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Analysis of multiplicative differential equation systems with constant coefficients: Solutions and stability

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Abstract

In this paper, we study the first-order system of multiplicative differential equations with constant coefficients. The multiplicative differential equations model the rate of change in relation to the quantity itself, while the standard additive differential equations cannot model this type of phenomenon or need more complicated calculations. In this paper we present some new results about the first-order system of multiplicative differential equations with constant coefficients like presenting an explicit closed-form solution formula based on matrix exponentiation; necessary and sufficient conditions for the asymptotic stability of the equilibrium solution $Y \equiv \mathbf{1}$; a characterization of periodic solutions in terms of purely imaginary eigenvalues; and a Lyapunov-type theorem for exponential stability. Examples are presented to demonstrate the efficiency of our findings.

Keywords. Multiplicative derivative, Dynamical systems, Existence of solution, Exponential matrix.
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1. INTRODUCTION

Classical calculus was developed in the 17th century by Newton and Leibniz based on the operation of addition, in which the ordinary derivative was introduced as an indicator of the rate of change relative to addition. But not all changes depend on addition, and some depend on the ratio of the changed quantity to the previous one, such as compound interest, population growth, and radioactive decay. In 1972, Grossman and Katz introduced non-Newtonian calculus, which employs operations beyond traditional addition and subtraction. A notable example is multiplicative calculus. The multiplicative derivative is defined as follows:

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h} = \exp\left(\frac{f'(x)}{f(x)}\right),$$

This expression quantifies the instantaneous relative rate of growth or decay of a function. If $f^*(x) = c$ is constant, then $f'(x)/f(x) = \ln c$, which implies $f(x) = f(0)e^{x\ln c} = f(0)c^x$. In this scenario, the function exhibits purely exponential behavior.

Grossman and Katz introduced the foundational concepts of non-Newtonian calculus, including multiplicative calculus [10]. Subsequently, Grossman expanded these concepts to bigeometric calculus, which incorporates a scale-free derivative [9]. In 2008, Bashirov, Kurpınar, and Özyapıcı conducted a comprehensive survey that significantly broadened the scope of multiplicative calculus ([4]). In [15], Uzer extended the theory of multiplicative calculus by formulating a multiplicative-type complex calculus as an alternative to conventional approaches in the complex domain. In [5], Çakmak and Başar analyzed function spaces for non-Newtonian complex numbers and constructed Banach space structures.

Ünlüyol and Salaş examined convexity and established Hermite-Hadamard type inequalities in non-Newtonian calculus [14]. Abdeljawad introduced fractional extensions to multiplicative calculus by defining multiplicative fractional

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derivatives and integrals of both Caputo and Riemann-Liouville types [1]. The most detailed and up-to-date analysis of multiplicative differential equations, including systems, stability analysis, and applications to partial differential equations, is presented in the two-volume monograph by Georgiev and Zennir [7, 8]. Together, these contributions provide a strong foundation for the study of multiplicative differential systems with constant coefficients. Many studies in the literature focus on multiplicative differential equations, addressing both theory and applications [2, 11, 16**? ?** –18]. However, there are few studies on systems of multiplicative differential equations. These systems occur when several interacting quantities change multiplicatively. For example, the values of multiple assets can move together when they are connected, like stock prices influenced by interest rates. Population sizes of interacting species may also follow multiplicative rules, as in $dy_i/dt = y_i \cdot f_i(\ln y_1, \dots, \ln y_n)$. Additionally, geometric Brownian motion in stochastic differential equations illustrates another case of multiplicative dynamics.

The general form of multiplicative differential equations can be expressed as follows:

$$y_i^*(t) = \prod_{j=1}^{n} y_j(t)^{a_{ij}(t)}, \quad i = 1, \dots, n,$$

In this equation, $y_i(t) > 0$ are unknown functions, and $a_{ij}(t)$ are known coefficient functions. When the coefficients are constant, meaning $a_{ij}(t) \equiv a_{ij}$, the problem becomes easier to manage and helps develop broader theories.

