider **Merge II** for Eqs. (2.4) and (2.5), one obtains

$$u(\eta) = \tau_0 + \tau_1\Gamma, \tag{2.42}$$

$$(u'(\eta))^2 = \frac{\tau_1^2(\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0+\zeta_1\Gamma}, \quad \xi_5 \neq 0, \ \zeta_1 \neq 0. \tag{2.43}$$

We receive the following results:

- **Results 2**:

$$\begin{aligned}
&k = k, \quad \nu = \nu, \quad p = p, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad \xi_0 = \xi_0,\\
&\xi_1 = \tfrac{k^2\tau_1\zeta_1\xi_0+4\gamma\tau_0\zeta_0^2-4\lambda\tau_0^3\zeta_0^2+2p^2\tau_0\zeta_0^2+2\nu\tau_0\zeta_0^2}{\tau_1k^2\zeta_0},\\
&\xi_2 = \tfrac{-4\lambda\tau_0^3\zeta_1+p^2\tau_1\zeta_0+2p^2\tau_0\zeta_1+4\gamma\tau_0\zeta_1+\nu\tau_1\zeta_0+2\nu\tau_0\zeta_1-6\lambda\tau_0^2\tau_1\zeta_0+2\gamma\tau_1\zeta_0}{k^2\tau_1},\\
&\xi_3 = \tfrac{p^2\zeta_1+\nu\zeta_1-6\lambda\tau_0^2\zeta_1+2\gamma\zeta_1-4\zeta_0\tau_1\tau_0\lambda}{k^2}, \xi_4 = -\tfrac{\lambda\tau_1(4\tau_0\zeta_1+\tau_1\zeta_0}{k^2},\\
&\xi_5 = \xi_5, \ \ \zeta_0 = \zeta_0, \quad \zeta_1 = \zeta_1.
\end{aligned} \tag{2.44}$$

Inserting the above parameters into Eqs. (2.5) and (2.10), one become

$$\pm(\eta-\eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\,d\Gamma}{\sqrt{\Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}}}. \tag{2.45}$$

Integrating Eq. (2.45), we get the Families for Eq. (1.2):

Family 1. Suppose $F(\Gamma) = \Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}$ can be written as:

$$F(\Gamma) = (\Gamma-\alpha_1)^5, \tag{2.46}$$

where α_1 is free. Then, we obtain

$$\pm(\eta-\eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\,d\Gamma}{(\Gamma-\alpha_1)^2\sqrt{\Gamma-\alpha_1}} = -\frac{2}{3}\frac{\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma\right)^{\frac{3}{2}}}{\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_1\right)(\Gamma-\alpha_1)^{\frac{3}{2}}}, \tag{2.47}$$

or

$$\Gamma = \frac{\frac{\zeta_0}{\xi_5}+\alpha_1\left[-\frac{3}{2}\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_1\right)(\eta-\eta_0)\right]^{\frac{3}{2}}}{\left[-\frac{3}{2}\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_1\right)(\eta-\eta_0)\right]^{\frac{3}{2}}-\frac{\zeta_1}{\xi_5}}. \tag{2.48}$$

Substituting (2.48) into (2.4), we get the following solution:

$$u_{13}(x,t) = e^{i(px+\nu t)}\left\{\tau_0+\tau_1\frac{\frac{\zeta_0}{\xi_5}+\alpha_1\left[-\frac{3}{2}\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_1\right)(kx-2pkt-\eta_0)\right]^{\frac{3}{2}}}{\left[-\frac{3}{2}\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_1\right)(kx-2pkt-\eta_0)\right]^{\frac{3}{2}}-\frac{\zeta_1}{\xi_5}}\right\}. \tag{2.49}$$

Family 2. Suppose $F(\Gamma)=\Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}$ can be written as:

$$F(\Gamma)=(\Gamma-\alpha_1)^4(\Gamma-\alpha_2), \tag{2.50}$$

where α_1 and α_2 are frees. Then, the results are

$$\begin{aligned}&\pm(\eta-\eta_0)=\int\frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\ d\Gamma}{(\Gamma-\alpha_1)^2\sqrt{\Gamma-\alpha_2}}=\\&-\frac{1}{2}\frac{(\Gamma-\alpha_1)(\Gamma-\alpha_2)\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\ [2\Pi_1\Pi_2+\Pi_3\ln(Y)]}{(\alpha_1-\alpha_2)\Pi_1\Pi_2\sqrt{(\Gamma-\alpha_1)^4(\Gamma-\alpha_2)}},\end{aligned} \tag{2.51}$$

where

$$\begin{aligned}&\Pi_1=\sqrt{(\alpha_1-\alpha_2)\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_1\right)},\quad \Pi_2=\sqrt{\frac{\zeta_1}{\xi_5}\Gamma^2+\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_2\right)\Gamma-\frac{\zeta_0}{\xi_5}\alpha_2},\\&\Pi_3=(\alpha_1-\Gamma)\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_2\right),\\&\Pi_4=\Gamma\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_2+2\frac{\zeta_1}{\xi_5}\alpha_1\right)+\frac{\zeta_0}{\xi_5}\alpha_1-2\frac{\zeta_0}{\xi_5}\alpha_2-\frac{\zeta_1}{\xi_5}\alpha_1\alpha_2,\qquad Y=\frac{\Pi_4+2\Pi_1\Pi_2}{\Gamma-\alpha_1}.\end{aligned} \tag{2.52}$$

Family 3. Suppose $F(\Gamma)=\Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}$ can be represented as:

$$F(\Gamma)=(\Gamma-\alpha_1)^3(\Gamma-\alpha_2)^2, \tag{2.53}$$

where α_1 and α_2 are frees. Then, the findings are

$$\begin{aligned}&\pm(\eta-\eta_0)=\int\frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\ d\Gamma}{\sqrt{(\Gamma-\alpha_1)^3(\Gamma-\alpha_2)^2}}=\\&-\frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\ [2(\alpha_1-\alpha_2)\Pi_2+(\Gamma-\alpha_1)\Pi_1\ln(Y)]}{(\alpha_1-\alpha_2)^2\Pi_2\sqrt{\Gamma-\alpha_1}},\end{aligned} \tag{2.54}$$

where

$$\Pi_1=\sqrt{-(\alpha_1-\alpha_2)\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_2\right)},\qquad \Pi_2=\sqrt{\frac{\zeta_1}{\xi_5}\Gamma^2+\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_1\right)\Gamma-\frac{\zeta_0}{\xi_5}\alpha_1}, \tag{2.55}$$

$$\Pi_3=\Gamma\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_1+2\frac{\zeta_1}{\xi_5}\alpha_2\right)+\frac{\zeta_0}{\xi_5}\alpha_2-2\frac{\zeta_0}{\xi_5}\alpha_1-\frac{\zeta_1}{\xi_5}\alpha_1\alpha_2,\qquad Y=\frac{\Pi_3+2\Pi_1\Pi_2}{\Gamma-\alpha_2}. \tag{2.56}$$

Family 4. Suppose $F(\Gamma)=\Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}$ can be represented as:

$$F(\Gamma)=(\Gamma-\alpha_1)^2(\Gamma-\alpha_2)^2(\Gamma-\alpha_3), \tag{2.57}$$

where α_1,α_2 and α_3 are frees. Then, the findings are:

$$\begin{aligned}&\pm(\eta-\eta_0)=\int\frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\ d\Gamma}{\sqrt{(\Gamma-\alpha_1)^3(\Gamma-\alpha_2)^2}}=\\&-\frac{\sqrt{\Gamma-\alpha_3}\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\ [(\alpha_2-\alpha_3)\Pi_1\ln(Y_1)-(\alpha_1-\alpha_3)\Pi_2\ln(Y_2)]}{\Pi_3\Pi_6},\end{aligned} \tag{2.58}$$

where

$$\begin{aligned}&\Pi_1=\sqrt{(\alpha_1-\alpha_3)\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_1\right)},\ \Pi_2=\sqrt{(\alpha_2-\alpha_3)\left(\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\alpha_2\right)},\\&\Pi_3=\sqrt{\frac{\zeta_1}{\xi_5}\Gamma^2+\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_3\right)\Gamma-\frac{\zeta_0}{\xi_5}\alpha_3},\\&\Pi_4=\Gamma\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_3+2\frac{\zeta_1}{\xi_5}\alpha_1\right)+\frac{\zeta_0}{\xi_5}\alpha_1-2\frac{\zeta_0}{\xi_5}\alpha_3-\frac{\zeta_1}{\xi_5}\alpha_1\alpha_3,\\&\Pi_5=\Gamma\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_3+2\frac{\zeta_1}{\xi_5}\alpha_2\right)+\frac{\zeta_0}{\xi_5}\alpha_2-2\frac{\zeta_0}{\xi_5}\alpha_3-\frac{\zeta_1}{\xi_5}\alpha_2\alpha_3,\\&\Pi_6=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3),\quad Y_1=\frac{\Pi_4+2\Pi_1\Pi_3}{\Gamma-\alpha_1},\quad Y_2=\frac{\Pi_5+2\Pi_2\Pi_3}{\Gamma-\alpha_2}.\end{aligned} \tag{2.59}$$

