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A hybrid Perturbation-Reproducing kernel method for numerical solution of Duffing oscillator

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Abstract

This paper introduces a novel semi-analytical technique for solving nonlinear ordinary differential equations with perturbed parameters. The proposed approach combines perturbation methods with reproducing kernel Hilbert space (RKHS) theory. The method transforms nonlinear periodic problems into systems of linear differential equations, which are then solved using RKHS techniques. We establish convergence properties and provide error estimates for the approximate solutions. Several numerical examples of Duffing oscillators demonstrate the effectiveness and accuracy of the proposed algorithm. Numerical results show that the method produces highly accurate approximate solutions, with performance comparable to established numerical techniques such as the fourth-order Runge-Kutta method.

Keywords. Duffing oscillator, Perturbation method, Reproducing kernel Hilbert space method, Error estimate, Nonlinear oscillations.

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1. Introduction

Nonlinear differential equations play a crucial role in modeling complex phenomena across science and engineering disciplines, describing how variables evolve with respect to one another. Many practical problems require solving initial-boundary value problems, finding solutions to differential equations that satisfy given initial and/or boundary conditions. These equations have attracted substantial research attention due to their applications in engineering, physics, economics, biology, and other fields [29]-[36].

In real-world scenarios, the nonlinear differential equations that model physical systems are often too complex for exact analytical solutions. One common approach involves simplifying the original problem to obtain an analytically tractable approximation, then using this approximate solution to estimate the solution of the original system. This paper examines an approach that combines perturbation methods for linearization with reproducing kernel Hilbert space (RKHS) methods for solving the resulting linear systems. This hybrid approach offers improved accuracy and reliable error estimates compared to traditional methods.

In 1918, Duffing introduced the differential equation now known as the Duffing equation for electronic circuit analysis [29, 30]. The classical Duffing equation represents the simplest oscillator exhibiting catastrophic changes in amplitude and phase when the forcing frequency varies gradually. The Duffing equation finds applications in diverse areas including signal processing [34], propagation of extremely short electromagnetic pulses in nonlinear media [23, 37], brain modeling [24], fuzzy modeling and adaptive control of uncertain chaotic systems [14, 33, 39], among others [20]-[22].

This paper presents a new computational method based on reproducing kernel Hilbert spaces for solving Duffing oscillator problems. We first apply perturbation methods to linearize the nonlinear Duffing equation, then employ RKHS techniques to solve the resulting linear systems. We investigate the convergence of approximate solutions and derive error estimates.

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Many conservative quasi-linear systems with one degree of freedom exhibit free oscillations described by second-order differential equations of the form:

$$\frac{d^2y}{d\tau^2} + \eta\frac{dy}{d\tau} + N(y) = \lambda\cos(\varpi\tau) + \mu\sin(\varpi\tau), \tag{1.1}$$

where y depends on τ, N is a nonlinear function of y, $\frac{dy}{d\tau}$ and $\frac{d^2y}{d\tau^2}$ represent system accelerations, $N(y)$ is the restoring force, and λ, μ, η, and ϖ are real parameters.

If N is analytic at $y = y_0$, its Taylor series expansion is:

$$N(y) = N(y_0) + (y-y_0)\frac{dN}{dy}(y_0) + \frac{(y-y_0)^2}{2!}\frac{d^2N}{dy^2}(y_0) + \frac{(y-y_0)^3}{3!}\frac{d^3N}{dy^3}(y_0) + \dots. \tag{1.2}$$

Assuming $y = y_0$ is an equilibrium position ($N(y_0) = 0$), equation (1.1) becomes:

$$\frac{d^2y}{d\tau^2} + \eta\frac{dy}{d\tau} + \kappa_1(y-y_0) + \kappa_2(y-y_0)^2 + \kappa_3(y-y_0)^3 + \dots = \lambda\cos(\varpi\tau) + \mu\sin(\varpi\tau), \tag{1.3}$$

where $\kappa_j = \frac{1}{j!}\frac{d^jN}{dy^j}(y_0),\ j = 1, 2, \dots.$

Introducing $\Phi = y - y_0$ transforms (1.3) to:

$$\frac{d^2\Phi}{d\tau^2} + \eta\frac{d\Phi}{d\tau} + \kappa_1\Phi + \kappa_2\Phi^2 + \kappa_3\Phi^3 + \dots = \lambda\cos(\varpi\tau) + \mu\sin(\varpi\tau). \tag{1.4}$$

This paper focuses on the special case:

$$\frac{d^2\Phi}{d\tau^2} + \eta\frac{d\Phi}{d\tau} + \kappa_1\Phi + \kappa_3\Phi^3 = \lambda\cos(\varpi\tau) + \mu\sin(\varpi\tau), \tag{1.5}$$

where $\kappa_1 > 0$ and κ_3 is arbitrary. Equation (1.5) is the Duffing equation.

Following [30], we nondimensionalize the equation using characteristic length L and time T, with $t = \frac{\tau}{T}$ and $u = \frac{\Phi}{L}$. According to the chain rule for differentiation, we have:

$$\frac{d}{d\tau} = \frac{1}{T}\frac{d}{dt}, \quad \frac{d^2}{d\tau^2} = \frac{1}{T^2}\frac{d^2}{dt^2}.$$

Equation (1.5) becomes:

$$\frac{d^2u}{dt^2} + \eta T\frac{du}{dt} + \kappa_1 T^2 u + \kappa_3 T^2 L^2 u^3 = \frac{T^2\lambda}{L}\cos(\varpi Tt) + \frac{T^2\mu}{L}\sin(\varpi Tt). \tag{1.6}$$

Choosing $T = \frac{\omega}{\sqrt{\kappa_1}}$ and defining $\varepsilon = \frac{\kappa_3\omega^2L^2}{\kappa_1}$, $\delta = \frac{\eta\omega}{\sqrt{\kappa_1}}$, $\Omega = \frac{\varpi\omega}{\sqrt{\kappa_1}}$, $A = \frac{\omega^2\lambda}{\kappa_1 L}$, and $B = \frac{\omega^2\mu}{\kappa_1 L}$ yields:

$$\frac{d^2u}{dt^2} + \delta\frac{du}{dt} + \omega^2 u + \varepsilon u^3 = A\cos(\Omega t) + B\sin(\Omega t). \tag{1.7}$$

Here, ε is a dimensionless parameter measuring nonlinearity strength. Initial conditions are:

$$u(0) = a, \quad \frac{du}{dt}(0) = b. \tag{1.8}$$

The solution u depends on both t and ε, denoted as $u = u_\varepsilon(t)$.

Several oscillators can be transformed into Duffing oscillator form:

- **(i)** Simple pendulum: $\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$.
- **(ii)** Second oscillator: $\frac{d^2\theta}{dt^2} = \Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$.
- **(iii)** Third oscillator: $\frac{d^2u}{dt^2} + \frac{\omega^2 u}{1+u^2} = 0$.

Appendix A details the transformation of problems (i)-(iii) to Equation (1.3).

This paper is organized as follows: Section 2 introduces perturbation methods for transforming the Duffing equation into linear systems. Sections 3 and 4 present reproducing kernel Hilbert space fundamentals and their application to our problem. Section 5 provides numerical examples demonstrating algorithm performance. Section 6 concludes the paper.

2. PERTURBATION METHOD FOR LINEARIZATION

This section investigates solutions to Duffing-type differential equations with small perturbed parameter ε $(0 < \varepsilon \ll 1)$ using perturbation methods. For background on perturbation theory, see [7, 18, 29, 30].