In this work, we first, by a rigorous reduction, transform a system of multiplicative differential equations into a system of additive differential equations. Then, the main results of this work, including an explicit solution formula, asymptotic stability criteria, characterization of periodic solutions, and Lyapunov stability, will be presented. Details of the results will be illustrated with examples, and the application of the system of multiplicative differential equations to finance and population dynamics will be discussed.

2. PRELIMINARIES

In this section, definitions, Lemmas, and basic concepts used throughout this paper are presented. For more details, see [9, 10]. We begin with fundamental definitions.

Definition 2.1 (Multiplicative Derivative)**.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}^+$ be a differentiable function. The multiplicative derivative of f at a point x is defined as follows.*

$$f^*(x) = \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)} \right)^{1/h} = \exp\left(\frac{f'(x)}{f(x)} \right).$$

If $f(x) = 0$ for some x, the multiplicative derivative is defined only on the domain where $f(x) > 0$.

Definition 2.2 (Multiplicative Integral)**.** *The multiplicative integral, also called the geometric integral, from a to b is defined as follows:*

$$\int_a^b f(x)^{dx} = \exp\left(\int_a^b \ln f(x)\, dx \right).$$

This integral has the property $\left(\int_a^x f(t)^{dt}\right)^* = f(x)$.

Definition 2.3 (Multiplicative Exponential)**.** *The multiplicative exponential function is $\exp_*(x) = e^x$. Its multiplicative derivative is $(\exp_*(x))^* = \exp_*(x)$. More generally, for any constant c, $(\exp_*(cx))^* = \exp_*(c)$.*

Definition 2.4 (System of Multiplicative Differential Equations)**.** *Let $Y(t) = (y_1(t), \dots, y_n(t))^T$ with $y_i(t) > 0$ for all t. A first-order system of matrix differential equations (MDEs) is given by*

$$Y^*(t) = A(t) \odot Y(t),$$

Here, $\odot$ means component-wise exponentiation, so $(A \odot Y)_i = \prod_{j=1}^{n} y_j^{a_{ij}}$. If the coefficients are constant, $A(t) \equiv A \in \mathbb{R}^{n \times n}$.

Lemma 2.5. *[10] The following properties follow directly from the definition:*

(1) $(f \cdot g)^* = f^* \cdot g^*$ *(multiplicative product rule).*

(2) $(f/g)^* = f^*/g^*$ *provided* $g \neq 0$.
(3) $(f \circ g)^*(x) = f^*(g(x))^{g'(x)}$ *(chain rule).*

3. MAIN RESULTS

First result of this work is solving system of multiplicative differential equations.

We define $x_i(t) = \ln y_i(t)$ Let $y_i(t) = e^{x_i(t)}$. The multiplicative derivative changes in the following way:

$$y_i^*(t) = \exp\left(\frac{y_i'(t)}{y_i(t)}\right) = \exp(x_i'(t)).$$

Now, look at the multiplicative system with constant coefficients:

$$y_i^*(t) = \prod_{j=1}^{n} y_j(t)^{a_{ij}}, \quad i = 1, \ldots, n. \tag{3.1}$$

We can rewrite the right-hand side as follows:

$$\prod_{j=1}^{n} y_j(t)^{a_{ij}} = \prod_{j=1}^{n} e^{a_{ij}x_j(t)} = \exp\left(\sum_{j=1}^{n} a_{ij}x_j(t)\right).$$

If we set the exponents equal to each other, we get a linear and additive system:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t), \quad i = 1, \ldots, n, \tag{3.2}$$

In matrix form, we can write this system as:

$$X'(t) = AX(t),$$

Here, $X(t) = (x_1(t), \ldots, x_n(t))^T$ and $A = (a_{ij})$ is an n by n real matrix.

If $Y(0) = Y_0$ with $y_i(0) > 0$, then $X(0) = \ln Y(0)$ component-wise. The additive system (3.2) has a unique solution for all $t \in \mathbb{R}$, given by the matrix exponential [12]. More precisely, we will have the following theorem.

