Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-21
DOI:10.22034/cmde.2026.70778.3535

Derivatives of Humbert confluent hypergeometric functions with respect to their parameters

Recep Şahin[1], **Ayman Shehata**[2], **Shimaa I. Moustafa**[2] **and Oğuz Yağcı**[1,*]

[1]Department of Mathematics, Faculty of Arts and Sciences, Kırıkkale University, Kırıkkale 71450, Turkey.

[2]Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt.

Abstract

Humbert confluent hypergeometric functions of two variables arise in many problems of mathematical physics and applied analysis; however, their behaviour with respect to parameters has not been studied systematically. In this paper, we investigate derivatives with respect to numerator and denominator parameters for the seven classical Humbert functions Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1 and Ξ_2. Using their double-series representations, together with elementary properties of the Gamma and digamma functions, we derive explicit formulas for first-order parameter derivatives and express them compactly in terms of Srivastava's triple hypergeometric function $F^{(3)}$. By differentiating the underlying partial differential equations, we further obtain simple operator recurrences for derivatives of arbitrary order; these recurrences yield closed differentiation and reduction formulas in terms of contiguous Humbert functions. Finally, we show how these results lead to Taylor-type parameter expansions and illustrate their use with basic numerical examples and plots.

Keywords. Generalized hypergeometric functions, Srivastava's hypergeometric function, Humbert confluent hypergeometric functions.

2000 Mathematics Subject Classification. Primary 33C15, 33C20; Secondary 33C65, 33D70, 05A30.

1. Introduction

Hypergeometric functions and their multivariable analogues play a central role in the theory of special functions, as well as in many areas of mathematical physics, engineering and applied analysis. Starting from the classical Gauss and Kummer functions of one variable, various generalizations have been introduced, including the Appell and Lauricella families of two or more variables (see, for example, [8, 22]). Among their confluent limits, the seven functions introduced by P. Humbert form a distinguished class of two-variable confluent hypergeometric functions, now customarily denoted by Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1 and Ξ_2. They admit simple double-series representations in terms of Pochhammer symbols and Gamma functions and satisfy systems of linear partial differential equations with polynomial coefficients.

In many applications, the parameters of these functions carry direct physical, geometric, or probabilistic meaning. Consequently, one is interested not only in the functions themselves but also in their variation with respect to the parameters. Derivatives with respect to numerator or denominator parameters arise naturally in sensitivity analysis, perturbation methods, analytic continuation and the derivation of asymptotic expansions. For single-variable hypergeometric functions such as ${}_2F_1$, ${}_1F_1$ and more general ${}_pF_q$ series, parameter derivatives have been studied extensively and can be expressed in terms of polygamma functions and shifted hypergeometric functions (see, e.g., [4–6] and the references therein). Parameter derivatives of several two-variable Horn and related hypergeometric functions have also been investigated in recent work [2, 7]. By contrast, a systematic treatment of parameter derivatives for the seven classical Humbert confluent hypergeometric functions of two variables has been lacking.

Recent studies show that Humbert-type functions continue to attract attention in several directions, including (p, q)- and basic Humbert functions, bibasic Humbert series, k-Humbert extensions, and large-argument asymptotics

Received: 17 December 2025; Accepted: 07 July 2026.
∗ Corresponding author. Email: oguzyagci26@gmail.com.

for Ψ_1 and Ψ_2; see, for example, [3, 9, 12, 13, 18–20]. These developments motivate a unified parameter-differentiation framework for the classical Humbert family.

The aim of this work is to provide such a systematic treatment. Starting from the double-series definitions of the seven Humbert functions and using elementary properties of the Gamma, digamma and polygamma functions, together with standard identities for the Pochhammer symbol, we first derive explicit formulas for the derivatives with respect to each numerator and denominator parameter. We then recast these first-order parameter derivatives in a compact, unified form in terms of Srivastava's triple hypergeometric function $F^{(3)}$, which plays a natural role as a building block for multivariable parameter differentiation.

A second ingredient of our approach is the use of the systems of partial differential equations satisfied by the Humbert functions. By differentiating these PDEs with respect to the parameters, we obtain simple operator recurrence relations that express nth-order parameter derivatives in terms of lower-order ones. Combining these recurrences with shift identities for Pochhammer symbols leads to closed differentiation and reduction formulas relating parameter derivatives to contiguous Humbert functions and to higher-order derivatives with respect to the variables.

Finally, to illustrate the applicability of the theoretical results, we present a short numerical study for selected Humbert functions. In particular, we compute sample values and two- and three-dimensional plots for Φ_1 and its derivative with respect to a numerator parameter, using the series representations implied by our formulas. These numerical illustrations confirm that the parameter-derivative calculus developed in this paper can be implemented efficiently and can provide a practical tool for applications.

The paper is organized as follows. In Section 2, we recall the basic notation and properties of the Gamma and polygamma functions, the Pochhammer symbol, the Humbert confluent hypergeometric functions, and Srivastava's triple hypergeometric function $F^{(3)}$. The same section also records the explicit first-order parameter derivative formulas, including the complete list of $F^{(3)}$-forms for the first-order parameter derivatives. In Section 3, we derive general recurrence relations for nth-order derivatives with respect to the parameters by differentiating the underlying systems of partial differential equations. Section 4 contains explicit differentiation and reduction formulas in terms of contiguous Humbert functions, while Section 5 discusses representative applications. Numerical examples, error estimates, and graphical illustrations are presented in Section 6. Concluding remarks and some perspectives for further work are given in Section 7.

2. Preliminaries

In this section, we collect the basic notation and auxiliary results that will be used throughout the paper. Unless otherwise stated, all parameters are complex and are chosen to avoid poles of the Gamma function; the variables x and y are complex numbers lying in the domains of convergence explicitly indicated below.

2.1. **Gamma function, Pochhammer symbol and polygamma functions.** We recall that the Euler Gamma function $\Gamma(z)$ is defined for $\Re(z) > 0$ by (see, e.g., [1, 10, 16])

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

and is extended to a meromorphic function on $\mathbb{C}$ with simple poles at the non-positive integers. The rising factorial, or Pochhammer symbol, $(a)_n$ is given by (see, e.g., [15, 17])

$$(a)_0 := 1, \qquad (a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n \in \mathbb{N}.$$

The logarithmic derivative of the Gamma function is the digamma function (see, e.g., [1, 16])

$$\Psi(z) := \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{2.1}$$

and its higher derivatives

$$\Psi_r(z) := \frac{d^r}{dz^r}\Psi(z), \qquad r \in \mathbb{N},$$

are called the polygamma functions. Using the well-known representation of Ψ as a convergent series, one obtains the identity

$$\Psi(z+n)-\Psi(z)=\sum_{k=0}^{n-1}\frac{1}{z+k},\qquad n\in\mathbb{N}, \tag{2.2}$$

and, more generally,

$$\Psi_r(z+n)-\Psi_r(z)=(-1)^r r!\sum_{k=0}^{n-1}\frac{1}{(z+k)^{r+1}},\qquad r\in\mathbb{N}_0,\ n\in\mathbb{N}. \tag{2.3}$$

From (2.1)–(2.2) and the representation of $(z)_n$ in terms of Gamma functions, we readily obtain the derivative of the Pochhammer symbol with respect to its parameter:

$$\frac{d}{dz}(z)_n=(z)_n\big[\Psi(z+n)-\Psi(z)\big]=(z)_n\sum_{k=0}^{n-1}\frac{1}{z+k},\qquad n\in\mathbb{N}. \tag{2.4}$$

Similarly, differentiation of the reciprocal of a Pochhammer symbol yields

$$\frac{d}{dz}\frac{1}{(z)_n}=-\frac{1}{(z)_n}\sum_{k=0}^{n-1}\frac{1}{z+k},\qquad n\in\mathbb{N}, \tag{2.5}$$

which will be used below for derivatives with respect to denominator parameters.

We shall also use the following simple rearrangement formula for double series:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A\big(k,n-k\big), \tag{2.6}$$

valid whenever both sides converge absolutely. This identity allows us to convert sums over independent indices into sums over triangular regions, a step that will be convenient when expressing derivatives in terms of triple hypergeometric series.

2.2. **Humbert confluent hypergeometric functions of two variables.** We now recall the seven Humbert confluent hypergeometric functions of two variables, which are confluent forms of the classical Appell functions (see, e.g., [11, 14, 22]). They are defined in terms of double power series as follows:

$$\Phi_1(a,b;c;x,y):=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_{m+n}\,m!\,n!}\,x^m y^n,\qquad |x|<1,\ |y|<\infty, \tag{2.7}$$

$$\Phi_2(a,b;c;x,y):=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}\,m!\,n!}\,x^m y^n,\qquad |x|<\infty,\ |y|<\infty, \tag{2.8}$$

$$\Phi_3(a;b;x,y):=\sum_{m,n=0}^{\infty}\frac{(a)_m}{(b)_{m+n}\,m!\,n!}\,x^m y^n,\qquad |x|<\infty,\ |y|<\infty. \tag{2.9}$$

The two Humbert functions of Ψ-type are given by

$$\Psi_1(a,b;c,d;x,y):=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_m(d)_n\,m!\,n!}\,x^m y^n,\qquad |x|<1,\ |y|<\infty, \tag{2.10}$$

$$\Psi_2(a;b,c;x,y):=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(b)_m(c)_n\,m!\,n!}\,x^m y^n,\qquad |x|<\infty,\ |y|<\infty, \tag{2.11}$$

and the two Humbert functions of Ξ-type are defined by

$$\Xi_1(a,b,c;d;x,y) := \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n (c)_m}{(d)_{m+n}\, m!\, n!}\, x^m y^n, \qquad |x|<1,\ |y|<\infty, \tag{2.12}$$

$$\Xi_2(a,b;c;x,y) := \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_m}{(c)_{m+n}\, m!\, n!}\, x^m y^n, \qquad |x|<1,\ |y|<\infty. \tag{2.13}$$

Each of these double series defines an analytic function in the indicated domain of convergence and admits analytic continuation in the parameters a, b, c, d, provided that poles of the Gamma function are avoided. The functions (2.7)–(2.13) satisfy systems of linear partial differential equations with polynomial coefficients; these systems will later be used to derive recurrence relations for higher-order derivatives with respect to the parameters.

