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Adaptive stepsize optimized three-step hybrid block method for first-order initial value problems

Adebayo Abiodun Aderogba[1,*] **and Higinio Ramos**[2]

[1]Department of Mathematics, Obafemi Awolowo University, Ile-Ife, 220005, Nigeria.

[2]Escuela Politécnica Superior de Zamora, University of Salamanca, Salamanca, Spain.

Abstract

In this paper, an efficient numerical solver based on a hybrid optimized block method for approximating solutions of first-order initial value problems is constructed and analysed. The scheme is designed in such a way that each block contains three steps with three offset points. To enhance its efficiency, we introduce simplified formulations of the underlying equations and employ an adaptive step size strategy based on local error estimates. We establish the stability region, order of consistency, and convergence properties of the method. The effectiveness of the proposed approach is demonstrated through its application to several test problems.

Keywords. ODEs, Hybrid method, Block method, Embedded-type procedure, Variable step-size formulation.

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1. Introduction

Many real-world problems are modeled through differential equations, which may be linear or nonlinear, partial or ordinary, with or without delay. Mathematical analysis of these equations provides insights into their qualitative behavior. The boundedness of non-oscillatory solutions to some linear second order ordinary differential equations was the subject of [30] while [29] discussed the stability of linear second-order differential equation with constant delay based on the Lyapunov-Krasovskii functional. The same approach was employed in [7] to discuss the continuability and boundedness of solutions to integro differential equation with multiple delays, and was later extended to analyze boundedness, stability and integrability of integro-differential equations with time-delay retardation in [31].

Although some of these equations admit exact analytical solutions, many are so complex that obtaining their exact solutions is difficult or even impossible. In such cases, it is necessary to use numerical methods to provide approximate solutions. With technological advancements and widespread computer availability,, numerical methods have become the dominant approach to solving differential equations.

Our interest is to approximate the solution of the following first-order initial value problem (IVP)

$$\frac{dy}{dt} = f(t,y),\ t > t_0,\ t_0 \in \mathbf{R},$$
$$y(t_0) = y_0, \tag{1.1}$$

where $y : [t_0, T] \to \mathbf{R}^n$ and $f : [t_0, T] \times \mathbf{R}^n \to \mathbf{R}^n$. Problem (1.1) is assumed to satisfy conditions ensuring the existence of a unique solution and continuous dependence on initial data. Although first-order initial value problems may appear simple, many other types of differential equation can be reformulated as first-order IVPs. Consequently, numerical methods developed for (1.1) can often be extended to other differential equations. Runge-Kutta methods and linear multistep methods [11, 20] are some of the traditional numerical approaches to approximate the solution

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∗ Corresponding author. Email: aaderogba@oauife.edu.ng .

of Equation (1.1). In order to improve accuracy and stability, especially for challenging initial value problems such as stiff or singularly perturbed equations, several non-classical methods have been developed.

The block method, introduced in [12, 25], analyzed for stability in [33] and improved in [32] is one of those methods. These methods are well known for their inherent stability and computational efficiency. Block methods have been applied to several problems in Physics and Mathematics. Its application to problems in fluid dynamics can be found in [13] while [1, 10, 16, 18, 35] showcase some of the applications of the methods to solve higher-order differential equations. Recent studies by Brugnano and co-authors [4–6] has explored the convergence, stability, implementation, and blending of block methods. A blended scheme is obtained by combining two basic linear multistep methods in a way that enhances accuracy and computational speed while enabling efficient parallel implementation. The accuracy of block methods can be further improved by introducing off-step points, as discussed in [8]. Optimizing the location of off-step points, along with other scheme parameters, has been found to significantly improve performance [15]. An optimized version of a two-step hybrid block method has been applied to second-order differential equations in [17].

The solution of ordinary differential equations can vary considerably across different regions; some regions exhibit rapid variations requiring very small step sizes, while others change more gradually, allowing larger steps. Traditionally, authors have used uniformly small step sizes throughout the simulation, often resulting in excessive computational costs. Adaptive step-size methods address this issue by dynamically adjusting the step size based on the local behavior of the solution, a major advantage of such approaches.

Recently, the robustness of block methods has been improved through variable step-size strategies. For instance, a variable stepsize block method without optimization was designed for third-order initial value problems in [19], while it was proposed for a one-step optimized block method in [27] for first-order initial value problems and in [28] for second-order initial value problems.

This work introduces a 3-step optimized hybrid block method with three off-grid points for solving first-order initial value problems (IVPs). The proposed method differs from previous efforts such as [14] in three key aspects:

(1) It incorporates adaptive step-size control, following the foundational approach of [23, 24], allowing for better handling of stiff or rapidly changing solutions;
(2) It utilizes a custom 7-degree polynomial basis (as opposed to Chebyshev polynomials), which offers flexibility in node selection and potentially improves approximation accuracy in certain regimes [15];
(3) It introduces three strategically chosen off-grid points inspired by [8] to improve accuracy, along with simplified formulations of the computational scheme.

Section 2 gives a detailed description of the derivation of the scheme, while the properties of the method are discussed in section 3. Section 4 presents numerical experiments to demonstrate the effectiveness of our approach and section 5 concludes the paper with observations and recommendations for future research.

2. Development of the numerical scheme

In this section, the steps for the derivation of a three-step optimized block method are discussed. The focus of the work is to provide an accurate numerical solver for the first-order system of differential equations in (1.1). The domain of integration $t_0 \le t \le T,\ T \in \mathbb{R}$, is subdivided into non-overlapping subintervals $t_0 < t_1 < t_2 < \cdots < t_n = T$ where $t_i = t_0 + i\Delta t,\ i = 1,\ 2, \cdots, n$, where $\Delta t = h = \frac{T-t_0}{n}$ is the fixed step size. The approximations $y_{n+i} \simeq y(t_{n+i}),\ i = 1, 2, 3$, are obtained within every block containing three steps simultaneously, that is, on $[t_n,\ t_{n+3}]$. The solutions at three off-step points $t_{n+r},\ t_{n+s},\ t_{n+j}$ with $0 < r < 1 < s < 2 < j < 3$ are also determined in order to improve accuracy. Hence, within every block, the approximations at $t_{n+r}, t_{n+1}, t_{n+s}, t_{n+2}, t_{n+j}, t_{n+3}$ are obtained simultaneously.

To derive the method, firstly an appropriate polynomial function is given as an ansatz to approximate the solution. Considering on each block the initial value as an interpolation condition, and seven collocation conditions concerning the first derivative of the solution, we assume that

$$y(t) \simeq p(t) = \sum_{i=0}^{7} a_i t^i. \tag{2.1}$$

The polynomial $p(t)$ and its first derivative are collocated at the points within the block so as to obtain the coefficients $a_i,\ i = 0, 1, 2, \cdots, 7$. Specifically, assume $y(t_n) \approx p(t_n) = y_n$ and $y'(t_n) \approx p'(t_n) = f_n$, that is,

$$\sum_{i=0}^{7} a_i t_n^i = y_n,$$
$$\sum_{i=1}^{7} i a_i t_{n+v}^{i-1} = f_{n+v},\ v = 0, r, 1, s, 2, j, 3. \tag{2.2}$$

This results in a system of eight linear equations $T\vec{a} = \vec{f}$ with

$$T = \begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 1 & 2t_{n+r} & 3t_{n+r}^2 & 4t_{n+r}^3 & 5t_{n+r}^4 & 6t_{n+r}^5 & 7t_{n+r}^6 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 \\ 0 & 1 & 2t_{n+s} & 3t_{n+s}^2 & 4t_{n+s}^3 & 5t_{n+s}^4 & 6t_{n+s}^5 & 7t_{n+s}^6 \\ 0 & 1 & 2t_{n+2} & 3t_{n+2}^2 & 4t_{n+2}^3 & 5t_{n+2}^4 & 6t_{n+2}^5 & 7t_{n+2}^6 \\ 0 & 1 & 2t_{n+j} & 3t_{n+j}^2 & 4t_{n+j}^3 & 5t_{n+j}^4 & 6t_{n+j}^5 & 7t_{n+j}^6 \\ 0 & 1 & 2t_{n+3} & 3t_{n+3}^2 & 4t_{n+3}^3 & 5t_{n+3}^4 & 6t_{n+3}^5 & 7t_{n+3}^6 \end{pmatrix},$$

and

$$\vec{a} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{pmatrix}^T,$$

$$\vec{f} = \begin{pmatrix} y_n & f_n & f_{n+r} & f_{n+1} & f_{n+s} & f_{n+2} & f_{n+j} & f_{n+3} \end{pmatrix}^T,\ f_{n+v} = f(t_{n+v}, y_{n+v}).$$

Remark 1. The coefficient matrix in the above system is a Vandermonde-type matrix, hence it is always invertible. However, the degree k of the considered polynomial affects the condition of the matrix. The matrix can become ill-conditioned for large values of k. Nevertheless, since the polynomial degree used in our method is less than 13, the matrix remains well-conditioned [26].