3. The improved Bernoulli sub-ODE Technique

Consider the following stages of IBSOM as
Step 1. Consider the following Function

$$\mathcal{F}_1(\psi, \psi_x, \psi_y, \psi_t, \psi_{xx}, \psi_{tt}, ...) = 0, \tag{3.1}$$

which is converted to

$$\mathcal{F}_2(U, r_1U', r_2U', r_1^2U'', r_2^2U'', ...) = 0, \quad \xi = r_1x - r_2t. \tag{3.2}$$

Step 2. Consider the following solution function

$$U(\zeta) = \frac{\sum_{k=0}^{n} a_k F^k(\zeta)}{\sum_{k=0}^{m} a_k F^k(\zeta)} = \frac{a_0 + a_1F(\zeta) + a_2F^2(\zeta) + ... + a_nF^n(\zeta)}{b_0 + b_1F(\zeta) + b_2F^2(\zeta) + ... + b_mF^m(\zeta)}, \tag{3.3}$$

so that

$$F'(\zeta) = bF(\zeta) + dF^{\theta}(\zeta), \quad b \neq 0, \quad d \neq 0, \quad \theta \in \mathbb{R} - \{0, 1, 2\}, \tag{3.4}$$

and $F(\xi)$ is Bernoulli differential. We get the following relation as:

$$\Psi(F(\zeta)) = \eta_p F^p(\zeta) + ...\eta_1 F(\zeta) + \eta_0 = 0. \tag{3.5}$$

Step 3. Let the equation system as the mentioned below:

$$\eta_l = 0, \quad l = 0, 1, ..., p. \tag{3.6}$$

Step 4. Solve the equation (3.4). Then, we obtain

$$F(\zeta) = \left[\frac{-d}{b} + \frac{E}{e^{b(\theta-1)\zeta}}\right]^{\frac{1}{1-\theta}}, \quad b \neq d, \tag{3.7}$$

$$F(\zeta) = \left[\frac{E - 1 + (E+1)\tanh\left(\frac{b(1-\theta)\zeta}{2}\right)}{1 - \tanh\left(\frac{b(1-\theta)\zeta}{2}\right)}\right]^{\frac{1}{1-\theta}}, \quad b = d, \quad E \in \mathbb{R}. \tag{3.8}$$

3.1. **Implementations of IBSOM.** Balancing u^3 and u'' in Eq. (2.14), we can find the below relation as:

$$\theta + m = n + 1. \tag{3.9}$$

We have some cases including:
Merge A: $\theta = n = 3, m = 1$.
By inserting **Merge A** in Eq. (2.14), we reach

$$U(\zeta) = \frac{a_0 + a_1F(\zeta) + a_2F^2(\zeta) + a_3F^3(\zeta)}{b_0 + b_1F(\zeta)} = \frac{\Theta_1(\zeta)}{\Phi_1(\zeta)}, \tag{3.10}$$

$$U'(\zeta) = \frac{\Theta_1'(\zeta)\Phi_1(\zeta) - \Theta_1(\zeta)\Phi_1'(\zeta)}{\Phi_1^2(\zeta)}, \tag{3.11}$$

$$U''(\zeta) = \frac{\Theta_1''(\zeta)\Phi_1(\zeta) - \Theta_1(\zeta)\Phi_1'(\zeta)}{\Phi_1^2(\zeta)} - \frac{[\Theta_1(\zeta)\Phi_1'(\zeta)]'\Phi_1^2(\zeta) - 2\Theta_1(\zeta)[\Phi_1'(\zeta)]^2\Phi_1(\zeta)}{\Phi_1^4(\zeta)}, \tag{3.12}$$

where $a_3 \neq 0$ and $b_1 \neq 0$. When we use Eqs. (3.10) and (3.12) in Eq. (2.14). The results are:
Case I:

$$k = \frac{a_3\sqrt{-\lambda}}{2b_1 d}, \quad \nu = -\frac{4\gamma d^2 b_1^2 + 2p^2 d^2 b_1^2 - b^2 a_3^2 \lambda}{2d^2 b_1^2}, \quad p = p, \quad a_0 = \frac{ba_2}{2d}, \quad a_1 = \frac{ba_3}{2d}, \tag{3.13}$$

$$a_2 = a_2, \quad a_3 = a_3, \quad b_0 = \frac{a_2 b_1}{a_3}, \quad b_1 = b_1.$$

After replacing the above parameters in solution function we reach

$$U(\zeta) = \frac{a_3(b + 2F(\xi)^2 d)}{2b_1 d}. \tag{3.14}$$

Result : If we take (3.7) and (3.8), then we have the following solutions as:

$$u_1(x,t) = \frac{a_3}{b_1}\, e^{i\left(px - \frac{4\gamma d^2 b_1^2 + 2p^2 d^2 b_1^2 - b^2 a_3^2 \lambda}{2d^2 b_1^2} t\right)} \left\{ \frac{b}{2d} + \left[\frac{-d}{b} + \frac{E}{e^{\frac{ba_3\sqrt{-\lambda}}{b_1 d}(x-2pt)}} \right]^{-1} \right\}, \tag{3.15}$$

$$u_2(x,t) = \frac{a_3}{b_1}\, e^{i\left(px - \frac{4\gamma b_1^2 + 2p^2 b_1^2 - a_3^2 \lambda}{2b_1^2} t\right)} \left\{ \frac{1}{2} + \left[\frac{E - 1 - (E+1)\tanh\left(\frac{a_3\sqrt{-\lambda}}{2b_1}(x - 2pt)\right)}{1 + \tanh\left(\frac{a_3\sqrt{-\lambda}}{2b_1}(x - 2pt)\right)} \right]^{-1} \right\}, \tag{3.16}$$

where b, d and E are constant free.

Merge B: $\theta = n = 4, m = 1$.

By inserting **Merge B** in Eq. (2.14), one get

$$U(\zeta) = \frac{a_0 + a_1 F(\zeta) + a_2 F^2(\zeta) + a_3 F^3(\xi) + a_4 F^4(\zeta)}{b_0 + b_1 F(\zeta)} = \frac{\Theta_1(\zeta)}{\Phi_1(\zeta)}, \tag{3.17}$$

where $a_4 \neq 0$ and $b_1 \neq 0$. The results are:
Case I:

$$k = \frac{a_4\sqrt{-\lambda}}{3b_1 d}, \quad \nu = -\frac{4\gamma d^2 b_1^2 + 2p^2 d^2 b_1^2 - b^2 a_4^2 \lambda}{2d^2 b_1^2}, \quad p = p, \quad a_0 = \frac{ba_3}{2d}, \quad a_1 = \frac{ba_4}{2d}, \tag{3.18}$$

$$a_2 = 0, \quad a_3 = a_3, \quad a_4 = a_4, \quad b_0 = \frac{a_3 b_1}{a_4}, \quad b_1 = b_1.$$

We have the following solution

$$U(\xi) = \frac{a_4(b + 2F(\xi)^3 d)}{2b_1 d}. \tag{3.19}$$

Result : If we take (3.7) and (3.8), then we have the following solutions as:

$$u_3(x,t) = e^{i\left(px - \frac{4\gamma d^2 b_1^2 + 2p^2 d^2 b_1^2 - b^2 a_3^2 \lambda}{2d^2 b_1^2} t\right)} \left\{ \frac{a_3 b}{2b_1 d} + \frac{2a_4}{2b_1} \left[\frac{-d}{b} + \frac{E}{e^{\frac{ba_4\sqrt{-\lambda}}{b_1 d}(x-2pt)}} \right]^{-1} \right\}, \tag{3.20}$$

$$u_4(x,t) = e^{i\left(px - \frac{4\gamma b_1^2 + 2p^2 b_1^2 - a_4^2 \lambda}{2b_1^2} t\right)} \left\{ \frac{a_3}{2b_1} + \frac{2a_4}{2b_1} \left[\frac{E - 1 - (E+1)\tanh\left(\frac{a_4\sqrt{-\lambda}}{2b_1}(x - 2pt)\right)}{1 + \tanh\left(\frac{a_4\sqrt{-\lambda}}{2b_1}(x - 2pt)\right)} \right]^{-1} \right\}, \tag{3.21}$$

where b, d and E are constant free.