When $\varepsilon = 0$, Equation (1.7) reduces to the linear non-homogeneous differential equation:

$$\frac{d^2u}{dt^2} + \delta\frac{du}{dt} + \omega^2 u = A\cos(\Omega t) + B\sin(\Omega t). \tag{2.1}$$

Assuming $0 \le \delta < 2\omega$ and initial conditions (1.8), the solution is:

$$u(t) = \exp\left(-\frac{\delta}{2}t\right)\Big(a\cos(\sigma t) + b\sin(\sigma t)\Big) + \Big(C\cos(\Omega t) + D\sin(\Omega t)\Big), \tag{2.2}$$

where $\sigma = \sqrt{\omega^2 - \frac{\delta^2}{4}}$ and $C = \dfrac{A(\omega^2-\Omega^2)+B\delta\Omega}{(\omega^2-\Omega^2)^2+\delta^2\Omega^2}$, $D = \dfrac{B(\omega^2-\Omega^2)-A\delta\Omega}{(\omega^2-\Omega^2)^2+\delta^2\Omega^2}$ are parameters..

For small $\varepsilon \neq 0$, the solution is no longer given by (2.2). We consider a correction expressed as a power series in ε:

$$u(t) = \sum_{k=0}^{\infty} u_k(t)\varepsilon^k = u_0(t) + u_1(t)\varepsilon + u_2(t)\varepsilon^2 + \cdots, \tag{2.3}$$

with initial conditions:

$$u_0(0) = a,\ u_0'(0) = b, \quad u_k(0) = u_k'(0) = 0,\ k = 1, 2, 3, \ldots. \tag{2.4}$$

Now, differentiating both sides of Equation (2.3) twice yields:

$$u'(t) = \sum_{k=0}^{\infty} u_k'(t)\varepsilon^k, \tag{2.5}$$

$$u''(t) = \sum_{k=0}^{\infty} u_k''(t)\varepsilon^k. \tag{2.6}$$

Substituting equations (2.3), (2.5), and (2.6) into Equation (1.7), we obtain:

$$\Big(\sum_{k=0}^{\infty} u_k''(t)\varepsilon^k\Big) + \delta\Big(\sum_{k=0}^{\infty} u_k'(t)\varepsilon^k\Big) + \omega^2\Big(\sum_{k=0}^{\infty} u_k(t)\varepsilon^k\Big) + \varepsilon\Big(\sum_{k=0}^{\infty} u_k(t)\varepsilon^k\Big)^3 = A\cos(\Omega t) + B\sin(\Omega t). \tag{2.7}$$

Expanding the cubic term $\Big(\sum_{k=0}^{\infty} u_k(t)\varepsilon^k\Big)^3$ via the binomial theorem and moving it to the right-hand side, then collecting coefficients of like powers of ε, we arrive at:

$$\sum_{k=0}^{\infty}\big(u_k''(t) + \delta u_k'(t) + \omega^2 u_k(t)\big)\varepsilon^k = \big(A\cos(\Omega t) + B\sin(\Omega t)\big) + \sum_{k=1}^{\infty} h_k(t)\,\varepsilon^k, \tag{2.8}$$

where

$$h_1(t) = -\,u_0(t)^3,$$
$$h_2(t) = -\,3u_0(t)^2 u_1(t),$$
$$h_3(t) = -\,3u_0(t)\Big(u_0(t)u_2(t) + u_1(t)^2\Big),$$
$$h_4(t) = -\,3u_0(t)\Big(u_0(t)u_3(t) + 2u_1(t)u_2(t)\Big) - u_1(t)^3,$$
$$h_5(t) = -\,3u_0(t)\Big(u_0(t)u_4(t) + 2u_1(t)u_3(t) + u_2(t)^2\Big) - 3u_1(t)^2u_2(t),$$
$$h_6(t) = -\,3u_0(t)\Big(u_0(t)u_5(t) + 2u_1(t)u_4(t) + 2u_2(t)u_3(t)\Big) - 3u_1(t)\Big(u_1(t)u_3(t) + u_2(t)^2\Big),$$

$$h_7(t) = -3u_0(t)\Big(u_0(t)u_6(t) + 2u_1(t)u_5(t) + 2u_2(t)u_4(t) + u_3(t)^2\Big) - 3u_1(t)\Big(u_1(t)u_4(t) + 2u_2(t)u_3(t)\Big) - u_2(t)^3.$$

Equating coefficients of ε^k to zero yields the system:

$$\begin{aligned} &\varepsilon^0: \quad u_0''(t) + \delta u_0'(t) + \omega^2 u_0(t) = A\cos(\Omega t) + B\sin(\Omega t), \quad u_0(0) = a,\ u_0'(0) = b,\\ &\varepsilon^k: \quad u_k''(t) + \delta u_k'(t) + \omega^2 u_k(t) = h_k(t), \quad u_k(0) = 0,\ u_k'(0) = 0, \quad k = 1, 2, 3, \ldots. \end{aligned} \tag{2.9}$$

Thus, solving the cubic Duffing oscillator reduces to recursively solving the linear system (2.9). We employ the reproducing kernel Hilbert space method (RKHSM) for this purpose, introduced in the next section.

3. Preliminaries and Reproducing Kernel Hilbert Spaces

This section presents the fundamentals of reproducing kernel Hilbert spaces (RKHS) [11]-[10] (see, in particular, [3]). A Hilbert space is a complete inner product space. An RKHS H is a Hilbert space of functions defined on a set X with a unique bi-variable function K on $X \times X$ with the following properties:

$$K(\cdot, x) \in H, \quad \forall x \in X,$$

$$u(x) = \langle u, K(\cdot, x)\rangle, \quad \forall x \in X,\ \forall u \in H.$$

The function K is the reproducing kernel of H. Equivalently, an RKHS is a Hilbert space where evaluation functionals are continuous.

Considering (2.9), we examine the general second-order linear differential equation:

$$\begin{cases} \alpha(x)v'' + \beta(x)v' + \gamma(x)v = f(x), \\ v(0) = 0, \qquad\qquad v'(0) = 0. \end{cases} \tag{3.1}$$

We define RKHS spaces $W_{2,0}^3[0,1]$ and $W_2^1[0,1]$:

$$W_{2,0}^3[0,1] = \{v \mid v, v', v'' \in AC[0,1],\ v(0) = v'(0) = 0,\ v''' \in L^2[0,1]\},$$

with inner product:

$$\langle u, v\rangle = u(0)v(0) + u'(0)v'(0) + u''(0)v''(0) + \int_0^1 u'''(t)v'''(t)dt, \tag{3.2}$$

and norm $\|v\| = \sqrt{\langle v, v\rangle}$. Here, $AC[0,1]$ denotes absolutely continuous functions on $[0,1]$.

Let $K(\cdot, x)$ be the reproducing kernel of $W_{2,0}^3[0,1]$, satisfying:

$$\langle v(\cdot), K(\cdot, x)\rangle = v(x).$$

The unique reproducing kernel is [28]:

$$K(t,x) = \begin{cases} \frac{1}{4}x^2t^2 + \frac{1}{12}x^2t^3 - \frac{1}{24}xt^4 + \frac{1}{120}t^5, & t \le x, \\ \frac{1}{4}x^2t^2 + \frac{1}{12}x^3t^2 - \frac{1}{24}x^4t + \frac{1}{120}x^5, & t > x. \end{cases} \tag{3.3}$$

Similarly, define $W_2^1[0,1]$ as:

$$W_2^1[0,1] = \{u \mid u \in AC[0,1],\ u' \in L^2[0,1]\},$$

with inner product:

$$\langle u, v\rangle = u(0)v(0) + \int_0^1 u'(t)v'(t)dt, \tag{3.4}$$

and norm $\|u\| = \sqrt{\langle u, u\rangle}$. Its reproducing kernel $R(\cdot, x)$ is [28]:

$$R(t,x) = \begin{cases} 1 + t, & t \le x, \\ 1 + x, & t > x. \end{cases} \tag{3.5}$$

Theorem 3.1. *[9] Let $\{x_i\}_{i=1}^{\infty}$ be dense in $[0,1]$. Then $\{R(\cdot, x_i)\}_{i=1}^{\infty}$ is a basis for $W_2^1[0,1]$.*

4. NUMERICAL SCHEME BASED ON REPRODUCING KERNEL HILBERT SPACES

This section develops a numerical method for solving differential Equation (3.1) using RKHS techniques. We define the operator L, derive its adjoint L^*, and analyze convergence with error estimates.