2.3. **Srivastava's triple hypergeometric function.** A central role in our analysis is played by Srivastava's triple hypergeometric function, which provides a natural framework for the expressions arising from parameter differentiation of the Humbert functions [21, 22]. Following the notation used in the literature, we write

$$F^{(3)}\left[\begin{array}{c}(a) :: (b); (b'); (b'') : (c); (c'); (c'') \\ (e) :: (g); (g'); (g'') : (h); (h'); (h'')\end{array}\middle|\, x, y, z\right] := \sum_{m,n,p=0}^{\infty} \frac{A(m,n,p)}{m!\, n!\, p!}\, x^m y^n z^p, \tag{2.14}$$

where the coefficient $A(m,n,p)$ is given by

$$A(m,n,p) = \frac{\prod_{i=1}^{A}(a_i)_{m+n+p}\prod_{i=1}^{B}(b_i)_{m+n}\prod_{i=1}^{B'}(b'_i)_{n+p}\prod_{i=1}^{B''}(b''_i)_{m+p}\prod_{i=1}^{C}(c_i)_m\prod_{i=1}^{C'}(c'_i)_n\prod_{i=1}^{C''}(c''_i)_p}{\prod_{i=1}^{E}(e_i)_{m+n+p}\prod_{i=1}^{G}(g_i)_{m+n}\prod_{i=1}^{G'}(g'_i)_{n+p}\prod_{i=1}^{G''}(g''_i)_{m+p}\prod_{i=1}^{H}(h_i)_m\prod_{i=1}^{H'}(h'_i)_n\prod_{i=1}^{H''}(h''_i)_p}. \tag{2.15}$$

Here (a) denotes the collection of parameters $a_1, \ldots, a_A$, and similarly for the other grouped parameters (b), (b'), (b''), (c), (c'), (c''), (e), (g), (g'), (g''), (h), (h') and (h''). The precise convergence conditions for the triple series (2.14) can be found in the standard references on multiple hypergeometric functions and will not be repeated here. In all subsequent applications, (x, y, z) will be chosen so that the corresponding series is absolutely convergent.

In what follows, we shall use several specializations of (2.14) in which many of the parameter groups are empty or contain only a single parameter. In particular, the first-order parameter derivatives of the Humbert functions in subsection 2.4 will be expressed in terms of $F^{(3)}$ with carefully chosen parameter arrays, whereas higher-order derivatives will be related to iterated differential operators acting on such triple hypergeometric series.

2.4. **Explicit first-order parameter derivatives and $F^{(3)}$ forms.** We now record the first-order parameter derivatives stated in the abstract. These formulas follow directly from (2.4) and (2.5); they are included explicitly so that the later recurrence relations are anchored in concrete series identities. Define

$$\mathcal{H}_\alpha(N) := \Psi(\alpha+N) - \Psi(\alpha) = \sum_{j=0}^{N-1} \frac{1}{\alpha+j}, \qquad \mathcal{H}_\alpha(0) := 0. \tag{2.16}$$

For compactness, let the coefficient of $x^m y^n$ in each Humbert series be denoted by

$$A^{\Phi_1}_{m,n} := \frac{(a)_{m+n}(b)_m}{(c)_{m+n} m! n!} x^m y^n, \qquad A^{\Phi_2}_{m,n} := \frac{(a)_m (b)_n}{(c)_{m+n} m! n!} x^m y^n, \tag{2.17}$$

$$A^{\Phi_3}_{m,n} := \frac{(a)_m}{(b)_{m+n} m! n!} x^m y^n, \qquad A^{\Psi_1}_{m,n} := \frac{(a)_{m+n}(b)_m}{(c)_m (d)_n m! n!} x^m y^n, \tag{2.18}$$

$$A^{\Psi_2}_{m,n} := \frac{(a)_{m+n}}{(b)_m (c)_n m! n!} x^m y^n, \qquad A^{\Xi_1}_{m,n} := \frac{(a)_m (b)_n (c)_m}{(d)_{m+n} m! n!} x^m y^n, \tag{2.19}$$

$$A^{\Xi_2}_{m,n} := \frac{(a)_m (b)_m}{(c)_{m+n} m! n!} x^m y^n. \tag{2.20}$$

Within the domains of absolute convergence, termwise differentiation gives

$$\frac{\partial \Phi_1}{\partial a} = \sum_{m,n\geq 0} A^{\Phi_1}_{m,n} \mathcal{H}_a(m+n), \quad \frac{\partial \Phi_1}{\partial b} = \sum_{m,n\geq 0} A^{\Phi_1}_{m,n} \mathcal{H}_b(m), \quad \frac{\partial \Phi_1}{\partial c} = -\sum_{m,n\geq 0} A^{\Phi_1}_{m,n} \mathcal{H}_c(m+n), \tag{2.21}$$

$$\frac{\partial \Phi_2}{\partial a} = \sum_{m,n\geq 0} A^{\Phi_2}_{m,n} \mathcal{H}_a(m), \quad \frac{\partial \Phi_2}{\partial b} = \sum_{m,n\geq 0} A^{\Phi_2}_{m,n} \mathcal{H}_b(n), \quad \frac{\partial \Phi_2}{\partial c} = -\sum_{m,n\geq 0} A^{\Phi_2}_{m,n} \mathcal{H}_c(m+n), \tag{2.22}$$

$$\frac{\partial \Phi_3}{\partial a} = \sum_{m,n\geq 0} A^{\Phi_3}_{m,n} \mathcal{H}_a(m), \quad \frac{\partial \Phi_3}{\partial b} = -\sum_{m,n\geq 0} A^{\Phi_3}_{m,n} \mathcal{H}_b(m+n), \tag{2.23}$$

and

$$\frac{\partial \Psi_1}{\partial a} = \sum_{m,n\geq 0} A^{\Psi_1}_{m,n} \mathcal{H}_a(m+n), \quad \frac{\partial \Psi_1}{\partial b} = \sum_{m,n\geq 0} A^{\Psi_1}_{m,n} \mathcal{H}_b(m), \tag{2.24}$$

$$\frac{\partial \Psi_1}{\partial c} = -\sum_{m,n\geq 0} A^{\Psi_1}_{m,n} \mathcal{H}_c(m), \quad \frac{\partial \Psi_1}{\partial d} = -\sum_{m,n\geq 0} A^{\Psi_1}_{m,n} \mathcal{H}_d(n), \tag{2.25}$$

$$\frac{\partial \Psi_2}{\partial a} = \sum_{m,n\geq 0} A^{\Psi_2}_{m,n} \mathcal{H}_a(m+n), \quad \frac{\partial \Psi_2}{\partial b} = -\sum_{m,n\geq 0} A^{\Psi_2}_{m,n} \mathcal{H}_b(m), \quad \frac{\partial \Psi_2}{\partial c} = -\sum_{m,n\geq 0} A^{\Psi_2}_{m,n} \mathcal{H}_c(n), \tag{2.26}$$

$$\frac{\partial \Xi_1}{\partial a} = \sum_{m,n\geq 0} A^{\Xi_1}_{m,n} \mathcal{H}_a(m), \quad \frac{\partial \Xi_1}{\partial b} = \sum_{m,n\geq 0} A^{\Xi_1}_{m,n} \mathcal{H}_b(n), \tag{2.27}$$

$$\frac{\partial \Xi_1}{\partial c} = \sum_{m,n\geq 0} A^{\Xi_1}_{m,n} \mathcal{H}_c(m), \quad \frac{\partial \Xi_1}{\partial d} = -\sum_{m,n\geq 0} A^{\Xi_1}_{m,n} \mathcal{H}_d(m+n), \tag{2.28}$$

$$\frac{\partial \Xi_2}{\partial a} = \sum_{m,n\geq 0} A^{\Xi_2}_{m,n} \mathcal{H}_a(m), \quad \frac{\partial \Xi_2}{\partial b} = \sum_{m,n\geq 0} A^{\Xi_2}_{m,n} \mathcal{H}_b(m), \quad \frac{\partial \Xi_2}{\partial c} = -\sum_{m,n\geq 0} A^{\Xi_2}_{m,n} \mathcal{H}_c(m+n). \tag{2.29}$$

The positive signs correspond to numerator parameters, while the negative signs correspond to denominator parameters. Equations (2.21)–(2.29) are often the most convenient forms for numerical computation, because they only require the same double summation as the original Humbert function with an additional digamma-difference factor.

The same identities can be encoded as special cases of Srivastava's triple hypergeometric function $F^{(3)}$. To make this statement explicit, we give the full $F^{(3)}$-forms for the three parameter derivatives of Φ_1. Empty parameter groups are denoted by "$-$", and a group such as $(a,1)$ means that both parameters a and 1 occur in that group. For $c \notin \{0, -1, -2, \ldots\}$,

$$\begin{aligned}\frac{\partial}{\partial a}\Phi_1(a,b;c;x,y) &= \frac{bx}{c} F^{(3)}\left[\begin{array}{c} (a+1) :: -;-;(b+1) : (1);-;(a,1) \\ (c+1) :: -;-;(2) : -;-;(a+1) \end{array}\middle| x,y,x\right] \\ &\quad + \frac{y}{c} F^{(3)}\left[\begin{array}{c} (a+1) :: -;-;(a) : (b);(1);(1) \\ (c+1) :: -;(2);(a+1) : -;-;- \end{array}\middle| x,y,y\right],\end{aligned} \tag{2.30}$$

$$\frac{\partial}{\partial b}\Phi_1(a,b;c;x,y) = \frac{ax}{c} F^{(3)}\left[\begin{array}{c} (a+1) :: -;-;(b+1) : (1);-;(b,1) \\ (c+1) :: -;-;(2) : -;-;(b+1) \end{array}\middle| x,y,x\right], \tag{2.31}$$

$$\begin{aligned}\frac{\partial}{\partial c}\Phi_1(a,b;c;x,y) &= -\frac{abx}{c^2} F^{(3)}\left[\begin{array}{c} (a+1) :: -;-;(b+1) : (1);-;(c,1) \\ (c+1) :: -;-;(2) : -;-;(c+1) \end{array}\middle| x,y,x\right] \\ &\quad - \frac{ay}{c^2} F^{(3)}\left[\begin{array}{c} (a+1) :: -;-;(c) : (b);(1);(1) \\ (c+1) :: -;(2);(c+1) : -;-;- \end{array}\middle| x,y,y\right].\end{aligned} \tag{2.32}$$

These formulas are obtained from (2.21) by writing, for example, $\mathcal{H}_a(m+n) = \mathcal{H}_a(m)+\mathcal{H}_{a+m}(n)$, expanding the finite sums in (2.16), and then applying the triangular rearrangement (2.6). For completeness, we now give the corresponding full $F^{(3)}$-forms for the remaining six Humbert functions. The denominator parameters are again assumed to avoid non-positive integers, and the identities extend by analytic continuation wherever both sides are defined.