Merge C: $\theta = 3, n = 4, m = 2$.

By inserting **Merge C** for Eq. (2.14), we offer

$$U(\zeta) = \frac{a_0 + a_1 F(\zeta) + a_2 F^2(\zeta) + a_3 F^3(\zeta) + a_4 F^4(\zeta)}{b_0 + b_1 F(\zeta) + b_2 F^2(\zeta)} = \frac{\Theta_1(\zeta)}{\Phi_1(\zeta)}, \tag{3.22}$$

where $a_4 \neq 0$ and $b_2 \neq 0$. The findings are:
Case I:

$$\begin{aligned} &k=\frac{\sqrt{p^2+2\gamma+\nu}}{2b}, \quad a_0=a_1=a_3=b_1=0, \quad a_2=\frac{ba_4}{d}, \quad b_0=\frac{b^2\sqrt{-\lambda}\,a_4}{2d^2\sqrt{p^2+2\gamma+\nu}},\\ &b_2=\frac{\sqrt{-\lambda}\,a_4 b}{d\sqrt{p^2+2\gamma+\nu}}, \quad b=b, \quad d=d, \quad p=p, \quad \nu=\nu. \end{aligned} \tag{3.23}$$

We have the following solution

$$U(\zeta)=\frac{2F(\zeta)^2\left(dF(\zeta)^2+b\right)d\sqrt{-\lambda}}{b\sqrt{p^2+2\gamma+\nu}\left(b+2dF(\zeta)^2\right)}, \quad \zeta=\frac{\sqrt{p^2+2\gamma+\nu}}{2b}\,(x-2p\,t)\,. \tag{3.24}$$

Result: If we consider (3.7) and (3.8), then we have the following solutions as:

$$\begin{aligned} &u_5(x,t)=e^{i(px+\nu\,t)}\left\{\frac{2\left[\frac{-d}{b}+\frac{E}{e^{\chi}}\right]^{-1}\left(d\left[\frac{-d}{b}+\frac{E}{e^{\chi}}\right]^{-1}+b\right)d\sqrt{-\lambda}}{b\sqrt{p^2+2\gamma+\nu}\left(b+2d\left[\frac{-d}{b}+\frac{E}{e^{\chi}}\right]^{-1}\right)}\right\},\\ &\chi=\sqrt{p^2+2\gamma+\nu}\,(x-2p\,t)\,, \quad \lambda<0, \end{aligned} \tag{3.25}$$

$$u_6(x,t)=e^{i(px+\nu\,t)}\left\{\frac{2\left[\frac{E-1+(E+1)\tanh\left(\frac{-\chi}{2}\right)}{1-\tanh\left(\frac{-\chi}{2}\right)}\right]^{-1}\left(d\left[\frac{E-1+(E+1)\tanh\left(\frac{-\chi}{2}\right)}{1-\tanh\left(\frac{-\chi}{2}\right)}\right]^{-1}+b\right)d\sqrt{-\lambda}}{b\sqrt{p^2+2\gamma+\nu}\left(b+2d\left[\frac{E-1+(E+1)\tanh\left(\frac{-\chi}{2}\right)}{1-\tanh\left(\frac{-\chi}{2}\right)}\right]^{-1}\right)}\right\}, \tag{3.26}$$

$$\chi=\sqrt{p^2+2\gamma+\nu}\,(x-2p\,t)\,, \quad \lambda<0,$$

where b, d and E are constant free.

Case II:

$$\begin{aligned} &k=\frac{\sqrt{-p^2-2\gamma-\nu}}{2\sqrt{2}\,b}, \quad a_1=a_3=b_1=0, \quad a_0=\frac{b^2a_4}{2d^2}, \quad a_2=\frac{ba_4}{d},\\ &b_0=\frac{b^2\sqrt{2\lambda}\,a_4}{2\sqrt{-p^2-2\gamma-\nu}\,d^2}, \quad b_2=\frac{a_4 b\sqrt{2\lambda}}{d\sqrt{-p^2-2\gamma-\nu}}, \quad b=b, \quad d=d, \quad p=p, \quad \nu=\nu. \end{aligned} \tag{3.27}$$

We have the following solution

$$\begin{aligned} &U(\xi)=\frac{\left(2F(\xi)^4d^2+2dF(\xi)^2b+b^2\right)\sqrt{p^2+2\gamma+\nu}}{b\sqrt{-2\lambda}\left(b+2dF(\xi)^2\right)},\\ &\xi=\frac{\sqrt{-p^2-2\gamma-\nu}}{2\sqrt{2}\,b}\,(x-2p\,t)\,. \end{aligned} \tag{3.28}$$

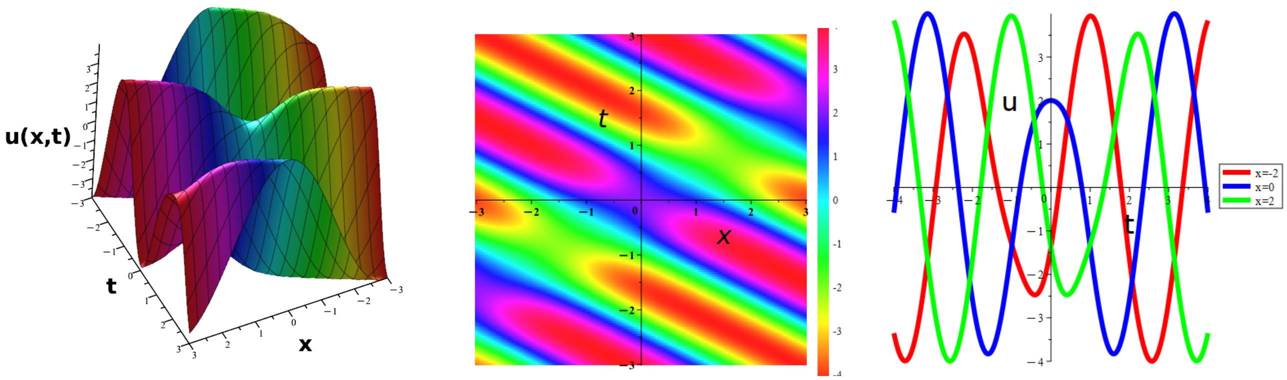
Result : If we consider (3.7) and (3.8), then we have the following solutions as:

$$u_7(x,t)=e^{i(px+\nu\,t)}\left\{\frac{\left(2\left[\frac{-d}{b}+\frac{E}{e^{\chi}}\right]^{-2}d^2+2dF(\xi)^2\,b+b^2\right)\sqrt{p^2+2\gamma+\nu}}{b\sqrt{-2\lambda}\left(b+2d\left[\frac{-d}{b}+\frac{E}{e^{\chi}}\right]^{-1}\right)}\right\}, \tag{3.29}$$

$$\chi=\frac{\sqrt{-p^2-2\gamma-\nu}}{\sqrt{2}}\,(x-2p\,t)\,, \quad \lambda<0,$$

$$\begin{aligned} &u_8(x,t)=e^{i(px+\nu\,t)}\times\\ &\left\{\frac{\left(2\left[\frac{E-1+(E+1)\tanh\left(\frac{-\chi}{2}\right)}{1-\tanh\left(\frac{-\chi}{2}\right)}\right]^{-2}d^2+2d\left[\frac{E-1+(E+1)\tanh\left(\frac{-\chi}{2}\right)}{1-\tanh\left(\frac{-\chi}{2}\right)}\right]^{-1}b+b^2\right)\sqrt{p^2+2\gamma+\nu}}{b\sqrt{-2\lambda}\left(b+2d\left[\frac{E-1+(E+1)\tanh\left(\frac{-\chi}{2}\right)}{1-\tanh\left(\frac{-\chi}{2}\right)}\right]^{-1}\right)}\right\}, \end{aligned} \tag{3.30}$$

$$\chi=\frac{\sqrt{-p^2-2\gamma-\nu}}{\sqrt{2}}\,(x-2p\,t)\,, \quad \lambda<0,$$



(a) (b) (c)

FIGURE 1. Real solution of dark soliton solution (2.40) u_{11} with values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$.

where b, p, d and E are constant free.