The unknown coefficients a_i are obtained, by solving the system of equations above, as functions of y_n and $f_{n+v},\ v = r, 1, s, 2, j, 3$. Putting $t = t_n + xh$, the approximating polynomial, after substituting the $a_i's$, can be written as

$$p(t_n + xh) = y_n + h(\alpha_0 f_n + \alpha_r f_{n+r} + \alpha_1 f_{n+1} + \alpha_s f_{n+s} + \alpha_2 f_{n+2} + \alpha_j f_{n+j} + \alpha_3 f_{n+3}), \tag{2.3}$$

where the $\alpha_i's$ are functions of x and are given by

$$\begin{aligned}\alpha_0 = -\frac{x}{2520\,jrs}&(105\,jrsx^3 - 84\,jrx^4 - 84\,jsx^4 + 70\,jx^5 - 84\,rsx^4 + 70\,rx^5 + 70\,sx^5 \\ &- 60\,x^6 - 840\,jrsx^2 + 630\,jrx^3 + 630\,jsx^3 - 504\,jx^4 + 630\,rsx^3 - 504\,rx^4 \\ &- 504\,sx^4 + 420\,x^5 + 2310\,jrsx - 1540\,jrx^2 - 1540\,jsx^2 + 1155\,jx^3 - 1540\,rsx^2 \\ &+ 1155\,rx^3 + 1155\,sx^3 - 924\,x^4 - 2520\,jrs + 1260\,jrx + 1260\,jsx - 840\,jx^2 \\ &+ 1260\,rsx - 840\,rx^2 - 840\,sx^2 + 630\,x^3),\end{aligned} \tag{2.4}$$

$$\begin{aligned}\alpha_r = -\frac{x^2}{420(r-3)r(r-1)(-s+r)(r-2)(j-r)}&(84\,jsx^3 - 70\,jx^4 - 70\,sx^4 + 60\,x^5 \\ &- 630\,jsx^2 + 504\,jx^3 + 504\,sx^3 - 420\,x^4 + 1540\,jsx - 1155\,jx^2 - 1155\,sx^2 \\ &+ 924\,x^3 - 1260\,js + 840\,jx + 840\,sx - 630\,x^2),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\alpha_1 = {} & \frac{x^2}{840(r-1)(s-1)(j-1)}(105\, jrsx^2 - 84\, jrx^3 - 84\, jsx^3 + 70\, jx^4 - 84\, rsx^3 \\ & + 70\, rx^4 + 70\, sx^4 - 60\, x^5 - 700\, jrsx + 525\, jrx^2 + 525\, jsx^2 - 420\, jx^3 \\ & + 525\, rsx^2 - 420\, rx^3 - 420\, sx^3 + 350\, x^4 + 1260\, jrs - 840\, jrx - 840\, jsx \\ & + 630\, jx^2 - 840\, rsx + 630\, rx^2 + 630\, sx^2 - 504\, x^3),\end{aligned} \tag{2.6}$$

$$\begin{aligned}\alpha_s = {} & \frac{x^2}{420(s-3)s(rs-s^2-r+s)(s-2)(j-s)}(84\, jrx^3 - 70\, jx^4 - 70\, rx^4 \\ & + 60\, x^5 - 630\, jrx^2 + 504\, jx^3 + 504\, rx^3 - 420\, x^4 + 1540\, jrx - 1155\, jx^2 \\ & - 1155\, rx^2 + 924\, x^3 - 1260\, jr + 840\, jx + 840\, rx - 630\, x^2),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\alpha_2 = {} & -\frac{x^2}{840(rs-2\,r-2\,s+4)(j-2)}(105\, jrsx^2 - 84\, jrx^3 - 84\, jsx^3 + 70\, jx^4 \\ & - 84\, rsx^3 + 70\, rx^4 + 70\, sx^4 - 60\, x^5 - 560\, jrsx + 420\, jrx^2 + 420\, jsx^2 - 336\, jx^3 \\ & + 420\, rsx^2 - 336\, rx^3 - 336\, sx^3 + 280\, x^4 + 630\, jrs - 420\, jrx - 420\, jsx + 315\, jx^2 \\ & - 420\, rsx + 315\, rx^2 + 315\, sx^2 - 252\, x^3),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\alpha_j = {} & \left(\frac{x^2}{420j(j-3)(j^4 - j^3r - j^3s + j^2rs - 3\,j^3 + 3\,j^2r + 3\,j^2s - 3\,jrs + 2\,j^2 - 2\,jr - 2\,js + 2\,rs)}\right) \\ & (84\, rsx^3 - 70\, rx^4 - 70\, sx^4 + 60\, x^5 - 630\, rsx^2 + 504\, rx^3 + 504\, sx^3 - 420\, x^4 + 1540\, rsx - 1155\, rx^2 \\ & - 1155\, sx^2 + 924\, x^3 - 1260\, rs + 840\, rx + 840\, sx - 630\, x^2),\end{aligned} \tag{2.9}$$

$$\begin{aligned}\alpha_3 = {} & \left(\frac{x^2}{2520\, jrs - 7560\, jr - 7560\, js - 7560\, rs + 22680\, j + 22680\, r + 22680\, s - 68040}\right)(105\, jrsx^2 \\ & - 84\, jrx^3 - 84\, jsx^3 + 70\, jx^4 - 84\, rsx^3 + 70\, rx^4 + 70\, sx^4 - 60\, x^5 \\ & - 420\, jrsx + 315\, jrx^2 + 315\, jsx^2 - 252\, jx^3 + 315\, rsx^2 - 252\, rx^3 - 252\, sx^3 \\ & + 210\, x^4 + 420\, jrs - 280\, jrx - 280\, jsx + 210\, jx^2 - 280\, rsx + 210\, rx^2 + 210\, sx^2 - 168\, x^3).\end{aligned} \tag{2.10}$$

Equation (2.3) implies that the value of x determines the formula that approximates each point within the block. That is, when $x = r$, y_{n+r} is obtained and so on.

2.1. **Optimization of the scheme.** To determine the values of the parameters $r,\ s,\ j$ we optimize the local truncation errors for the formulas corresponding to $y_{n+1},\ y_{n+2}$ and y_{n+3}. These errors are given by

$$\begin{aligned}\mathcal{L}(y(t_{n+1});h) = {} & \frac{h^8}{4233600}[\frac{10583}{378}[19jrs + \frac{378}{882}[11j + 11r + 11s] - \frac{1}{2}[17jr + 17js + 17rs]] - 83]y^{(8)}(t_n) \\ & + \frac{h^9}{50803200}(jrs[798(j+r+s) + 4074] - 357(j^2r + jr^2 + j^2s + js^2 + r^2s + rs^2) \\ & - 1944(jr + rs + js) + 198(j^2 + r^2 + s^2) + 1188(j+r+s) - 832)y^{(9)}(t_n),\end{aligned}$$

$$\begin{aligned}\mathcal{L}(y(t_{n+2});h) = {} & \frac{h^8}{33075}\left(8 + \frac{7}{4}(jrs + jr + js + rs) - \frac{9}{2}(j+r+s)\right)y^{(8)}(t_n) \\ & + \frac{h^9}{3175200}(21[jrs(j+r+s+8) + j^2r + j^2s + jr^2 + js^2r^2s + s^2r] \\ & - 54(j^2 + r^2 + s^2) + 72(jr + js + rs) - 324(j+r+s) + 736)y^{(9)}(t_n),\end{aligned}$$

and

$$\mathcal{L}(y(t_{n+3});h) = \frac{h^8}{156800}\left(14(2jrs - 3(jr+js+rs)) + 108(j+r+s) - 297\right) y^{(8)}(t_n)$$
$$\frac{h^9}{627200}(7[jrs(2(j+r+s)+6) - 3(j^2(r+s) + r^2(j+s) + s^2(r+j))]$$
$$+ 54(j^2+s^2+r^2) - 72(jr+js+rs) + 324(j+r+s) - 1296)y^{(9)}(t_n).$$

