Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-19
DOI:10.22034/cmde.2026.70600.3520

B-spline-based fuzzy transform method for the numerical solution of variable coefficients multi-term fractional differential equations

Sedaghat Shahmorad[1,2,*], Fariba Bahrami[1], and Masumeh Zeinali[1]
[1]Department of Applied Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, University of Tabriz, Tabriz, Iran.
[2]Department of Applied Mathematics and Informatics, Faculty of Sciences, Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan.

Abstract

In this paper, an effective numerical algorithm based on the fuzzy transform with cubic B-splines as basic functions is designed to solve multi-terms fractional differential equations (FDEs, for short) with variable coefficients. An analytical approximation with desirable regularity is given to the involving initial value problem. The proposed method provides a non-recursive approximate solution based on an operational matrix and a vector of known quantities. The convergence analysis of the method is carried out. An estimation error is also provided, which enables us to report the error for the problems whose exact solutions are not available. The good performance of our method over the available methods is shown by numerical results.

Keywords. Fuzzy transform, B-spline functions, Multi-term Fractional differential equations, Estimation error.
2010 Mathematics Subject Classification. 26A33, 34A08, 65L10, 65R20, 65L70.

1. Introduction

The fuzzy transform (F-transform for short) that was introduced for the first time by Perfilieva in 2006 [25], converts every continuous function to a vector of real numbers. Each component of this vector is a minimizer of the weighted least squares error. The inverse F-transform is defined using direct F-transform. Despite its fuzzy origin, since the inverse F-transform is an approximation to the primary function, the subject of F-transform can be studied in the classical approximation theory. Thus, many researchers have made considerable efforts to improve its approximation properties [1, 4, 20, 30, 36]. The approximation properties of F-transform in both continuous and discrete forms depend on two items: basic functions as weight functions and basis functions as the basis of the approximation subspace for the direct F^m-transform (the generalization of F-transform) components. By reviewing the associated literature, it is realized that there are two approaches for developing the F-transform. Some researchers have focused on the basis functions [1, 36], and some others have made efforts on modifying the basic functions [4, 30]. In the area of modifying the basic functions, two essential works have been done by Stefanini [30] and Bede et al. [4]. Stefanini [30] improved the smoothing properties of F-transform by introducing a parametric form of the basic functions and generalizing the support of basic functions. Bede et al. [4] introduced a new type of F-transform by generalizing the support of basic functions. They developed new varieties of F-transforms using the B-splines, Shepard kernel, Bernstein polynomials, and Favard-Szasz-Mirakjan kernel. Furthermore, they proved Korovkin's theorem for the F-transform approximation, an important theorem in the classical approximation theory. Besides these researches, valuable works have been done to generalize F-transform components. In [27], Perfilieva et al. generalized F-transform components from constants functions to polynomials of degree up to m. To this end, they extended the basis functions from $\{1\}$ to $\{1, x, x^2, \ldots, x^m\}$. They called this new type of F-transform as F^m-transform. Similarly, Alikhani et al. [1] replaced the polynomial basis functions with trigonometric polynomials of degree up to m. Finally, Zeinali et al. [36] extended the basis functions to the most general form.

Received: 08 December 2025; Accepted: 20 June 2026.
∗ Corresponding author. Email: shahmorad@tabrizu.ac.ir, sedaghat.shahmorad@manas.edu.kg.

Besides the significant applications of F-transform in engineering and computer sciences [29, 34], it has been used in the numerical solution of differential, and integral equations [19, 26, 33, 36, 37]. In [19], a Cauchy problem has been solved using a continuous F-transform method, where they made a comparison between the F-transform method and classical methods such as finite difference and Runge – Kutta. Perfilieva et al. [26], applied the F-transform method for solving two-point boundary value problems with constant coefficients. They conclude that the F-transform method is more accurate for noisy problems than the classical methods such as finite difference. In [33], the discrete version of the F-transform was used to introduce a Picard-like numerical method for solving a delay differential equation. Numerical solutions of the two-point boundary value problems with variable coefficients based on continuous F^m-transform have been studied in [36]. It seems that the research in this field is still too young, and more studies can be done.

In recent years, fractional calculus has been attracting the attention of many researchers because of its more accurate description of physical problems such as damping laws, electromagnetic acoustics, viscoelasticity, electroanalytical chemistry neuron modeling, diffusion processing, and material science(see for example [6]). Also, many efforts have been made to find solutions for the fractional differential and integral equations in the analytical and numerical forms. The efforts included in introducing finite difference [35], collocation-shooting [2], operational matrix [28] and many other methods [3, 17, 22]. Recently, differential equations of multi-term have been the focus of many researches [13, 14, 16, 18, 23].

There is a long history of research on the Abel integral equation, a special equivalent form of fractional differential equations. A number of its numerical solution methods such as algorithms based on relevant mesh shift [5, 8], quadrature based methods [11, 24], collocation approximation methods [9, 10, 32] etc. exist in the literature.

The main goal of this paper is to propose a numerical method to solve multi-term differential equation of fractional order. To do this, at first Abel integral equation of the second kind is solved numerically. Then the proposed algorithm is generalized to solve single and multi-term fractional differential equations with variable coefficients. The proposed algorithm is based on a discrete form of F-transform with generalized B-spline basic functions.

The most important advantages of the proposed method may be summarized as follow:

- the error bound depends only on the modulus of continuity of approximate solution;
- the method is flexible in implementation;
- it gives sufficiently smooth piecewise best approximation in a small support;
- it doesn't require any starting point or auxiliary function for starting;
- it is as accurate as of the most of existing numerical methods.

This paper is organized as follows. In section 2, the basic concepts of fractional calculus and the fuzzy transform are introduced. Section 3 is divided into two subsections. In subsection 3.1, a numerical method based on the discrete fuzzy transform is discussed for solving Abel integral equations. The same technique is used in subsection 3.2 to find numerical solutions to linear single-term fractional differential equations with variable coefficients. Section 4 is devoted to the generalization of this method for solving multi-term fractional differential equations. The stability of the method with respect to data perturbation is discussed in section 5. An error estimate is also derived in section 6 for problems whose exact solutions are not available. Illustrative examples are presented in section 7 to confirm the theoretical results and the effectiveness of the method. Finally, a brief conclusion is given in section 8.

2. Preliminaries

Here, the basic definitions of fractional operators and fuzzy transforms, and their fundamental properties are presented that will be used throughout the next sections.

2.1. Fractional operators.

Definition 2.1. [21] Let $\mu \in \mathbb{R}$; we say $f \in C_\mu[a,b]$ if there exists a real number $\gamma > \mu$ such that $f(x) = (x-a)^\gamma g(x)$ for $x \in [a,b]$, where $g \in C[a,b]$.

Definition 2.2. [21] Let $\mu \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{0\}$. We say $f \in C^m_\mu[a,b]$, if $f^{(m)} \in C_\mu[a,b]$.

Definition 2.3. [21] Let $\alpha \geq 0$, $\mu \geq -1$ and $f \in C_\mu[a,b]$. Then the operator J_a^α, defined by

$$J_a^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, & \alpha > 0, \quad x \geq a \\ f(x), & \alpha = 0, \end{cases} \tag{2.1}$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order α.

Definition 2.4. [21] Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $f \in C_{-1}^n[a,b]$. Then the operator D_a^α, defined by

$$D_a^\alpha f(x) = J_a^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{1-(n-\alpha)}} dt, \tag{2.2}$$

for $a \leq x \leq b$, is called the Caputo fractional differential operator of order α.

Proposition 2.5. *[7, 12, 21] Let $f \in C_\mu[a,b]$, $\mu \geq -1$. Then*

(1) $J_a^\alpha J_a^\nu f = J_a^{\alpha+\nu} f, \quad \alpha, \nu > 0.$
(2) $J_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}, \quad \alpha > 0, \quad \beta > -1, \quad x \geq a.$
(3) $J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k, \quad n-1 < \alpha \leq n, \quad x \geq a \quad f \in C_{-1}^m[a,b].$
(4) $D_a^\alpha J_a^\alpha f(x) = f(x), \quad x \geq a, \quad n-1 < \alpha \leq n.$
(5) $D_a^\alpha c = 0$, *where c is a constant.*
(6) $D_a^\alpha (x-a)^\beta = \begin{cases} 0, & \beta \in \{0, 1, \ldots, n-1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (x-a)^{\beta-\alpha}, & \beta \geq n, \end{cases}$
where $n-1 < \alpha \leq n$.

2.2. **Discrete F- transform.**

Definition 2.6. Let $[a,b] \subset \mathbb{R}$, $n \geq 2$ and $a = x_1 < \ldots < x_n = b$ be partition of $[a,b]$. Considering $x_0 = x_1$ and $x_{n+1} = x_n$, the fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions, form a fuzzy partition of $[a,b]$ if the following conditions are satisfied for $k = 1, \ldots, n$.