For Φ_2 one obtains

$$\frac{\partial}{\partial a}\Phi_2(a,b;c;x,y) = \frac{x}{c}F^{(3)}\left[\begin{matrix} - :: -;-;(a+1) : (1);(b);(a,1) \\ (c+1) :: -;-;(2) : -;-;(a+1) \end{matrix}\middle| x,y,x\right], \tag{2.33}$$

$$\frac{\partial}{\partial b}\Phi_2(a,b;c;x,y) = \frac{y}{c}F^{(3)}\left[\begin{matrix} - :: -;(b+1);- : (a);(1);(b,1) \\ (c+1) :: -;(2);- : -;-;(b+1) \end{matrix}\middle| x,y,y\right], \tag{2.34}$$

$$\begin{aligned}\frac{\partial}{\partial c}\Phi_2(a,b;c;x,y) = &-\frac{ax}{c^2}F^{(3)}\left[\begin{matrix} - :: -;-;(a+1) : (1);(b);(c,1) \\ (c+1) :: -;-;(2) : -;-;(c+1) \end{matrix}\middle| x,y,x\right] \\ &-\frac{by}{c^2}F^{(3)}\left[\begin{matrix} - :: -;(b+1);(c) : (a);(1);(1) \\ (c+1) :: -;(2);(c+1) : -;-;- \end{matrix}\middle| x,y,y\right].\end{aligned} \tag{2.35}$$

For Φ_3 the two parameter derivatives are

$$\frac{\partial}{\partial a}\Phi_3(a;b;x,y) = \frac{x}{b}F^{(3)}\left[\begin{matrix} - :: -;-;(a+1) : (1);-;(a,1) \\ (b+1) :: -;-;(2) : -;-;(a+1) \end{matrix}\middle| x,y,x\right], \tag{2.36}$$

$$\begin{aligned}\frac{\partial}{\partial b}\Phi_3(a;b;x,y) = &-\frac{ax}{b^2}F^{(3)}\left[\begin{matrix} - :: -;-;(a+1) : (1);-;(b,1) \\ (b+1) :: -;-;(2) : -;-;(b+1) \end{matrix}\middle| x,y,x\right] \\ &-\frac{y}{b^2}F^{(3)}\left[\begin{matrix} - :: -;-;(b) : (a);(1);(1) \\ (b+1) :: -;(2);(b+1) : -;-;- \end{matrix}\middle| x,y,y\right].\end{aligned} \tag{2.37}$$

For Ψ_1, the four first-order parameter derivatives can be written as

$$\begin{aligned}\frac{\partial}{\partial a}\Psi_1(a,b;c,d;x,y) = &\frac{bx}{c}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;(b+1) : (1);-;(a,1) \\ - :: -;-;(2,c+1) : -;(d);(a+1) \end{matrix}\middle| x,y,x\right] \\ &+\frac{y}{d}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;(a) : (b);(1);(1) \\ - :: -;(2,d+1);(a+1) : (c);-;- \end{matrix}\middle| x,y,y\right],\end{aligned} \tag{2.38}$$

$$\frac{\partial}{\partial b}\Psi_1(a,b;c,d;x,y) = \frac{ax}{c}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;(b+1) : (1);-;(b,1) \\ - :: -;-;(2,c+1) : -;(d);(b+1) \end{matrix}\middle| x,y,x\right], \tag{2.39}$$

$$\frac{\partial}{\partial c}\Psi_1(a,b;c,d;x,y) = -\frac{abx}{c^2}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;(b+1) : (1);-;(c,1) \\ - :: -;-;(2,c+1) : -;(d);(c+1) \end{matrix}\middle| x,y,x\right], \tag{2.40}$$

$$\frac{\partial}{\partial d}\Psi_1(a,b;c,d;x,y) = -\frac{ay}{d^2}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;- : (b);(1);(d,1) \\ - :: -;(2,d+1);- : (c);-;(d+1) \end{matrix}\middle| x,y,y\right]. \tag{2.41}$$

For Ψ_2, the corresponding formulas are

$$\begin{aligned}\frac{\partial}{\partial a}\Psi_2(a;b,c;x,y) = &\frac{x}{b}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;- : (1);-;(a,1) \\ - :: -;-;(2,b+1) : -;(c);(a+1) \end{matrix}\middle| x,y,x\right] \\ &+\frac{y}{c}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;(a) : -;(1);(1) \\ - :: -;(2,c+1);(a+1) : (b);-;- \end{matrix}\middle| x,y,y\right],\end{aligned} \tag{2.42}$$

$$\frac{\partial}{\partial b}\Psi_2(a;b,c;x,y) = -\frac{ax}{b^2}F^{(3)}\left[\begin{matrix} (a+1) :: -;-;- : (1);-;(b,1) \\ - :: -;-;(2,b+1) : -;(c);(b+1) \end{matrix}\middle| x,y,x\right], \tag{2.43}$$

$$\frac{\partial}{\partial c}\Psi_2(a;b,c;x,y)=-\frac{ay}{c^2}F^{(3)}\left[\begin{matrix}(a+1)::-;-;-:-;(1);(c,1)\\ -::-;(2,c+1);-:(b);-;(c+1)\end{matrix}\,\middle|\,x,y,y\right]. \tag{2.44}$$

For Ξ_1, one has

$$\frac{\partial}{\partial a}\Xi_1(a,b,c;d;x,y)=\frac{cx}{d}F^{(3)}\left[\begin{matrix}-::-;-;(a+1,c+1):(1);(b);(a,1)\\ (d+1)::-;-;(2):-;-;(a+1)\end{matrix}\,\middle|\,x,y,x\right], \tag{2.45}$$

$$\frac{\partial}{\partial b}\Xi_1(a,b,c;d;x,y)=\frac{y}{d}F^{(3)}\left[\begin{matrix}-::-;(b+1);-:(a,c);(1);(b,1)\\ (d+1)::-;(2);-:-;-;(b+1)\end{matrix}\,\middle|\,x,y,y\right], \tag{2.46}$$

$$\frac{\partial}{\partial c}\Xi_1(a,b,c;d;x,y)=\frac{ax}{d}F^{(3)}\left[\begin{matrix}-::-;-;(a+1,c+1):(1);(b);(c,1)\\ (d+1)::-;-;(2):-;-;(c+1)\end{matrix}\,\middle|\,x,y,x\right], \tag{2.47}$$

$$\begin{aligned}\frac{\partial}{\partial d}\Xi_1(a,b,c;d;x,y)=&-\frac{acx}{d^2}F^{(3)}\left[\begin{matrix}-::-;-;(a+1,c+1):(1);(b);(d,1)\\ (d+1)::-;-;(2):-;-;(d+1)\end{matrix}\,\middle|\,x,y,x\right]\\ &-\frac{by}{d^2}F^{(3)}\left[\begin{matrix}-::-;(b+1);(d):(a,c);(1);(1)\\ (d+1)::-;(2);(d+1):-;-;-\end{matrix}\,\middle|\,x,y,y\right].\end{aligned} \tag{2.48}$$

Finally, for Ξ_2, the complete $F^{(3)}$-forms are

$$\frac{\partial}{\partial a}\Xi_2(a,b;c;x,y)=\frac{bx}{c}F^{(3)}\left[\begin{matrix}-::-;-;(a+1,b+1):(1);-;(a,1)\\ (c+1)::-;-;(2):-;-;(a+1)\end{matrix}\,\middle|\,x,y,x\right], \tag{2.49}$$

$$\frac{\partial}{\partial b}\Xi_2(a,b;c;x,y)=\frac{ax}{c}F^{(3)}\left[\begin{matrix}-::-;-;(a+1,b+1):(1);-;(b,1)\\ (c+1)::-;-;(2):-;-;(b+1)\end{matrix}\,\middle|\,x,y,x\right], \tag{2.50}$$

$$\begin{aligned}\frac{\partial}{\partial c}\Xi_2(a,b;c;x,y)=&-\frac{abx}{c^2}F^{(3)}\left[\begin{matrix}-::-;-;(a+1,b+1):(1);-;(c,1)\\ (c+1)::-;-;(2):-;-;(c+1)\end{matrix}\,\middle|\,x,y,x\right]\\ &-\frac{y}{c^2}F^{(3)}\left[\begin{matrix}-::-;-;(c):(a,b);(1);(1)\\ (c+1)::-;(2);(c+1):-;-;-\end{matrix}\,\middle|\,x,y,y\right].\end{aligned} \tag{2.51}$$

The double-series formulas (2.21)–(2.29) and the explicit $F^{(3)}$-forms (2.30)–(2.51) give two equivalent representations of the same first-order parameter derivatives: the former are often the most convenient for numerical computation, whereas the latter exhibit the promised triple-hypergeometric structure in full detail.

3. Higher-order derivatives of Humbert confluent hypergeometric functions with respect to parameters

In this section, we derive recursive formulas for derivatives of arbitrary order with respect to the parameters of the Humbert confluent hypergeometric functions. The key idea is to make systematic use of the linear partial differential equations (PDEs) satisfied by each Humbert function and to differentiate these PDEs with respect to the parameters. Because the operators involved are linear in the parameters, this procedure leads to simple recurrence relations that express the nth parameter derivative in terms of the corresponding $(n-1)$st derivative.