To determine the values of the off-step parameters r, s, j, the coefficients of the principal terms in the above local truncation errors were set to zero, yielding the system

$$-\frac{83}{4233600} + \frac{19\,jrs}{151200} - \frac{17\,jr}{302400} - \frac{17\,js}{302400} - \frac{17\,rs}{302400} + \frac{11\,j}{352800} + \frac{11\,r}{352800} + \frac{11\,s}{352800} = 0,$$
$$\frac{8}{33075} + \frac{jrs}{18900} + \frac{jr}{18900} + \frac{js}{18900} + \frac{rs}{18900} - \frac{j}{7350} - \frac{r}{7350} - \frac{s}{7350} = 0,$$
$$-\frac{297}{156800} + \frac{jrs}{5600} - \frac{3\,jr}{11200} - \frac{3\,js}{11200} - \frac{3\,rs}{11200} + \frac{27\,j}{39200} + \frac{27\,r}{39200} + \frac{27\,s}{39200} = 0.$$

Solving this system gives

$$r = \frac{1}{2}(3-\sqrt{5}),\ \ s = \frac{3}{2},\ \ j = \frac{1}{2}(3+\sqrt{5}).$$

Substituting these values into Equations $y_{n+x} = p(t_n + xh)$, $x = r, 1, s, 2, u, 3$, we obtain the formulas for the optimized three step hybrid scheme as:

$$\begin{aligned} y_{n+r} = y_n + \frac{h}{3780(\sqrt{5}+3)}&(516\sqrt{5}f_n - 270\sqrt{5}f_{n+j} + 1242\sqrt{5}f_{n+r} + 384\sqrt{5}f_{n+s} \\ &- 1089\sqrt{5}f_{n+1} - 774\sqrt{5}f_{n+2} - 9\sqrt{5}f_{n+3} + 1498\,f_n + 1008\,f_{n+j} + 3528\,f_{n+r} \\ &+ 1792\,f_{n+s} - 567\,f_{n+1} + 378\,f_{n+2} - 77\,f_{n+3}), \end{aligned} \tag{2.11}$$

$$\begin{aligned} y_{n+1} = y_n &- \frac{2}{15}\sqrt{5}hf_{n+j} + \frac{2}{15}\sqrt{5}hf_{n+r} + \frac{106\,hf_n}{945} + \frac{41\,f_{n+j}h}{140} + \frac{41\,f_{n+r}h}{140} \\ &- \frac{16\,f_{n+s}h}{189} + \frac{151\,f_{n+1}h}{420} + \frac{11\,f_{n+2}h}{420} + \frac{f_{n+3}h}{945}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} y_{n+s} = y_n &- \frac{81\sqrt{5}hf_{n+j}}{640} + \frac{81\sqrt{5}hf_{n+r}}{640} + \frac{1037\,hf_n}{8960} + \frac{81\,f_{n+j}h}{280} + \frac{81\,f_{n+r}h}{280} \\ &+ \frac{8\,f_{n+s}h}{35} + \frac{5427\,f_{n+1}h}{8960} - \frac{243\,f_{n+2}h}{8960} - \frac{13\,f_{n+3}h}{8960}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} y_{n+2} = y_n &- \frac{2}{15}\sqrt{5}hf_{n+j} + \frac{2}{15}\sqrt{5}hf_{n+r} + \frac{107\,hf_n}{945} + \frac{2}{7}f_{n+j}h + \frac{2}{7}f_{n+r}h \\ &+ \frac{512\,f_{n+s}h}{945} + \frac{58\,f_{n+1}h}{105} + \frac{23\,f_{n+2}h}{105} + \frac{2\,f_{n+3}h}{945}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} y_{n+j} = y_n + \frac{h}{3780\sqrt{5} - 11340}&(516\sqrt{5}f_n + 1242\sqrt{5}f_{n+j} - 270\sqrt{5}f_{n+r} \\ &+ 384\sqrt{5}f_{n+s} - 1089\sqrt{5}f_{n+1} - 774\sqrt{5}f_{n+2} - 9\sqrt{5}f_{n+3} - 1498\,f_n \\ &- 3528\,f_{n+j} - 1008\,f_{n+r} - 1792\,f_{n+s} + 567\,f_{n+1} - 378\,f_{n+2} + 77\,f_{n+3}), \end{aligned} \tag{2.15}$$

$$y_{n+3} = y_n + \frac{4\,hf_n}{35} + \frac{81\,f_{n+j}h}{140} + \frac{81\,f_{n+r}h}{140} + \frac{16\,f_{n+s}h}{35} + \frac{81\,f_{n+1}h}{140} + \frac{81\,f_{n+2}h}{140} + \frac{4\,f_{n+3}h}{35}. \tag{2.16}$$

3. Properties of the method

In this section, consistency, zero-stability, and linear stability and consequently, the convergence of the designed method, are discussed.

3.1. Consistency of the method.

The block method given in Equations (2.11) to (2.16) can be written in matrix form as

$$Y_{n+\nu} = A_0 Y_n + h(A_1 F_n + A_2 F_{n+\nu}), \tag{3.1}$$

where

$$Y_{n+\nu} = \begin{pmatrix} y_{n+r} \\ y_{n+1} \\ y_{n+s} \\ y_{n+2} \\ y_{n+j} \\ y_{n+3} \end{pmatrix},\ Y_n = \begin{pmatrix} y_{n+r-3} \\ y_{n-2} \\ y_{n+s-3} \\ y_{n-1} \\ y_{n+j-3} \\ y_n \end{pmatrix},\ F_n = \begin{pmatrix} f_{n+r-3} \\ f_{n-2} \\ f_{n+s-3} \\ f_{n-1} \\ f_{n+r-3} \\ f_n \end{pmatrix},\ F_{n+\nu} = \begin{pmatrix} f_{n+r} \\ f_{n+1} \\ f_{n+s} \\ f_{n+2} \\ f_{n+j} \\ f_{n+3} \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{516\sqrt{5}+1498}{3780(3+\sqrt{5})} \\ 0 & 0 & 0 & 0 & 0 & \frac{106}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{1037}{8960} \\ 0 & 0 & 0 & 0 & 0 & \frac{107}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{516\sqrt{5}-1498}{3780(-3+\sqrt{5})} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{35} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \frac{1242\sqrt{5}+3528}{3780(3+\sqrt{5})} & -\frac{1089\sqrt{5}+567}{3780(3+\sqrt{5})} & \frac{384\sqrt{5}+1792}{3780(3+\sqrt{5})} & -\frac{774\sqrt{5}-378}{3780(3+\sqrt{5})} & -\frac{270\sqrt{5}-1008}{3780(3+\sqrt{5})} & -\frac{9\sqrt{5}+77}{3780(3+\sqrt{5})} \\ \frac{2\sqrt{5}}{15}+\frac{41}{140} & \frac{151}{140} & \frac{16}{189} & \frac{11}{420} & \frac{41}{140}-\frac{2\sqrt{5}}{15} & \frac{106}{945} \\ \frac{81\sqrt{5}}{640}+\frac{81}{280} & \frac{5427}{960} & \frac{8}{35} & -\frac{243}{8960} & -\frac{81\sqrt{5}}{640}+\frac{81}{280} & -\frac{13}{8960} \\ \frac{2\sqrt{5}}{15}+\frac{2}{7} & \frac{53}{105} & \frac{512}{945} & \frac{23}{105} & \frac{2}{7}-\frac{2\sqrt{5}}{15} & \frac{2}{945} \\ \frac{270\sqrt{5}-1008}{3780(-3+\sqrt{5})} & \frac{567-1089\sqrt{5}}{3780(-3+\sqrt{5})} & \frac{384\sqrt{5}-1792}{3780(-3+\sqrt{5})} & -\frac{774\sqrt{5}+378}{3780(-3+\sqrt{5})} & \frac{1242\sqrt{5}-3528}{3780(-3+\sqrt{5})} & \frac{77-9\sqrt{5}}{3780(-3+\sqrt{5})} \\ \frac{81}{140} & \frac{81}{140} & \frac{64}{140} & \frac{81}{140} & \frac{81}{140} & \frac{16}{140} \end{pmatrix}.$$