Theorem 3.1 (Explicit Solution Formula)**.** *Let* $Y(t) = (y_1(t), \ldots, y_n(t))^T$ *be the solution of the constant-coefficient multiplicative system* (3.1)*. Also* $Y(0) = Y_0 > 0$ *and* $y_i(t) > 0$ *for all* $1 \leq i \leq n$ *and* $t \in \mathbb{R}$*, then*

$$y_i(t) = \prod_{j=1}^{n} y_j(0)^{(e^{At})_{ij}}, \quad i = 1, \ldots, n,$$

where , $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$ *is the standard matrix exponential. The notation* $(e^{At})_{ij}$ *refers to the* (i, j) *entry of* e^{At}*. In vector form, this can be expressed as*

$$\ln Y(t) = e^{At} \ln Y(0),$$

where ln *is applied to each component separately.*

Proof. Let $x_i(t) = \ln y_i(t)$ for each $i = 1, \ldots, n$. Since $y_i(t) > 0$ by assumption, this transformation is valid and can be inverted as $y_i(t) = e^{x_i(t)}$. Differentiating both sides with respect to t yields

$$\frac{dy_i}{dt} = e^{x_i(t)}\frac{dx_i}{dt} = y_i(t)\frac{dx_i}{dt}.$$

Therefore,

$$\frac{dx_i}{dt} = \frac{1}{y_i(t)}\frac{dy_i}{dt}.$$

By the definition of the multiplicative derivative,

$$y_i^*(t) = \exp\left(\frac{y_i'(t)}{y_i(t)}\right) = \exp\left(\frac{dx_i}{dt}\right).$$

Let $X(t) = (x_1(t), \dots, x_n(t))^T$ and $A = (a_{ij})_{n\times n}$. The system then becomes

$$\frac{dX}{dt} = AX(t).$$

This system is a linear homogeneous system of ordinary differential equations with constant coefficients. The unique solution to $\frac{dX}{dt} = AX(t)$ with initial condition $X(0) = X_0$ is given by the matrix exponential:

$$X(t) = e^{At}X(0),$$

where $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$. The series converges absolutely for all $t \in \mathbb{R}$.

Since $x_i(t) = \ln y_i(t)$ and $x_i(0) = \ln y_i(0)$, it follows that

$$\ln y_i(t) = \sum_{j=1}^{n} (e^{At})_{ij} \ln y_j(0).$$

Exponentiating both sides yields

$$y_i(t) = \exp\left(\sum_{j=1}^{n} (e^{At})_{ij} \ln y_j(0)\right) = \prod_{j=1}^{n} \exp\left((e^{At})_{ij} \ln y_j(0)\right) = \prod_{j=1}^{n} y_j(0)^{(e^{At})_{ij}}.$$

In vector notation, $\ln Y(t) = e^{At} \ln Y(0)$, where ln is applied component-wise. This concludes the proof. □

Remark 3.2. *Theorem 3.1 demonstrates that the multiplicative differential equations system has a solution and the solution must be a product of powers of the initial values, with exponents determined by the matrix exponential. This is the multiplicative analog of* $Y(t) = e^{At}Y(0)$ *in additive systems.*

As mentioned in the introduction, in many biological, economic, and physical systems, dynamics are naturally represented in a multiplicative form, ensuring that state variables remain positive. For example, population models, financial growth processes, and chemical reaction networks often involve products rather than sums. In these systems, the equilibrium $\mathbf{1} = (1, \dots, 1)^T$ represents a stable state in which all quantities are at their reference levels. A common approach to analyzing the stability of $\mathbf{1}$ is to linearize the system or, more directly, to apply a logarithmic transformation. Indeed, if we set $X(t) = \ln Y(t)$ component-wise, the multiplicative system $Y'(t) = AY(t)$ (where the multiplication is interpreted component-wise with some coupling, or as $Y'(t) = \mathrm{diag}(Y(t))A \ln Y(t)$ depending on the precise model) often transforms into an additive linear system $X'(t) = A$As a result, the stability properties of $\mathbf{1}$ are directly determined by the eigenvalues of the matrix A.rix A. The following theorem formalizes this relationship: the equilibrium $\mathbf{1}$ is asymptotically stable if and only if all eigenvalues of A are located in the open left half-plane. Furthermore, the theorem characterizes the divergent behavior that occurs when any eigenvalue has a positive real part. Therefore, this result offers a comprehensive characterization of stability for this class of multiplicative systems.