The operator L is defined as a mapping from the Sobolev space $W^3_{2,0}[0,1]$ to $W^1_2[0,1]$ as follows:

$$\begin{cases} L : W^3_{2,0}[0,1] \longrightarrow W^1_2[0,1], \\ Lv(x) = f(x), \qquad 0 < x < 1, \\ v(0) = 0, \quad v'(0) = 0, \end{cases} \tag{4.1}$$

where

$$Lv(x) = \alpha(x)v''(x) + \beta(x)v'(x) + \gamma(x)v(x), \qquad x \in [0,1].$$

For the specific linear problems derived from the Duffing oscillator after rescaling (see Remark below), the coefficients become constant, and the homogeneous initial-value problem $Lv = 0$ with $v(0) = v'(0) = 0$ has only the trivial solution; hence L is invertible.

Theorem 4.1. *Let $L : W^3_{2,0}[0,1] \to W^1_2[0,1]$ be invertible with $Lv(x) = \alpha(x)v'' + \beta(x)v' + \gamma(x)v$. The adjoint operator $L^* : W^1_2[0,1] \to W^3_{2,0}[0,1]$ is:*

$$\begin{cases} L^*u(x) = \alpha(x)u''(x) + \big(2\alpha'(x) - \beta(x)\big)u'(x) + \big(\alpha''(x) - \beta'(x) + \gamma(x)\big)u(x), & 0 < x < 1, \\ u(1) = 0, & u'(1) = 0. \end{cases} \tag{4.2}$$

Proof. We begin with the definition of the operator L:

$$Lv(x) = \alpha(x)v''(x) + \beta(x)v'(x) + \gamma(x)v(x).$$

By definition of the adjoint operator L^*, we have

$$\langle Lv, u\rangle = \langle v, L^*u\rangle,$$

where the inner product is taken in $L^2[0,1]$. Thus,

$$\begin{aligned} \langle Lv, u\rangle &= \int_0^1 Lv(x)\, u(x)\, dx \\ &= \int_0^1 \Big[\alpha(x)v''(x) + \beta(x)v'(x) + \gamma(x)v(x)\Big] u(x)\, dx \\ &= \int_0^1 \alpha(x)u(x)\, v''(x)\, dx + \int_0^1 \beta(x)u(x)\, v'(x)\, dx + \int_0^1 \gamma(x)u(x)\, v(x)\, dx. \end{aligned}$$

We now integrate the first two integrals by parts. For the first integral,

$$\int_0^1 \alpha(x)u(x)\, v''(x)\, dx = \Big[\alpha(x)u(x)\, v'(x)\Big]_0^1 - \int_0^1 \frac{d}{dx}\big[\alpha(x)u(x)\big] v'(x)\, dx.$$

Integrating the remaining term by parts again gives

$$\int_0^1 \alpha(x)u(x)\, v''(x)\, dx = \Big[\alpha(x)u(x)\, v'(x)\Big]_0^1 - \Big[\frac{d}{dx}\big(\alpha(x)u(x)\big)\, v(x)\Big]_0^1 + \int_0^1 \frac{d^2}{dx^2}\big[\alpha(x)u(x)\big] v(x)\, dx.$$

For the second integral,

$$\int_0^1 \beta(x)u(x)\, v'(x)\, dx = \Big[\beta(x)u(x)\, v(x)\Big]_0^1 - \int_0^1 \frac{d}{dx}\big[\beta(x)u(x)\big] v(x)\, dx.$$

Substituting these results back and collecting terms, we obtain

$$\langle Lv, u\rangle = \Big[\alpha(x)u(x)v'(x)\Big]_0^1 - \Big[\big(\alpha'(x)u(x) + \alpha(x)u'(x)\big)v(x)\Big]_0^1 + \Big[\beta(x)u(x)v(x)\Big]_0^1$$

$$+\int_0^1 \left\{ \frac{d^2}{dx^2}\big[\alpha(x)u(x)\big] - \frac{d}{dx}\big[\beta(x)u(x)\big] + \gamma(x)u(x) \right\} v(x)\,dx.$$

Computing the derivatives explicitly yields

$$\frac{d^2}{dx^2}\big[\alpha(x)u(x)\big] = \alpha''(x)u(x) + 2\alpha'(x)u'(x) + \alpha(x)u''(x),$$
$$\frac{d}{dx}\big[\beta(x)u(x)\big] = \beta'(x)u(x) + \beta(x)u'(x).$$

Hence,

$$\langle Lv, u\rangle = \Big[\alpha(x)u(x)v'(x) - \big(\alpha'(x)u(x) + \alpha(x)u'(x)\big)v(x) + \beta(x)u(x)v(x)\Big]_0^1$$
$$+ \int_0^1 \Big[\alpha(x)u''(x) + \big(2\alpha'(x) - \beta(x)\big)u'(x) + \big(\alpha''(x) - \beta'(x) + \gamma(x)\big)u(x)\Big] v(x)\,dx.$$

Recall that $v \in W^3_{2,0}[0,1]$ satisfies $v(0) = 0$ and $v'(0) = 0$. To ensure that the boundary terms vanish, we impose the conditions $u(1) = 0$ and $u'(1) = 0$. Under these assumptions,

$$\langle Lv, u\rangle = \int_0^1 \Big[\alpha(x)u''(x) + \big(2\alpha'(x) - \beta(x)\big)u'(x) + \big(\alpha''(x) - \beta'(x) + \gamma(x)\big)u(x)\Big] v(x)\,dx.$$

Since this must equal $\langle v, L^*u\rangle = \int_0^1 v(x)\,L^*u(x)\,dx$ for all admissible v, we identify

$$L^*u(x) = \alpha(x)u''(x) + \big(2\alpha'(x) - \beta(x)\big)u'(x) + \big(\alpha''(x) - \beta'(x) + \gamma(x)\big)u(x).$$

This completes the derivation of the adjoint operator L^*. □

Define $\varphi_i(x) = R(x, x_i)$ for $i = 1, 2, \ldots$, where $R(x, x_i)$ is the reproducing kernel of the Sobolev space $W^1_2[0,1]$, and set $\psi_i(x) = L^*\varphi_i(x)$, where L^* denotes the adjoint operator of L. It follows directly from the definitions that each ψ_i belongs to the space $W^3_{2,0}[0,1]$.

Theorem 4.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of the interval $[0,1]$. Then the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ forms a complete system in $W^3_{2,0}[0,1]$.*

Proof. Take an element $v \in W^3_{2,0}[0,1]$ satisfying $\langle v, \psi_i\rangle_{W^3_{2,0}} = 0$ for every i. By the definition of the adjoint operator and the reproducing property, we obtain

$$\langle v, \psi_i\rangle_{W^3_{2,0}} = \langle v, L^*\varphi_i\rangle_{W^3_{2,0}} = \langle Lv, \varphi_i\rangle_{W^1_2} = \langle Lv, R(\cdot, x_i)\rangle_{W^1_2} = Lv(x_i).$$

Hence, $Lv(x_i) = 0$ for all i. Since the set $\{x_i\}$ is dense in $[0,1]$ and Lv is continuous, it follows that $Lv(x) = 0$ for every $x \in [0,1]$. By the uniqueness of the solution to Equation (4.1) and the invertibility of L, we conclude that $v \equiv 0$. Therefore, the system $\{\psi_i\}_{i=1}^{\infty}$ is complete in $W^3_{2,0}[0,1]$. (The density of $\{x_i\}$ is a hypothesis; it guarantees that vanishing at all points implies $Lv \equiv 0$ on the whole interval.) □

Applying the Gram–Schmidt orthogonalization process to the complete system $\{\psi_i\}_{i=1}^{\infty}$ yields an orthonormal basis $\{\tilde{\psi}_i\}_{i=1}^{\infty}$ of $W^3_{2,0}[0,1]$. Explicitly,

$$\tilde{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\,\psi_k(x), \qquad i = 1, 2, 3, \ldots, \tag{4.3}$$

where the coefficients β_{ik} are obtained from the orthogonalization procedure.