The associated linear difference operator to the method in Equations (2.11) to (2.16), denoted $\mathbf{L}$, is defined as

$$\mathbf{L}[z(x_n);h] = \sum_{i=r,\,1,\,s,\,2,\,j,\,3} [\alpha_i^1 z(x_n+ih) - \alpha_i^0 z(x_n-(i-3)h) - h(\beta_i^1 z'(x_n+ih) + \beta_i^0 z'(x_n-(i-3)h))],$$

given that $z(x)$ is sufficiently differentiable and $\alpha_i^1,\ \alpha_i^0,\ \beta_i^1,\ \beta_i^0$, for each i in the summation, are columns of the matrices $I_6,\ A_0,\ A_1,\ A_2$ respectively. The method in Equations (2.11) to (2.16) is said to be consistent of order p with the system of differential equations (1.1) if after the expansion of the terms $z(x_n+ih)$, $z(x_n-(i-3)h)$, $z'(x_n+ih)$, $z'(x_n-(i-3)h)$ in Taylor series about x_n we get

$$\mathbf{L}[z(x_n);h] = C_0 z(x_n) + C_1 h z'(x_n) + C_2 h^2 z''(x_n) + \cdots + C_q h^q z^{(q)}(x_n) + \cdots,$$

with $C_0 = C_1 = C_2 = \cdots = C_p = 0,\ C_{p+1} \neq 0$. The C_i are constant vectors of length 6 and C_{p+1} is referred to as the error constant. The error constant for the proposed method is

$$C_8 = \left(\frac{\sqrt{5}}{32256} - \frac{1}{10752},\ \ \frac{1}{40320},\ \ -\frac{3}{114688},\ \ \frac{1}{20160},\ \ -\frac{\sqrt{5}}{32256} - \frac{1}{10752},\ \ \frac{3}{4480}\right).$$

This implies that the proposed method is consistent of order 7.

3.2. **Zero-stability.** The difference scheme (2.11)–(2.16) is said to be zero-stable if, as the step size h approaches zero, the scheme is stable. Observe that as $h \to 0$, the formulas (2.11) to (2.16) reduce to a system written in matrix form as

$$Y_{n+\nu} - A_0 Y_n = 0. \tag{3.2}$$

The expression $|I_6 R - A_0|$ gives the first characteristic polynomial of the system (3.2). This polynomial is given by $R^5(R-1)$, whose roots are $R = 0$ and $R = 1$. Clearly, the multiplicity of $R = 1$ is one. Since all the roots lie within the unit disk, the scheme represented by Equations (2.11)–(2.16) is zero-stable.

3.3. **Linear stability.** Using the Dahlquist test equation, this section compares the behavior of the approximate solution provided by the optimized three-step block method as given in section 2 to the behaviour of the exact solution of the test equation

$$y'(t) = \lambda y(t), \;\; \lambda \in \mathbb{C}, \;\; Re(\lambda) < 0. \tag{3.3}$$

The solutions of Equation (3.3) decay as $t \to \infty$. The region within which the approximation given by the designed block method (2.11)–(2.16) behaves in the same way is of utmost interest. The scheme is therefore applied to Equation (3.3). This gives, in matrix form, the system of linear equations

$$(I_6 - h\lambda A_2)Y_{n+\nu} = (A_0 + h\lambda A_1)Y_n,$$

whose solution is given by

$$Y_{n+\nu} = [(I_6 - \tilde{h}A_2)^{-1}(A_0 + \tilde{h}A_1)]Y_n, \;\; \tilde{h} = \lambda h.$$

The linear stability of the scheme is therefore determined by the matrix $Q(\tilde{h}) = (I_6 - \tilde{h}A_2)^{-1}(A_0 + \tilde{h}A_1)$ referred to as the stability matrix of the hybrid block method. The linear stability region is the region in the complex $\tilde{h}$-plane where the spectral radius verifies $|\rho(Q)| \leq 1$. The leading eigenvalue of the matrix Q is

$$\rho(Q) = 2\frac{P(\tilde{h})}{D(\tilde{h})},$$

where

$$\begin{aligned} P(\tilde{h}) = {} & (29495670145515 + 5977634439315\sqrt{5})\tilde{h}^6 \\ & + (28223566716315 + 52513097973654\sqrt{5})\tilde{h}^5 + (876276237734140 + 103478357498492\sqrt{5})\tilde{h}^4 \\ & + (1837749589939760 + 326469663245600\sqrt{5})\tilde{h}^3 + (168518315040264 \\ & + 192909577718480\sqrt{5})\tilde{h}^2 + (726735158553600 + 102166772467200\sqrt{5})\tilde{h} \\ & + 930575507904000 + 130823306208000\sqrt{5}, \end{aligned}$$

and

$$\begin{aligned} D(\tilde{h}) = {} & 48962952664770\tilde{h}^6 - (284525066721975 + 17684764444383\sqrt{5})\tilde{h}^5 \\ & + (453764539078700 + 76344426925696\sqrt{5})\tilde{h}^4 - (1954396707935840 \\ & + 278539631137040\sqrt{5})\tilde{h}^3 + (2339140799338560 + 344817628210720\sqrt{5})\tilde{h}^2 \\ & - (4129982730316800 + 580606292313600\sqrt{5})\tilde{h} + 1861151015808000 \\ & + 261646612416000\sqrt{5}. \end{aligned}$$

The linear stability region is shown in Figure 1.

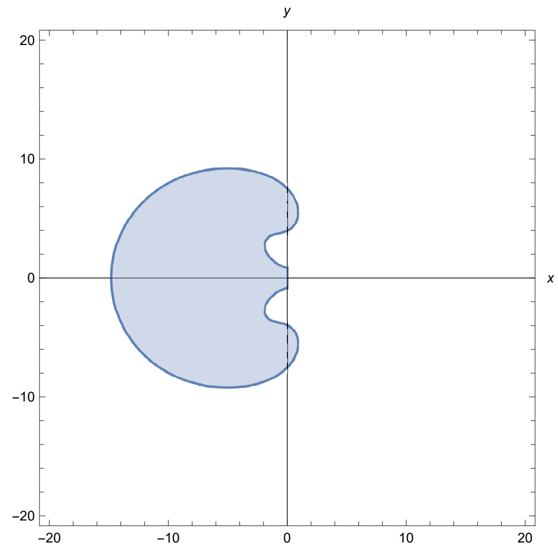


FIGURE 1. Stability region of the block method (2.11)–(2.16).

3.4. **Improving the scheme.** To enhance the performance of the proposed method, we consider varying the step-size taking into account an estimate of the local error. This approach is based on the strategy given by Shampine [24]. A lower order formula is employed to approximate the solution of the differential equation within each and every block. This is done to estimate the local error upon which the choice of the step-size is based until a threshold (a set tolerance) is reached. Hence, the lower-order method is used to estimate the local error while the higher-order method is used for advancing the integration. To ensure that this procedure does not introduce additional computational costs, the lower-order method is designed to utilize function evaluations that have already been computed by the higher-order method. In this article, a fifth-order linear multi-step formula is used, given by

$$\tilde{y}_{n+3} = y_n + \frac{1}{10}\left(1323 + 621\sqrt{5}\right) y_{n+r} + \frac{1}{2}\left(513 + 135\sqrt{5}\right) y_{n+1} - \frac{1}{5}\left(1944 + 648\sqrt{5}\right) y_{n+s}$$
$$+ h\left[\left(27 + \frac{54}{5}\sqrt{5}\right) f_{n+r} + \frac{1}{2}\left(351 + 135\sqrt{5}\right) f_{n+1} + \left(84 + \frac{108}{5}\sqrt{5}\right) f_{n+s}\right],$$

with local truncation error $\left(\frac{21}{640} + \frac{3}{128}\sqrt{5}\right) f^{(6)}h^6 + O(h^7)$. The local error estimate (EST) is obtained at the endpoint t_{n+3} by the formula

$$EST = \|y_{n+3} - \tilde{y}_{n+3}\|.$$

This estimate is then used to adapt the step size according to

$$h_{new} = \nu h_{old}\left(\frac{tol}{EST}\right)^{\frac{1}{q+1}},$$

where q is the order of the lower order method, h_{old} is the previous step-size, $0 \leq \nu < 1$ is referred to as the safety factor (this is set in order to reduce the number of failed steps) and tol is the tolerance set by the user. This approach effectively adapts the step-size according to the local error estimate.