1. (locality) $A_k(x) = 0$ for all $x \in [a, x_{k-1}] \bigcup [x_{k+1}, b]$;
2. (continuity) $A_k(x)$ is continuous;
3. (positivity) $A_k(x) > 0$ for all $x \in (x_{k-1}, x_{k+1})$. The membership functions $A_1, \ldots, A_n$ are called basic functions.

Let the nodes $x_1, \ldots, x_n$ are equidistant with step size h, and in addition to the conditions 1–3 we have

4. $A_k(x)$ is strictly increasing on $[x_{k-1}, x_k]$, for $k = 2, \ldots, n$; and it is strictly decreasing on $[x_k, x_{k+1}]$ for all $k = 1, \ldots, n-1$;
5. $A_k(x_k - x) = A_k(x_k + x)$ for all $k = 2, \ldots, n-1$, and all $x \in [0,h]$;
6. $A_k(x) = A_{k-1}(x-h)$ for all $k = 2, \ldots, n$, and all $x \in [x_{k-1}, x_{k+1}]$.

Then $A_1, \ldots, A_n$, $n \geq 2$, is called an h-uniform fuzzy partition of $[a,b]$.

According to [25], the discrete direct F-transform of f is defined as

$$F_k = \frac{\sum_{j=1}^M f(t_j) A_k(t_j)}{\sum_{j=1}^M A_k(t_j)}, \quad k = 1, \ldots, n,$$

where $\{t_1, t_2, \ldots, t_M\}$ is a sufficiently dense set with respect to the fuzzy partition of $[a,b]$, i.e.,

$$\forall k, \exists j, \quad A_k(t_j) > 0. \tag{2.3}$$

The discrete direct F- transform can be represented by the vector

$$F[f] = [F_1, \ldots, F_n],$$

where $F_k, k = 1, \ldots, n$ are called the components of F- transform.

The inverse F- transform of f is defined by

$$\hat{f}(x) = \sum_{k=1}^{n} F_k A_k(x).$$

2.3. **F-transform with cubic B-splines.** Bede and Stefanini introduced a generalization of fuzzy partition by extending the support of A_i from (x_{i-1}, x_{i+1}) to (x_{i-r}, x_{i+r}) for $r \in \mathbb{N}$ [4, 30]. One of the most important examples is the generalization of basic functions to B-spline functions. The type of fuzzy transform that is introduced here has very good smoothness properties.

Let $P : a = x_1 < x_2 < \ldots < x_n = b$, and $x_{-r+1} < \ldots < x_0 < a$ and $b < x_{n+1} < \ldots < x_{n+r}$ be the auxiliary knots. Then the kth B-spline of order $r-1$ with $r \geq 1$ is defined by

$$B_k^r(t) = r(x_{r+k} - x_k)[x_k, \ldots, x_{r+k}](. - t)_+^{r-1}, \quad k \in \{-r+1, \ldots, n\},$$

where $(. - t)_+(x) = (x - t)_+ = \max\{0, x - t\}$ and $[x_0, \ldots, x_r]f$ stands for the divided difference of order r for the function f (see [31]). Due to the properties of B-splines, (P, B_k^r) defines a fuzzy r-partition, i.e., $\sum_{k=-r+1}^{n} B_k^r(t) = r$. In order to simplify the calculation, we will use the normalized form $N_k^r(t) = \frac{1}{r} B_k^r(t)$ of B-splines, which can be created recursively using the following simple recurrence formula

$$\begin{cases} N_k^1(t) = \begin{cases} 1, & x_k \leq t < x_{k+1}, \\ 0, & otherwise, \end{cases} \\ N_k^r(t) = \frac{t - x_k}{x_{k+r-1} - x_k} N_k^{r-1}(t) + \frac{x_{k+r} - t}{x_{k+r} - x_{k+1}} N_{k+1}^{r-1}(t), \quad r \geq 2, \end{cases} \tag{2.4}$$

for $k = -r+1, \ldots, n$. Then the following identity holds

$$\sum_{k=-r+1}^{n} N_k^r(x) = 1, \tag{2.5}$$

i.e., the normalized B-splines form a fuzzy partition. For $r = 4$, the recurrence relation implies the formula of normalized cubic B-splines as follows

$$N_k^4(t) = \frac{1}{6h^3} \begin{cases} (t - x_k)^3, & x_k \leq t < x_{k+1}, \\ (t - x_k)^2(x_{k+2} - t) + (t - x_k)(t - x_{k+1})(x_{k+3} - t) + (x_{k+4} - t)(t - x_{k+1})^2, & x_{k+1} \leq t < x_{k+2}, \\ (t - x_k)(x_{k+3} - t)^2 + (x_{k+4} - t)(t - x_{k+1})(x_{k+3} - t) + (x_{k+4} - t)^2(t - x_{k+2}), & x_{k+2} \leq t < x_{k+3}, \\ (x_{k+4} - t)^3, & x_{k+3} \leq t < x_{k+4}, \\ 0, & t < x_k \text{ or } t \geq x_{k+4}, \end{cases}$$

for $k = -3, -2, -1, \ldots, n$.

The inverse F- transform with respect to the normalized B-splines is modified as

$$\hat{f}_r(x) = \sum_{k=-r+1}^{n} F_{k,r} N_k^r(x),$$

where

$$F_{k,r} = \frac{\sum_{j=1}^{M} N_k^r(t_j) f(t_j)}{\sum_{j=1}^{M} N_k^r(t_j)}, \quad k = -r+1, \ldots, n.$$

The components of the discrete F-transform are the minimizer of the following discrete L^2-error:

$$\sum_{j=1}^{M} (f(t_j) - y)^2 N_k^r(t_j), \quad k = -r+1, \ldots, n,$$

on $\mathbb{R}$. In addition to satisfying with the above best approximation property, the following theorem is met.

Theorem 2.7. *[4] For any continuous function $f : [a, b] \to \mathbb{R}$, we have*

$$f(x) = \hat{f}_r(x) + E(r, f),$$

where $|E(r, f)| \leq r\omega(f, \delta)$, $\delta = \max_{1 \leq k \leq n-1} |x_{k+1} - x_k|$ and $\omega(f, \delta)$ denotes the modulus of continuity for the function f on $[a, b]$.

Remark 2.8. Due to the fact that the above mentioned approximation is exact for $f(x) = 1, \forall x \in [a, b]$, we have

$$\frac{1}{\sum_{j=1}^{M} N_k^r(t_j)} \sum_{k=-r+1}^{n} \sum_{j=1}^{M} N_k^r(t_j) N_k^r(x) = 1.$$

3. The method to single-term fractional differential equations

In this section, we propose a method based on fuzzy transform with cubic B-splines to the FDEs of single-term. We give the formulation and analysis of the method in detail. To avoid confusion, we first go into detail regarding linear Abel integral equations, and then we discuss the single-term FDEs with variable coefficients.

3.1. **Linear Abel integral equations.** We consider an Abel integral equation of the form

$$y(x) = f(x) + \mu(x) \int_a^x (x - t)^{\nu - 1} y(t) dt, \quad 0 < \nu < 1, \quad x \in [a, b], \tag{3.1}$$

where f and μ are given continuous functions. Existence and uniqueness of a continuous solution for such equations are guaranteed by Theorem 7.2.1 in [15].

Let $\{t_1, t_2, \ldots, t_M\}$ be a sufficiently dense set with respect to the fuzzy partition of $[a, b]$ that satisfies (2.3). We look for an approximation of the form

$$y(t) \simeq y_n(t) = \sum_{k=-r+1}^{n} F_{k,r}(y_n) N_k^r(t), \tag{3.2}$$

where $F_{k,r}(y_n) = \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j) y_n(t_j)$ and $s_k = \sum_{j=1}^{M} N_k^r(t_j)$. Substituting from (3.2) in (3.1), we obtain

$$\begin{aligned} y_n(x) &= f(x) + \mu(x) \int_a^x (x - t)^{\nu - 1} \sum_{k=-r+1}^{n} F_{k,r}(y_n) N_k^r(t) dt \\ &= f(x) + \mu(x) \sum_{k=-r+1}^{n} F_{k,r}(y_n) \int_a^x (x - t)^{\nu - 1} N_k^r(t) dt \\ &= f(x) + \mu(x) \sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j) y_n(t_j) \int_a^x (x - t)^{\nu - 1} N_k^r(t) dt \\ &= f(x) + \mu(x) \sum_{j=1}^{M} y_n(t_j) \sum_{k=-r+1}^{n} \frac{1}{s_k} N_k^r(t_j) \int_a^x (x - t)^{\nu - 1} N_k^r(t) dt. \end{aligned}$$

Then, by putting $x = t_i$ for $i = 1, \ldots, M$, we derive a linear system of algebraic equations as follows

$$y_n(t_i) = f(t_i) + \sum_{j=1}^{M} b_{ij} y_n(t_j), \tag{3.3}$$

with

$$b_{ij} = \mu(t_i) \sum_{k=-r+1}^{n} \frac{1}{s_k} N_k^r(t_j) \int_a^{t_i} (t_i - t)^{\nu - 1} N_k^r(t) dt, \quad i, j = 1, 2, \ldots M.$$

We write the system (3.3) in the compact form

$$(I - B) \underline{y} = \underline{f}, \tag{3.4}$$

and solve it for the unknown vector $\underline{y}$, where

$$\underline{y} = \begin{bmatrix} y_n(t_1) \\ \vdots \\ y_n(t_M) \end{bmatrix}, \quad \underline{f} = \begin{bmatrix} f(t_1) \\ \vdots \\ f(t_M) \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \dots & b_{1M} \\ \vdots & \ddots & \vdots \\ b_{M1} & \dots & b_{MM} \end{bmatrix}.$$

Proposition 3.1. *Let B be given as in (3.4). Then*

$$\|B\| \leq \|\mu\|_\infty \frac{(b-a)^\nu}{\nu},$$

where ν is given as in (3.1).