Theorem 4.3. *Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of the interval $[0,1]$, and assume that the solution to Equation (4.1) is unique. Then the exact solution can be represented by the series*

$$v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\,f(x_k)\,\tilde{\psi}_i(x), \tag{4.4}$$

provided that the series converges.

Proof. By Theorem 4.2 and the construction in (4.3), the set $\{\tilde{\psi}_i\}_{i=1}^{\infty}$ forms an orthonormal basis for $W_{2,0}^3[0,1]$. For any $v \in W_{2,0}^3[0,1]$, we can expand it in this basis:

$$v(x) = \sum_{i=1}^{\infty} \langle v, \tilde{\psi}_i \rangle \, \tilde{\psi}_i(x).$$

Substituting the expression for $\tilde{\psi}_i$ from (4.3) gives

$$v(x) = \sum_{i=1}^{\infty} \Big\langle v, \sum_{k=1}^{i} \beta_{ik}\psi_k \Big\rangle \tilde{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle v, \psi_k\rangle \tilde{\psi}_i(x).$$

Since $\psi_k = L^*\varphi_k$ and L is the operator in (4.1), we have

$$\langle v, \psi_k\rangle = \langle v, L^*\varphi_k\rangle = \langle Lv, \varphi_k\rangle = Lv(x_k) = f(x_k).$$

Thus,

$$v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\tilde{\psi}_i(x), \tag{4.5}$$

which is exactly (4.4). ☐

We now establish the convergence of the series representation (4.4). Let v_n denote the n-term partial sum of the series, i.e.,

$$v_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\tilde{\psi}_i(x). \tag{4.6}$$

Theorem 4.4. *Let* $\{x_i\}_{i=1}^{\infty}$ *be a dense subset of* $[0,1]$, $f \in W_2^1[0,1]$, *and define*

$$A_i = \sum_{k=1}^{i} \beta_{ik} f(x_k), \qquad i = 1, 2, \ldots.$$

Suppose the approximate solutions $v_n(x) = \sum_{i=1}^{n} A_i\tilde{\psi}_i(x)$ *satisfy* $\|v_n\|_{W_{2,0}^3} \leq M$ *for some constant* M *independent of* n. *Then:*

(1) v_n *converges in* $W_{2,0}^3[0,1]$ *to a function* v, *and*

$$v(x) = \sum_{i=1}^{\infty} A_i\tilde{\psi}_i(x).$$

(2) *The derivatives* v_n' *and* v_n'' *converge uniformly on* $[0,1]$ *to* v' *and* v'', *respectively.*

Proof. (1) From the orthonormality of $\{\tilde{\psi}_i\}$,

$$\begin{aligned}
||v_{n+1}||^2_{W_{2,0}^3} &= ||v_n||^2_{W_{2,0}^3} + A_{n+1}^2 \\
&= ||v_{n-1}||^2_{W_{2,0}^3} + A_n^2 + A_{n+1}^2 \\
&= ||v_{n-2}||^2_{W_{2,0}^3} + A_{n-1}^2 + A_n^2 + A_{n+1}^2 \\
&= \cdots
\end{aligned} \tag{4.7}$$

Iterating this relation gives

$$\|v_n\|^2_{W_{2,0}^3} = \sum_{i=1}^{n} A_i^2.$$

Since $\|v_n\|_{W^3_{2,0}}$ is bounded, the series $\sum_{i=1}^{\infty} A_i^2$ converges; hence $\{A_i\} \in \ell^2$.

For $m > n$,

$$\begin{aligned}||v_m - v_n||^2_{W^3_{2,0}} &= ||v_m - v_{m-1} + v_{m-1} - \cdots + v_{n+1} - v_n||^2_{W^3_{2,0}} \\ &\leq ||v_m - v_{m-1}||^2_{W^3_{2,0}} + ||v_{m-1} - v_{m-2}||^2_{W^3_{2,0}} + \cdots + ||v_{n+1} - v_n||^2_{W^3_{2,0}} \\ &= \sum_{i=n+1}^{m} A_i^2 \to 0, \ \ m, n \to \infty.\end{aligned}$$

Thus $\{v_n\}$ is a Cauchy sequence in the complete space $W^3_{2,0}[0,1]$, and so converges to some $v \in W^3_{2,0}[0,1]$.

(2) For $i = 1, 2$ and for each x in $[0,1]$,

$$\begin{aligned}\left|\frac{d^i v(x)}{dx^i} - \frac{d^i v_n(x)}{dx^i}\right| &= \left|\left\langle v - v_n, \frac{\partial^i K_x(\cdot)}{\partial x^i}\right\rangle_{W^3_{2,0}}\right| \\ &\leq \left\|\frac{\partial^i K_x(\cdot)}{\partial x^i}\right\|_{W^3_{2,0}} \|v - v_n\|_{W^3_{2,0}} \\ &= \sqrt{\left.\frac{\partial^{2i} K_x(t)}{\partial x^i \partial t^i}\right|_{t=x}} \, \|v - v_n\|_{W^3_{2,0}}.\end{aligned}$$

Because the reproducing kernel K is smooth, the norm $\left\|\frac{\partial^i K_x}{\partial x^i}\right\|_{W^3_{2,0}}$ is bounded uniformly for $x \in [0,1]$. Since $\|v - v_n\|_{W^3_{2,0}} \to 0$, we obtain uniform convergence of v'_n to v' and v''_n to v''. □

Theorem 4.5. *Let* $\Lambda_n = \{x_0, x_1, \ldots, x_n\}$ *be a partition of the interval* $[0,1]$ *with* $0 = x_0 < x_1 < \cdots < x_n = 1$, *and let* $\Delta = \max_{1 \leq k \leq n}(x_k - x_{k-1})$ *denote the mesh size. Suppose* v *is the exact solution of Equation* (4.1) *and* v_n *its* n*-term approximation obtained via* (4.6)*, both belonging to* $W^3_{2,0}[0,1]$*. If* v'' *is Lipschitz continuous on* $[0,1]$ *and* v'''_n *is uniformly bounded, then there exists a constant* $C > 0$ *such that*

$$\|v - v_n\|_\infty \leq C\,\Delta^3.$$

Proof. Fix an arbitrary $x \in [0,1]$. Then x belongs to some subinterval $[x_{k-1}, x_k]$ of the partition. We start by decomposing the second derivative error as follows:

$$v''(x) - v''_n(x) = \big[v''(x) - v''(x_{k-1})\big] + \big[v''(x_{k-1}) - v''_n(x_{k-1})\big] + \big[v''_n(x_{k-1}) - v''_n(x)\big]. \tag{4.8}$$

Since $v_n \in W^3_{2,0}[0,1]$, the fundamental theorem of calculus gives

$$v''_n(x) - v''_n(x_{k-1}) = \int_{x_{k-1}}^{x} v'''_n(t)\,dt. \tag{4.9}$$

Moreover, the triangle inequality yields the following integral bounds:

$$|v(x) - v_n(x)| \leq |v(x_{k-1}) - v_n(x_{k-1})| + \int_{x_{k-1}}^{x} |v'(t) - v'_n(t)|\,dt, \tag{4.10}$$

$$|v'(x) - v'_n(x)| \leq |v'(x_{k-1}) - v'_n(x_{k-1})| + \int_{x_{k-1}}^{x} |v''(t) - v''_n(t)|\,dt. \tag{4.11}$$