The variable step-size approach can be summarized in the following steps:

- Obtain EST.
- If $EST < tol$, accept the solution and double the step-size to proceed with the integration to the next time level.
- If $EST > tol$, reject the solution and repeat the computation using the new step-size h_{new}.

To prevent large fluctuations in step sizes, we set maximum (h_{maxi}) and minimum step-sizes (h_{mini}) and if $h_{mini} \leq h_{new} \leq h_{maxi}$, then $h_{old} = h_{new}$.

The next section presents numerical experiments comparing fixed-step and variable-step formulations. Additionally, a simplified formulation of the method is introduced to improve efficiency. The advantages of using those approaches are obvious as can be seen from the tables provided.

4. Numerical experiments

In this section, the method is applied to some differential equations and systems of differential equations to show its efficiency. In these experiments, a simpler formulation of the formulas in Equations (2.11)–(2.16) has been also considered. In this formulation the of occurrences of f_{n+v} is reduced and is given as follows:

$$\begin{aligned} hf_{n+r} = -\frac{1}{540}&((1050+75\sqrt{5})y_n + 150hf_n + (54\sqrt{5}-810)y_{n+r} + 675(1-\sqrt{5})y_{n+1} \\ &-(1280-768\sqrt{5})y_{n+s} + (1350-675\sqrt{5})y_{n+2} - (810-378\sqrt{5})y_{n+j} - (175-75\sqrt{5})y_{n+3}), \end{aligned} \tag{4.1}$$

$$\begin{aligned} hf_{n+1} = -\frac{1}{270}&(-195y_n - 30hf_n + 108(2+\sqrt{5})y_{n+r} + 135y_{n+1} - 512y_{n+s} \\ &+135y_{n+2} + 108(2-\sqrt{5})y_{n+j} + 5y_{n+3}), \end{aligned} \tag{4.2}$$

$$\begin{aligned} hf_{n+s} = -\frac{1}{1920}&(925y_n + 150hf_n - (810+486\sqrt{5})y_{n+r} + 4050y_{n+1} - 1280y_{n+s} \\ &-2025y_{n+2} - (810-486\sqrt{5})y_{n+j} - 50y_{n+3}), \end{aligned} \tag{4.3}$$

$$\begin{aligned} hf_{n+2} = \frac{1}{135}&(90y_n + 15hf_n - 54(1+\sqrt{5})y_{n+r} + 270y_{n+1} - 512y_{n+s} \\ &+270y_{n+2} - 54(1-\sqrt{5})y_{n+j} - 10y_{n+3}), \end{aligned} \tag{4.4}$$

$$\begin{aligned} hf_{n+j} = -\frac{1}{540}&((1050-75\sqrt{5})y_n + 150hf_n - (810+378\sqrt{5})y_{n+r} + 675(1+\sqrt{5})y_{n+1} \\ &-(1280+768\sqrt{5})y_{n+s} + (1350+675\sqrt{5})y_{n+2} - (810+54\sqrt{5})y_{n+j} - (175+75\sqrt{5})y_{n+3}), \end{aligned} \tag{4.5}$$

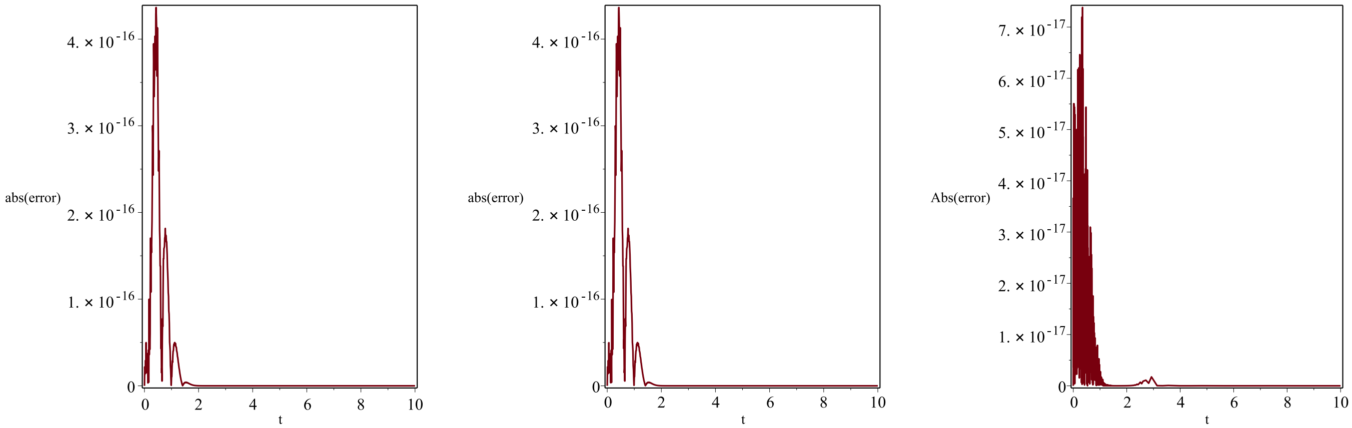
$$hf_{n+3} = -\frac{1}{30}(-175y_n - 30hf_n + 324y_{n+r} - 405y_{n+1} + 512y_{n+s} - 405y_{n+2} + 324y_{n+j} - 175y_{n+3}). \tag{4.6}$$

Experiments 1 and 2 illustrate the application of the proposed scheme to non stiff linear and nonlinear problems. The purpose of these examples is to demonstrate that the proposed method can compete with existing public domain solvers. The results are, therefore, compared with Matlab ode15 solver. Application of the method to linear non stiff system is shown in Experiment 3. The result is compared with a recently published existing method in the literature. Experiments 4 and 5 show the application of the proposed scheme to systems of stiff linear initial value problems. The comparison of the results with existing methods in the literature is also presented. Experiments 6 and 7 showcase the robustness of the method in approximating the solutions of homogeneous and nonhomogeneous stiff nonlinear systems. The efficiency of the method is shown by comparing its results with those of some recently presented methods. In order to demonstrate that the optimized three-step hybrid block method with three intra-step points can compete with known initial value solvers, the method was applied to a nonhomogeneous stiff nonlinear problem in Experiment 8. The result is more accurate than the result obtained via the Matlab ode15.

Experiment 1. Approximate the solution of

$$y'(t) = -10ty(t),\ t \in [0,10],\ y(0) = 1.$$

The exact solution of the equation is $y(t) = \exp(-5t^2)$.



(A) Regular formulation. (B) Simpler formulation. (C) Adaptive stepsize.

FIGURE 2. Comparison of the error due to the block method (2.11)–(2.16) and its simpler formulation (4.1)–(4.6) for Experiment 1.

TABLE 1. Comparison of final time of integration (FT), number of steps taken (NS), maximum absolute error (MAE) and CPU time for the regular formulation (RF); simpler formulation (SF), regular formulation in variable step size (Var RF) and simpler formulation in variable step size (Var SF) approaches for Experiment 1.

Approach	FT	NS	MAE	CPU Time
RF	10.0	81	$1.872E-7$	$0.0468\,Sec$
SF	10.0	81	$1.872E-7$	$0.0468\,Sec$
Var RF	10.0	81	$8.190E-13$	$0.0468\,Sec$
Var SF	10.0	81	$8.190E-13$	$0.0468\,Sec$
ode15s	10.0		$3.540E-4$	$0.9905\,Sec$

TABLE 2. Comparison of maximum absolute error (MAE) with existing method for Experiment 1. The simpler formulation (SF) of the formulas was employed here and in all other examples.