Proof. We know that

$$\|B\| = \max_{1\leq i\leq M} \sum_{j=1}^{M} |b_{ij}|.$$

On the other hand, we have

$$\sum_{j=1}^{M} |b_{ij}| = |\mu(t_i)| \int_a^{t_i} (t_i - t)^{\nu-1} \Big(\sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j) N_k^r(t) \Big) dt,$$

for $i = 1, 2, \dots M$. Since $\sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j) N_k^r(t) = 1$ (see Proposition 2.8), we deduce

$$\|B\| \leq \|\mu\|_\infty \frac{(b-a)^\nu}{\nu},$$

where $\|\mu\|_\infty = \sup_{a\leq t\leq b} |\mu(t)|$. This completes the proof. □

Theorem 3.2. *Let $\{y_n(t_j)\}_{j=1}^{M}$ be the solution set of (3.3) and $y(x)$ satisfies Eq. (3.1). If $1 - \|\mu\|_\infty \frac{(b-a)^\nu}{\nu} > 0$, then*

$$\max_{1\leq j\leq M} |y(t_j) - y_n(t_j)| \leq C\omega(y, h) \tag{3.5}$$

for some constant $C \geq 0$.

Proof. We first notice that, by the assumption

$$1 - \|\mu\|_\infty \frac{(b-a)^\nu}{\nu} > 0,$$

the system (3.3) is solvable and has a unique solution for each $M > 2$. Put $x = t_i$ in Eq.(3.1), i.e.,

$$y(t_i) = f(t_i) + \mu(t_i) \int_a^{t_i} (t_i - t)^{\nu-1} y(t) dt. \tag{3.6}$$

Since

$$y(t) = \sum_{k=-r+1}^{n} \left(\frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j) y(t_j) \right) N_k^r(t) + E(r, y), \tag{3.7}$$

with $|E(r, y)| \leq r\omega(y, h)$, we arrive at

$$y(t_i) = f(t_i) + \mu(t_i) \int_a^{t_i} (t_i - t)^{\mu-1} \left(\sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j) y(t_j) N_k^r(t) + E(r, y) \right) dt. \tag{3.8}$$

On the other hand, from (3.3) we have

$$y_n(t_i) = f(t_i) + \mu(t_i) \sum_{j=1}^{M} y_n(t_j) \sum_{k=-r+1}^{n} \frac{1}{s_k} N_k^r(t_j) \int_a^{t_i} (t_i - t)^{\nu-1} N_k^r(t) dt. \tag{3.9}$$

Subtracting (3.9) from (3.8) implies

$$|y(t_i) - y_n(t_i)| = \left|\mu(t_i)\int_a^{t_i}(t_i-t)^{\nu-1}\left(\sum_{k=-r+1}^{n}\frac{1}{s_k}\sum_{j=1}^{M}N_k^r(t_j)\Big(y(t_j)-y_n(t_j)\Big)N_k^r(t)+E(r,y)\right)dt\right|$$
$$\leq |\mu(t_i)|\max_{1\leq j\leq M}|y(t_j)-y_n(t_j)|\int_a^{t_i}(t_i-t)^{\nu-1}\Big(\sum_{k=-r+1}^{n}\frac{1}{s_k}\sum_{j=1}^{M}N_k^r(t_j)N_k^r(t)\Big)dt$$
$$+\int_a^{t_i}(t_i-t)^{\nu-1}|E(r,y)|dt$$
$$\leq \max_{1\leq j\leq M}|y(t_j)-y_n(t_j)|\|\mu\|_\infty\frac{(b-a)^\nu}{\nu}+r\omega(y,h)\frac{(b-a)^\nu}{\nu}.$$

(In the last inequality we used the identity $\sum_{k=-r+1}^{n}\frac{1}{s_k}\sum_{j=1}^{M}N_k^r(t_j)N_k^r(t)=1$).

Since we have this inequality for $i=1,\dots,m$, it yields

$$\max_{1\leq i\leq M}|y(t_i)-y_n(t_i)|\leq \max_{1\leq j\leq M}|y(t_j)-y_n(t_j)|\|\mu\|_\infty\frac{(b-a)^\nu}{\nu}+r\omega(y,h)\frac{(b-a)^\nu}{\nu}. \tag{3.10}$$

Finally, since $1-\|\mu\|_\infty\frac{(b-a)^\nu}{\nu}>0$, we conclude that

$$\max_{1\leq j\leq M}|y(t_j)-y_n(t_j)|\leq C\omega(y,h),$$

where $C=\frac{r(b-a)^\nu}{\nu-\|\mu\|_\infty(b-a)^\nu}$. □

3.2. **Fractional differential equations with a single fractional term.** Consider a fractional initial value problem (FIVP) of the from

$$\begin{cases} D_a^\alpha y(x)+p(x)y(x)=f(x), & n-1<\alpha<n,\\ y^{(k)}(a)=b_k, & k=0,1,\dots,n-1,\end{cases} \tag{3.11}$$

where $n=\lceil\alpha\rceil$, p and f are given continuous functions, b_ks are known real numbers and D_a^α denotes a fractional differential operator in the Caputo sense.

To solve Problem (3.11) by the method of subsection 3.1, we need to write it in the form of Eq. (3.1). To do so, we set

$$\phi(x)=D_a^\alpha y(x),$$

and use Proposition 2.5, poperty 3 to get

$$y(x)=\sum_{k=0}^{n-1}\frac{b_k}{k!}(x-a)^k+\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\phi(t)dt,\quad n-1<\alpha<n. \tag{3.12}$$

Then Problem (3.11) reduces to the equivalent integral equation:

$$\phi(x)+\frac{p(x)}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\phi(t)dt=f(x)-p(x)\sum_{k=0}^{n-1}\frac{b_k}{k!}(x-a)^k,\quad n-1<\alpha<n. \tag{3.13}$$

It is then straightforward to use the method of subsection 3.1 to Eq. (3.13) to get an approximation $\phi_n(x)$ to $\phi(x)$. We use then $\phi_n(x)$ on the right hand side of (3.12) to obtain the approximate solution

$$y_n(x)=\sum_{k=0}^{n-1}\frac{b_k}{k!}(x-a)^k+\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\phi_n(t)dt, \tag{3.14}$$

to the Problem (3.11).

At this stage we are ready to discuss the convergence of the proposed method for FDEs with single fractional term. Let $y(x)$ and $y_n(x)$ be the exact and approximate solutions of the Problem (3.11) respectively, where $y_n(x)$ is obtained by the aforementioned method. Now, we are going to prove that $\|y - y_n\|_\infty \to 0$ as $n \to \infty$.

Theorem 3.3. *Let* $P = \max\limits_{x\in[a,b]} |p(x)|$ *and* $L = 1 - \frac{P(b-a)^\alpha}{\alpha\Gamma(\alpha+1)} > 0$. *Then*

$$\|y - y_n\|_\infty \leq D\omega(\phi, h), \tag{3.15}$$

where $D = \frac{r(b-a)^{2\alpha}}{\alpha\Gamma(\alpha+1) - P\alpha(b-a)^\alpha} + \frac{r(b-a)^\alpha}{\Gamma(\alpha+1)}$.

Proof. From (3.12) and (3.14), we have

$$|y(x) - y_n(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |\phi(t) - \phi_n(t)| dt, \tag{3.16}$$

where ϕ verifies (3.13) and

$$\phi_n(t) = \sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j)\phi_n(t_j)N_k^r(t). \tag{3.17}$$

By Theorem 2.7, we have

$$\phi(t) = \sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j)\phi(t_j)N_k^r(t) + E(r,\phi), \tag{3.18}$$

where $|E(r,\phi)| \leq r\omega(\phi,h)$. Subtracting (3.17) from (3.18) yields

$$\begin{aligned}
|\phi(t) - \phi_n(t)| &\leq \sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j)|\phi(t_j) - \phi_n(t_j)| N_k^r(t) + |E(r,\phi)| \\
&\leq \max_{1\leq j\leq M} |\phi(t_j) - \phi_n(t_j)| \sum_{k=-r+1}^{n} \frac{1}{s_k} \sum_{j=1}^{M} N_k^r(t_j)N_k^r(t) + r\omega(\phi,h) \\
&\leq \max_{1\leq j\leq M} |\phi(t_j) - \phi_n(t_j)| + r\omega(\phi,h).
\end{aligned} \tag{3.19}$$

and so, by using Theorem 3.2, we obtain

$$\max_{t\in[a,b]} |\phi(t) - \phi_n(t)| \leq C\omega(\phi,h), \tag{3.20}$$

with $C = \frac{r(b-a)^\alpha}{\alpha - \frac{P}{\Gamma(\alpha)}(b-a)^\alpha} + r$.