Throughout this section, we use the shorthand notation

$$p:=\frac{\partial}{\partial x},\qquad q:=\frac{\partial}{\partial y},\qquad r:=\frac{\partial^2}{\partial x^2},\qquad s:=\frac{\partial^2}{\partial x\,\partial y},\qquad t:=\frac{\partial^2}{\partial y^2}.$$

3.1. **The case of Φ_1.** We begin with the function

$$\Phi_1(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_{m+n}\,m!\,n!}\,x^m y^n.$$

It is known that Φ_1 satisfies a pair of second-order linear PDEs of hypergeometric type; see, for example, [8, 22]. For our purposes it is convenient to write these equations in the compact operator form

$$D\Phi_1(a,b;c;x,y) = 0, \tag{3.1}$$

$$M\Phi_1(a,b;c;x,y) = 0, \tag{3.2}$$

where

$$D = x(1-x)\,r + y(1-x)\,s + y(1-y)\,t + \big[c-(a+b+1)x\big]\,p - by\,q - ab, \tag{3.3}$$

$$M = y\,t + x\,s + (c-y)\,q - x\,p - a. \tag{3.4}$$

Here D and M act on functions of (x,y), but their coefficients depend linearly on the parameters a,b,c. This dependence allows us to generate parameter-derivative relations in a straightforward way.

First-order parameter derivatives. To illustrate the method, first consider the derivative with respect to a. Differentiating (3.1) with respect to a and using that $\partial/\partial a$ commutes with D except through the coefficients, we obtain

$$D\left(\frac{\partial\Phi_1}{\partial a}\right) + \frac{\partial D}{\partial a}\,\Phi_1 = 0.$$

From (3.3) we compute

$$\frac{\partial D}{\partial a} = -x\,p - b,$$

so that

$$D\left(\frac{\partial\Phi_1}{\partial a}\right) = (x\,p + b)\,\Phi_1. \tag{3.5}$$

The right-hand side is particularly simple: it consists of a first-order differential operator in (x,y) applied to the original function Φ_1. Using similar calculations for the derivatives with respect to b and c, we obtain

$$D\left(\frac{\partial\Phi_1}{\partial b}\right) = (x\,p + y\,q + a)\,\Phi_1, \tag{3.6}$$

$$D\left(\frac{\partial\Phi_1}{\partial c}\right) = -p\,\Phi_1. \tag{3.7}$$

On the other hand, differentiating the second PDE (3.2) with respect to a,b,c yields

$$M\left(\frac{\partial\Phi_1}{\partial a}\right) = \Phi_1, \tag{3.8}$$

$$M\left(\frac{\partial\Phi_1}{\partial b}\right) = 0, \tag{3.9}$$

$$M\left(\frac{\partial\Phi_1}{\partial c}\right) = -q\,\Phi_1. \tag{3.10}$$

Equations (3.5)–(3.10) are the basic relations for the first-order parameter derivatives of Φ_1 obtained purely from the PDEs.

In many applications, it is convenient to rewrite the right-hand sides in terms of contiguous Humbert functions such as $\Phi_1(a+1,b;c;x,y)$ and $\Phi_1(a,b+1;c;x,y)$. Such representations will be used later in connection with the differentiation formulas of section 4; for the moment we keep the simpler operator form, which is sufficient to obtain recurrence relations for higher-order derivatives.

Recursive formulas for the nth derivatives. We now differentiate the relations (3.5)–(3.7) repeatedly with respect to the parameters. For instance, applying $\partial^{n-1}/\partial a^{n-1}$ to (3.5) gives

$$D\left(\frac{\partial^n\Phi_1}{\partial a^n}\right) + (n-1)\frac{\partial D}{\partial a}\frac{\partial^{n-1}\Phi_1}{\partial a^{n-1}} = (x\,p+b)\,\frac{\partial^{n-1}\Phi_1}{\partial a^{n-1}}.$$

Since $\partial D/\partial a = -(x\,p+b)$, this becomes

$$D\left(\frac{\partial^n\Phi_1}{\partial a^n}\right) = n(x\,p+b)\,\frac{\partial^{n-1}\Phi_1}{\partial a^{n-1}}.$$

The same argument, applied to the derivatives with respect to b and c, produces the corresponding recurrence formulas. For later reference, we write them compactly as

$$\begin{aligned} D\left(\frac{\partial^n\Phi_1}{\partial a^n}\right) &= n\,(x\,p+b)\,\frac{\partial^{n-1}\Phi_1}{\partial a^{n-1}},\\ D\left(\frac{\partial^n\Phi_1}{\partial b^n}\right) &= n\,(x\,p+y\,q+a)\,\frac{\partial^{n-1}\Phi_1}{\partial b^{n-1}},\\ D\left(\frac{\partial^n\Phi_1}{\partial c^n}\right) &= -n\,p\,\frac{\partial^{n-1}\Phi_1}{\partial c^{n-1}}. \end{aligned} \tag{3.11}$$

Similarly, repeated differentiation of (3.8)–(3.10) leads to

$$\begin{aligned} M\left(\frac{\partial^n\Phi_1}{\partial a^n}\right) &= n\,\frac{\partial^{n-1}\Phi_1}{\partial a^{n-1}},\\ M\left(\frac{\partial^n\Phi_1}{\partial b^n}\right) &= 0,\\ M\left(\frac{\partial^n\Phi_1}{\partial c^n}\right) &= -n\,q\,\frac{\partial^{n-1}\Phi_1}{\partial c^{n-1}}. \end{aligned} \tag{3.12}$$

Equations (3.11) and (3.12) constitute a system of simple recurrence relations that can be used inductively to generate the nth derivatives of Φ_1 with respect to a, b and c, once the $(n-1)$st derivatives are known. When desired, the operators $(x\,p+b)$ and $(x\,p+y\,q+a)$ acting on Φ_1 or its derivatives can be replaced by contiguous combinations of Humbert functions, using the differentiation formulas in section 4.

3.2. **Other Humbert functions.** For the remaining Humbert confluent hypergeometric functions $\Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$, we follow the same strategy. Each of these functions satisfies a pair of second-order PDEs of hypergeometric type (see, e.g., [11, 14, 22]). Denoting by (P,Q), $(\widetilde{D},\widetilde{M})$, (D,M), etc., the corresponding pairs of differential operators, we again differentiate the PDEs with respect to the parameters and obtain recursive relations. Because the calculations are entirely analogous to those carried out for Φ_1, we state only the resulting formulas.

The function Φ_2*.* Let P and Q be the two PDE operators satisfied by Φ_2. Then, for $n \geq 1$, we have

$$\begin{aligned} P\left(\frac{\partial^n\Phi_2}{\partial a^n}\right) &= n\,\frac{\partial^{n-1}\Phi_2}{\partial a^{n-1}},\\ P\left(\frac{\partial^n\Phi_2}{\partial b^n}\right) &= 0,\\ P\left(\frac{\partial^n\Phi_2}{\partial c^n}\right) &= -n\,p\,\frac{\partial^{n-1}\Phi_2}{\partial c^{n-1}}, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} Q\left(\frac{\partial^n \Phi_2}{\partial a^n}\right) &= 0, \\ Q\left(\frac{\partial^n \Phi_2}{\partial b^n}\right) &= n\,\frac{\partial^{n-1}\Phi_2}{\partial b^{n-1}}, \\ Q\left(\frac{\partial^n \Phi_2}{\partial c^n}\right) &= -n\,q\,\frac{\partial^{n-1}\Phi_2}{\partial c^{n-1}}. \end{aligned} \tag{3.14}$$

The function Φ_3*.* For the function $\Phi_3(a;b;x,y)$, let $\widetilde{D}$ and $\widetilde{M}$ denote the corresponding PDE operators. Then

$$\begin{aligned} \widetilde{D}\left(\frac{\partial^n \Phi_3}{\partial a^n}\right) &= n\,\frac{\partial^{n-1}\Phi_3}{\partial a^{n-1}}, \\ \widetilde{D}\left(\frac{\partial^n \Phi_3}{\partial b^n}\right) &= -n\,p\,\frac{\partial^{n-1}\Phi_3}{\partial b^{n-1}}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \widetilde{M}\left(\frac{\partial^n \Phi_3}{\partial a^n}\right) &= 0, \\ \widetilde{M}\left(\frac{\partial^n \Phi_3}{\partial b^n}\right) &= -n\,q\,\frac{\partial^{n-1}\Phi_3}{\partial b^{n-1}}. \end{aligned} \tag{3.16}$$

The functions Ψ_1 *and* Ψ_2*.* If we again use the notation (D, M) for the pair of PDE operators of Ψ_1 (the symbols are the same as for Φ_1, but they act now on Ψ_1), we obtain

$$\begin{aligned} D\left(\frac{\partial^n \Psi_1}{\partial a^n}\right) &= n\,(x\,p+b)\,\frac{\partial^{n-1}\Psi_1}{\partial a^{n-1}}, \\ D\left(\frac{\partial^n \Psi_1}{\partial b^n}\right) &= n\,(x\,p+y\,q+a)\,\frac{\partial^{n-1}\Psi_1}{\partial b^{n-1}}, \\ D\left(\frac{\partial^n \Psi_1}{\partial c^n}\right) &= n\,p\,\frac{\partial^{n-1}\Psi_1}{\partial c^{n-1}}, \\ D\left(\frac{\partial^n \Psi_1}{\partial d^n}\right) &= 0, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} M\left(\frac{\partial^n \Psi_1}{\partial a^n}\right) &= n\,\frac{\partial^{n-1}\Psi_1}{\partial a^{n-1}}, \\ M\left(\frac{\partial^n \Psi_1}{\partial b^n}\right) &= 0, \\ M\left(\frac{\partial^n \Psi_1}{\partial c^n}\right) &= 0, \\ M\left(\frac{\partial^n \Psi_1}{\partial d^n}\right) &= -n\,q\,\frac{\partial^{n-1}\Psi_1}{\partial d^{n-1}}. \end{aligned} \tag{3.18}$$

For $\Psi_2(a;b,c;x,y)$ we obtain analogous formulas:

$$D\left(\frac{\partial^n \Psi_2}{\partial a^n}\right) = n\,\frac{\partial^{n-1}\Psi_2}{\partial a^{n-1}},$$
$$D\left(\frac{\partial^n \Psi_2}{\partial b^n}\right) = -n\,p\,\frac{\partial^{n-1}\Psi_2}{\partial b^{n-1}}, \tag{3.19}$$
$$D\left(\frac{\partial^n \Psi_2}{\partial c^n}\right) = 0,$$

and

$$M\left(\frac{\partial^n \Psi_2}{\partial a^n}\right) = n\,\frac{\partial^{n-1}\Psi_2}{\partial a^{n-1}},$$
$$M\left(\frac{\partial^n \Psi_2}{\partial b^n}\right) = 0, \tag{3.20}$$
$$M\left(\frac{\partial^n \Psi_2}{\partial c^n}\right) = -n\,q\,\frac{\partial^{n-1}\Psi_2}{\partial c^{n-1}}.$$