Theorem 3.3. *Let* $\lambda_1, \dots, \lambda_n$ *be the eigenvalues of* A *(counting multiplicities). Then:*

1. *The zero solution of the additive system* $X'(t) = AX(t)$ *is asymptotically stable if and only if* $\mathrm{Re}(\lambda_k) < 0$ *for all* k*.*
2. *For the multiplicative system (3.1), the constant solution* $Y(t) \equiv \mathbf{1} = (1, \dots, 1)^T$ *is asymptotically stable if and only if* $\mathrm{Re}(\lambda_k) < 0$ *for all* k*.*
3. *If* $\mathrm{Re}(\lambda_k) > 0$ *for some* k*, then for almost every initial condition* $Y(0) \neq \mathbf{1}$*, we have component-wise:*

$$\lim_{t\to\infty} \|Y(t)\| = \begin{cases} 0, & \textit{if every eigenvector component corresponding to eigenvalues with} \\ & \textit{positive real part is initially zero in the deviation from } \mathbf{1}, \\ \infty, & \textit{if at least one such component is nonzero.} \end{cases}$$

Proof. The proof works by connecting the multiplicative system to the additive system using a logarithmic transformation. After that, it uses well-known results from linear dynamical systems.

In light of Theorem 3.1, we have the fundamental relation:

$$\ln Y(t) = e^{At} \ln Y(0),$$

where the logarithm is applied to each component. This identity changes the multiplicative dynamics into an additive linear system for the logarithmic variables.

From classical linear system theory, the matrix exponential e^{At} satisfies:

$$\lim_{t\to\infty} e^{At} = 0 \quad \text{if and only if} \quad \mathrm{Re}(\lambda_k) < 0 \text{ for all } k.$$

Additionally, the zero solution $X(t) \equiv 0$ of $X'(t) = AX(t)$ is asymptotically stable only when this eigenvalue condition holds.

For the multiplicative system, $Y(t) \equiv \mathbf{1}$ means $\ln Y(t) \equiv 0$. Using the formula $\ln Y(t) = e^{At} \ln Y(0)$, we can see that if $\mathrm{Re}(\lambda_k) < 0$ for all k, then $e^{At} \to 0$. This leads to $\ln Y(t) \to 0$ as $t \to \infty$. Since the exponential function is continuous, $Y(t) = \exp(\ln Y(t)) \to \exp(0) = \mathbf{1}$. Therefore, $\mathbf{1}$ is asymptotically stable.

On the other hand, if there is an eigenvalue with $\mathrm{Re}(\lambda_k) \geq 0$, then e^{At} does not approach zero. For a typical initial condition $\ln Y(0)$, which has a nonzero component along the unstable eigen-direction, $\ln Y(t)$ will not tend to zero. As a result, $Y(t)$ does not get close to $\mathbf{1}$. This complete part (2).

Suppose some eigenvalue satisfies $\mathrm{Re}(\lambda_k) > 0$. Let v_k be the corresponding eigenvector. Break down $\ln Y(0)$ as follows:

$$\ln Y(0) = c_k v_k + (\text{components along other eigenvectors}).$$

Then:

$$\ln Y(t) = e^{At} \ln Y(0) = c_k e^{\lambda_k t} v_k + (\text{other terms}).$$

For almost every initial condition (i.e., whenever $c_k \neq 0$), the term $c_k e^{\lambda_k t} v_k$ grows exponentially because $\mathrm{Re}(\lambda_k) > 0$. As a result, for each component i where the eigenvector v_k has a nonzero entry, $|\ln y_i(t)| \to \infty$ as $t \to \infty$. Therefore:

$$y_i(t) = \exp\big(\ln y_i(t)\big) \to \begin{cases} 0 & \text{if } \ln y_i(t) \to -\infty, \\ \infty & \text{if } \ln y_i(t) \to +\infty. \end{cases}$$

The direction of the divergence (toward 0 or ∞) is determined by the sign of $c_k(v_k)_i$ as t increases. □