Finally, from Eqs. (3.16) and (3.20), we deduce

$$\begin{aligned}
|y(x) - y_n(x)| &\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}|\phi(t) - \phi_n(t)|dt \leq \frac{1}{\Gamma(\alpha+1)} \max_{t\in[a,b]} |\phi(t) - \phi_n(t)|(b-a)^\alpha \\
&\leq \frac{C\omega(\phi,h)(b-a)^\alpha}{\Gamma(\alpha+1)} = D\omega(\phi,h),
\end{aligned}$$

which completes the proof. □

Corollary 3.4. *By the hypothesis of Theorem 3.3, we have* $\|y - y_n\|\infty \to 0$ *as* $n \to \infty$., *i.e., the approximate solution* $y_n(x)$ *converges to the exact solution* $y(x)$.

4. Multi-term fractional differential equations

In this section, we discuss the formulation of $F-$transform method for solving a class of multi-term fractional initial value problems of the form

$$\begin{cases} D_a^{\alpha} y(x) + \sum_{i=1}^{L} p_i(x) D_a^{\alpha_i} y(x) = f(x), & m-1 < \alpha \leq m, \ x \in [a,b], \\ y^{(k)}(a) = b_k, & k = 0, 1, \dots, m-1, \end{cases} \tag{4.1}$$

where $0 \leq \alpha_L \leq \alpha_{L-1} \leq \ldots \leq \alpha_1 < \alpha$, $m = \lceil \alpha \rceil$, p_i and f are given continuous functions, D_a^{α} and $D_a^{\alpha_i}$ are fractional differential operators in the Caputo sense, and b_ks are given real numbers.

To solve problem (4.1) by the method of section 3, we try to find an equivalent integral equation to this problem. Let

$$\phi(x) = D_a^{\alpha} y(x). \tag{4.2}$$

Then by Proposition 2.5 (3), we have

$$y(x) = \sum_{k=0}^{m-1} \frac{b_k}{k!}(x-a)^k + J_a^{\alpha}\phi(x), \quad m-1 < \alpha < m, \tag{4.3}$$

and hence

$$D_a^{\alpha_i} y(x) = \sum_{k=\lceil \alpha_i \rceil}^{m-1} \frac{b_k}{\Gamma(k+1-\alpha_i)}(x-a)^{k-\alpha_i} + J_a^{\alpha-\alpha_i}\phi(x), \quad m-1 < \alpha < m. \tag{4.4}$$

for $\alpha_i > 0$. Using (4.2)-(4.4) in (4.1), it is reduced to a fractional integral equation of the form

$$\phi(x) + \sum_{i=1}^{L} p_i(x) J_a^{\alpha-\alpha_i}\phi(x) = f(x) - \sum_{i=1}^{L} \sum_{k=\lceil \alpha_i \rceil}^{m-1} b_k \frac{p_i(x)(x-a)^{k-\alpha_i}}{\Gamma(k+1-\alpha_i)}, \tag{4.5}$$

and using definition 2.3, it yields the weakly singular integral equation

$$\phi(x) + \sum_{i=1}^{L} \frac{p_i(x)}{\Gamma(\alpha-\alpha_i)} \int_a^x (x-t)^{\alpha-\alpha_i-1}\phi(t)dt = g(x), \tag{4.6}$$

for the function $\phi(x)$, where

$$g(x) := f(x) - \sum_{i=1}^{L} \sum_{k=\lceil \alpha_i \rceil}^{m-1} b_k \frac{p_i(x)(x-a)^{k-\alpha_i}}{\Gamma(k+1-\alpha_i)}. \tag{4.7}$$

To find an approximate solution to Eq. (4.6), we formulate the F- transform technique as follows.

Let

$$\phi(x) \simeq \phi_n(x) = \sum_{k=-r+1}^{n} F_k(\phi_n) N_k^r(x), \tag{4.8}$$

where

$$F_k(\phi_n) = \frac{\sum_{j=1}^{M} \phi_n(t_j) N_k^r(t_j)}{\sum_{j=1}^{M} N_k^r(t_j)}, \quad k = -r+1, \dots, n, \tag{4.9}$$

defines the discrete F-transform for the function ϕ_n. To get the approximation (4.8) to the function ϕ_n, it is sufficient to find the set $\{\phi_n(t_j)\}_{j=1}^{M}$ for a dense subset of points $\{t_j\}_{j=1}^{M}$ in $[a,b]$. To do so, we replace $\phi(x)$ by the right hand side of (4.8) under the integral sign in Eq. (4.6) and arrive at

$$\phi(x)+\sum_{i=1}^{L}\frac{p_i(x)}{\Gamma(\alpha-\alpha_i)}\sum_{k=-r+1}^{n}F_k(\phi)\int_a^x(x-t)^{\alpha-\alpha_i-1}N_k^r(t)dt=g(x), \tag{4.10}$$

which can be simplified as

$$\phi(x)+\sum_{k=-r+1}^{n}F_k(\phi)Q_k(x)=g(x), \quad x\in[a,b], \tag{4.11}$$

with

$$Q_k(x):=\sum_{i=1}^{L}\frac{p_i(x)}{\Gamma(\alpha-\alpha_i)}\int_a^x(x-t)^{\alpha-\alpha_i-1}N_k^r(t)dt. \tag{4.12}$$

To find the unknown values $\{\phi(t_j)\}_{j=1}^{M}$ from Eq. (4.11), we set $s_k=\sum_{j=1}^{M}N_k^r(t_j)$, then

$$F_k(\phi)=\frac{1}{s_k}\sum_{j=1}^{M}\phi(t_j)N_k^r(t_j).$$

Substituting this in Eq. (4.11) and rearranging the summations yield

$$\phi(x)+\sum_{j=1}^{M}\left(\sum_{k=-r+1}^{n}\frac{1}{s_k}N_k^r(t_j)Q_k(x)\right)\phi(t_j)=g(x), \quad x\in[a,b], \tag{4.13}$$

Now, set $x=t_i$ for $i=1,\dots,M$ and

$$b_{ij}=\sum_{k=-r+1}^{n}\frac{1}{s_k}Q_k(t_i)N_k^r(t_j), \quad i,j=1,\dots,M. \tag{4.14}$$

This leads to determining the unknown values $\{\phi(t_j)\}_{j=1}^{M}$ as the solution of the linear system

$$\phi(t_i)+\sum_{j=1}^{M}b_{ij}\phi(t_j)=g(t_i), \quad i=1,\dots,M, \tag{4.15}$$

which can be written in the compact form

$$(I+B)\,\Phi=G \tag{4.16}$$

with $\Phi=[\phi(t_1),\dots,\phi(t_M)]^T$, $G=[g(t_1),\dots,g(t_M)]^T$ and $B=[b_{ij}]_{i,j=1}^{M}$.

Combining the relations (4.8) and (4.3), an approximate solution to the original multi-term fractional problem (4.1) will be obtained from:

$$y(x)\simeq y_n(x)=\sum_{k=0}^{n-1}\frac{b_k}{k!}(x-a)^k+J_a^{\alpha}\phi_n(x), \tag{4.17}$$

where $\phi_n(x)$ is computed by (4.8).

Remark 4.1. The integrals included in the system (4.16) may be evaluated numerically.

Algorithm: The steps of our proposed method for solving the general multi-term FIVP (4.1) can be summarized as follows:

1: Input $a,b,\ r,n,M,\alpha,\ L,\ \alpha_i,\ i=1,\dots,L$.
2: set $m=\lceil\alpha\rceil$.
3: Input the functions $f(x)$, $p_i(x)$, $i=1,\dots,L$.

4: Use a uniform partition for the interval $[a, b]$ to create the points $x_j,\ j = -r+1, \dots, n$.
5: Use the recurrence relation (2.4) to calculate $N_k^r(t)$ for $k = -r+1, \dots, n$.
6: Input the initial values $b_k,\ i = 0, \dots, m-1$.
7: Use the formula (4.7) to calculate $g(x)$.
8: Calculate $Q_k(x)$ from (4.12).
9: Select the points t_j from $[a, b]$ for $j = 0, \dots, M$.
10: Compute b_{ij} from (4.14).
11: Set $G = [g(t_1), \dots, g(t_M)]^T$ and $B = [b_{ij}]_{i,j=1}^M$.
12: Solve the linear system (4.16) for the vector Φ.
13: Compute $F_k(\phi_n)$ from (4.9) for $k = -r+1, \dots, n$.
14: Obtain the function $\phi_n(x)$ from (4.8).
15: Use the relation (4.17) to find an approximate solution to (4.1).
16: End.