The functions Ξ_1 *and* Ξ_2. Finally, for $\Xi_1(a,b,c;d;x,y)$, using again a suitable pair of PDE operators (D,M), we obtain

$$D\left(\frac{\partial^n \Xi_1}{\partial a^n}\right) = n\,(x\,p+c)\,\frac{\partial^{n-1}\Xi_1}{\partial a^{n-1}}, \tag{3.21}$$
$$D\left(\frac{\partial^n \Xi_1}{\partial b^n}\right) = 0, \tag{3.22}$$
$$D\left(\frac{\partial^n \Xi_1}{\partial c^n}\right) = n\,(x\,p+a)\,\frac{\partial^{n-1}\Xi_1}{\partial c^{n-1}}, \tag{3.23}$$
$$D\left(\frac{\partial^n \Xi_1}{\partial d^n}\right) = -n\,p\,\frac{\partial^{n-1}\Xi_1}{\partial d^{n-1}}, \tag{3.24}$$

while

$$M\left(\frac{\partial^n \Xi_1}{\partial a^n}\right) = 0, \tag{3.25}$$
$$M\left(\frac{\partial^n \Xi_1}{\partial b^n}\right) = n\,\frac{\partial^{n-1}\Xi_1}{\partial b^{n-1}}, \tag{3.26}$$
$$M\left(\frac{\partial^n \Xi_1}{\partial c^n}\right) = 0, \tag{3.27}$$
$$M\left(\frac{\partial^n \Xi_1}{\partial d^n}\right) = -n\,q\,\frac{\partial^{n-1}\Xi_1}{\partial d^{n-1}}. \tag{3.28}$$

For $\Xi_2(a,b;c;x,y)$ we similarly obtain

$$D\left(\frac{\partial^n \Xi_2}{\partial a^n}\right) = n\,(x\,p+b)\,\frac{\partial^{n-1}\Xi_2}{\partial a^{n-1}}, \tag{3.29}$$
$$D\left(\frac{\partial^n \Xi_2}{\partial b^n}\right) = n\,(x\,p+a)\,\frac{\partial^{n-1}\Xi_2}{\partial b^{n-1}}, \tag{3.30}$$
$$D\left(\frac{\partial^n \Xi_2}{\partial c^n}\right) = -n\,p\,\frac{\partial^{n-1}\Xi_2}{\partial c^{n-1}}, \tag{3.31}$$

and

$$M\left(\frac{\partial^n \Xi_2}{\partial a^n}\right) = 0, \tag{3.32}$$

$$M\left(\frac{\partial^n \Xi_2}{\partial b^n}\right) = 0, \tag{3.33}$$

$$M\left(\frac{\partial^n \Xi_2}{\partial c^n}\right) = -n\,q\,\frac{\partial^{n-1}\Xi_2}{\partial c^{n-1}}. \tag{3.34}$$

To summarise, all Humbert confluent hypergeometric functions admit simple and parallel recurrence relations for derivatives of arbitrary order with respect to their parameters. These relations are obtained by a uniform procedure based on differentiating the underlying PDEs and are especially convenient when combined with the contiguous-relation formulas of the next section, where the operators $(x\,p+\lambda)$ and $(x\,p+y\,q+\lambda)$ are rewritten in terms of Humbert functions with shifted parameters.

4. Differentiation formulas for the Humbert confluent hypergeometric functions

In this section, we derive differentiation and contiguous-relation formulas for the Humbert confluent hypergeometric functions of two variables. We also obtain several reduction formulas with respect to the variables that express higher-order derivatives in terms of contiguous Humbert functions. Throughout, we continue to use the notation

$$p := \frac{\partial}{\partial x}, \qquad q := \frac{\partial}{\partial y},$$

so that, for example, xp stands for the differential operator $x\,\partial/\partial x$.

The results obtained here may be viewed as explicit realizations of the operator recurrence relations derived in section 3. For instance, in the case of Φ_1, the relations (3.11)–(3.12) express the nth parameter derivatives in terms of the corresponding $(n-1)$st derivatives by means of the differential operators $xp+b$, $xp+yq+a$ and p, acting on Φ_1 and its parameter derivatives. In the present section, we show that these operators can be rewritten in a simple way as combinations of contiguous Humbert functions with shifted parameters by acting on the double-series definitions recalled in section 2. In this way, the abstract operator recurrences of section 3 are converted into closed formulas for parameter and variable derivatives.

The proofs of the differentiation formulas follow a common pattern. One starts from a series representation such as (2.7)–(2.13), applies a simple shift identity for Pochhammer symbols (for example, $(a+1)_n = (a)_n\big(1+n/a\big)$), and then rearranges the resulting series to recognize the defining series of a contiguous Humbert function. For this reason, we give a detailed proof only in a prototype case (Theorem 4.1); for the remaining theorems, we provide only brief proof indications or omit the proofs when they are identical.

Theorem 4.1. *Let $\Phi_1 = \Phi_1(a,b;c;x,y)$ be the Humbert function defined in (2.7). Then the following differentiation and contiguous relations hold:*

$$(xp+yq+a)\,\Phi_1(a,b;c;x,y) = a\,\Phi_1(a+1,b;c;x,y), \tag{4.1}$$

$$(xp+b)\,\Phi_1(a,b;c;x,y) = b\,\Phi_1(a,b+1;c;x,y), \tag{4.2}$$

$$(xp+yq+c-1)\,\Phi_1(a,b;c;x,y) = (c-1)\,\Phi_1(a,b;c-1;x,y). \tag{4.3}$$

Consequently,

$$(a-c+1)\,\Phi_1(a,b;c;x,y) = a\,\Phi_1(a+1,b;c;x,y) - (c-1)\,\Phi_1(a,b;c-1;x,y). \tag{4.4}$$

Proof. Using the identity

$$(a+1)_{m+n} = (a)_{m+n}\left(1+\frac{m+n}{a}\right),$$

in the defining series of $\Phi_1(a+1,b;c;x,y)$ and comparing with the series for $\Phi_1(a,b;c;x,y)$, we obtain (4.1) after a straightforward rearrangement. In the same way, using

$$(b+1)_m = (b)_m \left(1+\frac{m}{b}\right),$$

we obtain (4.2). Finally, by means of

$$\frac{1}{(c-1)_{m+n}} = \left(1+\frac{m+n}{c-1}\right)\frac{1}{(c)_{m+n}},$$

we arrive at (4.3). The relation (4.4) then follows by eliminating the operator $xp+yq$ from (4.1) and (4.3). In view of (3.11), these formulas identify the operators occurring on the right-hand side of the recurrence relations of Section 3 with explicit shifts in the parameters. □

Theorem 4.2. *The mixed parameter-derivative formulas*

$$(xp+yq+a)\,\frac{\partial}{\partial b}\Phi_1(a,b;c;x,y) = a\,\frac{\partial}{\partial b}\Phi_1(a+1,b;c;x,y), \tag{4.5}$$

$$(xp+b)\,\frac{\partial}{\partial a}\Phi_1(a,b;c;x,y) = b\,\frac{\partial}{\partial a}\Phi_1(a,b+1;c;x,y), \tag{4.6}$$

hold. More generally, for every integer $n \geq 1$ we have

$$(xp+yq+a)\,\frac{\partial^n}{\partial b^n}\Phi_1(a,b;c;x,y) = a\,\frac{\partial^n}{\partial b^n}\Phi_1(a+1,b;c;x,y), \tag{4.7}$$

$$(xp+b)\,\frac{\partial^n}{\partial a^n}\Phi_1(a,b;c;x,y) = b\,\frac{\partial^n}{\partial a^n}\Phi_1(a,b+1;c;x,y). \tag{4.8}$$

Proof. Differentiating (4.1) and (4.2) with respect to b and a, respectively, and observing that $xp+yq+a$ and $xp+b$ do not depend on these parameters, we obtain (4.5)–(4.6). Repeated differentiation yields (4.7) and (4.8), which are compatible with the recurrences (3.11)–(3.12). □

Theorem 4.3. *For every integer $r \geq 1$, the derivatives of Φ_1 with respect to the variables satisfy the reduction formulas*

$$\frac{\partial^r}{\partial x^r}\Phi_1(a,b;c;x,y) = \frac{(a)_r(b)_r}{(c)_r}\,\Phi_1(a+r,b+r;c+r;x,y), \tag{4.9}$$

$$\frac{\partial^r}{\partial y^r}\Phi_1(a,b;c;x,y) = \frac{(a)_r}{(c)_r}\,\Phi_1(a+r,b;c+r;x,y). \tag{4.10}$$

Proof. Termwise differentiation of the defining double series of Φ_1 with respect to x gives

$$\frac{\partial^r}{\partial x^r}\Phi_1(a,b;c;x,y) = \sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_m}{(c)_{m+n}\,m!\,n!}\,(m)_r\,x^{m-r}y^n,$$

where $(m)_r = m(m-1)\cdots(m-r+1)$ and the terms with $m<r$ vanish. Writing $(m)_r = \Gamma(m+1)/\Gamma(m+1-r)$ and shifting the summation index, we obtain (4.9). The proof of (4.10) is similar and therefore omitted. When these relations are combined with (3.11)–(3.12), they yield explicit expressions for all mixed derivatives with respect to parameters and variables. □

Theorem 4.4. *Let $\Phi_2 = \Phi_2(a,b;c;x,y)$ be the Humbert function defined in (2.8). Then*

$$(xp+a)\,\Phi_2(a,b;c;x,y) = a\,\Phi_2(a+1,b;c;x,y), \tag{4.11}$$

$$(yq+b)\,\Phi_2(a,b;c;x,y) = b\,\Phi_2(a,b+1;c;x,y), \tag{4.12}$$

$$(xp+yq+c-1)\,\Phi_2(a,b;c;x,y) = (c-1)\,\Phi_2(a,b;c-1;x,y). \tag{4.13}$$

In particular,

$$(a+b-c+1)\,\Phi_2(a,b;c;x,y) = a\,\Phi_2(a+1,b;c;x,y) + b\,\Phi_2(a,b+1;c;x,y) - (c-1)\,\Phi_2(a,b;c-1;x,y). \tag{4.14}$$

Theorem 4.5. *For every integer $n \geq 1$, we have*

$$(xp + a)\,\frac{\partial^n}{\partial b^n}\Phi_2(a, b; c; x, y) = a\,\frac{\partial^n}{\partial b^n}\Phi_2(a + 1, b; c; x, y), \tag{4.15}$$

$$(yq + b)\,\frac{\partial^n}{\partial a^n}\Phi_2(a, b; c; x, y) = b\,\frac{\partial^n}{\partial a^n}\Phi_2(a, b + 1; c; x, y). \tag{4.16}$$

These relations are the explicit counterparts of the recurrences (3.13)–(3.14) for Φ_2.