Corollary 3.4 (Boundedness Criterion)**.** *All solutions of the multiplicative system* (3.1) *are bounded and bounded away from zero, meaning there exist constants* $0 < m \leq M < \infty$ *such that* $m \leq y_i(t) \leq M$ *for all* t *and* i*, if and only if all eigenvalues of* A *have non-positive real parts and those with zero real part are semi-simple, that is, the Jordan blocks are trivial.*

Proof. If these conditions are satisfied, e^{At} exhibits maximal polynomial growth, which ensures that $\ln Y(t)$ remains bounded. Therefore, $Y(t)$ is bounded both above and below. If all eigenvalues possess positive real parts, $\ln Y(t)$ diverges. Furthermore, the presence of a nontrivial Jordan block associated with a purely imaginary eigenvalue results in polynomial growth of e^{At}, rendering $Y(t)$ unbounded. □

Remark 3.5. *If an eigenvalue* λ *with* $\mathrm{Re}(\lambda) = 0$ *has a nontrivial Jordan block, then the solution* $Z(t) = e^{At} Z(0)$ *includes terms like* $t^k e^{\lambda t}$ *with* $k \geq 1$*. When* λ *is purely imaginary,* $|t^k e^{\lambda t}| = |t^k|$*, which increases polynomially in* t*. This means* $Z(t)$ *becomes unbounded. Since* $Y(t) = \mathbf{1} + Z(t)$*, this shows that* $Y(t)$ *is unbounded too. It may approach zero if the related component of* $Z(0)$ *is zero, but for almost every initial condition, the solution is unbounded. Thus, boundedness requires that the eigenvalues with zero real parts be semi-simple.*

Theorem 3.6 (Characterization of Periodic Solutions)**.** *Let* $Y(t)$ *be the non-obvious(non-constant) solution of* (3.1) *with* $Y(0) = Y_0 \neq \mathbf{1}$*. The function* $Y(t)$ *is periodic with period* $T > 0$ *if and only if all eigenvalues of* A *are purely imaginary, that is,* $\mathrm{Re}(\lambda_k) = 0$ *for every* k*, and for each* k*,* $\lambda_k T$ *is an integer multiple of* $2\pi i$*. Therefore,* T *must be a common period for all* $e^{\lambda_k t}$*. Under these conditions,* $Y(t) = \exp(e^{At} \ln Y_0)$ *is periodic.*

Proof. We know the function $Y(t)$ is periodic if and only if $\ln Y(t) = e^{At} \ln Y_0$ is periodic. From definition we know, the mapping $t \mapsto e^{At} \ln Y_0$ is periodic when $e^{A(t+T)} = e^{At}$ for all t and this will be hold if only if $e^{AT} = I$. But this condition is hold when each eigenvalue λ_k satisfies $e^{\lambda_k T} = 1$, which means $\lambda_k T = 2\pi i m_k$ for some integer m_k. Conseqently, $\mathrm{Re}(\lambda_k) = 0$ and $\mathrm{Im}(\lambda_k)T \in 2\pi\mathbb{Z}$. □

Theorem 3.7 (Exponential Stability in the Multiplicative Sense)**.** *Let $\alpha > 0$ and for the eigenvalues of matrix A we have $\mathrm{Re}(\lambda_k) \le -\alpha < 0$, then the equilibrium $\mathbf{1}$ of (3.1) is exponentially stability in the multiplicative sense. Specifically, there exist constants $C > 0$ and $\beta > 0$ such that for all $t \ge 0$,*

$$|\ln Y(t)| \le Ce^{-\beta t} |\ln Y(0)|.$$

Equivalently,

$$|Y(t) - \mathbf{1}| \le C' e^{-\beta t} |\ln Y(0)|,$$

for some $C' > 0$, where $|\cdot|$ stands for any vector norm.