5. Effects of Data Perturbations

In this section, we examine how perturbed data influences the variation of solutions. To do so, we first present Lemma 6.19, as cited in [14]:

Lemma 5.1. *Let $\nu, T, \epsilon_1, \epsilon_2 > 0$, and $\delta : [0, T] \to \mathbb{R}$ be continuous. Assume that for all $x \in [0, T]$,*

$$|\delta(x)| \le \epsilon_1 + \frac{\epsilon_2}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} |\delta(t)|\, dt.$$

Then, for all $x \in [0, T]$,

$$|\delta(x)| \le \epsilon_1 \, E_\nu\big(\epsilon_2 x^\nu\big),$$

where E_ν denotes the Mittag–Leffler function of order ν,

$$E_\nu(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + 1)}.$$

The following theorem demonstrates the effect of variations in the initial value and the right-hand side function on the solution of Problem (4.1). Without loss of generality, we prove the stability concept for the case $a = 0$. The general case can be addressed by applying a suitable transformation to reduce it to this specific scenario. Furthermore, the influence of the variation of coefficient functions on the solution can be investigated using a similar approach.

Theorem 5.2. *Let y be a solution of Problem (4.1) on the interval $[0, T]$, where $T > 0$. Let z be the solution to the following fractional initial value problem:*

$$\begin{cases} D_0^\alpha z(x) + \sum_{i=1}^{L} p_i(x) D_0^{\alpha_i} z(x) = g(x), & m-1 < \alpha \le m, \\ z^{(k)}(0) = c_k, \qquad k = 0, 1, \dots, m-1, \end{cases} \tag{5.1}$$

assuming g is a continuous function and c_k are given real numbers. Then, the following inequality holds:

$$\max_{0 \le x \le T} |y(x) - z(x)| \le C_1 \max_{1 \le k \le m} |b_k - c_k| + C_2 \max_{0 \le x \le T} |f(x) - g(x)|, \tag{5.2}$$

for some constants $C_1, C_2 \ge 0$.

Proof. Let $\phi(x) = D_0^\alpha y(x)$ and $\psi(x) = D_0^\alpha z(x)$. Then, recalling (4.5) for ψ, we have

$$\psi(x) + \sum_{i=1}^{L} p_i(x) J_0^{\alpha - \alpha_i} \psi(x) = g(x) - \sum_{i=1}^{L} \sum_{k=\lceil \alpha_i \rceil}^{m-1} c_k \frac{p_i(x) x^{k-\alpha_i}}{\Gamma(k+1-\alpha_i)}. \tag{5.3}$$

Subtracting equations (4.5) and (5.3), we obtain the following inequality:

$$|\psi(x)-\phi(x)| \le \sum_{i=1}^{L} |p_i(x)| J_0^{\alpha-\alpha_i} |\psi(x)-\phi(x)| + |g(x)-f(x)| + \sum_{i=1}^{L}\sum_{k=\lceil \alpha_i \rceil}^{m-1} |c_k-b_k| \frac{p_i(x)x^{k-\alpha_i}}{\Gamma(k+1-\alpha_i)}.$$

For the first term on the right-hand side of the inequality, we have:

$$\sum_{i=1}^{L} |p_i(x)| J_0^{\alpha-\alpha_i} |\psi(x)-\phi(x)| \le M_1 \sum_{i=1}^{L} \frac{1}{\Gamma(\alpha-\alpha_i)} \int_0^x (x-t)^{\alpha-\alpha_i-1} |\psi(t)-\phi(t)| dt$$
$$\le \frac{M_1}{\Gamma(\alpha-\alpha_1)} \int_0^x (x-t)^{\alpha-\alpha_1-1} |\psi(t)-\phi(t)| \left(\sum_{i=1}^{L} \frac{\Gamma(\alpha-\alpha_1)}{\Gamma(\alpha-\alpha_i)} (x-t)^{\alpha_1-\alpha_i} \right)$$
$$\le \frac{M}{\Gamma(\alpha-\alpha_1)} \int_0^x (x-t)^{\alpha-\alpha_1-1} |\psi(t)-\phi(t)| dt.$$

Where M_1, M depend only on T, α, α_i and $\max_i(\max_{x\in[0,T]} |p_i(x)|)$. Here we used the fact that $\alpha_i \le \alpha_1 < \alpha$ by assumptions in Problem (4.1). Combining these inequalities, we can find $N_1 > 0$ such that

$$|\psi(x)-\phi(x)| \le |f(x)-g(x)| + N_1 \sum_{k=0}^{m-1} |c_k-b_k| + \frac{M}{\Gamma(\alpha-\alpha_1)} \int_0^x (x-t)^{\alpha-\alpha_1-1} |\psi(t)-\phi(t)| dt$$
$$\le \max_{x\in[0,T]} |f(x)-g(x)| + N \max_k |c_k-b_k| + \frac{M}{\Gamma(\alpha-\alpha_1)} \int_0^x (x-t)^{\alpha-\alpha_1-1} |\psi(t)-\phi(t)| dt,$$

for some $N > 0$. At this stage, we can employ Lemma 5.1 to arrive at:

$$\max_{x\in[0,T]} |\psi(x)-\phi(x)| \le \left(\max_{x\in[0,T]} |f(x)-g(x)| + N \max_k |c_k-b_k| \right) E_{\alpha-\alpha_1}(Mx^{(\alpha-\alpha_1)}),$$

for $x \in [0,T]$. Thus , under the conditions that $\max_{x\in[0,T]} |f(x)-g(x)| \le \epsilon$ and $\max_k |c_k-b_k| \le \epsilon$, it follows that $\max_{x\in[0,T]} |\psi(x)-\phi(x)| = O(\epsilon)$. □

6. Error estimation

Let us define the error function $e_n(x)$ associated to the approximation $y_n(x)$ as $e_n(x) = y_n(x) - y(x)$, where $y(x)$ denotes the exact solution of the Problem (4.1). Since $y_n(x)$ is an approximate solution of (4.1), it then satisfies the perturbed problem

$$\begin{cases} D_a^{\alpha} y_n(x) + \sum_{i=1}^{L} p_i(x) D_a^{\alpha_i} y_n(x) = f(x) + H_n(x), & m-1 < \alpha < m, \\ y_n^{(k)}(a) = b_k, \quad k = 0,1,\dots,m-1, \end{cases} \tag{6.1}$$

where the perturbation term $H_n(x)$ can be computed from

$$H_n(x) = D_a^{\alpha} y_n(x) + \sum_{i=1}^{L} p_i(x) D_a^{\alpha_i} y_n(x) - f(x), \qquad m-1 < \alpha < m. \tag{6.2}$$

Subtracting both sides of (4.1) and (6.1), we conclude the error function $e_n(x)$ satisfying the initial value problem

$$\begin{cases} D_a^{\alpha} e_n(x) + \sum_{i=1}^{L} p_i(x) D_a^{\alpha_i} e_n(x) = H_n(x), & m-1 < \alpha < m, \\ e_n^{(k)}(a) = 0, \quad k = 0,1,\dots,m-1. \end{cases} \tag{6.3}$$

Table 1. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for the Example 7.1.

	n=10	n=10	n=20	n=20	n=50	n=50
x	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$
0	0	0	0	0	0	0
0.2	3.8667e-04	9.6651e-03	9.9018e-04	9.9977e-05	6.2101e-05	5.8689e-05
0.4	3.0025e-04	1.4351e-04	6.8009e-04	3.4872e-04	3.3608e-04	1.7338e-04
0.6	1.6769e-03	1.3184e-03	2.8415e-04	6.7050e-05	3.6042e-04	3.5202e-04
0.8	6.4026e-04	1.8750e-04	2.9125e-04	2.6894e-04	7.0364e-04	2.8981e-04
1	2.0579e-03	1.2354e-03	1.4246e-03	3.4434e-04	1.3032e-03	5.9596e-07

This is the same problem as defined in (4.1) only with different right hand side and homogeneous initial conditions. Therefore, by setting

$$\psi_n(x) = D_a^{\alpha} e_n(x), \tag{6.4}$$

the equivalent integral equation takes the form

$$\psi_n(x) + \sum_{i=1}^{L} \frac{p_i(x)}{\Gamma(\alpha-\alpha_i)} \int_a^x (x-t)^{\alpha-\alpha_i-1}\psi_n(t)dt = H_n(x), \quad m-1<\alpha<m. \tag{6.5}$$

Solving (6.5) by the same way of (3.13), gives an estimation $\widetilde{e}_n(x)$ to the error function $e_n(x)$ according to the relation (4.17) with $b_k = 0$ for $k = 1, \dots, m-1$, i.e.,

$$e_n(x) \simeq \widetilde{e}_n(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\widetilde{\psi}_n(t)dt, \tag{6.6}$$

where $\widetilde{\psi}_n$ denotes the approximate solution of Eq. (6.5).