Theorem 4.6. *For every integer $r \geq 1$, the reduction formulas*

$$\frac{\partial^r}{\partial x^r}\Phi_2(a, b; c; x, y) = \frac{(a)_r}{(c)_r}\,\Phi_2(a + r, b; c + r; x, y), \tag{4.17}$$

$$\frac{\partial^r}{\partial y^r}\Phi_2(a, b; c; x, y) = \frac{(b)_r}{(c)_r}\,\Phi_2(a, b + r; c + r; x, y), \tag{4.18}$$

hold.

Theorem 4.7. *Let $\Phi_3 = \Phi_3(a; b; x, y)$ be the Humbert function defined in (2.9). Then*

$$(xp + a)\,\Phi_3(a; b; x, y) = a\,\Phi_3(a + 1; b; x, y), \tag{4.19}$$

$$(xp + yq + b - 1)\,\Phi_3(a; b; x, y) = (b - 1)\,\Phi_3(a; b - 1; x, y). \tag{4.20}$$

Consequently,

$$(a - b + 1)\,\Phi_3(a; b; x, y) = a\,\Phi_3(a + 1; b; x, y) - (b - 1)\,\Phi_3(a; b - 1; x, y). \tag{4.21}$$

Theorem 4.8. *For every integer $n \geq 1$,*

$$(xp + a)\,\frac{\partial^n}{\partial b^n}\Phi_3(a; b; x, y) = a\,\frac{\partial^n}{\partial b^n}\Phi_3(a + 1; b; x, y). \tag{4.22}$$

Theorem 4.9. *For every integer $r \geq 1$ we have*

$$\frac{\partial^r}{\partial x^r}\Phi_3(a; b; x, y) = \frac{(a)_r}{(b)_r}\,\Phi_3(a + r; b + r; x, y), \tag{4.23}$$

$$\frac{\partial^r}{\partial y^r}\Phi_3(a; b; x, y) = \frac{1}{(b)_r}\,\Phi_3(a; b + r; x, y). \tag{4.24}$$

Theorem 4.10. *Let $\Psi_1 = \Psi_1(a, b; c, d; x, y)$ be the Humbert function of Ψ-type. Then*

$$(xp + yq + a)\,\Psi_1 = a\,\Psi_1(a + 1, b; c, d; x, y), \tag{4.25}$$

$$(xp + b)\,\Psi_1 = b\,\Psi_1(a, b + 1; c, d; x, y), \tag{4.26}$$

$$(xp + c - 1)\,\Psi_1 = (c - 1)\,\Psi_1(a, b; c - 1, d; x, y), \tag{4.27}$$

$$(yq + d - 1)\,\Psi_1 = (d - 1)\,\Psi_1(a, b; c, d - 1; x, y). \tag{4.28}$$

These imply the contiguous relations

$$(a - b - d + 1)\,\Psi_1 = a\,\Psi_1(a + 1, b; c, d; x, y) - b\,\Psi_1(a, b + 1; c, d; x, y) - (d - 1)\,\Psi_1(a, b; c, d - 1; x, y), \tag{4.29}$$

$$(a - c - d + 2)\,\Psi_1 = a\,\Psi_1(a + 1, b; c, d; x, y) - (c - 1)\,\Psi_1(a, b; c - 1, d; x, y) - (d - 1)\,\Psi_1(a, b; c, d - 1; x, y), \tag{4.30}$$

$$(b - c + 1)\,\Psi_1 = b\,\Psi_1(a, b + 1; c, d; x, y) - (c - 1)\,\Psi_1(a, b; c - 1, d; x, y). \tag{4.31}$$

Theorem 4.11. *For every integer $n \geq 1$, we have*

$$(xp + yq + a)\,\frac{\partial^n}{\partial b^n}\Psi_1 = a\,\frac{\partial^n}{\partial b^n}\Psi_1(a + 1, b; c, d; x, y), \tag{4.32}$$

$$(xp + b)\,\frac{\partial^n}{\partial a^n}\Psi_1 = b\,\frac{\partial^n}{\partial a^n}\Psi_1(a, b + 1; c, d; x, y). \tag{4.33}$$

Theorem 4.12. *For every integer $r \geq 1$,*

$$\frac{\partial^r}{\partial x^r}\Psi_1(a,b;c,d;x,y) = \frac{(a)_r(b)_r}{(c)_r}\,\Psi_1(a+r,b+r;c+r,d;x,y), \tag{4.34}$$

$$\frac{\partial^r}{\partial y^r}\Psi_1(a,b;c,d;x,y) = \frac{(a)_r}{(d)_r}\,\Psi_1(a+r,b;c,d+r;x,y). \tag{4.35}$$

Theorem 4.13. *Let $\Psi_2 = \Psi_2(a;b,c;x,y)$ be the second Humbert function of Ψ-type. Then*

$$(xp+yq+a)\,\Psi_2 = a\,\Psi_2(a+1;b,c;x,y), \tag{4.36}$$

$$(xp+b-1)\,\Psi_2 = (b-1)\,\Psi_2(a;b-1,c;x,y), \tag{4.37}$$

$$(yq+c-1)\,\Psi_2 = (c-1)\,\Psi_2(a;b,c-1;x,y). \tag{4.38}$$

In particular,

$$(a-b-c+2)\,\Psi_2 = a\,\Psi_2(a+1;b,c;x,y) - (b-1)\,\Psi_2(a;b-1,c;x,y) - (c-1)\,\Psi_2(a;b,c-1;x,y). \tag{4.39}$$

Theorem 4.14. *For every integer $n \geq 1$,*

$$(xp+yq+a)\,\frac{\partial^n}{\partial b^n}\Psi_2 = a\,\frac{\partial^n}{\partial b^n}\Psi_2(a+1;b,c;x,y). \tag{4.40}$$

Theorem 4.15. *For every integer $r \geq 1$,*

$$\frac{\partial^r}{\partial x^r}\Psi_2(a;b,c;x,y) = \frac{(a)_r}{(b)_r}\,\Psi_2(a+r;b+r,c;x,y), \tag{4.41}$$

$$\frac{\partial^r}{\partial y^r}\Psi_2(a;b,c;x,y) = \frac{(a)_r}{(c)_r}\,\Psi_2(a+r;b,c+r;x,y). \tag{4.42}$$

Theorem 4.16. *Let $\Xi_1 = \Xi_1(a,b,c;d;x,y)$ be the Humbert function of Ξ-type. Then*

$$(xp+a)\,\Xi_1 = a\,\Xi_1(a+1,b,c;d;x,y), \tag{4.43}$$

$$(yq+b)\,\Xi_1 = b\,\Xi_1(a,b+1,c;d;x,y), \tag{4.44}$$

$$(xp+c)\,\Xi_1 = c\,\Xi_1(a,b,c+1;d;x,y), \tag{4.45}$$

$$(xp+yq+d-1)\,\Xi_1 = (d-1)\,\Xi_1(a,b,c;d-1;x,y). \tag{4.46}$$

Consequently,

$$(a-c)\,\Xi_1 = a\,\Xi_1(a+1,b,c;d;x,y) - c\,\Xi_1(a,b,c+1;d;x,y), \tag{4.47}$$

$$(a+b-d+1)\,\Xi_1 = a\,\Xi_1(a+1,b,c;d;x,y) + b\,\Xi_1(a,b+1,c;d;x,y) - (d-1)\,\Xi_1(a,b,c;d-1;x,y), \tag{4.48}$$

$$(b+c-d+1)\,\Xi_1 = b\,\Xi_1(a,b+1,c;d;x,y) + c\,\Xi_1(a,b,c+1;d;x,y) - (d-1)\,\Xi_1(a,b,c;d-1;x,y). \tag{4.49}$$

Theorem 4.17. *For every integer $n \geq 1$ the following differentiation formulas hold:*

$$(xp+a)\,\frac{\partial^n}{\partial b^n}\Xi_1 = a\,\frac{\partial^n}{\partial b^n}\Xi_1(a+1,b,c;d;x,y), \tag{4.50}$$

$$(xp+a)\,\frac{\partial^n}{\partial c^n}\Xi_1 = a\,\frac{\partial^n}{\partial c^n}\Xi_1(a+1,b,c;d;x,y), \tag{4.51}$$

$$(yq+b)\,\frac{\partial^n}{\partial a^n}\Xi_1 = b\,\frac{\partial^n}{\partial a^n}\Xi_1(a,b+1,c;d;x,y), \tag{4.52}$$

$$(yq+b)\,\frac{\partial^n}{\partial c^n}\Xi_1 = b\,\frac{\partial^n}{\partial c^n}\Xi_1(a,b+1,c;d;x,y), \tag{4.53}$$

$$(xp+c)\,\frac{\partial^n}{\partial a^n}\Xi_1 = c\,\frac{\partial^n}{\partial a^n}\Xi_1(a,b,c+1;d;x,y), \tag{4.54}$$

$$(xp+c)\,\frac{\partial^n}{\partial b^n}\Xi_1 = c\,\frac{\partial^n}{\partial b^n}\Xi_1(a,b,c+1;d;x,y). \tag{4.55}$$

Theorem 4.18. *For every integer $r \geq 1$,*

$$\frac{\partial^r}{\partial x^r}\Xi_1(a,b,c;d;x,y) = \frac{(a)_r(c)_r}{(d)_r}\,\Xi_1(a+r,b,c+r;d+r;x,y), \tag{4.56}$$

$$\frac{\partial^r}{\partial y^r}\Xi_1(a,b,c;d;x,y) = \frac{(b)_r}{(d)_r}\,\Xi_1(a,b+r,c;d+r;x,y). \tag{4.57}$$