Step size	Maximum Error		
	[14]	Proposed	Proposed (Variable Step-Size)
0.01	$E = 4.69 \times 10^{-16}$	$E = 4.36 \times 10^{-16}$	$h_0 = 0.25,\ T = 10,\ E = 1.84 \times 10^{-16}$

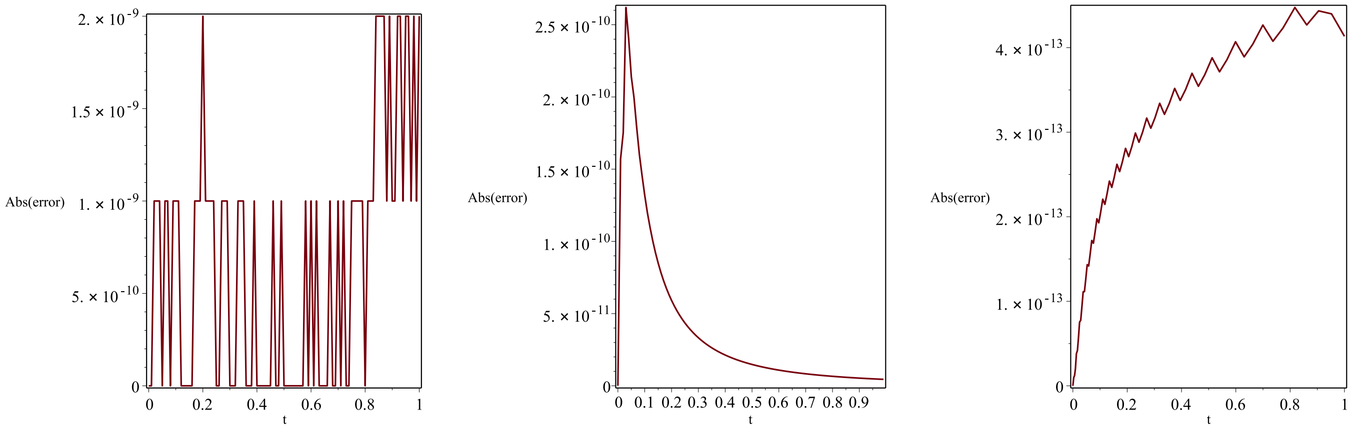
It is observed from Figure 2 and Table 1 that the simpler formulation produces the same results as the regular formulation while the adaptive step-size approach reduces the error significantly. Comparing the approximation by the proposed method with the result in [14] in Table 2, it is obvious that the proposed scheme performs favorably and the adaptive step-size scheme outperforms the fixed step-size methods.

Experiment 2. Approximate the solution of the nonlinear equation

$$y'(t) = -10(1-y(t))^2,\ t \in [0,\,10],\ y(0) = 2.$$

Its exact solution is $y(t) = \dfrac{2+10t}{1+10t}$.

Figure 3 and Table 3 show that the simpler formulation reduces the computational time while the adaptive step-size approach reduces the error significantly without extra computational cost for Experiment 2.



(A) Regular formulation. (B) Simpler formulation. (C) Adaptive stepsize.

Figure 3. Comparison of the error due to the block method (2.11)–(2.16) and its simpler formulation (4.1)–(4.6).

Table 3. Comparison of final time of integration (FT), number of steps taken (NS), maximum absolute error (MAE) and CPU time for the regular formulation (RF); simpler formulation (SF), regular formulation in variable step size (Var RF) and simpler formulation in variable step size (Var SF) approaches.

Approach	FT	NS	MAE	CPU Time
RF	10.0	111	$1.616E-4$	$0.078\,Sec$
SF	10.0	111	$1.616E-4$	$0.063\,Sec$
Var RF	10.0	111	$2.619E-10$	$0.078\,Sec$
Var SF	10.0	111	$2.619E-10$	$0.063\,Sec$
ode15s	10.0		$3.986E-4$	$0.787\,Sec$

Experiment 3. Obtain the approximate solution of the linear system

$$\begin{aligned} y_1'(t) &= y_1(t) + y_2(t),\ \ y_1(0) = 0,\\ y_2'(t) &= -y_1(t) + y_2(t),\ \ y_2(0) = 1. \end{aligned}$$

The exact solution of this system is

$$\begin{aligned} y_1(t) &= e^t \sin(t),\\ y_2(t) &= e^t \cos(t). \end{aligned}$$

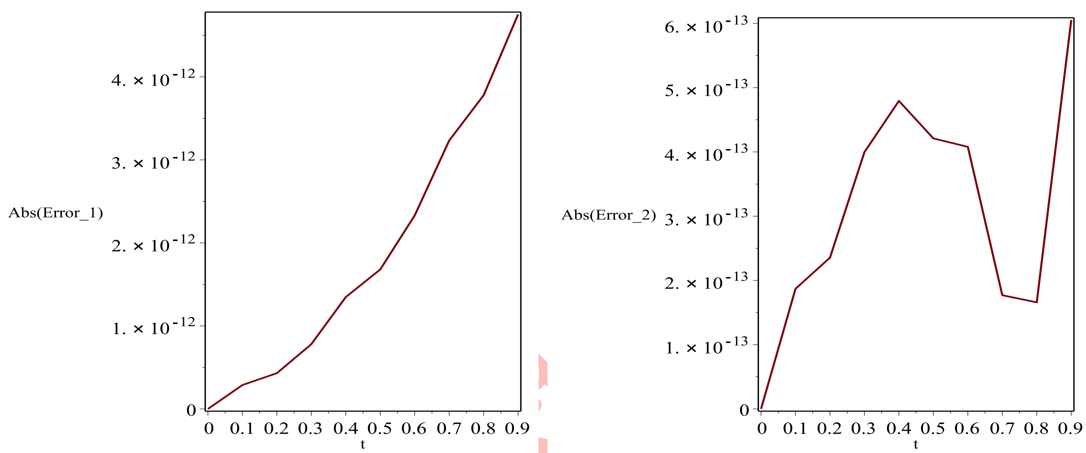
Obviously, from Table 4 and Figures (4–5), the proposed method performs better than the result in [22]. In addition, the variable step-size method is more accurate than the fixed step-size method.

Experiment 4. Obtain the approximate solution of the linear system

$$\begin{aligned} y_1'(t) &= -21y_1(t) + 19y_2(t) - 20y_3(t),\ \ y_1(0) = 1,\\ y_2'(t) &= 19y_1(t) - 21y_2(t) + 20y_3(t),\ \ y_2(0) = 0,\\ y_3'(t) &= 40y_1(t) - 40y_2(t) + 40y_3(t),\ \ y_3(0) = -1. \end{aligned} \tag{4.7}$$

TABLE 4. Comparison of maximum absolute error (MAE) and rate of convergence of the proposed method with existing method. The simpler formulation (SF) of the formulas are employed here and in all other examples.

Step size	Maximum Error		
	[22]	Proposed	Proposed (Variable Step-Size)
0.2	$E_1 = 2.01 \times 10^{-5}$	$E_1 = 1.12 \times 10^{-9}$	$h_0 = 0.25$
	$E_2 = 1.89 \times 10^{-5}$	$E_2 = 1.39 \times 10^{-10}$	
0.167	$E_1 = 4.86 \times 10^{-7}$	$E_1 = 2.42 \times 10^{-10}$	$Tol = 10^{-9}$
	$E_2 = 1.32 \times 10^{-6}$	$E_2 = 3.07 \times 10^{-11}$	
0.1	$E_1 = 6.97 \times 10^{-11}$	$E_2 = 4.75 \times 10^{-12}$	$Max\ Error_1 = 1.78 \times 10^{-15}$
	$E_2 = 4.64 \times 10^{-11}$	$E_2 = 6.05 \times 10^{-13}$	$Max\ Error_2 = 6.66 \times 10^{-15}$



(A) $|y_1 - y_{1exact}|$ (B) $|y_2 - y_{2exact}|$

FIGURE 4. Error plot of the fixed step-size block method's simpler formulation (4.1)–(4.6) for Experiment 3.

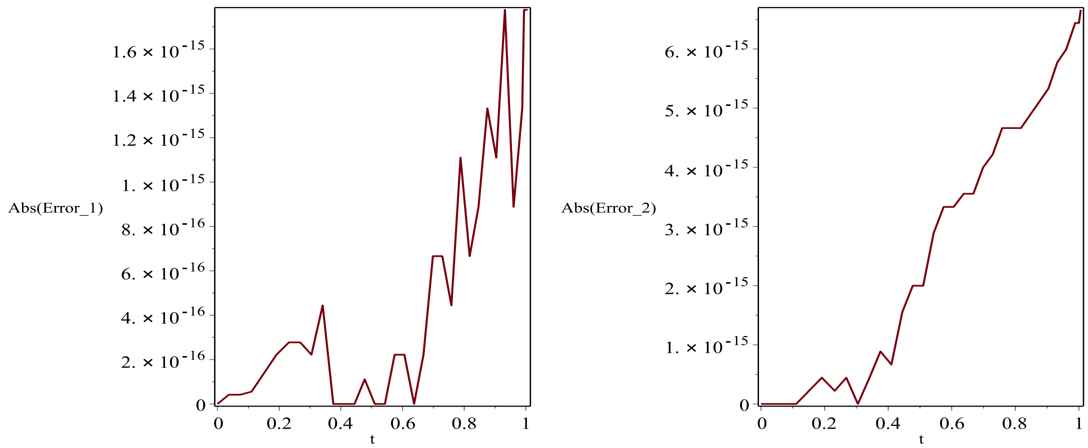
The exact solution to this system is

$$y_1(t) = \frac{1}{2}\left(e^{-2t} + e^{-40t}(\cos 40\,t + \sin 40\,t)\right),$$
$$y_2(t) = \frac{1}{2}\left(e^{-2t} - e^{-40t}(\cos 40\,t + \sin 40\,t)\right),$$
$$y_3(t) = e^{-40t}(\cos 40\,t - \sin 40\,t).$$

For each of the step-sizes considered, the proposed scheme is more accurate than the method in [34]. The numerical rates of convergence (shown in parenthesis in Table 5) agree with the order of consistency of the method as computed in section 3.1. The numerical rate of convergence p is computed from the formula

$$p = \frac{\log(E_1) - \log(E_2)}{\log(h_1) - \log(h_2)},$$

where $E_i,\ i = 1, 2$ is the maximum absolute error obtained when the method is used with step-size h_i.