7. Numerical experiments

In this section, we consider the following examples in order to illustrate the accuracy of the proposed method, confirm the presented theoretical results, and compare our results with those in the literature.

Example 7.1. Consider the multi-term FIVP

$$\begin{cases} D_0^{\alpha} y(x) + y(x) = x^{\lambda} + \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} x^{\lambda-\alpha}, & 0 \le \alpha \le \lambda \le 1, \quad x \in [0,1], \\ y(0) = 0, \end{cases} \tag{7.1}$$

with the exact solution $y(x) = x^{\lambda}$.

For this problem we have $L = 1,\ p_1(x) = 1,\ \alpha_1 = 0$. We choose $\alpha = \frac{1}{4}$ and $\lambda = \frac{1}{2}$, then $m = \lceil \alpha \rceil = 1$ and the right hand side is $f(x) = \sqrt{x} + \frac{2\sqrt{\pi}}{\Gamma(0.25)} x^{0.25}$. Hence, the integral equation in (4.6) takes the form

$$\phi(x) + \frac{1}{\Gamma(0.25)} \int_0^x \frac{\phi(t)}{(x-t)^{0.75}} dt = \sqrt{x} + \frac{2\sqrt{\pi}}{\Gamma(0.25)} x^{0.25}, \tag{7.2}$$

where $\phi(x) = D_0^{\frac{1}{2}} y(x)$.

Now, we implement our method on the Eq. (7.2), and we report the results of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for $n = 10, 20, 50$ in Table 1. The related plots for the function $|e_n(x)|$ are shown in Figure 1. These results show a good accuracy of our method. The approximate solution for the Problem (7.1) is computed by (4.17), where $\phi_n(x)$ is the approximate solution of Eq. (7.2). Table 1 shows the corresponding results after perturbing the right-hand side of Eq. (7.1) with $\epsilon = 10^{-3}$.

TABLE 2. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for the Example 7.1 with perturbation $\epsilon = 10^{-3}$.

	n=10	n=10	n=20	n=20	n=50	n=50
x	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$
0	0	0	0	0	0	0
0.2	4.6107e-05	9.6590e-03	1.4232e-03	1.0031e-04	4.9568e-04	5.8613e-05
0.4	7.7774e-04	1.4268e-04	1.1576e-03	3.4853e-04	1.4185e-04	1.7321e-04
0.6	2.1802e-03	1.3176e-03	2.1977e-04	6.6943e-05	1.4366e-04	3.5178e-04
0.8	1.1773e-04	1.8748e-04	2.3122e-04	2.6871e-04	1.8089e-04	2.8959e-04
1	1.5207e-03	1.2351e-03	8.8747e-04	3.4443e-04	7.6594e-04	4.4773e-07

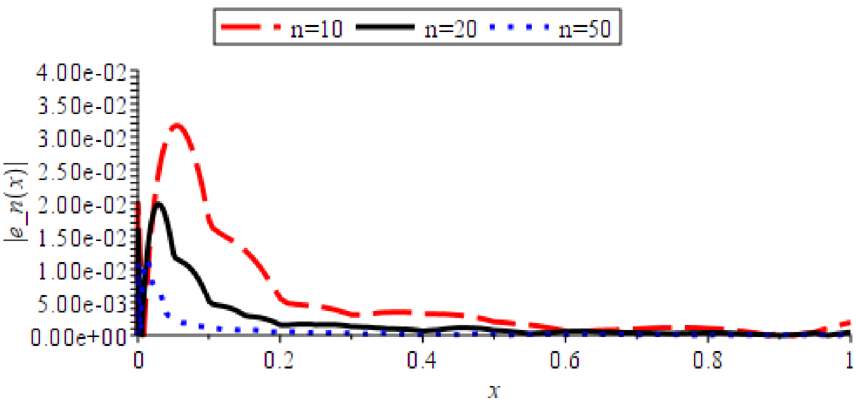

FIGURE 1. The plots of the absolute error for Example 7.1 ($\alpha = \frac{1}{4}$, $\lambda = \frac{1}{2}$).

Example 7.2. Consider the multi-term FIVP

$$\begin{cases} D^2y(x) + sin(x)D_0^{\frac{1}{2}}y(x) + xy(x) = f(x), \\ y(0) = y'(0) = 0, \\ f(x) = x^9 - x^8 + 56x^6 - 42x^5 + sin(x)\left(\frac{32768}{6435}x^{\frac{15}{2}} - \frac{2048}{429}x^{\frac{13}{2}}\right), \end{cases} \tag{7.3}$$

with exact solution $y(x) = x^8 - x^7$. Let $\phi(x) = D^2y(x)$. Then

$$\begin{cases} y(x) = J_0^2\phi(x) = \int_0^x (x-t)\phi(t)dt, \\ D^{\frac{1}{2}}y(x) = J_0^{\frac{3}{2}}\phi(x) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^x (x-t)^{\frac{1}{2}}\phi(t)dt, \end{cases} \tag{7.4}$$

and so the integral equation equivalent to the Problem (7.3) is obtained in the form

$$\phi(x) + \int_0^x k(x,t)\phi(t)dt = f(x),$$

where $k(x,t) = \left(\frac{2sin(x)}{\sqrt{\pi}}(x-t)^{-\frac{1}{2}} + x\right)(x-t)$.

Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ are presented in Table 3 for $n = 10, 20, 50$. Figure 2 shows the related plots for the function $|e_n(x)|$.

Example 7.3. Consider the multi-term FIVP

$$\begin{cases} Dy(x) + \sqrt{x}D_0^{0.5}y(x) + xy(x) = \frac{3}{2}\sqrt{x} + \frac{3\sqrt{\pi}}{4}x^{1.5} + x^{2.5}, \\ y(0) = 0 \end{cases} \tag{7.5}$$

Table 3. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for Example 7.2.

	n=10	n=10	n=20	n=20	n=50	n=50
x	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$
0	0	0	0	0	0	0
0.2	4.9784e-05	6.7091e-05	9.4213e-06	9.9242e-06	1.3499e-06	1.1824e-06
0.4	7.0575e-04	6.0500e-04	1.7267e-04	1.4481e-04	2.9483e-05	2.1644e-05
0.6	2.0289e-03	1.0157e-03	6.2222e-04	3.7960e-04	1.7987e-04	4.3261e-05
0.8	3.5611e-03	9.4190e-03	4.7652e-05	1.2039e-03	7.7764e-04	3.4689e-04
1	5.6656e-02	8.0877e-02	1.0886e-02	1.3305e-02	1.4235e-03	2.0112e-03

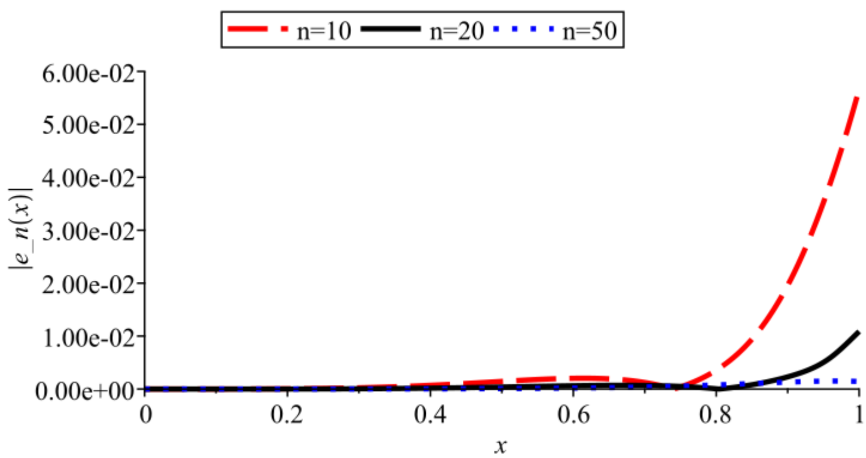

Figure 2. Plots of the error function $|e_n(x)|$ for the Example 7.2.

Table 4. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for Example 7.3.

	n=10	n=10	n=20	n=20	n=50	n=50
x	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$
0	0	0	0	0	0	0
0.2	7.9893e-03	9.3433e-04	3.4016e-03	9.4745e-04	9.7265e-04	4.4882e-04
0.4	8.4089e-03	5.1484e-03	3.2530e-03	3.6658e-03	8.9399e-04	1.6315e-03
0.6	6.5945e-03	1.5873e-02	2.5306e-03	1.0170e-02	6.9271e-04	4.3873e-03
0.8	2.4917e-03	3.2213e-02	1.0825e-03	2.0652e-02	3.1742e-04	8.9018e-03
1	7.3497e-04	5.4186e-02	3.8342e-04	3.5276e-02	1.2178e-04	1.5246e-02

with the exact solution $y(x) = x^{1.5}$. Here, we have $\alpha = 1,\ m = \lceil \alpha \rceil = 1,\ L = 2,\ p_1(x) = x^{1/2},\ p_2(x) = x, \alpha_1 = 0.5, \alpha_2 = 0$ and $f(x) = \frac{3}{2}\sqrt{x} + \frac{3\sqrt{\pi}}{4}x^{1.5} + x^{2.5}$. Hence, letting $\phi(x) = Dy(x)$, the integral equation in (4.6) reduces to

$$\phi(x) + \int_0^x k(x,t)\phi(t)dt = \frac{3}{2}\sqrt{x} + \frac{3\sqrt{\pi}}{4}x^{1.5} + x^{2.5}, \tag{7.6}$$

where $k(x,t) = x + \sqrt{\frac{x}{\pi(x-t)}}$.