Case III:

$$\begin{aligned} &k = \frac{\sqrt{p^2+2\gamma+\nu}}{b\sqrt{2}}, \quad a_1 = a_3 = b_1 = 0, \quad a_0 = \frac{b^2 a_4}{4d^2}, \quad a_2 = \frac{b a_4}{d}, \\ &b_0 = \frac{b^2 a_4\sqrt{\lambda}}{2d^2\sqrt{2p^2+4\gamma+2\nu}}, \quad b_2 = \frac{a_4 b\sqrt{\lambda}}{d\sqrt{2p^2+4\gamma+2\nu}}, \quad b = b, \quad d = d, \quad p = p, \quad \nu = \nu. \end{aligned} \tag{3.31}$$

We have the following solution

$$U(\xi) = \frac{\sqrt{2p^2+4\gamma+2\nu}\,\left(b + 2dF(\xi)^2\right)}{2b\sqrt{\lambda}}, \quad \xi = \frac{\sqrt{p^2+2\gamma+\nu}}{b\sqrt{2}}\,(x - 2p\,t)\,. \tag{3.32}$$

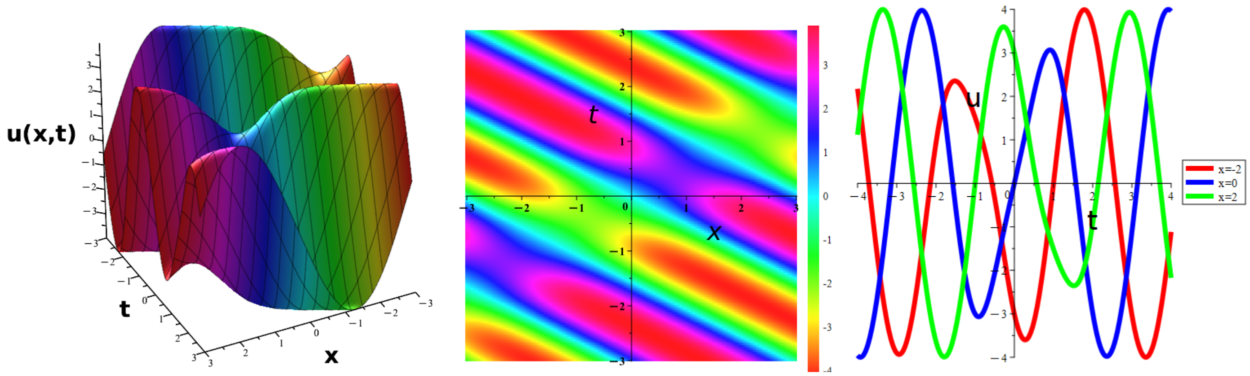
Result : If we take (3.7) and (3.8), then we have the following solutions as:

$$\begin{aligned} &u_9(x,t) = e^{i(px+\nu\,t)}\left\{\frac{\sqrt{2p^2+4\gamma+2\nu}\,\left(b+2d\left[\frac{-d}{b}+\frac{E}{e^{X}}\right]^{-1}\right)}{2b\sqrt{\lambda}}\right\}, \\ &\xi = \frac{2\sqrt{p^2+2\gamma+\nu}}{\sqrt{2}}\,(x - 2p\,t)\,, \end{aligned} \tag{3.33}$$

$$u_{10}(x,t) = e^{i(px+\nu\,t)}\left\{\frac{\sqrt{2p^2+4\gamma+2\nu}\,\left(b+2d\left[\frac{E-1+(E+1)\tanh\left(\frac{-X}{2}\right)}{1-\tanh\left(\frac{-X}{2}\right)}\right]^{-1}\right)}{2b\sqrt{\lambda}}\right\}, \tag{3.34}$$

$$\xi = \frac{2\sqrt{p^2+2\gamma+\nu}}{\sqrt{2}}\,(x - 2p\,t)\,, \quad \lambda < 0,$$

where b, p, d and E are constant free.



(a) (b) (c)

FIGURE 2. Imaginary solution of dark soliton solution (2.40) u_{11} with values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$.

4. Stability Properties

4.1. Stability Properties on equation (2.37). This section provide the stability of our traveling wave solutions for Eq. (2.37) as follows

$$\begin{aligned} &u_8(x,t) = e^{i(px+\nu t)}\left\{2\,X_1\,(p)\,\tau_1\left(X_2\,(p) + X_3\,(p)\cosh\left(\frac{\sqrt{X_1(p)}k(-pt+x)}{\Pi}\right)\right)^{-1}\right\},\\ &X_1\,(p) = (\alpha_1 - \alpha_2)\,(\alpha_1 - \alpha_3)\,,\\ &X_2\,(p) = 2\,\alpha_1 - \alpha_2 - \alpha_3,\\ &X_3\,(p) = \alpha_3 - \alpha_2, \end{aligned} \tag{4.1}$$

was examine by using the Hamiltonian technique [28] as

$$\begin{aligned} F(p) &= \int_{-\infty}^{+\infty}\frac{1}{2}|u_8^2(\omega,t)|d\omega\\ &= \lim_{x\to+\infty} 2\frac{(X_1(p))^{3/2}{\tau_1}^2\Pi}{X_2(p)-X_3(p)}\left(-2\frac{Y_3(p)}{\sqrt{Y_1(p)}} - 2\frac{X_3(p)}{X_2(p)+X_3(p)}\left(\frac{Y_2(p)}{(Y_2(p))^2(X_2(p)-X_3(p)) - X_2(p) - X_3(p)} - \frac{Y_3(p)}{\sqrt{Y_1(p)}}\right)\right)\\ &- \lim_{x\to-\infty} 2\frac{(X_1(p))^{3/2}{\tau_1}^2\Pi}{X_2(p)-X_3(p)}\left(-2\frac{Y_3(p)}{\sqrt{Y_1(p)}} - 2\frac{X_3(p)}{X_2(p)+X_3(p)}\left(\frac{Y_2(p)}{(Y_2(p))^2(X_2(p)-X_3(p)) - X_2(p) - X_3(p)} - \frac{Y_3(p)}{\sqrt{Y_1(p)}}\right)\right),\\ Y_1\,(p) &= (X_2\,(p) + X_3\,(p))\,(X_2\,(p) - X_3\,(p))\,,\\ Y_2\,(p) &= \tanh\left(1/2\,\frac{\sqrt{X_1\,(p)}kpt}{\Pi} - 1/2\,\frac{\sqrt{X_1\,(p)}kx}{\Pi}\right),\\ Y_3\,(p) &= \operatorname{arctanh}\left(\frac{(X_2\,(p) - X_3\,(p))\,Y_2\,(p)}{\sqrt{Y_1\,(p)}}\right), \end{aligned} \tag{4.2}$$

in which $F(p)$ symbolizes the momentum function and also ϕ_4 is utilized to express the wave speed and $u_8(x,t)$ are analytical solutions. The stability condition for solitary waves is given by

$$\frac{\partial F_1(p)}{\partial p} > 0. \tag{4.3}$$

Inserting (4.1) into (4.2), we obtain

$$
\begin{aligned}
F(p) &= \int_{-5}^{+5} \frac{1}{2} |u_8^2(\omega, t)|\, d\omega \\
&= 2\,\frac{(X_1(p))^{3/2}{\tau_1}^2\Pi}{X_2(p) - X_3(p)} \left(-2\,\frac{Y_{31}(p)}{\sqrt{Y_1(p)}} - 2\,\frac{X_3(p)}{X_2(p) + X_3(p)} \left(\frac{Y_{21}(p)}{(Y_{21}(p))^2 (X_2(p) - X_3(p)) - X_2(p) - X_3(p)} - \frac{Y_{31}(p)}{\sqrt{Y_1(p)}} \right) \right) \\
&\quad - 2\,\frac{(X_1(p))^{3/2}{\tau_1}^2\Pi}{X_2(p) - X_3(p)} \left(-2\,\frac{Y_{32}(p)}{\sqrt{Y_1(p)}} - 2\,\frac{X_3(p)}{X_2(p) + X_3(p)} \left(\frac{Y_{22}(p)}{(Y_{22}(p))^2 (X_2(p) - X_3(p)) - X_2(p) - X_3(p)} - \frac{Y_{32}(p)}{\sqrt{Y_1(p)}} \right) \right), \\
Y_{21}(p) &= \tanh\left(1/2\,\frac{\sqrt{X_1(p)}k(pt-5)}{\Pi} \right), \qquad Y_{22}(p) = \tanh\left(1/2\,\frac{\sqrt{X_1(p)}k(pt+5)}{\Pi} \right), \\
Y_{31}(p) &= \operatorname{arctanh}\left(\frac{(X_2(p) - X_3(p))\,Y_{21}(p)}{\sqrt{Y_1(p)}} \right), \qquad Y_{32}(p) = \operatorname{arctanh}\left(\frac{(X_2(p) - X_3(p))\,Y_{22}(p)}{\sqrt{Y_1(p)}} \right).
\end{aligned} \tag{4.4}
$$