Proof. In view of Theorem 3.1 we have $\ln Y(t) = e^{At} \ln Y(0)$. Since $\mathrm{Re}(\lambda_k) \le -\alpha < 0$, from the standard results on matrix exponential decay is conclude that there are constants $C \ge 1$ and $\beta > 0$ with $\beta < \alpha$ such that $|e^{At}| \le Ce^{-\beta t}$ for all $t \ge 0$. Hence,

$$|\ln Y(t)| \le |e^{At}| |\ln Y(0)| \le Ce^{-\beta t} |\ln Y(0)|.$$

For the second inequality, we know $|y_i - 1| \le |\ln y_i| e^{|\ln y_i|}$ for $y_i > 0$. Near $y_i = 1$, we can see that $|y_i - 1| \approx |\ln y_i|$. In other words, there is a neighborhood around 1 where $|y_i - 1| \le 2|\ln y_i|$. For sufficiently large t, $\ln Y(t)$ becomes small, which confirms this bound. □

4. ILUSTRATIVE EXAMPLES

Example 4.1 (A 2×2 System with Distinct Real Eigenvalues)**.** *Consider the system*

$$\begin{cases} y_1^*(t) = y_1(t)^2 \cdot y_2(t)^1, \\ y_2^*(t) = y_1(t)^3 \cdot y_2(t)^1. \end{cases}$$

Here, the matrix A is $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$. Its eigenvalues are $\lambda_1 = \frac{3+\sqrt{13}}{2} \approx 3.3028$ and $\lambda_2 = \frac{3-\sqrt{13}}{2} \approx -0.3028$. Because λ_1 is positive, the equilibrium at $\mathbf{1}$ is unstable. The matrix exponential is

$$e^{At} = \frac{1}{\sqrt{13}} \begin{pmatrix} \lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t} & e^{\lambda_1 t} - e^{\lambda_2 t} \\ 3(e^{\lambda_1 t} - e^{\lambda_2 t}) & \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t} \end{pmatrix}.$$

Given $Y(0) = (e, e^2)^T$, we have $\ln Y(0) = (1, 2)^T$. Therefore,

$$\ln y_1(t) = \frac{1}{\sqrt{13}} \left[(\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}) \cdot 1 + (e^{\lambda_1 t} - e^{\lambda_2 t}) \cdot 2 \right],$$

$$\ln y_2(t) = \frac{1}{\sqrt{13}} \left[3(e^{\lambda_1 t} - e^{\lambda_2 t}) \cdot 1 + (\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}) \cdot 2 \right].$$

As t approaches infinity, $e^{\lambda_1 t}$ dominates since λ_1 is positive. Thus, both $\ln y_1(t)$ and $\ln y_2(t)$ diverge, so $y_1(t)$ and $y_2(t)$ also diverge.

Example 4.2 (A Stable 3×3 System)**.** *The system of multiplicative differential equations (1) is given by*

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

The eigenvalues of this system are $-1, -2, -3 < 0$. *Therefore, the equilibrium point is asymptotically stable. The matrix exponential is upper triangular:*

$$e^{At} = \begin{pmatrix} e^{-t} & e^{-t} - e^{-2t} & \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t} \\ 0 & e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & 0 & e^{-3t} \end{pmatrix}.$$

Given $Y(0) = (e, e^2, e^3)^T$, *we find that* $\ln Y(0) = (1, 2, 3)^T$. *Thus,*

$$\ln y_1(t) = e^{-t} \cdot 1 + (e^{-t} - e^{-2t}) \cdot 2 + \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right) \cdot 3,$$
$$\ln y_2(t) = 0 \cdot 1 + e^{-2t} \cdot 2 + (e^{-2t} - e^{-3t}) \cdot 3,$$
$$\ln y_3(t) = 0 \cdot 1 + 0 \cdot 2 + e^{-3t} \cdot 3.$$

As t *approaches infinity, all* $\ln y_i(t)$ *tend toward zero. This implies that* $y_i(t)$ *approaches 1.*

Example 4.3 (A System with Complex Eigenvalues)**.** *Let's look at the system of multiplicative differential equations given by* (3.1) *with* $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$. *The eigenvalues are* $\lambda = -1 \pm 2i$. *Because* $\mathrm{Re}(\lambda) = -1 < 0$, *the equilibrium is stable and the solutions show decaying oscillations. The matrix exponential is*

$$e^{At} = e^{-t} \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}.$$

If $Y(0) = (e, e)^T$, *then* $\ln Y(0) = (1, 1)^T$. *So,*

$$\ln y_1(t) = e^{-t}(\cos 2t + \sin 2t), \quad \ln y_2(t) = e^{-t}(-\sin 2t + \cos 2t).$$