For this example the absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for $n = 10, 20, 50$ are reported in Table 4. Figure 3 shows the related plots for the function $|e_n(x)|$.

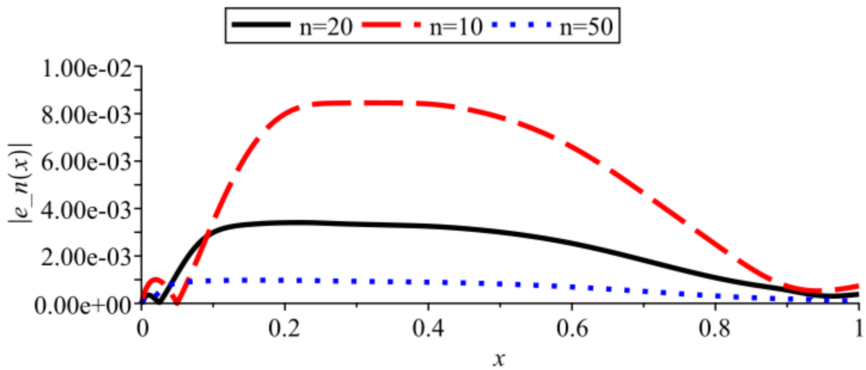

FIGURE 3. Plots of the function $|e_n(x)|$ for Example 7.3.

Example 7.4. Consider the multi-term FIVP

$$\begin{cases} D_0^{3/2}y(x) + 2Dy(x) + 3\sqrt{x}D_0^{1/2}y(x) + (1-x)y(x) = f(x), \\ y(0)=0, \quad y'(0) = b = \begin{cases} 0, & m = 5,7,9, \\ 1, & m = 3, \end{cases} \end{cases} \tag{7.7}$$

where $f(x)$ is determined such that $y(x) = x^{m/3}\cos(x^{m/3})$ to be exact solution of the problem for $m = 3,5,7,9$.

Let $\phi(x) = D_0^{3/2}y(x)$. Then

$$\begin{cases} y(x) = bx + J_0^{\frac{3}{2}}\phi(x) = bx + \frac{2}{\sqrt{\pi}}\int_0^x (x-t)^{\frac{1}{2}}\phi(t)dt, \\ Dy(x) = b + J_0^{\frac{1}{2}}\phi(x) = b + \frac{1}{\sqrt{\pi}}\int_0^x (x-t)^{-\frac{1}{2}}\phi(t)dt, \\ D_0^{\frac{1}{2}}y(x) = \frac{b\sqrt{x}}{\Gamma(\frac{3}{2})} + J_0^1\phi(x) = \frac{2b\sqrt{x}}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\int_0^x \phi(t)dt. \end{cases} \tag{7.8}$$

Substituting from above for the terms $D_0^{3/2}y(x), Dy(x), D_0^{1/2}y(x), y(x)$ in Eq. (7.7), the equivalent integral equation for the function ϕ is written in the form

$$\phi(x) + \int_0^x k(x,t)\phi(t)dt = f(x) - 2b - \frac{6bx}{\sqrt{\pi}} - bx(1-x), \quad 0 \le x \le 1,$$

where

$$k(x,t) = \frac{2}{\sqrt{\pi(x-t)}} + 3\sqrt{x} + \frac{2}{\sqrt{\pi}}(1-x)\sqrt{x-t}.$$

For $n = 10, 20, 50$ and $m = 3,5,7,9$, the absolute values of exact and estimate errors have been reported in Tables 5-8 for $x = 0, 0.2, \dots, 1$. Figure 4 shows the plots of absolute errors for $n = 10, 20, 50$.

TABLE 5. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for Example 7.4(m=3).

	n=10	n=10	n=20	n=10	n=50	n=10
x	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$	$\lvert e_n(x)\rvert$	$\lvert\widetilde{e}_n(x)\rvert$
0	0	0	0	0	0	0
0.2	6.4771e-04	4.4610e-04	1.8598e-04	8.9730e-05	3.4101e-05	3.1511e-05
0.4	1.0969e-03	2.9068e-04	3.0333e-04	5.1742e-04	5.2531e-05	6.1540e-04
0.6	1.2792e-03	3.8638e-03	3.4688e-04	2.7896e-03	5.7150e-05	2.5571e-03
0.8	1.2355e-03	1.1883e-02	3.3029e-04	7.7064e-03	5.9399e-05	6.6860e-03
1	1.5571e-03	2.5428e-02	3.8931e-04	1.5959e-02	6.8577e-05	1.3621e-02

8. Conclusion

In this paper, a new simple and effective numerical algorithm is designed for solving a class of multi-term fractional differential equations with variable coefficients. The new algorithm uses fuzzy transforms with cubic B-splines, which possess the following important advantages:

- The error bound is dependent only on the modulus of continuity of the approximate solution.
- The method is flexible in implementation.
- A sufficiently smooth piecewise best approximation is given in a small support.
- No starting point or auxiliary function is required for starting.
- Accuracy comparable to that of most existing numerical methods is achieved.

Acknowledgement

This project is supported by a research grant of University of Tabriz (number 898).

References

[1] R. Alikhani, M. Zeinali, F. Bahrami, S. Shahmorad, and I. Perfilieva, *Trigonometric F^m-transform and its approximative properties*, Soft Computing, *21*(13) (2017), 3567–3577.

Table 6. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for Example 7.4 (m=5).

	n=10	n=10	n=20	n=20	n=50	n=50
x	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$
0	0	0	0	0	0	0
0.2	3.7143e-03	7.8546e-04	1.6481e-03	4.5824e-04	4.5446e-04	3.3420e-04
0.4	5.7198e-03	1.3483e-03	2.1291e-03	2.5285e-03	5.5116e-04	1.2428e-03
0.6	6.2129e-03	1.2430e-02	2.0756e-03	6.8301e-03	4.7099e-04	2.4152e-03
0.8	5.4843e-03	1.7086e-02	1.2265e-03	3.9794e-03	3.0612e-05	1.4430e-05
1	6.7874e-03	4.1553e-02	1.4774e-03	2.3720e-02	9.8281e-05	1.2716e-02

Table 7. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for Example 7.4 (m=7).

	n=10	n=10	n=20	n=20	n=50	n=50
x	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$
0	0	0	0	0	0	0
0.2	3.8555e-05	6.6616e-04	1.3895e-04	3.7381e-04	4.6855e-05	2.3482e-04
0.4	1.0989e-03	2.4046e-03	4.0016e-04	4.2639e-03	8.2547e-05	1.7842e-03
0.6	2.1657e-03	1.3480e-02	7.5112e-04	1.3807e-02	1.9041e-04	5.1292e-03
0.8	2.5185e-03	3.6927e-02	1.0845e-03	2.5262e-02	3.5849e-04	7.6432e-03
1	8.0620e-04	1.6900e-02	2.1999e-03	3.4353e-03	6.7670e-04	3.1214e-03

Table 8. Absolute values of the error functions $|e_n(x)|$ and $|\widetilde{e}_n(x)|$ for Example 7.4 (m=9).

	n=10	n=10	n=20	n=20	n=50	n=50
x	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$	$\|e_n(x)\|$	$\|\widetilde{e}_n(x)\|$
0	0	0	0	0	0	0
0.2	1.3214e-03	7.6646e-04	3.7891e-04	6.7908e-06	6.8619e-005	1.6721e-004
0.4	2.2531e-03	1.4046e-03	5.9615e-04	2.7904e-03	1.1165e-004	2.3602e-003
0.6	1.2418e-03	1.1105e-02	3.8615e-04	1.2376e-02	1.1501e-004	9.4286e-003
0.8	7.9334e-03	3.2064e-02	1.3119e-03	3.0425e-02	1.1613e-004	2.1017e-002
1	4.9161e-02	1.9545e-03	5.3174e-03	6.5815e-03	7.6314e-004	3.7343e-003