Then, $\frac{\partial F(p)}{\partial p}$ will be as

$$
\begin{aligned}
\frac{\partial F(p)}{\partial p} &= 3\,\frac{\sqrt{X_1(p)}\left(\frac{d}{dp}X_1(p)\right)\Pi\,{\tau_1}^2(Z_1(p)-Z_2(p))}{A_1(p)} - 2\,\frac{(X_1(p))^{3/2}\left(\frac{d}{dp}A_1(p)\right)\Pi\,{\tau_1}^2(Z_1(p)-Z_2(p))}{(A_1(p))^2} \\
&+2\,\frac{(X_1(p))^{3/2}{\tau_1}^2\Pi}{A_1(p)} \left(-2\,\frac{\frac{d}{dp}Y_{31}(p)}{\sqrt{Y_1(p)}} + \frac{Y_{31}(p)\frac{d}{dp}Y_1(p)}{(Y_1(p))^{3/2}} - 2\,\frac{X_3(p)C_1(p)+\left(\frac{d}{dp}X_3(p)\right)Z_3(p)}{A_2(p)} + 2\,\frac{X_3(p)Z_3(p)\frac{d}{dp}A_2(p)}{(A_2(p))^2} \right) \\
&-2\,\frac{(X_1(p))^{3/2}{\tau_1}^2\Pi}{A_1(p)} \left(-2\,\frac{\frac{d}{dp}Y_{32}(p)}{\sqrt{Y_1(p)}} + \frac{Y_{32}(p)\frac{d}{dp}Y_1(p)}{(Y_1(p))^{3/2}} - 2\,\frac{X_3(p)C_2(p)+\left(\frac{d}{dp}X_3(p)\right)Z_4(p)}{A_2(p)} + 2\,\frac{X_3(p)Z_4(p)\frac{d}{dp}A_2(p)}{(A_2(p))^2} \right), \\
Z_1(p) &= -2\,\frac{Y_{31}(p)}{\sqrt{Y_1(p)}} - 2\,\frac{X_3(p)}{X_2(p)+X_3(p)} \left(\frac{Y_{21}(p)}{(Y_{21}(p))^2(X_2(p)-X_3(p))-X_2(p)-X_3(p)} - \frac{Y_{31}(p)}{\sqrt{Y_1(p)}} \right), \\
Z_2(p) &= -2\,\frac{Y_{32}(p)}{\sqrt{Y_1(p)}} - 2\,\frac{X_3(p)}{X_2(p)+X_3(p)} \left(\frac{Y_{22}(p)}{(Y_{22}(p))^2(X_2(p)-X_3(p))-X_2(p)-X_3(p)} - \frac{Y_{32}(p)}{\sqrt{Y_1(p)}} \right), \\
Z_3(p) &= \frac{Y_{21}(p)}{(Y_{21}(p))^2(X_2(p)-X_3(p))-X_2(p)-X_3(p)} - \frac{Y_{31}(p)}{\sqrt{Y_1(p)}}, \\
Z_4(p) &= \frac{Y_{22}(p)}{(Y_{22}(p))^2(X_2(p)-X_3(p))-X_2(p)-X_3(p)} - \frac{Y_{32}(p)}{\sqrt{Y_1(p)}}, \\
A_1(p) &= X_2(p) - X_3(p), \qquad A_2(p) = X_2(p) + X_3(p), \\
B_1(p) &= \frac{Y_{21}(p)\left(2\,Y_{21}(p)A_1(p)\frac{d}{dp}Y_{21}(p)+(Y_{21}(p))^2\frac{d}{dp}A_1(p)-\frac{d}{dp}A_2(p)\right)}{\left((Y_{21}(p))^2A_1(p)-A_2(p)\right)^2}, \\
B_2(p) &= \frac{Y_{22}(p)\left(2\,Y_{22}(p)A_1(p)\frac{d}{dp}Y_{22}(p)+(Y_{22}(p))^2\frac{d}{dp}A_1(p)-\frac{d}{dp}A_2(p)\right)}{\left((Y_{22}(p))^2A_1(p)-A_2(p)\right)^2}, \\
C_1(p) &= \frac{\frac{d}{dp}Y_{21}(p)}{(Y_{21}(p))^2A_1(p)-A_2(p)} - B_1(p) - \frac{\frac{d}{dp}Y_{31}(p)}{\sqrt{Y_1(p)}} + 1/2\,\frac{Y_{31}(p)\frac{d}{dp}Y_1(p)}{(Y_1(p))^{3/2}}, \\
C_2(p) &= \frac{\frac{d}{dp}Y_{22}(p)}{(Y_{22}(p))^2A_1(p)-A_2(p)} - B_2(p) - \frac{\frac{d}{dp}Y_{32}(p)}{\sqrt{Y_1(p)}} + 1/2\,\frac{Y_{32}(p)\frac{d}{dp}Y_1(p)}{(Y_1(p))^{3/2}}.
\end{aligned} \tag{4.5}
$$

According to the values $\lambda = 2, \gamma = 2, \nu = 1, \tau_0 = 2, \tau_1 = 1, \Pi = 1, k = 2$ and by considering $t = 0$, then we get

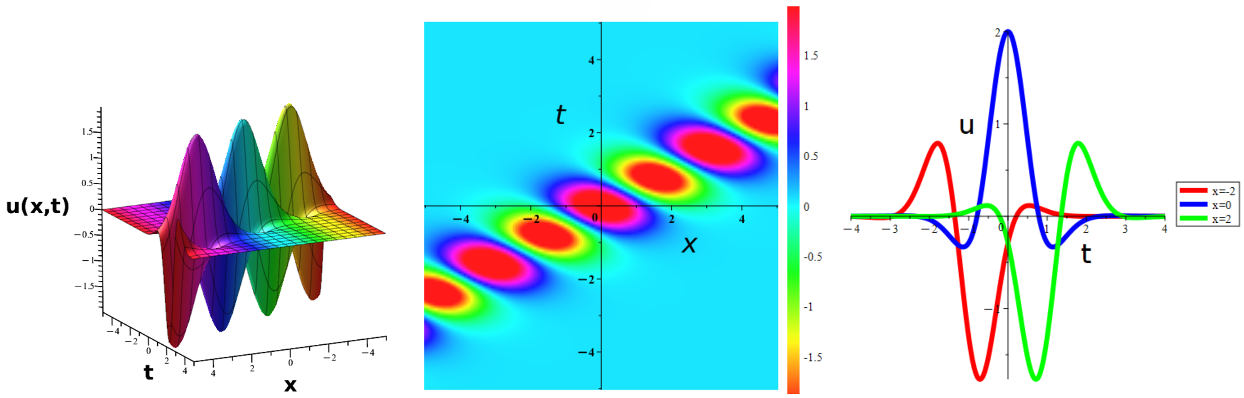
$$
\left| \frac{\partial F(p)}{\partial p} \right|_{p=5} = 0.6979299840 > 0. \tag{4.6}
$$

which denotes that $u_8(x, t)$ is stable.