Theorem 4.19. *Let $\Xi_2 = \Xi_2(a,b;c;x,y)$ be the second Humbert function of Ξ-type. Then*

$$(xp+a)\,\Xi_2 = a\,\Xi_2(a+1,b;c;x,y), \tag{4.58}$$

$$(xp+b)\,\Xi_2 = b\,\Xi_2(a,b+1;c;x,y), \tag{4.59}$$

$$(xp+yq+c-1)\,\Xi_2 = (c-1)\,\Xi_2(a,b;c-1;x,y). \tag{4.60}$$

In particular,

$$(a-b)\,\Xi_2(a,b;c;x,y) = a\,\Xi_2(a+1,b;c;x,y) - b\,\Xi_2(a,b+1;c;x,y). \tag{4.61}$$

Theorem 4.20. *For every integer $n \geq 1$,*

$$(xp+a)\,\frac{\partial^n}{\partial b^n}\Xi_2 = a\,\frac{\partial^n}{\partial b^n}\Xi_2(a+1,b;c;x,y), \tag{4.62}$$

$$(xp+b)\,\frac{\partial^n}{\partial a^n}\Xi_2 = b\,\frac{\partial^n}{\partial a^n}\Xi_2(a,b+1;c;x,y). \tag{4.63}$$

Theorem 4.21. *For every integer $r \geq 1$,*

$$\frac{\partial^r}{\partial x^r}\Xi_2(a,b;c;x,y) = \frac{(a)_r(b)_r}{(c)_r}\,\Xi_2(a+r,b+r;c+r;x,y), \tag{4.64}$$

$$\frac{\partial^r}{\partial y^r}\Xi_2(a,b;c;x,y) = \frac{1}{(c)_r}\,\Xi_2(a,b;c+r;x,y). \tag{4.65}$$

5. Applications

The differentiation formulas obtained in the preceding sections can be used in a variety of problems arising in physics, applied mathematics, engineering, and related areas. In particular, many applications require the evaluation of Humbert confluent hypergeometric functions for parameter values that are close to, but not exactly equal to, a prescribed set of numerator or denominator parameters. In such situations, the explicit formulas for derivatives with respect to the parameters provide a convenient tool for constructing local parameter expansions and for analysing special parameter configurations.

We first illustrate how the relations derived in sections 2.4–4 simplify when the parameters satisfy certain algebraic constraints.

Special parameter configurations. Consider the Humbert function

$$\Phi_1(a,b;c;x,y),$$

whose defining series has been recalled in section 2. Along the diagonal $a = c$ in the parameter space, the function effectively depends on a reduced number of parameters. Combining the first-order derivative formulas with respect to a and c (see subsection 2.4), one finds that

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial c}\right)\Phi_1(a,b;c;x,y)\bigg|_{a=c} = 0. \tag{5.1}$$

Thus, when a and c are varied simultaneously along the diagonal $a = c$, the value of Φ_1 remains unchanged. In other words, Φ_1 is locally constant under the combined variation of a and c subject to $a = c$.

A completely analogous phenomenon occurs for the Humbert function

$$\Psi_1(a,b;c,d;x,y).$$

If we restrict to the diagonal $b = c$, then the first-order parameter derivative formulas for Ψ_1 imply that

$$\left(\frac{\partial}{\partial b} + \frac{\partial}{\partial c}\right) \Psi_1(a, b; c, d; x, y)\bigg|_{b=c} = 0. \tag{5.2}$$

Hence, in this case Ψ_1 is invariant under simultaneous variations of b and c along the line $b = c$ in the (b, c)-plane.

Relations of the type (5.1) and (5.2) are typical in problems where the physical or geometric model singles out particular combinations of numerator and denominator parameters.

Taylor expansions with respect to parameters. An important class of applications of the parameter-derivative formulas obtained in sections 2.4–4 consists of Taylor expansions with respect to the parameters. Such expansions allow one, for instance, to approximate Humbert confluent hypergeometric functions near a given set of parameter values or to study their sensitivity with respect to small perturbations of those parameters.

Let us fix a reference value $a_0 \in \mathbb{C}$ and expand Φ_1 with respect to the parameter a around $a = a_0$. Using the existence of the nth-order derivatives $\partial^n \Phi_1/\partial a^n$ and the formulas established in section 2.4, we obtain the Taylor expansion

$$\Phi_1(a, b; c; x, y) = \sum_{n=0}^{\infty} \frac{(a - a_0)^n}{n!} \frac{\partial^n}{\partial a^n} \Phi_1(a, b; c; x, y)\bigg|_{a=a_0}, \tag{5.3}$$

whenever the series converges. In the same way, fixing a reference value $c_0 \in \mathbb{C}$ and expanding with respect to the denominator parameter c gives

$$\Phi_1(a, b; c; x, y) = \sum_{n=0}^{\infty} \frac{(c - c_0)^n}{n!} \frac{\partial^n}{\partial c^n} \Phi_1(a, b; c; x, y)\bigg|_{c=c_0}. \tag{5.4}$$

The explicit expressions for the derivatives with respect to a and c, obtained earlier in terms of Srivastava's triple hypergeometric function $F^{(3)}$, turn the formal Taylor series (5.3)–(5.4) into computable expansions. Similar Taylor expansions can be written with respect to any other numerator or denominator parameter, for Φ_1 as well as for the remaining Humbert functions Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1 and Ξ_2, by making use of the corresponding nth-order parameter derivatives derived in section 3.

Together with the reduction formulas for variable derivatives obtained in section 4, these parameter expansions provide a flexible framework for analytical and numerical investigations of Humbert confluent hypergeometric functions in a broad range of applications. The first part of this discussion identifies parameter configurations with reduced sensitivity, whereas the second explains how parameter-derivative formulas enter Taylor expansions around prescribed reference values.

6. Numerical illustrations

In this section, we present numerical examples and graphical representations that illustrate the differentiation formulas obtained in the preceding sections. For concreteness we focus on the Humbert confluent hypergeometric function

$$\Phi_1(a, b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n}\, m!\, n!} x^m y^n,$$

together with its derivative with respect to the parameter a. All computations are carried out for the parameter choice

$$a = \tfrac{3}{2}, \qquad b = \tfrac{3}{4}, \qquad c = \tfrac{5}{2},$$

and for (x, y) in the square $[0, 0.8] \times [0, 0.8]$, which lies well inside the region of absolute convergence of the defining double series.

6.1. **Numerical evaluation.** For the numerical evaluation of Φ_1, we truncated the defining double series to all terms with $m+n \le N_{\max}$. Unless otherwise stated, the values reported below use $N_{\max} = 40$. The derivative with respect to a was computed by differentiating the series termwise and using

$$\frac{\partial}{\partial a}(a)_{m+n} = (a)_{m+n}\big[\Psi(a+m+n) - \Psi(a)\big], \tag{6.1}$$

where Ψ denotes the digamma function. Inserting (6.1) into the double series for Φ_1 yields the computational formula

$$\frac{\partial \Phi_1}{\partial a}(a,b;c;x,y) = \sum_{m,n\ge 0} \frac{(a)_{m+n}(b)_m}{(c)_{m+n} m! n!}\big[\Psi(a+m+n) - \Psi(a)\big] x^m y^n, \tag{6.2}$$

truncated with the same triangular condition $m+n \le N_{\max}$.

Table 1 displays representative numerical values of $\Phi_1(a,b;c;x,y)$ and $\partial\Phi_1/\partial a(a,b;c;x,y)$ at several points (x,y). All values are rounded to six decimal places.

TABLE 1. Sample numerical values of $\Phi_1(a,b;c;x,y)$ and $\partial\Phi_1/\partial a(a,b;c;x,y)$ for $a = 1.5$, $b = 0.75$, $c = 2.5$.

x	y	$\Phi_1(a,b;c;x,y)$	$\frac{\partial \Phi_1}{\partial a}(a,b;c;x,y)$
0.1	0.1	1.113819	0.079604
0.1	0.5	1.430456	0.325267
0.3	0.3	1.409546	0.318193
0.5	0.5	1.851703	0.749493
0.7	0.1	1.720944	0.717414
0.7	0.5	2.259061	1.278011

Table 1 shows that, for the chosen parameter values, both Φ_1 and $\partial\Phi_1/\partial a$ increase monotonically as either x or y increases. Moreover, the derivative $\partial\Phi_1/\partial a$ is strictly positive at all sample points, reflecting the increase of Φ_1 with respect to the parameter a in this range of variables.

6.2. **Truncation error, convergence and validation.** The numerical truncation used in (6.2) is triangular: all terms with total degree $m+n > N_{\max}$ are omitted. To estimate the tail, we compared successive triangular truncations against a high-precision reference computed with $N_{\max} = 100$. The most demanding point among the sample points in Table 1 is $(x,y) = (0.7, 0.5)$; the convergence at this point is shown in Table 2.

TABLE 2. Convergence of the triangular truncation for Φ_1 and $\partial\Phi_1/\partial a$ at $(x,y) = (0.7, 0.5)$. The reference values are computed with $N_{\max} = 100$.

$N_{\max}$	Φ_1	absolute error	$\partial\Phi_1/\partial a$	absolute error
10	2.252963220	6.10×10^{-3}	1.262287332	1.57×10^{-2}
15	2.258366455	6.94×10^{-4}	1.275998691	2.01×10^{-3}
20	2.258973836	8.69×10^{-5}	1.277737867	2.73×10^{-4}
30	2.259059188	1.59×10^{-6}	1.278005070	5.54×10^{-6}
40	2.259060743	3.25×10^{-8}	1.278010490	1.22×10^{-7}

As an independent validation, we used the limiting cases in which Φ_1 reduces to classical single-variable hypergeometric functions:

$$\Phi_1(a,b;c;0,y) = {}_1F_1(a;c;y), \qquad \Phi_1(a,b;c;x,0) = {}_2F_1(a,b;c;x). \tag{6.3}$$

For the same parameters and $N_{\max} = 40$, the comparison is summarized in Table 3.