By hypothesis, v'' is Lipschitz continuous; hence there exists $L > 0$ such that for all $x, y \in [0,1]$,

$$|v''(x) - v''(y)| \leq L\,|x - y|.$$

In particular,

$$|v''(x) - v''(x_{k-1})| \leq L\,\Delta. \tag{4.12}$$

Also, v_n''' is uniformly bounded: there exists $M > 0$ with $\|v_n'''\|_\infty \le M$. Consequently, from (4.9),

$$|v_n''(x) - v_n''(x_{k-1})| \le M\,\Delta. \tag{4.13}$$

Now, Theorem 4.4 guarantees that the approximate solution v_n converges to v in $W^3_{2,0}[0,1]$. Therefore, for any $\epsilon > 0$, we can choose n large enough so that at the node x_{k-1},

$$\max\bigl\{|v(x_{k-1}) - v_n(x_{k-1})|,\ |v'(x_{k-1}) - v_n'(x_{k-1})|,\ |v''(x_{k-1}) - v_n''(x_{k-1})|\bigr\} \le \epsilon. \tag{4.14}$$

Inserting the estimates (4.12), (4.13), and (4.14) into (4.8) gives

$$|v''(x) - v_n''(x)| \le L\Delta + \epsilon + M\Delta = (L+M)\Delta + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain the uniform bound

$$\|v'' - v_n''\|_\infty \le (L+M)\,\Delta. \tag{4.15}$$

Next, using (4.11), (4.14), and (4.15),

$$|v'(x) - v_n'(x)| \le \epsilon + \int_{x_{k-1}}^{x} \|v'' - v_n''\|_\infty\,dt \le \epsilon + (L+M)\,\Delta^2.$$

Again, letting $\epsilon \to 0$ yields

$$\|v' - v_n'\|_\infty \le (L+M)\,\Delta^2. \tag{4.16}$$

Finally, from (4.10), (4.14), and (4.16),

$$|v(x) - v_n(x)| \le \epsilon + \int_{x_{k-1}}^{x} \|v' - v_n'\|_\infty\,dt \le \epsilon + (L+M)\,\Delta^3.$$

Taking $\epsilon \to 0$ concludes the proof with $C = L + M$:

$$\|v - v_n\|_\infty \le C\,\Delta^3. \tag{4.17}$$

Note that in Equation (4.17), the error bound $O(\Delta^3)$ holds in the uniform norm on the interval $[0,1]$.

Remark on domain scaling. The reproducing kernel spaces $W^3_{2,0}[0,1]$ and $W^1_2[0,1]$ are defined on the unit interval. To apply the method to a problem posed on a general time interval $t \in [0,T]$, we rescale the independent variable as $s = t/T$, which maps $[0,T]$ onto $[0,1]$. Let $\tilde{u}(s) = u(Ts)$. Using the chain rule,

$$\frac{d}{dt} = \frac{1}{T}\frac{d}{ds}, \qquad \frac{d^2}{dt^2} = \frac{1}{T^2}\frac{d^2}{ds^2},$$

Equation (2.9) transforms into an equivalent linear differential equation for $\tilde{u}(s)$ on $[0,1]$ with appropriately rescaled coefficients. The RKHS method is then applied to this transformed problem, and the final solution in the original variable is recovered as $u(t) = \tilde{u}(t/T)$. For clarity, in the numerical examples we display the solutions directly in the original time variable.

5. NUMERICAL EXPERIMENTS

This section presents test problems demonstrating the efficiency and accuracy of our method. Table 1 summarizes experimental data for Examples 5.1-5.6. Figures 1-12 show close agreement between our approximate solutions and fourth-order Runge-Kutta solutions. Table 2 confirms theoretical error estimates. All computations used Maple 18 on a laptop with Intel Core i7-4700MQ processor (2.40GHz) and 8GB RAM.

For Examples 5.1-5.6, we compute:

$$u(t) \approx u_0(t) + u_1(t)\varepsilon + u_2(t)\varepsilon^2 + u_3(t)\varepsilon^3,$$

with each $u_i(t)$ approximated using RKHSM with $n = 100$. In all computations, a uniform grid $t_i = \frac{i-1}{n-1}$ $(i = 1, \dots, n)$ on $[0,1]$ is used.

Table 1. Test problems for cubic Duffing oscillator.

	δ	ω	ε	a	b	A	B	Ω
Example 5.1	0.00	1.0	0.01	2.0	0.0	0.0	0.0	$1+\frac{3}{2}\varepsilon-\frac{27}{16}\varepsilon^2$
Example 5.2	0.00	$\sqrt{2.0}$	0.1	0.0	0.0	1.0	0.0	1.0
Example 5.3	0.05	1.0	0.15	1.0	0.0	0.0	0.0	0.0
Example 5.4	0.03	1.0	0.03	1.0	0.0	0.0	0.15	0.8
Example 5.5	0.00	1.0	0.15	1.0	0.0	0.0	0.0	0.0
Example 5.6	0.00	1.0	0.15	0.5	-0.5	0.0	0.0	0.0

Table 2. Absolute errors between present method ($n = 100$) and Runge-Kutta for Examples 5.3-5.6.

x	Example 5.3	Example 5.4	Example 5.5	Example 5.6
0.0	0.0	0.0	0.0	0.0
0.1	2.40896×10^{-7}	2.34262×10^{-7}	2.30280×10^{-7}	2.37068×10^{-7}
0.2	4.84795×10^{-7}	4.71431×10^{-7}	4.65238×10^{-7}	4.12151×10^{-7}
0.3	7.23496×10^{-7}	7.11628×10^{-7}	6.96283×10^{-7}	6.54045×10^{-7}
0.4	9.52418×10^{-7}	9.46013×10^{-7}	9.18546×10^{-7}	8.17748×10^{-7}
0.5	1.17494×10^{-6}	1.15914×10^{-6}	1.13554×10^{-6}	1.05222×10^{-6}
0.6	1.33573×10^{-6}	1.39185×10^{-6}	1.29770×10^{-6}	1.21676×10^{-6}
0.7	1.37670×10^{-6}	1.58543×10^{-6}	1.33267×10^{-6}	1.40956×10^{-6}
0.8	1.27825×10^{-6}	1.77060×10^{-6}	1.21972×10^{-6}	1.58875×10^{-6}
0.9	7.82005×10^{-7}	1.97625×10^{-6}	7.21375×10^{-7}	1.73526×10^{-6}
1.0	1.48251×10^{-7}	2.07085×10^{-6}	2.59556×10^{-7}	1.87698×10^{-6}

Example 5.1. [29] Consider (1.7)-(1.8) with parameters from Table 1 (row 1). Exact solution:

$$u(t) = \left(2-\frac{\varepsilon}{2}\right)\cos(\Omega t) + \frac{\varepsilon}{4}\cos(3\Omega t).$$

Example 5.2. [38] Parameters from Table 1 (row 2).

Example 5.3. [2] Parameters from Table 1 (row 3). Modified differential transform method (MDTM) solution [2]:

$$u_{MDTM}(t) = 0.9962e^{-0.0262t}\cos(1.0551t) + 0.0038e^{-0.0572t}\cos(3.2685t).$$

Example 5.4. [2] Parameters from Table 1 (row 4). MDTM solution [2]:

$$u_{MDTM}(t) = -0.0236\cos(0.8t) + 0.3832\sin(0.8t) + e^{-0.015t}\big(1.0236\cos(1.0148t) - 0.287\sin(1.0148t)\big).$$

Example 5.5. [25] Parameters from Table 1 (row 5).

Example 5.6. [25] Parameters from Table 1 (row 6).