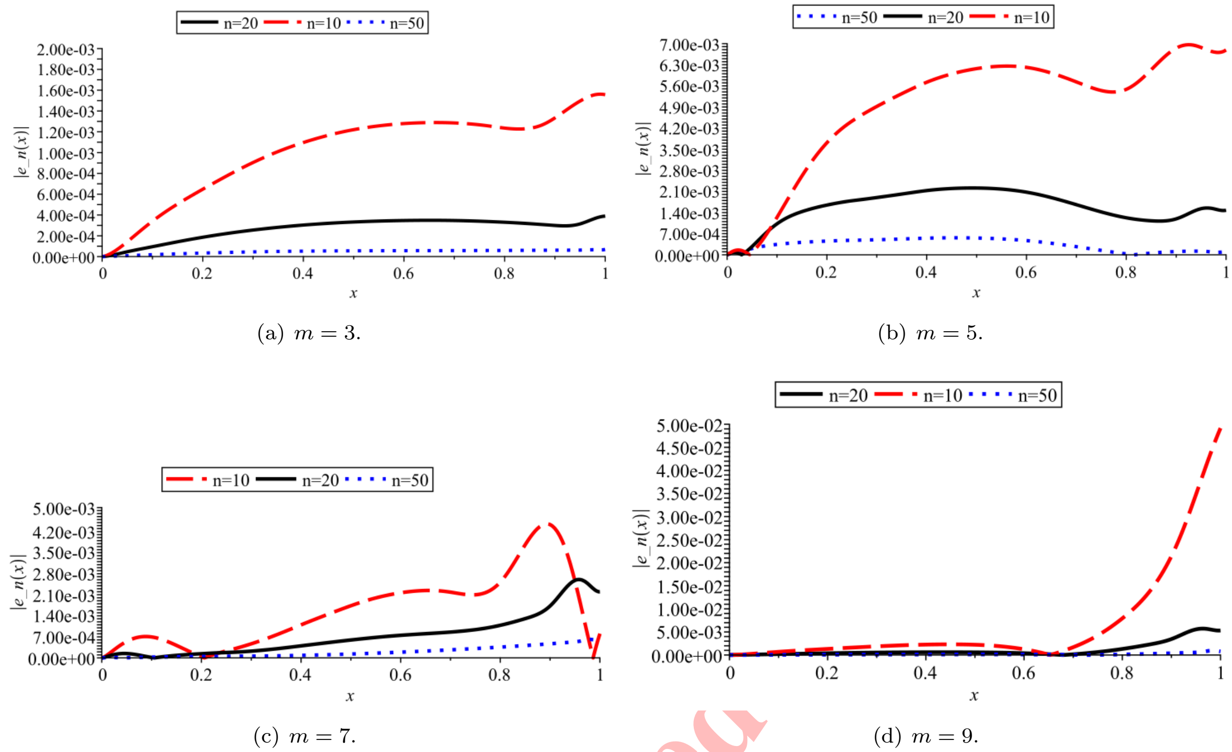

Figure 4. Plots of the error $e_n(x)$ for Example 7.4.

[2] O. M. Al-Mdallal, Muhammed I. Syam, and M. N. Anwar, *A collocation–shooting method for solving fractional boundary value problems*, Communications in Nonlinear Science and Numerical Simulation, *15*(12) (2010), 3814–3822.

[3] A. Anshima Singh, S. Kumar, and H. Ramos, *An efficient computational method based on exponential B-splines for a class of fractional sub-diffusion equations*, CMDE, *12*(4) (2024), 719-740.

[4] B. Bede and I. J. Rudas, *Approximation properties of fuzzy transforms*, Fuzzy Sets and Systems, *180*(1) (2011), 20–40.

[5] S. M. Belotserkovsky and I. K. Lifanov, *Method of discrete vortices*, CRC Press (1992).

[6] H. Beyer and S. Kempfle, *definition of physically consistent damping laws with fractional derivatives*, ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, *75*(8) (1995), 623–635.

[7] L. Blank, *Numerical treatment of differential equations of fractional order*, Numerical Analysis Report-University of Manchester Department of Mathematics (1996).

[8] I. V. Boykov and N. Y. Kudryashova, *Approximate methods for singular integral equations in exceptional cases*, Differentsial'nye Uravneniya, *36*(9) (2000), 1230–1237.

[9] H. Brunner, *The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes*, Mathematics of computation, *45*(172) (1985), 417–437.

[10] H. Brunner, M. R. Crisci, E. Russo, and A. Vecchio, *A family of methods for Abel integral equations of the second kind*, J. Coput. Appl. Math., *34*(2) (1991), 211–219.

[11] R. Cameron and S. McKee, *Product integration methods for second-kind Abel integral equations*, J. Coput. Appl. Math., *11*(1) (1984), 1–10.

[12] A. Carpinteri and F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, *378* (2014).

[13] H. Dehestani and Y. Ordokhani, *A highly accurate wavelet approach for multi-term variable-order fractional multi-dimensional differential equations*, CMDE, *13*(3) (2025), 850-869.

[14] K. Diethelm and Y. Luchko, *Numerical solution of linear multi-term initial value problems of fractional order*, J. Comput. Anal. Appl., *6*(3) (2004), 243-63.

[15] R. Gorenflo and S. Vessella, *Abel Integral Equations, Analysis and Applications*, Springer–Verlag (1991).

[16] H. Fazli, F. Bahrami, and S. Shahmorad, *Extremal solutions for multi-term nonlinear fractional differential equations with nonlinear boundary conditions*, CMDE, *11*(1) (2023), 32–41.

[17] Q. Hadi Haddam, E. Najafi, and S. Sohrabi, *Jacobi collocation method for numerical solution of nonlinear weakly singular Volterra integrodifferential equations: fractional and stochastic cases*, CMDE, *13*(3) (2025), 742-757.

[18] R. Katani, S. Shahmorad, and D. Conte, *Approximate solution of multi-term fractional differential equations via a block-by-block method*, J. Comput. Appl. Math., *453* (2025), 116135.

[19] A. Khastan, I. Perfilieva, and Z. Alijani, *A new fuzzy approximation method to Cauchy problems by fuzzy transform*, Fuzzy Sets and Systems, *288* (2016), 75–95.

[20] M. Kokainis and S. Asmuss, *Continuous and discrete higher-degree F-transforms based on B-splines*, Soft Computing, *21*(13) (2017), 615–3639.

[21] F. Mainardi, *Fractional Calculus: Some basic problems in continuum and statistical mechanics*, arXiv preprint arXiv:1201.0863 (2012).

[22] S. Malge and R. Kishun Lodhi, *Quintic B-spline method for numerical solution of second-order singularly perturbed delay differential equations*, CMDE, *14*(1) (2026), 36-49.

[23] M. Molavi-Arabshahi, J. Rashidinia, and S. Tanoomand, *Numerical solving of multi-term time fractional diffusion-wave equations using shifted Gegenbauer spectral collocation method*, CMDE, *13*(3) (2025), 815–827.

[24] A. Orsi, *Product integration for Volterra integral equations of the second kind with weakly singular kernels*, Mathematics of computation, *65*(215) (1996), 1201–1212.

[25] I. Perfilieva, *Fuzzy transforms: theory and applications*, Fuzzy sets and systems, *157*(8) (2006), 993–1023.

[26] I. Perfilieva, P. Stevuliakova, and R. Valasek, *F- Transform for numerical solution of two-point boundary value problem*, Iranian Journal of Fuzzy Systems, *14*(6) (2017), 1–13.

[27] I. Perfilieva, M. Daňková, and B. Bede, *Towards a higher degree F- Transform*, Fuzzy Sets and Systems, *180* (1) (2011), 3–19.

[28] A. Saadatmandi and M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Computers & mathematics with applications, *59*(3) (2010), 1326–1336.

[29] K. Solmann, R. Loffredo, and S. Tomasiello, *Using fuzzy transforms for neural networks-based wireless localization in outdoor environments*, Neural Comput. Applic., *36* (2024), 20027–20041.

[30] L. Stefanini, *F- transform with parametric generalized fuzzy partitions*, Fuzzy Sets and Systems, *180*(1) (2011), 98–120.

[31] J. Stoer and R. Bulirsch, *Introduction to numerical analysis*, *12* (2013).

[32] T. Tang, *A note on collocation methods for Volterra integro-differential equations with weakly singular kernels*, IMA journal of numerical analysis, *13*(1) (1993), 93–99.

[33] S. Tomasiello, *An alternative use of fuzzy transform with application to a class of delay differential equations*, I. J. Comput. Math., *94*(9) (2017), 1719–1726.

[34] S. Tomasiello, *Least-Squares Fuzzy Transforms and Autoencoders: Some Remarks and Application*, IEEE Transactions on Fuzzy Systems, *29*(1) (2021), 129-136.

[35] E. L. Vickie, A. Carreras Benjamin, D. Del-Castillo-Negrete, K. M. Ferreira-Mejias, and H. R. Hicks, *Numerical methods for the solution of partial differential equations of fractional order*, J. Comput. Physics, *192*(2) (2003), 406–421.

[36] M. Zeinali, R. Alikhani, S. Shahmorad, F. Bahrami, and I. Perfilieva, *On the structural properties of F^m-transform with applications*, Fuzzy Sets and Systems, *342* (2018), 32–52.

[37] M. Zeinali, F. Bahrami, and S. Shahmorad, *Recursive higher order fuzzy transform method for numerical solution of Volterra integral equation with singular and nonsingular kernels*, J. Comput. Appl. Math., *403* (2022), 113854.