(A) $|y_1 - y_{1exact}|$ (B) $|y_2 - y_{2exact}|$

Figure 5. Error plot of the variable step-size block method's simpler formulation (4.1)–(4.6) for Experiment 3. The initial step-size is $h = 0.25$, integration time is $T = 1.0$ and tolerance is $Tol = 10^{-9}$.

Table 5. Comparison of maximum absolute error (MAE) and rate of convergence of the proposed method with existing method. The simpler formulation (SF) of the formulas are employed here and in all other examples.

Step size	Maximum Error		
	[34] (Rate)	Proposed (Rate)	Proposed (Variable Step-Size)
0.05	1.26×10^{-1}	4.30×10^{-3}	$h_0 = 0.1$
0.025	1.45×10^{-2} (2.40)	2.13×10^{-5} (7.66)	$Tol = 10^{-9}$
0.0125	1.51×10^{-3} (5.70)	1.65×10^{-7} (7.01)	$Max\ Error = 1.67 \times 10^{-13}$
0.00625	1.14×10^{-4} (7.27)	1.29×10^{-9} (7.00)	-
0.003125	4.87×10^{-6} (7.46)	4.93×10^{-12} (8.03)	-

Experiment 5. Obtain the approximate solution of the stiff linear system

$$\begin{aligned} y_1'(t) &= -2y_1(t) + y_2(t) + 2\sin t, \quad y_1(0) = 2, \\ y_2'(t) &= 998y_1(t) - 999y_2(t) + 999(\cos t - \sin t), \quad y_2(0) = 3. \end{aligned} \tag{4.8}$$

The exact solution to the system is

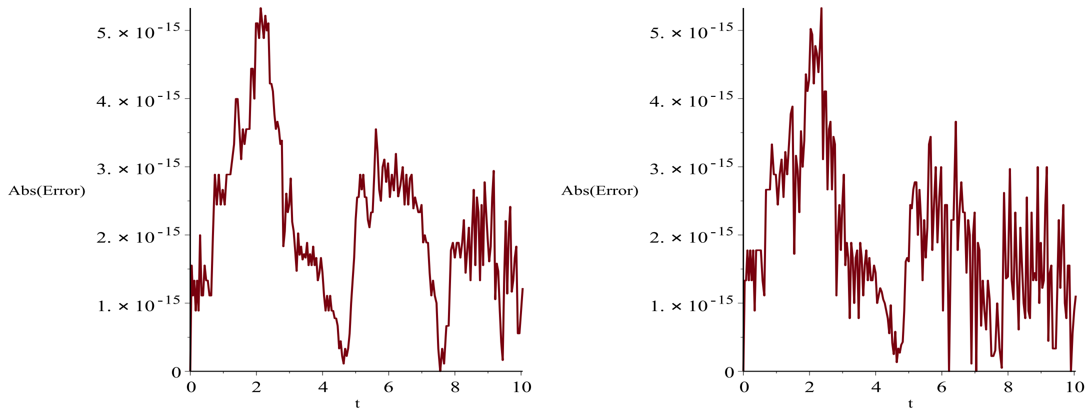
$$\begin{aligned} y_1(t) &= 2e^{-t} + \sin t, \\ y_2(t) &= 2e^{-t} + \cos t. \end{aligned}$$

When the method was applied to Experiment 5, the obtained result is more accurate than the one in [3]. We highlight that the method in [3] is of the same order as the proposed method. The variable step-size is more accurate than the fixed step-size approach as shown in Table 6 and Figure 6.

Experiment 6. Obtain the approximate solution of the homogeneous, nonlinear stiff system

$$\begin{aligned} y_1'(t) &= -1002\, y_1(t) + 1000\, (y_2(t))^2, \quad y_1(0) = 1, \\ y_2'(t) &= y_1(t) - y_2(t)(1 + y_2(t)), \quad y_2(0) = 1. \end{aligned}$$

The exact solution to this system is $y_1(t) = e^{-2t}$, $y_2(t) = e^{-t}$.



(A) $|y_1 - y_{1exact}|$. (B) $|y_2 - y_{2exact}|$.

FIGURE 6. Error plot of the variable step-size block method's simpler formulation (4.1)–(4.6) for Experiment 5. The initial step-size is $h = 0.4$, integration time is $T = 10$ and tolerance is $Tol = 10^{-9}$

TABLE 6. Comparison of maximum absolute error (MAE) and rate of convergence of the proposed method with existing method. The simpler formulation (SF) of the formulas are employed here and in all other examples.

Step size	Maximum Error		
	[3]	Proposed	Proposed (Variable Step-Size)
0.4	8.9924×10^{-7}	9.1391×10^{-9}	$h_0 = 0.4$
0.2	5.9042×10^{-9} (7.25)	3.5091×10^{-11} (8.02)	$Tol = 10^{-9}$
0.1	4.5695×10^{-11} (7.01)	2.2471×10^{-13} (7.29)	Max. Error $= 6.2 \times 10^{-15}$
0.05	2.9376×10^{-13} (7.28)	5.1868×10^{-15} (5.44)	-

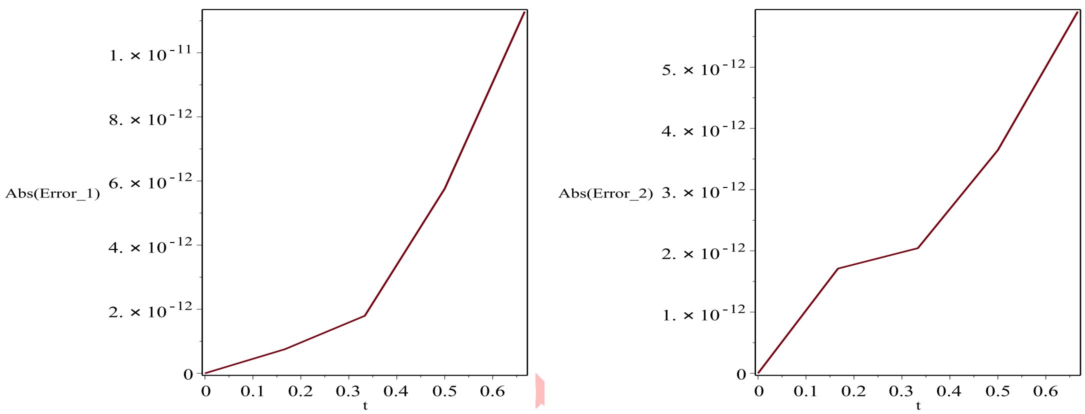
TABLE 7. Comparison of maximum absolute error (MAE) and rate of convergence of the proposed method with existing method. The simpler formulation (SF) of the formulas are employed here and in all other examples.