5. Graphical interpretation

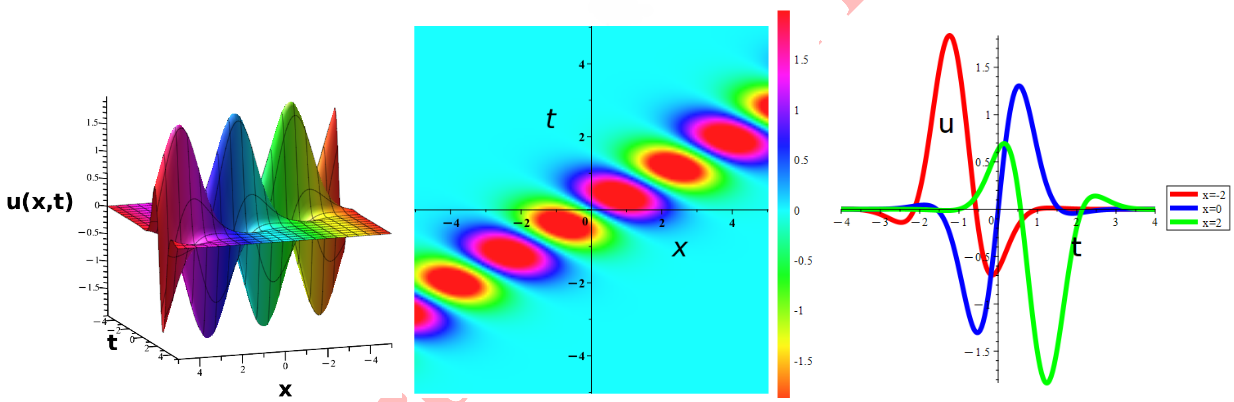
In this study, we have successfully obtained a plethora of novel solitary wave solutions, which hold significant intuitive importance for elucidating phenomena related to unstable nonlinear Schrödinger equation. These solutions encompass a variety of waveforms, including solitary wave solutions represented by the hyperbolic function exp and tanh, kink wave solutions depicted by tanh, periodic wave solutions expressed through trigonometric functions such as sin, tan, and *sec*. Additionally, there are other singular solitary wave solutions represented by sinh, exp, and rational functions. These solutions not only enrich our understanding of unstable NLSE phenomena but also provide powerful mathematical tools for describing solitary waves, periodic waves, and other complex waveforms.

The dark soliton is plotted in Figures 1 and 2 including real and imaginary solutions with the values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$. Also, periodic wave solution is plotted in Figures 3 and 4 including real and imaginary solutions



(a) (b) (c)

FIGURE 3. Real solution of periodic wave solution (2.41) u_{12} with values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$.



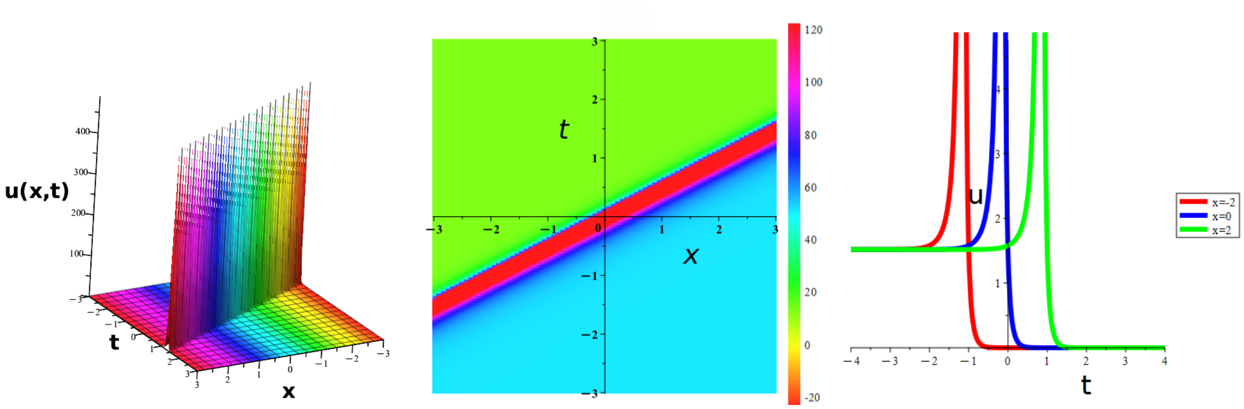
(a) (b) (c)

FIGURE 4. Imaginary solution of periodic wave solution (2.41) u_{12} with values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$.

with the values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$.
To provide a more intuitive illustration of the physical significance of the solutions obtained, we selected two representative solutions for graphical interpretation: u_5 and u_6, represented by Eqs. (3.25) and (3.26), respectively. u_5 is a soliton solution expressed using the singular solitary wave solutions exp, while u_6 is a kink wave solution represented by the hyperbolic function tanh. To clearly demonstrate their solitary characteristics, we present three-dimensional waveform plots, density plots, and two-dimensional waveform plots for these solutions. In Figures 5 and 6, the parameter values are set as follows: $d = 2, b = 1, E = 2, p = 1, \gamma = 1, \nu = 2, \lambda = -2$.

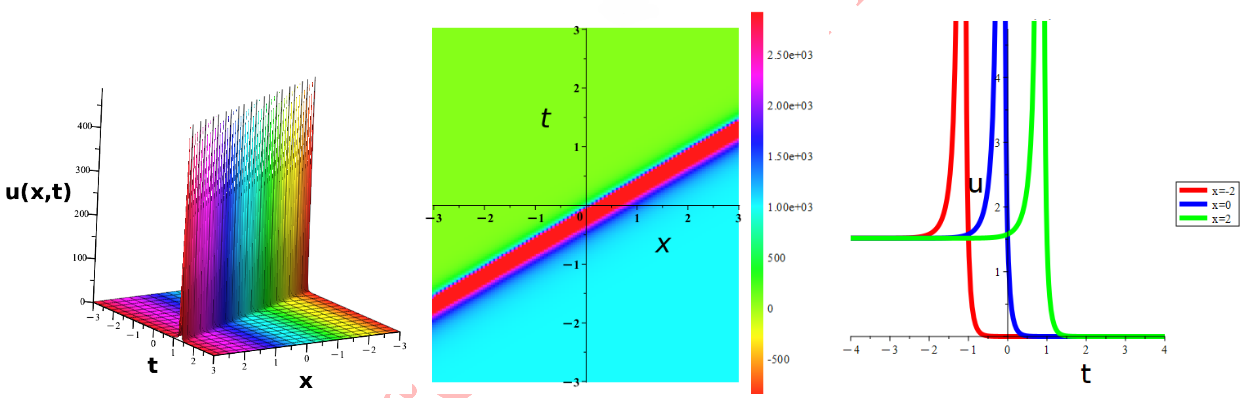
6. Result and discussions

The comprehensive analysis of unstable nonlinear Schrödinger equation has yielded a rich spectrum of the exact analytical solutions, each exhibiting distinct physical characteristics and dynamical behaviors. Our systematic investigation using the generalized extended trial equation scheme and the improved Bernoulli sub-ODE scheme have successfully generated multiple families



(a) (b) (c)

FIGURE 5. Graphs of breather wave solution (3.25) u_5 with values $d = 2, b = 1, E = 2, p = 1, \gamma = 1, \nu = 2, \lambda = -2$.



(a) (b) (c)

FIGURE 6. Graphs of breather wave solution (3.26) u_6 with values $d = 2, b = 1, E = 2, p = 1, \gamma = 1, \nu = 2, \lambda = -2$.

of solutions that capture the complex nonlinear wave phenomena inherent in this integrable system. The dark soliton solutions presented in Figures 1 and 2 including 3D, density and 2d plots demonstrate the behaviour of nonlinear wave interactions in the unstable nonlinear Schrödinger equation when Equation (2.40) u_{11} with values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$ are considered to more analysis. The periodic wave solutions presented in Figures 3 and 4 including 3D, density and 2d plots demonstrate the behaviour of periodic wave interactions in the unstable nonlinear Schrödinger equation when equation 2.41) u_{12} with values $\alpha_1 = 2, \alpha_2 = 1, \alpha_4 = 3, E = 2, p = 1, k = 2, \nu = 2$ are given to more analysis. The breather wave solutions presented in Figure 5 including 3D, density and 2d plots demonstrate the behaviour of breather wave interactions in the unstable nonlinear Schrödinger equation when Equation (3.25) u_5 with values $d = 2, b = 1, E = 2, p = 1, \gamma = 1, \nu = 2, \lambda = -2$ are given. In addition, the breather wave solutions offered in Figure (6) including 3D, density and 2d plots demonstrate the behaviour of breather wave interactions in the unstable nonlinear Schrödinger equation when equation (3.26) u_6 with values $d = 2, b = 1, E = 2, p = 1, \gamma = 1, \nu = 2, \lambda = -2$ are given. These comprehensive results demonstrate the remarkable richness and complexity of nonlinear wave dynamics captured by the unstable nonlinear Schrödinger equation. The various solution families reveal different aspects of wave behavior in dispersive nonlinear media, from the robust stability of soliton structures to the complex interaction dynamics of waves and the periodic modulation characteristics of breather solutions. The interaction scenarios showcase the fundamental principles governing energy and momentum exchange in nonlinear wave systems, while the fusion-fission phenomena reveal the existence of special resonance conditions that lead to dramatic modifications in wave behavior. These findings contribute significantly to our understanding of integrable nonlinear systems and provide valuable insights for applications in fluid dynamics, plasma physics, and nonlinear optics where such wave phenomena play crucial roles in energy transport and system dynamics.