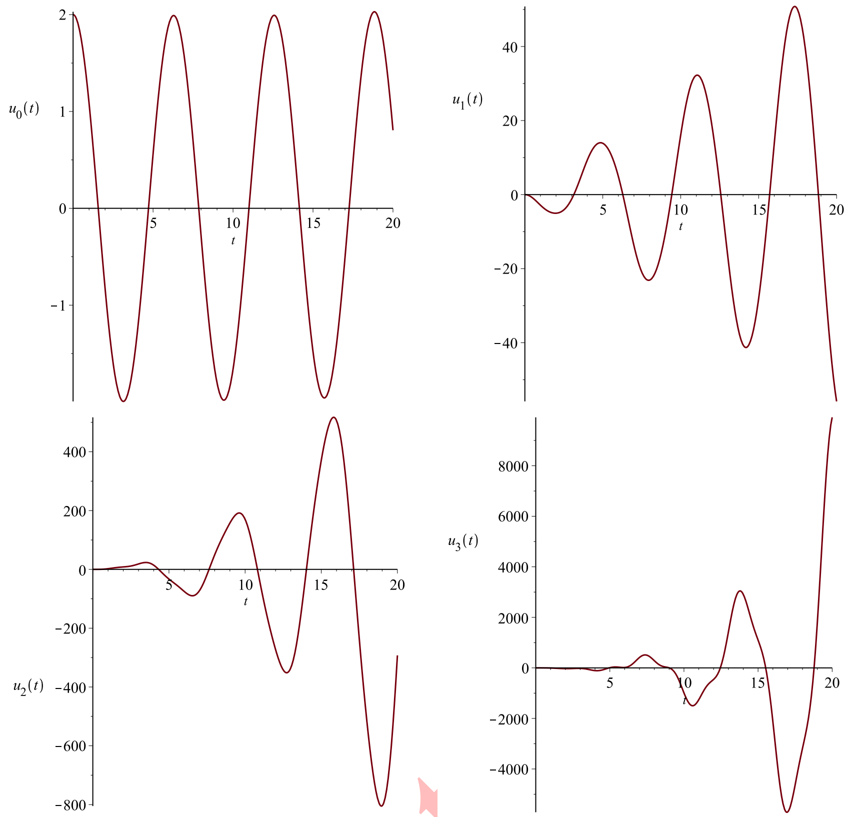

FIGURE 1. Approximations $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ obtained by the present method with $n = 100$ uniform nodes for Example 5.1.

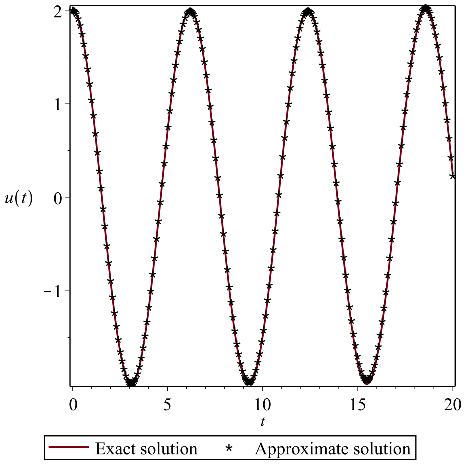

FIGURE 2. Comparison of the exact solution (solid line) and the approximate solution (dashed line) for Example 5.1. Excellent agreement is observed.

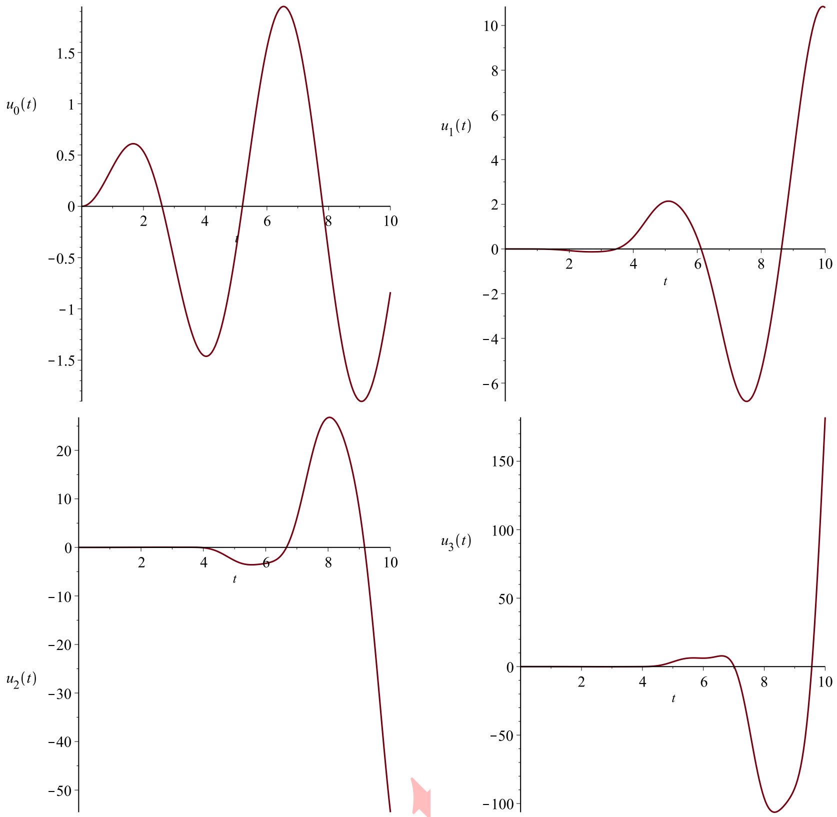

FIGURE 3. Approximations $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ using present method ($n = 100$) for Example 5.2

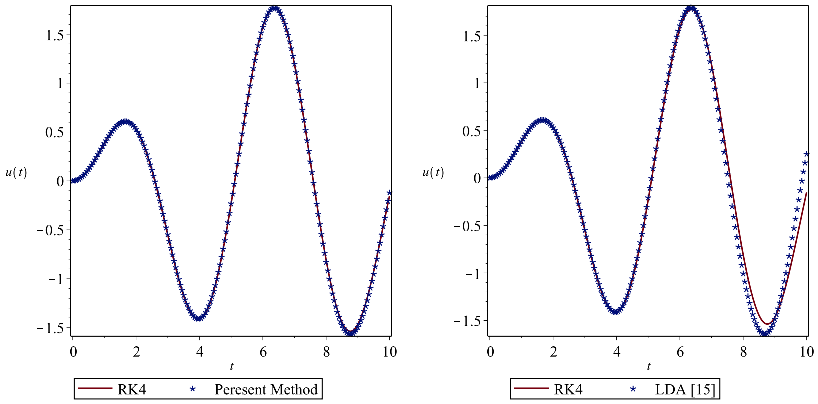

FIGURE 4. Comparison of the present method (left) and the LDA [38] (right) with the Runge–Kutta reference for Example 5.2. The present method shows closer agreement.