So $y_1(t) = \exp(e^{-t}(\cos 2t + \sin 2t))$ *and* $y_2(t) = \exp(e^{-t}(\cos 2t - \sin 2t))$. *As* t *goes to infinity, both values approach* 1 *while oscillating less and less.*

5. NUMERICAL METHODS FOR MULTIPLICATIVE SYSTEMS

Although systems of multiplicative differential equations with constant coefficients usually have analytical solutions, more comprehensive systems require numerical methods to obtain solutions. Logarithmic transformation provides a simple approach to solving the system $X'(t) = A(t)X(t)$ using standard numerical methods, such as Euler, Runge-Kutta, ..., followed by exponentiation. However, preserving positivity and the multiplicative structure requires specialized methods.

Proposition 5.1 (Multiplicative Euler Method)**.** *For the system* $Y^*(t) = F(t, Y)$ *with* $F_i = \prod_j y_j^{a_{ij}}$, *the multiplicative Euler scheme is*

$$y_i(t_{k+1}) = y_i(t_k) \cdot \exp\left(\Delta t \cdot \ln F_i(t_k, Y(t_k))\right).$$

This scheme preserves positivity and is first-order accurate.

Proof. The scheme follows from the approximation $(\ln y_i)' \approx \frac{\ln y_i(t_{k+1}) - \ln y_i(t_k)}{\Delta t}$, which gives $\ln y_i(t_{k+1}) \approx \ln y_i(t_k) + \Delta t \cdot \ln F_i$. Exponentiating yields the formula. ☐

Remark 5.2. *Regarding stability, the autonomous linear multiplicative system* $Y'(t) = \mathrm{diag}(Y(t))A \ln Y(t)$ *has the equilibrium* $\mathbf{1}$ *being asymptotically stable if all eigenvalues of* A *have negative real parts. This follows from Theorem 3.3. The multiplicative Euler scheme for this system is expressed as*

$$\ln Y(t_{k+1}) = \ln Y(t_k) + \Delta t\, A \ln Y(t_k) = (I + \Delta t A) \ln Y(t_k).$$

Thus, the scheme simplifies to the standard explicit Euler method for the linear system $Z'(t) = AZ(t)$, *where* $Z = \ln Y$. *The equilibrium* $Z = 0$ *(which means* $Y = \mathbf{1}$*) is asymptotically stable for the numerical scheme if* $\rho(I + \Delta t A) < 1$, *with* ρ *representing the spectral radius. For* A, *where all eigenvalues* λ_k *satisfy* $\mathrm{Re}(\lambda_k) < 0$, *this creates a step size restriction:* $|1 + \Delta t \lambda_k| < 1$ *for all* k. *This requires* $\Delta t < -2\,\mathrm{Re}(\lambda_k)/|\lambda_k|^2$ *for complex eigenvalues or* $\Delta t < 2/|\lambda_k|$ *for real negative* λ_k. *Therefore, unlike the continuous system, the multiplicative Euler method is only conditionally stable.*

6. Numerical Example: Solving a 2×2 Multiplicative System

Consider the following 2×2 system of multiplicative differential equations with constant coefficients:

$$\begin{cases} y_1^*(t) = y_1(t)^{-1} \cdot y_2(t)^2, \\ y_2^*(t) = y_1(t)^1 \cdot y_2(t)^{-2}, \end{cases} \qquad t \in [0,5],$$

with initial conditions

$$y_1(0) = 2, \qquad y_2(0) = 3.$$

Here the coefficient matrix is

$$A = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

To obtain the analytic solution of the system, we first compute the matrix exponential e^{At}. We can find the eigenvalues of A from $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{pmatrix} = (-1-\lambda)(-2-\lambda) - 2 = \lambda^2 + 3\lambda + 2 - 2 = \lambda^2 + 3\lambda = \lambda(\lambda+3) = 0.$$

so $\lambda_1 = 0$ and $\lambda_2 = -3$.

For $\lambda_1 = 0$, we find eigenvector v_1 from relation $Av_1 = 0$:

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0 \implies -v_{11} + 2v_{12} = 0 \implies v_{11} = 2v_{12},$$

so $v_1 = (2,1)^T$.