7. Conclusion

We checked the entire set of solutions for all cases with Maple, and found that all the obtained solutions satisfied the original equation (1.2). In this paper, we have considered the unstable NLSE. We used the ETEM and IBSOM to solve the unstable NLSE. We have found the general solutions of the mentioned equation. As solutions, we have obtained the solitary wave solution, the periodic wave solution, dark soliton solution, singular soliton solution, a kink soliton solution and a rational soliton solution. These techniques used here can be employed to other nonlinear partial differential equations. By using the Hamiltonian technique, the stability of the obtained solution was examined regarding the sufficient condition of stability, this study make sure that the obtained solutions were stable. As far as we know, these outcomes were achieved for this model for the first time. Future research directions will focus on further exploring the application of fractional derivatives in other nonlinear partial differential equations and investigating the properties of solutions in various physical contexts. Additionally, we aim to develop more efficient numerical methods to enhance the accuracy and efficiency of solving the complex nonlinear systems. Through these endeavors, we hope to contribute more to the development of nonlinear science and promote practical applications in related fields. Additionally, 2D and 3D graphs of some of the constructed solutions are presented for certain values of the free parameters. Compared to many other methods, these solutions demonstrate and highlight the proposed method's simplicity, reliability, and effectiveness.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. Aghazadeh, Y. Mahmoudi, and F. D. Saei, *Legendre approximation method for computing eigenvalues of fourth order fractional Sturm?Liouville problem*, Math. Comput. Simul. *206* (2023), 286-301.

[2] A. Aghazadeh, Y. Mahmoudi, and F. D. Saei, *Numerical method for solving fractional Sturm-Liouville eigenvalue problems of order two using Genocchi polynomials*, Iranian J. Num. Anal. Opt., *13* (2023), 121-140.

[3] A. Aghazadeh and M. Lakestani, *Application of cubic B-splines for second order fractional Sturm-Liouville problems*, Math. Comput. Simul., *238* (2025), 479-496.

[4] A. Aghazadeh and M. Lakestani, *Constructing optical soliton wave structure and modulation instability analysis for coupled fractional Lakshmanan-Porsezian-Daniel equation with Kerr's law nonlinearity*, Sci. Rep., *15* (2025), 45294.

[5] M. F. Aghdaei and J. Manafian, *Optical soliton wave solutions to the resonant Davey-Stewartson system*, Opt. Quant. Elec., *48* (2016), 413.

[6] S. Z. Ahmedov, R. A. Gasimov, and V. M. Habibov, *Solution of the mixed problem for the non-homogeneous parabolic equation with time derivative in the boundary condition*, Adv. Math. Models Appl., *10*(1), (2025) 184-193.

[7] M. H. Ali, H. M. Ahmed, A. Abd-Elmonem, N.A. A. Suoliman, K.K. Ahmed, and I. Samir, *Optical solitons for generalized perturbed nonlinear Schrödinger model in the presence of dual-power law nonlinear medium*, Optik 319 (2024), 172112.

[8] N. H. Ali, S.A. Mohammed, and J. Manafian, *New explicit soliton and other solutions of the Van der Waals model through the EShGEEM and the IEEM*, J. Mod. Technol. Eng., *8*(1) (2023), 5-18.

[9] I. Alraddadi, F. Alsharif, S. Malik, H. Ahmad, T. Radwan, and K. K. Ahmed, *Innovative soliton solutions for a (2+1)-dimensional generalized KdV equation using two effective approaches*, AIMS Math., *9* (12) (2024), 34966-34980.

[10] M. Arshad, A. R. Seadawy, D. Lu, and W. Jun, *Optical soliton solutions of unstable nonlinear Schrödinger dynamical equation and stability analysis with applications*, Optik, *157* (2018), 597-605.

[11] K. N. Asgarov, Y. N. Imamverdiyev, and M. M. Abutalibov, *Unsupervised machine learning methods for real-time anomaly detection in endpoints*, J. Modern Tech. Eng., *9*(3) (2024), 141-155.

[12] H .M. Baskonus and H. Bulut, *Exponential prototype structures for (2+1)-dimensional Boiti-Leon-Pempinelli systems in mathematical physics*, Waves Random Complex Media, *26* (2016), 201-208.

[13] L. A. Dawod, M. Lakestani, and J. Manafian, *Breather wave solutions for the (3+1)-D generalized shallow water wave equation with variable coefficients*, Qual. Theo. Dyn. Sys., 22 (2023), 127.

[14] M. Dehghan, J. Manafian, and A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Num. Meth. Partial Diff. Eq. J., *26* (2010), 486-498.

[15] M. Dehghan, J. Manafian, and A. Saadatmandi, *The solution of the linear fractional partial differential equations using the homotopy analysis method*, Z. Naturforsch, *65a* (2010), 935-949.

[16] M. Ekici, Q. Zhou, A. Sonmezoglu, J. Manafian, and M. Mirzazadeh, *The analytical study of solitons to the nonlinear Schrödinger equation with resonant nonlinearity*, Optik, *130* (2017), 378-382.

[17] A. R. Haghighi, A. Aghazedeh, and A. Abedini, *Comparison of integro quadratic and quartic spline interpolation*, TWMS J. Appl. Eng. Math. *10* (2020), 150-160.

[18] M. Inc, A.I. Aliyu, A. Yusuf, and D. Baleanu, *Optical solitons, nonlinear self-adjointness and conservation laws for the cubic nonlinear Schrödinger equation with repulsive delta potential*, Superlattices Microstruct., *111* (2017), 546-555.

[19] M. Inc, A. I. Aliyu, A. Yusuf, and D. Baleanu, *Gray optical soliton, linear stability analysis and conservation Laws via multipliers to the cubic nonlinear Schrödinger equation*, Optik, *164* (2018), 472-478.

[20] H. G. Kalt and M. K. Abd Al-Rahem, *Comparison of four methods to estimate the scale parameter for Extended Poisson Exponential distribution*, IOP Conference Series: Materials Sci. Eng., *571*(1) (2016), 012029.

[21] A. S. Khalifa, H. M. Ahmed, and K. K. Ahmed, *Construction of exact solutions for a higher-order stochastic modified Gerdjikov-Ivanov model using the IMETF method*, Phys. Scr., (2024)

[22] C. M. Khalique and A.Biswas, *A Lie symmetry approach to nonlinear Schrödinger's equation with non-Kerr law nonlinearity*, Commun. Nonlinear Sci. Num. Simul. *14*(12) (2009), 4033-4040.

[23] B. Koçak and M. Türk, *The Effect of Toxicode and Google Blockly Applications on 5th Grade Students' Algorithm Development and Computational Thinking Skills*, J. Modern Tech. Eng., *10*(1) (2025), 26-41.

[24] M. Lakestani, J. Manafian, A.R. Najafizadeh, and M. Partohaghighi, *Some new soliton solutions for the nonlinear the fifth-order integrable equations*, Comput. Meth. Diff. Eq., *10*(2) (2022), 445-460.

[25] X. Li, J. Manafian, O. A. Ilhan, A. Aghazadeh, B. Eslami, A. A. Fattah, K. H. Mahmoud, and Z. H. Ward, *Dynamical Behavior of the Soliton Solutions of the Sixth-order Benney-Luke Equation Using Hirota Bilinear Method Arising in Physical Sciences*, Qualit. Theo. Dyn. Sys. *25* (2026), 34.