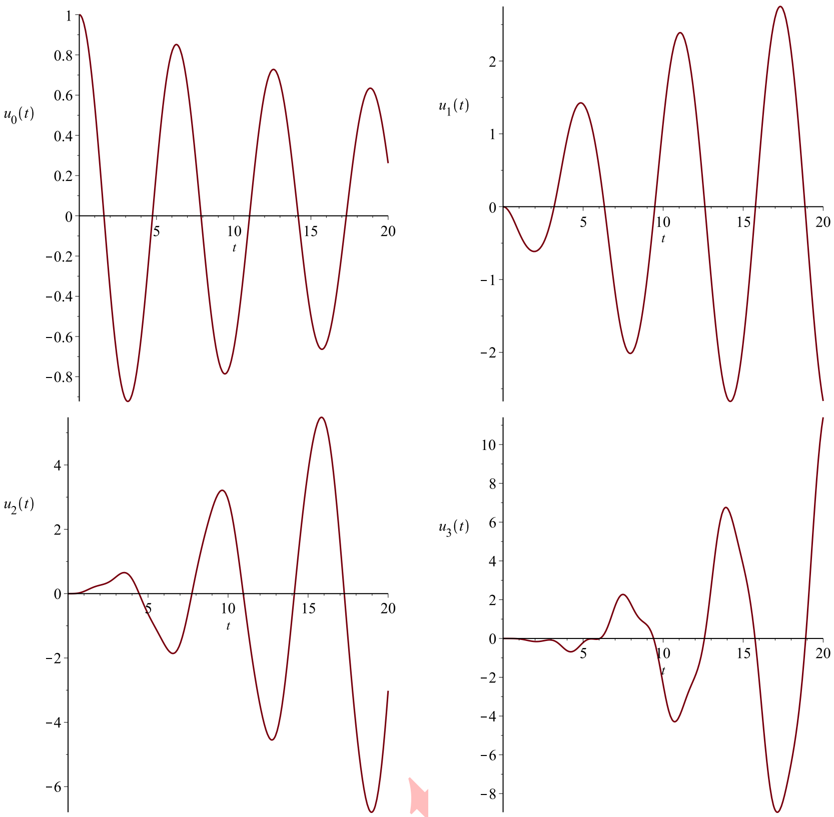

FIGURE 5. Approximations $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ using present method ($n = 100$) for Example 5.3.

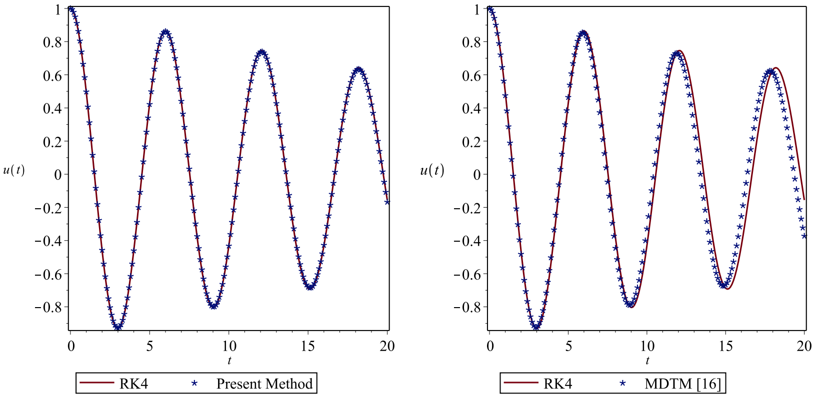

FIGURE 6. Comparison of the present method (left) and the MDTM [2] (right) with the Runge–Kutta reference for Example 5.3. The present method is more accurate, especially for larger times.

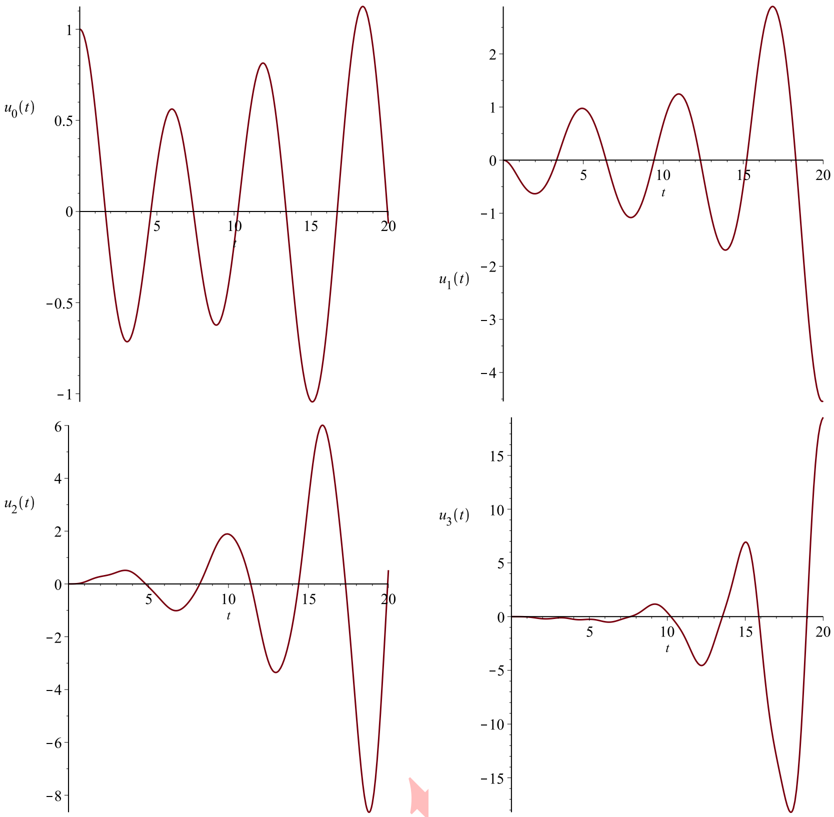

Figure 7. Approximations $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ using present method ($n = 100$) for Example 5.4.

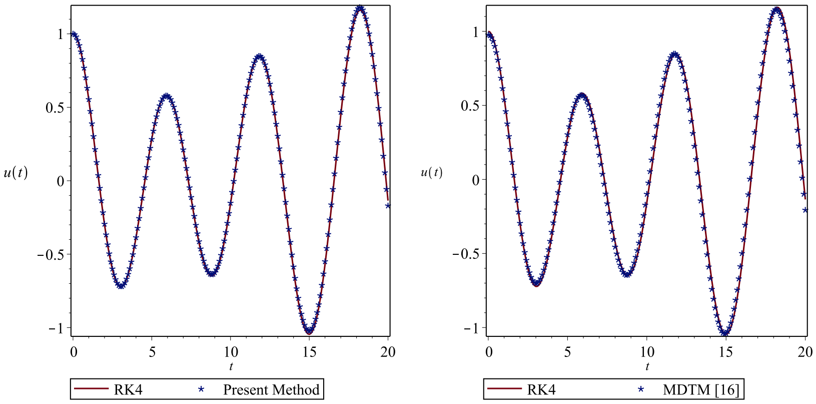

Figure 8. Comparison of the present method (left) and the MDTM [2] (right) with the Runge–Kutta reference for Example 5.4. The present method again yields a better approximation.

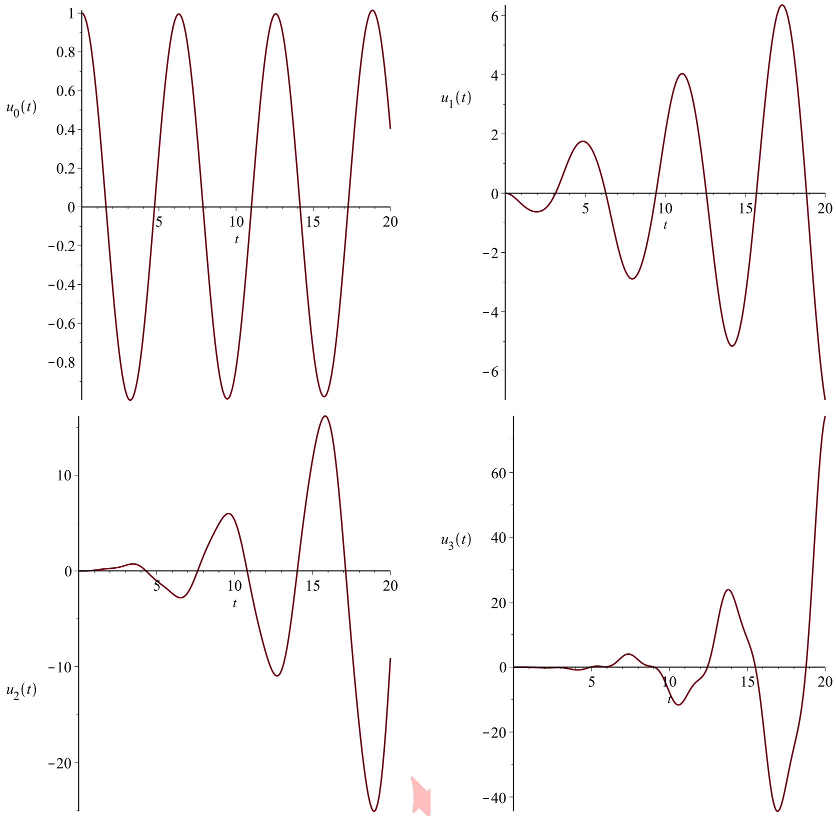

FIGURE 9. Approximations $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ using present method ($n = 100$) for Example 5.5.