Step size	Maximum Error (E)			
	[3]	[22]	Proposed	Proposed (Variable Step-Size)
1.25	$E_1 = 2.3 \times 10^{-9}$	-	$E_1 = 3.0 \times 10^{-6}$	$h_0 = 0.4$
	$E_2 = 2.8 \times 10^{-5}$	-	$E_2 = 1.6 \times 10^{-7}$	
0.83	$E_1 = 2.3 \times 10^{-9}$	-	$E_1 = 4.78 \times 10^{-7}$	$Tol = 10^{-7}$
	$E_2 = 2.8 \times 10^{-5}$	-	$E_2 = 2.34 \times 10^{-9}$	
0.625	$E_1 = 2.3 \times 10^{-9}$	-	$E_1 = 9.39 \times 10^{-7}$	Max. Error: $E_1 = 1.9 \times 10^{-12}$
	$E_2 = 2.9 \times 10^{-5}$	-	$E_2 = 7.11 \times 10^{-10}$	$E_2 = 1.4 \times 10^{-12}$
0.5	$E_1 = 2.3 \times 10^{-9}$	-	$E_1 = 2.85 \times 10^{-8}$	-
	$E_2 = 2.9 \times 10^{-5}$	-	$E_2 = 2.33 \times 10^{-12}$	-
0.167	-	$E_1 = 4.37 \times 10^{-6}$	$E_1 = 3.48 \times 10^{-10}$	-
	-	$E_2 = 4.08 \times 10^{-8}$	$E_2 = 2.29 \times 10^{-12}$	-

Remark 2. In [3], the problem in Experiment 6 was integrated from $t = 0$ to $t = 10.0$ using the corresponding step-sizes while it was integrated from $t = 0$ to $t = 1.0$ using six points in [22].

TABLE 8. Comparison of maximum absolute error (MAE) and rate of convergence of the proposed method with existing method. The simpler formulation (SF) of the formulas are employed here and in all other examples.

Step size	Maximum Error (E)		
	[22]	Proposed	Proposed (Variable Step-Size)
0.25	$E_1 = 1.79 \times 10^{-4}$ $E_2 = 6.70 \times 10^{-5}$	$E_1 = 3.17 \times 10^{-10}$ $E_2 = 1.44 \times 10^{-10}$	$h_0 = 0.25$
0.20	$E_1 = 6.83 \times 10^{-6}$ $E_2 = 2.82 \times 10^{-6}$	$E_1 = 7.84 \times 10^{-11}$ $E_2 = 3.20 \times 10^{-11}$	$Tol = 10^{-9}$
0.167	$E_1 = 1.96 \times 10^{-7}$ $E_2 = 1.04 \times 10^{-7}$	$E_1 = 1.13 \times 10^{-11}$ $E_2 = 5.91 \times 10^{-12}$	Max. Error: $E_1 = 5.33 \times 10^{-15}$ $E_2 = 1.78 \times 10^{-15}$



(A) $|y_1 - y_{1exact}|$ (B) $|y_2 - y_{2exact}|$

FIGURE 7. Error plot of the fixed step-size block method's simpler formulation (4.1)–(4.6) for Experiment 7.

Table 7 shows that the proposed method is more accurate than the method proposed earlier in [3] and [22] while the variable step-size method is more accurate than the fixed step-size methods when applied to the problem in Experiment 6.

Experiment 7. Obtain the approximate solution of the nonhomogeneous, nonlinear stiff system

$$y_1'(t) = -1002\, y_1(t) + 1000\, (y_2(t))^2 + 3003e^t + 2 - 1000e^{2t}, \;\; y_1(0) = 2,$$
$$y_2'(t) = y_1(t) - y_2(t)(1 + y_2(t)) - 5e^t + 1 + e^{2t}, \;\; y_2(0) = 0.$$

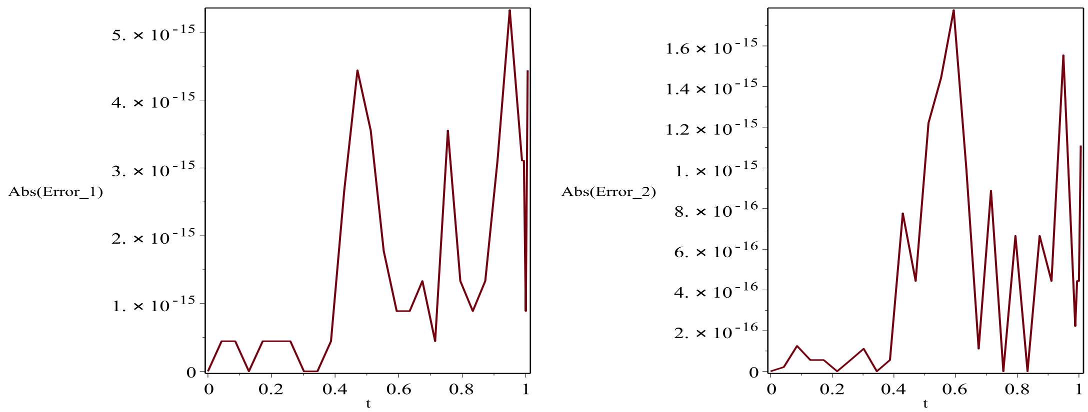
The exact solution to this system is $y_1(t) = 1 + e^t$, $y_2(t) = 1 - e^t$.

The solution of the system in Experiment 7 obtained in [22] is compared with the result obtained using the proposed method in Table 8. The proposed method is more accurate than the method in [22]. Both Table 8 and Figures 7 and 8 show that the variable step-size method is more accurate than the fixed step-size methods.

Experiment 8. Obtain the approximate solution of the nonlinear stiff system

$$y_1'(t) = y_2(t) - y_1(t)^2 - (1 + t), \quad t \in [0, 100],$$
$$y_2'(t) = 1 - 20(y_2(t)^2 - (1 + t)^2), \;\; y_1(0) = 1, \;\; y_2(0) = 1.$$

The exact solution to this system of equations is $y_1(t) = \dfrac{1}{1+t}$, $y_2(t) = 1 + t$.



(A) $|y_1 - y_{1exact}|$. (B) $|y_2 - y_{2exact}|$.

FIGURE 8. Error plot of the variable step-size block method's simpler formulation (4.1)–(4.6) for Experiment 7. The initial step-size is $h = 0.25$, integration time is $T = 1.0$ and tolerance is $Tol = 10^{-9}$.

TABLE 9. Comparison of final time of integration (FT), number of steps taken (NS), maximum absolute error (MAE) and CPU time for the regular formulation (RF); simpler formulation (SF), regular formulation in variable step size (Var RF) and simpler formulation in variable step size (Var SF) approaches for $y_1(t)$.

Approach	FT	NS	MAE	CPU Time
RF	100	168	$2.22E-4$	$0.2946\,Sec$
SF	100	168	$2.22E-4$	$0.1803\,Sec$
Var RF	100	168	$3.62E-14$	$0.1650\,Sec$
Var SF	100	168	$5.71E-14$	$0.1498\,Sec$
ode15s	100		0.001	$0.8500\,Sec$

TABLE 10. Comparison of final time of integration (FT), number of steps taken (NS), maximum absolute error (MAE) and CPU time for the regular formulation (RF); simpler formulation (SF), regular formulation in variable step size (Var RF) and simpler formulation in variable step size (Var SF) approaches for $y_2(t)$.

Approach	FT	NS	MAE	CPU Time
RF	100	168	$2.84E-14$	$0.2946\,Sec$
SF	100	168	$2.84E-14$	$0.1803\,Sec$
Var RF	100	168	$1.77E-15$	$0.1650\,Sec$
Var SF	100	168	$1.99E-15$	$0.1498\,Sec$
ode15s	100		$4.090E-5$	$0.8500\,Sec$

Though the simpler formulation does not reduce the computational error compared to the regular formulation, it does reduce the computational time, as shown in Tables 9 and 10. A reduction in both computational time and error is observed when the adaptive step-size method is applied to the problem.

5. Conclusion

This work presented the analysis of an optimized hybrid three step block method for system of first order differential equations. A-stability, linear stability, and consistency analyses were carried out and the convergence of the method was shown. The numerical robustness of the method was confirmed via some numerical experiments conducted both on scalar equations and systems of ordinary differential equations. An adaptive step size approach was employed to reduce error and computation time thereby improving the accuracy of the method. Overall, the simplified formulation reduces computational time in almost all the experiments considered. It was shown that the proposed method outperforms existing methods.

The authors are optimistic that the result in this article can be employed to numerically investigate the solutions of the fractional quantum-difference equations whose existence of unique solution has just been studied in [9] and in the literatures therein. In addition, applying the operator splitting approach, the proposed method can also be employed to the degenerated sub-equation of the boundary value problem of the bi-singular perturbed quasi-linear elliptic equation [21] while the other sub-equation may be approximated by the Micken's nonstandard finite difference method [2].

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