The estimates in Tables 2 and 3 justify the use of $N_{\max} = 40$ for the numerical figures and the rounded values reported in Table 1.

TABLE 3. Validation of the truncated Φ_1 series against the classical reductions in (6.3).

Point	Reduction	truncated value	absolute difference
$(0, 0.5)$	${}_1F_1(a; c; 0.5)$	1.361290826370	$< 10^{-60}$
$(0.7, 0)$	${}_2F_1(a, b; c; 0.7)$	1.610246558986	1.59×10^{-8}

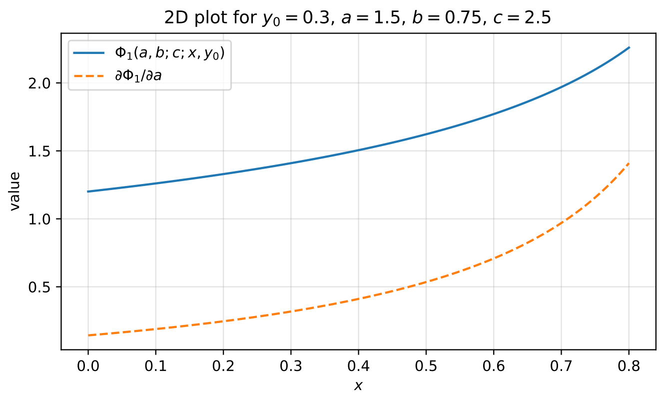

FIGURE 1. Plot of $\Phi_1(a, b; c; x, y_0)$ (solid line) and $\partial\Phi_1/\partial a(a, b; c; x, y_0)$ (dashed line) as functions of x for $a = 1.5$, $b = 0.75$, $c = 2.5$ and $y_0 = 0.3$.

6.3. **Two-dimensional plots.** To visualize more clearly the dependence of Φ_1 and its parameter derivative on the variable x, we consider the one-parameter family

$$x \longmapsto \Phi_1(a, b; c; x, y_0), \qquad x \longmapsto \frac{\partial \Phi_1}{\partial a}(a, b; c; x, y_0),$$

using the same parameter values as above and with $y_0 = 0.3$ fixed. Both functions were evaluated on a uniform grid in the interval $0 \leq x \leq 0.8$.

Figure 1 shows the resulting curves. The solid line corresponds to $\Phi_1(a, b; c; x, y_0)$, while the dashed line corresponds to $\partial\Phi_1/\partial a$. As expected, both functions increase smoothly with x, while the derivative with respect to a grows more rapidly than Φ_1 itself, indicating enhanced sensitivity to changes in a as x moves away from the origin.

6.4. **Three-dimensional surface plot.** We finally illustrate the joint dependence of Φ_1 on the variables (x, y) in the square $[0, 0.8] \times [0, 0.8]$. Using the same truncation and parameter values as before, we computed $\Phi_1(a, b; c; x, y)$ on a uniform 40×40 grid in this domain and constructed the corresponding surface plot.

The resulting graph is displayed in Figure 2. The surface is smooth and strictly increasing in both variables, with a moderate curvature near the origin and a steeper rise towards the corner $(x, y) = (0.8, 0.8)$. This behaviour is consistent with the positivity of the coefficients in the defining double series of Φ_1 for the present choice of parameters.

These numerical examples provide a concrete illustration of the analytical results obtained in the earlier sections and demonstrate that the parameter-derivative formulas can be implemented efficiently in practical computations.

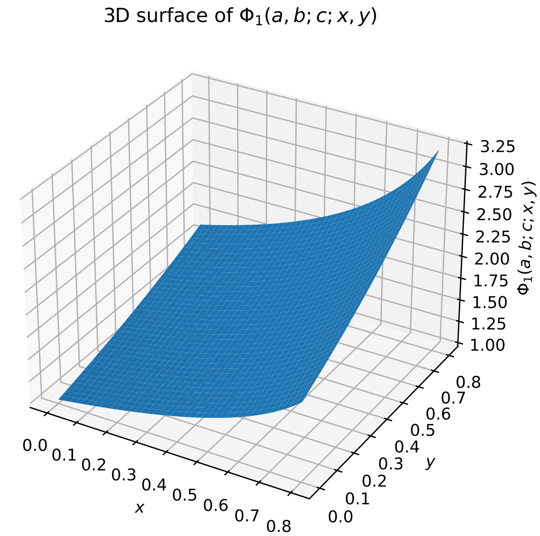

FIGURE 2. Three-dimensional surface plot of $\Phi_1(a, b; c; x, y)$ for $a = 1.5$, $b = 0.75$, $c = 2.5$ and $(x, y) \in [0, 0.8] \times [0, 0.8]$.

7. Concluding remarks

In this paper, we have carried out a systematic study of derivatives with respect to the parameters of the Humbert confluent hypergeometric functions of two variables. More precisely, we considered all seven classical Humbert functions Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1 and Ξ_2 and developed a unified framework for their differentiation with respect to numerator and denominator parameters.

Starting from the double-series representations recalled in section 2 and using elementary properties of the Gamma, digamma and polygamma functions, we first derived explicit formulas for the first-order derivatives with respect to each parameter. We then recast these formulas in a compact, uniform manner in terms of Srivastava's triple hypergeometric function $F^{(3)}$, which plays a natural role as a basic building block for multivariable parameter derivatives.

A second main ingredient of our approach is the use of the systems of linear partial differential equations satisfied by the Humbert functions. By differentiating these systems with respect to the parameters, we obtained simple operator recurrences for parameter derivatives of arbitrary order, as described in section 3. In section 4, these recurrences were combined with shift identities for Pochhammer symbols to produce explicit differentiation and reduction formulas that express parameter derivatives in terms of contiguous Humbert functions and higher-order derivatives with respect to the variables.

To complement the theoretical developments, section 6 presented numerical illustrations for the function Φ_1 and its derivative with respect to a numerator parameter. Sample values, convergence checks, validation against single-variable reductions, and two- and three-dimensional plots were obtained directly from the double-series representations, thereby demonstrating that the parameter-derivative formulas can be implemented in a straightforward and numerically stable way.

The results obtained here provide an analytic toolkit for working with Humbert confluent hypergeometric functions in contexts where parametric dependence is essential, such as sensitivity analysis, perturbation methods, and parameter fitting in applied models. Several directions for further research remain open. One natural extension is to consider generalized Humbert-type and related multivariable hypergeometric functions and to derive analogous parameter derivative formulas for them. Another direction is the development of dedicated numerical algorithms that exploit the present formulas to compute Humbert functions and their parameter derivatives efficiently over wider regions of the parameter and variable space. We hope that the results presented here will serve as a useful starting point for such investigations and for further applications in mathematical physics and applied analysis.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, 1964.

[2] P. Agarwal, A. Shehata, S. I. Moustafa, and S. Jain, *Derivatives of Horn's hypergeometric functions* G_1, G_2, Γ_1, *and* Γ_2 *with respect to their parameters*, in Fractional Order Systems and Applications in Engineering, Elsevier, (2023), 353–374.

[3] A. Al E'damat and A. Shehata, *On bibasic Humbert hypergeometric function* Φ_1, Malaysian Journal of Mathematical Sciences, *17*(1) (2023), 77–86.

[4] L. U. Ancarani and G. Gasaneo, *Derivatives of any order of the confluent hypergeometric function* ${}_1F_1(a, b, z)$ *with respect to the parameter* a *or* b, Journal of Mathematical Physics, *49*(6) (2008), 063508.

[5] L. U. Ancarani and G. Gasaneo, *Derivatives of any order of the Gaussian hypergeometric function* ${}_2F_1(a, b, c; z)$ *with respect to the parameters* a, b *and* c, Journal of Physics A: Mathematical and Theoretical, *42*(39) (2009), 395208.

[6] L. U. Ancarani and G. Gasaneo, *Derivatives of any order of the hypergeometric function* ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ *with respect to the parameters* a_i *and* b_i, Journal of Physics A: Mathematical and Theoretical, *43*(8) (2010), 085210.

[7] L. U. Ancarani, J. A. Del Punta, and G. Gasaneo, *Derivatives of Horn hypergeometric functions with respect to their parameters*, Journal of Mathematical Physics, *58*(7) (2017), 073504.

[8] P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques. Polynômes d'Hermite*, Gauthier-Villars, 1926.

[9] M. Arshad, M. Usman, M. Z. Iqbal, and A. Ali, *K-Humbert confluent hypergeometric functions* $\Phi_{1,k}$, $\Phi_{2,k}$, *and* $\Phi_{3,k}$, Journal of Mathematics and Computer Science, *35*(3) (2024), 348–361.

[10] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions, Vol. 1*, McGraw-Hill, 1953.

[11] H. Exton, *Multiple Hypergeometric Functions and Applications*, Ellis Horwood, 1976.

[12] P.-C. Hang and M.-J. Luo, *Asymptotics of the Humbert function* Ψ_1 *for two large arguments*, SIGMA, *20* (2024), 074.

[13] P.-C. Hang, M. Henkel, and M.-J. Luo, *Asymptotics of the Humbert functions* Ψ_1 *and* Ψ_2, Journal of Approximation Theory, *314* (2026), 106233.

[14] P. Humbert, *The confluent hypergeometric functions of two variables*, Proceedings of the Royal Society of Edinburgh, *41* (1922), 73–96.

[15] Y. L. Luke, *The Special Functions and Their Approximations, Vol. 2*, Academic Press, 1969.

[16] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

[17] E. D. Rainville, *Special Functions*, Macmillan, 1960.

[18] A. Shehata, *On the (p, q)-Humbert functions from the view point of the generating function method*, Journal of Function Spaces, *2020* (2020), Article ID 4794571.

[19] A. Shehata, S. I. Moustafa, P. Agarwal, and S. Jain, *On basic Humbert confluent hypergeometric functions*, in Fractional Order Systems and Applications in Engineering, Elsevier, (2023), 319–352.

[20] A. Shehata, *Certain new formulas for bibasic Humbert hypergeometric functions* Ψ_1 *and* Ψ_2, Indian Journal of Pure and Applied Mathematics, *56*(4) (2025), 1608–1623.

[21] H. M. Srivastava, *Some integrals representing triple hypergeometric functions*, Rendiconti del Circolo Matematico di Palermo, *16* (1967), 99–115.

[22] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood, 1985.