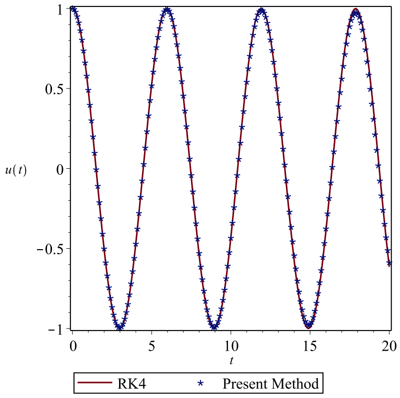

FIGURE 10. Comparison of present method with Runge-Kutta for Example 5.5.

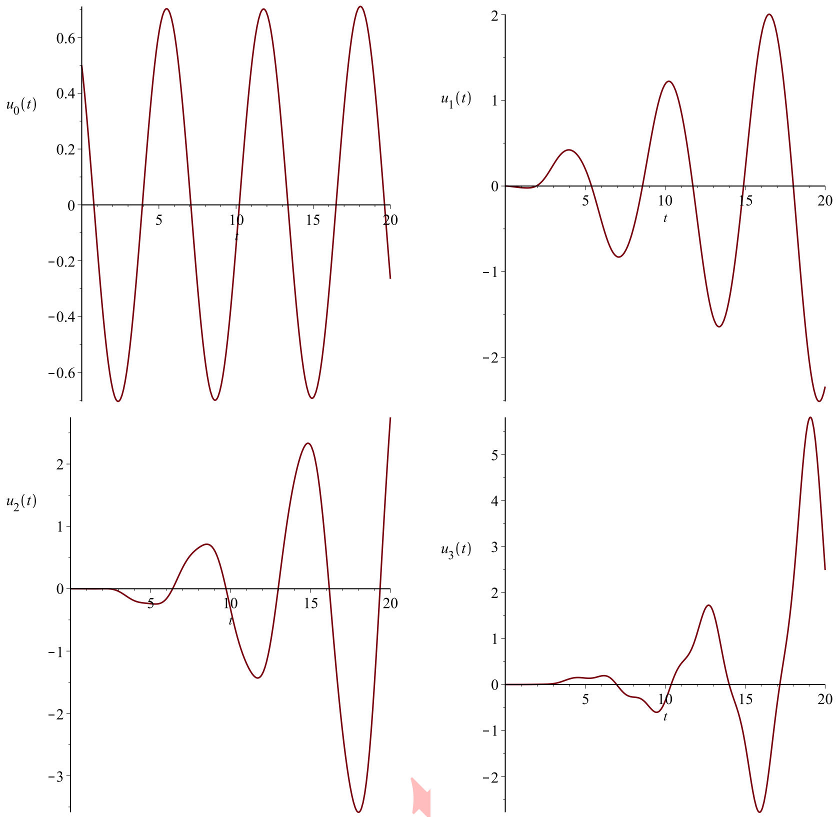

FIGURE 11. Approximations $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ using present method ($n = 100$) for Example 5.6.

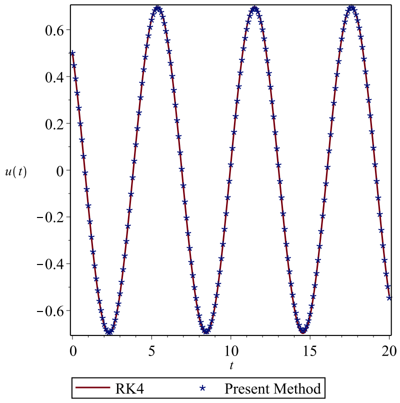

FIGURE 12. Comparison of present method with Runge-Kutta for Example 5.6.

6. CONCLUSIONS

This paper has presented a novel numerical algorithm for solving nonlinear differential equations with small perturbed parameters. We developed a hybrid method combining reproducing kernel Hilbert space techniques with perturbation methods for solving nonlinear Duffing oscillators. The method transforms the original nonlinear problem into a system of linear differential equations via perturbation expansion, then solves each linear equation using RKHS techniques.

Numerical experiments on various cubic Duffing oscillator examples demonstrate the method's effectiveness. Figures 1-10 show excellent agreement between our approximate solutions and fourth-order Runge-Kutta solutions. Comparisons with LDA [38] and MDTM [2] methods (Figures 4, 6, and 8) confirm the superior accuracy and efficiency of our approach. We established convergence properties of the RKHS method and derived error estimates, providing theoretical support for the numerical results.

The proposed method offers a powerful tool for solving nonlinear oscillatory problems with small parameters, combining the systematic linearization of perturbation methods with the accuracy and theoretical foundation of reproducing kernel techniques.

APPENDIX A. PHYSICAL OSCILLATORS REDUCIBLE TO DUFFING FORM

This appendix presents several physical oscillatory systems that can be transformed into the Duffing equation form, demonstrating the broader applicability of the proposed method. These examples are particularly relevant in weak signal detection problems [20, 37, 40].

Problem I: Simple Pendulum. The motion of a simple pendulum of length l under gravitational acceleration g is governed by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{6.1}$$

For small angular displacements θ, we use the Taylor expansion

$$\sin\theta \approx \theta - \frac{\theta^3}{6}, \tag{6.2}$$

keeping terms up to cubic order. Substituting (6.2) into (6.1) and defining $\omega^2 = g/l$ and $\varepsilon = \omega^2/6$ yields the Duffing-type equation

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = \varepsilon\theta^3, \tag{6.3}$$

with appropriate initial conditions such as $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$.

Problem II: Coupled Trigonometric Oscillator. Consider the oscillator described by

$$\frac{d^2\theta}{dt^2} = \Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta. \tag{6.4}$$

Using the identity $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, Equation (6.4) becomes

$$\frac{d^2\theta}{dt^2} = \frac{1}{2}\Omega^2\sin 2\theta - \frac{g}{R}\sin\theta. \tag{6.5}$$

Applying the small-angle approximations

$$\sin\theta \approx \theta - \frac{\theta^3}{6}, \tag{6.6}$$

$$\sin 2\theta \approx 2\theta - \frac{4\theta^3}{3}, \tag{6.7}$$

and substituting into (6.5), we obtain after simplification

$$\frac{d^2\theta}{dt^2} - \omega^2\theta + \varepsilon\theta^3 = 0, \tag{6.8}$$

where $\omega^2 = \Omega^2 - \frac{g}{R}$ and $\varepsilon = \frac{2}{3}\Omega^2 - \frac{g}{6R}$.

Problem III: Oscillator with Rational Nonlinearity. The equation

$$\frac{d^2u}{dt^2} + \frac{\omega^2 u}{1+u^2} = 0 \tag{6.9}$$

contains a rational nonlinearity. For small but finite u, we expand the denominator as

$$\frac{1}{1+u^2} \approx 1 - u^2, \tag{6.10}$$

retaining terms up to quadratic order. Substituting (6.10) into (6.9) gives the Duffing equation

$$\frac{d^2u}{dt^2} + \omega^2 u = \varepsilon u^3, \qquad \varepsilon = \omega^2. \tag{6.11}$$

These three examples illustrate how various nonlinear oscillators can be reduced to the Duffing form through appropriate approximations, thereby enabling the application of the perturbation-reproducing kernel method developed in this work.

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