[26] D. Lu, A. R. Seadawy, and M. Arshad, *Bright-dark solitary wave and elliptic function solutions of unstable nonlinear Schrödinger equation and their applications*, Opt. Quant. Elec. *50* (2018), 23.

[27] R. Mammadzada, *Various scheduling techniques used in human-robot collaborative system strategies for allocation and optimization of tasks*, J. Modern Tech. Eng., *8*(3) (2023), 244-257.

[28] J. Manafian, *On the complex structures of the Biswas-Milovic equation for power, parabolic and dual parabolic law nonlinearities*, Eur. Phys. J. Plus, *130* (2015), 255.

[29] J. Manafian, M. F. Aghdaei, and M. Zadahmad, *Analytic study of sixth-order thin-film equation by* $tan(\phi/2)$*-expansion method*, Opt. Quant. Elec., *48* (2016), 410.

[30] J. Manafian, *Optical soliton solutions for Schrödinger type nonlinear evolution equations by the* $tan(\phi/2)$*-expansion method*, Optik, *127* (2016), 4222-4245.

[31] J. Manafian, *Application of the ITEM for the system of equations for the ion sound and Langmuir waves*, Opt. Quant. Elec., *49* (2017), 17.

[32] J. Manafian and P. Bolgar, *Numerical idea to solve three-dimensional nonlinear Volterra integral equations with 3D-Legendre polynomials*, Comput. Meth. Diff. Eq., *14*(1) (2026), 361-373.

[33] J. Manafian, O.A. Ilhan, L. Avazpour, and A. Alizadeh, *Localized waves and interaction solutions to the fractional generalized CBS-BK equation arising in fluid mechanics*, Adv. Dif. Eq. *141* (2021), 2021.

[34] J. Manafian and M. Lakestani, *Dispersive dark optical soliton with Tzitzéica type nonlinear evolution equations arising in nonlinear optics*, Opt. Quant. Elec., *48* (2016), 116.

[35] J. Manafian and M. Lakestani, *Abundant soliton solutions for the Kundu-Eckhaus equation via* $tan(\phi/2)$*-expansion method*, Optik, *127* (2016), 5543-5551.

[36] J. Manafian and M. Lakestani, *Optical soliton solutions for the Gerdjikov-Ivanov model via* $tan(\phi/2)$*-expansion method*, Optik, *127* (2016), 9603-9620.

[37] J. Manafian, M. Lakestani, and A. Bekir, *Application of a new analytical method for the Richards' equation, based on the Brooks and Corey model*, J. Porous Media, *9*(11) (2016), 975-991.

[38] J. Manafian and M. Lakestani, *A new analytical approach to solve some the fractional-order partial differential equations*, Indian J. Phys., *91* (2017), 243-258.

[39] J. Manafian and M. Lakestani, *Solitary wave and periodic wave solutions for Burgers, Fisher, Huxley and combined forms of these equations by the G'/G-expansion method*, Pramana-J. Phys., *130* (2015), 31-52.
[40] M. Mirzazadeh and M. Eslami, *Exact multisoliton solutions of nonlinear Klein-Gordon equation in $1+2$ dimensions*, Eur. Phys. J. Plus, *128* (2015), 1-9.
[41] S. R. Moosavi, N. Taghizadeh, and J. Manafian, *Analytical approximations of one-dimensional hyperbolic equation with non-local integral conditions by reduced differential transform method*, Comput. Meth. Diff. Eq., *8*(3) (2020), 537-552.
[42] A.K. Mutashar, A. Hatem, and M. A. Hussein, *A new analytical and numerical solutions for biological population models by kamal adomian decomposition method*, Adv. Math. Models Appl., *10*(1) (2025), 194-207.
[43] S. D. Oloniiju, *New sets of soliton solutions for the generalized Whitham-Broer-Kaup-Boussinesq-Kupershmidt system*, Results Phys. *52* (2023), 106785.
[44] M. Pawlik and G. Rowlands, *The propagation of solitary waves in piezoelectric semiconductors*, J. Phys. C, *8* (1975), 1189-1204.
[45] Y. Qian, J. Manafian, M. Asiri, K. H. Mahmoud, A. I. Alanssari, and A. S. Alsubaie, *Nonparaxial solitons and the dynamics of solitary waves for the coupled nonlinear Helmholtz systems*, Opt. Quant. Elec., *55* (2023), 1022.
[46] M. N. Rafiq, M. H. Rafiq, and H. Alsaud, *Chaotic response, multistability and new wave structures for the generalized coupled Whitham-Broer-Kaup-Boussinesq-Kupershmidt system with a novel methodology*, Chaos Solitons Frac. *190* (2025), 115755.
[47] M. A. Rahem and M. Darrah, *A geometric approach for computing a measure of consensus for groups*, Int. Math. Forum *11*(20) (2016), 961-973.
[48] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys., *43* (2) (1992), 270-291.
[49] X. Rao, et al. *The nonlinear vibration and dispersive wave systems with extended homoclinic breather wave solutions*, Open Phys., *20* (2022), 795-821.
[50] D. Lu, A. Seadawy, and M. Arshad, *Applications of extended simple equation method on unstable nonlinear Schrödinger equations*, Optik, *140* (2017), 136-144.
[51] C. T. Sindi and J. Manafian, *Wave solutions for variants of the KdV-Burger and the K(n, n)-Burger equations by the generalized G'/G-expansion method*, Math. Meth. Appl. Sci., *87* (2016), 1-14.
[52] L. Sun, J. Manafian, O. A. Ilhan, M. Abotaleb, A. Y. Oudah, and A. S. Prakaash, *Theoretical analysis for miscellaneous soliton waves in metamaterials model by modification of analytical solutions*, Opt. Quant. Elec., *54* (2022), 651.
[53] E. Tala-Tebue, Z. I. Djoufack, E. Fendzi-Donfack, A. Kenfack-Jiotsa, and T.C. Kofan, *Exact solutions of the unstable nonlinear Schrödinger equation with the new Jacobi elliptic function rational expansion method and the exponential rational function method*, Optik, *127* (2016), 11124-11130.
[54] D. Talati and A. M. Wazwaz, *Some new integrable systems of two-component fifth-order equations*, Nonlinear Dyn., *87* (2017), 1111-1120.
[55] F. Tchier, A. Yusuf, A. I. Aliyu, and M. Inc, *Soliton solutions and conservation laws for lossy nonlinear transmission line equation*, Superlattices Microstructures, *107* (2017), 320-336.
[56] H. Triki, T. Hayat, O. M. Aldossary, and A. Biswas, *Bright and dark solitons for the resonant nonlinear Schrödinger's equation with time-dependent coefficients*, Opt. Laser Tech., *44* (2012), 2223-2231.
[57] M.M. Wadati, H. Segur, and M. J. Ablowitz, *A new Hamiltonian amplitude equation governing modulated wave instabilities,* J. Phys. Soc. Jpn., *61* (1992), 1187-1193.
[58] W. Wang, K. Hasanirokh, J. Manafian, M. Abotaleb, and Y. Yang, *Analytical approach for polar magnetooptics in multilayer spin-polarized light emitting diodes based on InAs quantum dots*, Opt. Quant. Elec., *54* (2022), 137.
[59] Y. Xu, et al. *Conservation Law, Stability Analysis, Degenerate Lump and Traveling Wave Solutions for (2+1)-Dimensional KP-BBM Equation*, Qual. Theory Dyn. Syst., *24* (2025), 246.
[60] M. Zhang, X. Xie, J. Manafian, O. A. Ilhan, and G. Singh, *Characteristics of the new multiple rogue wave solutions to the fractional generalized CBS-BK equation*, J. Adv. Research *38* (2022), 131-142.
[61] Q. Zhou, *Optical solitons in medium with parabolic law nonlinearity and higher order dispersion*, Waves Random Complex Media, 25 (2016), 52-59.
[62] Q. Zhou, M. Ekici, A. Sonmezoglu, J. Manafian, S. Khaleghizadeh, and M. Mirzazadeh, *Exact solitary wave solutions to the generalized Fisher equation*, Optik, *127* (2016), 12085-12092.
[63] Q. Zou, et al. *Exact breather waves solutions in a spatial symmetric nonlinear dispersive wave model in (2+1)-dimensions*, Sci. Rep., *14* (2024), 31718.