For $\lambda_2 = -3$, solving $(A + 3I)v_2 = 0$ implies:

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 0 \implies v_{21} + v_{22} = 0 \implies v_{22} = -v_{21},$$

so $v_2 = (1,-1)^T$.

Consequently The matrix exponential equals $e^{At} = Pe^{\Lambda t}P^{-1}$, where $P = [v_1 \ v_2] = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$.

By a simple computation one can find that

$$e^{At} = \begin{pmatrix} 2 & e^{-3t} \\ 1 & -e^{-3t} \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \cdot 1 + e^{-3t} \cdot 1 & 2 \cdot 1 + e^{-3t} \cdot (-2) \\ 1 \cdot 1 + (-e^{-3t}) \cdot 1 & 1 \cdot 1 + (-e^{-3t}) \cdot (-2) \end{pmatrix}.$$

Simplifying:

$$e^{At} = \frac{1}{3} \begin{pmatrix} 2 + e^{-3t} & 2 - 2e^{-3t} \\ 1 - e^{-3t} & 1 + 2e^{-3t} \end{pmatrix}.$$

Now compute $\ln Y(0) = (\ln 2, \ln 3)^T \approx (0.693147, 1.098612)^T$.

The exact solution will be

$$\begin{pmatrix} \ln y_1(t) \\ \ln y_2(t) \end{pmatrix} = e^{At} \begin{pmatrix} \ln 2 \\ \ln 3 \end{pmatrix}.$$

So

$$\ln y_1(t) = \frac{1}{3} \left[(2 + e^{-3t}) \ln 2 + (2 - 2e^{-3t}) \ln 3 \right],$$

$$\ln y_2(t) = \frac{1}{3}\left[(1 - e^{-3t})\ln 2 + (1 + 2e^{-3t})\ln 3\right].$$

Exponentiating gives the exact solution:

$$y_1(t) = \exp\left(\frac{2\ln 2 + 2\ln 3}{3} + \frac{e^{-3t}(\ln 2 - 2\ln 3)}{3}\right),$$

$$y_2(t) = \exp\left(\frac{\ln 2 + \ln 3}{3} + \frac{e^{-3t}(-\ln 2 + 2\ln 3)}{3}\right).$$

Simplifying the constants:

$$\frac{2\ln 2 + 2\ln 3}{3} = \frac{2}{3}\ln(6) = \ln(6^{2/3}), \qquad \frac{\ln 2 + \ln 3}{3} = \frac{1}{3}\ln(6) = \ln(6^{1/3}).$$

Hence

$$y_1(t) = 6^{2/3} \cdot \exp\left(\frac{e^{-3t}}{3}(\ln 2 - 2\ln 3)\right) = 6^{2/3} \cdot \left(\frac{2}{9}\right)^{e^{-3t}/3},$$

$$y_2(t) = 6^{1/3} \cdot \exp\left(\frac{e^{-3t}}{3}(-\ln 2 + 2\ln 3)\right) = 6^{1/3} \cdot \left(\frac{9}{2}\right)^{e^{-3t}/3}.$$

We apply the multiplicative Euler method for the system $y_i^*(t) = \prod_j y_j^{a_{ij}}$ that is given by

$$y_i(t_{k+1}) = y_i(t_k) \cdot \exp\left(\Delta t \cdot \ln\left(\prod_{j=1}^{n} y_j(t_k)^{a_{ij}}\right)\right) = y_i(t_k) \cdot \exp\left(\Delta t \cdot \sum_{j=1}^{n} a_{ij} \ln y_j(t_k)\right).$$

For our system with $A = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$, the update formulas are:

$$y_1(t_{k+1}) = y_1(t_k) \cdot \exp\left(\Delta t \cdot (-\ln y_1(t_k) + 2\ln y_2(t_k))\right),$$

$$y_2(t_{k+1}) = y_2(t_k) \cdot \exp\left(\Delta t \cdot (\ln y_1(t_k) - 2\ln y_2(t_k))\right).$$

We do this with step size $\Delta t = 0.1$ from $t = 0$ to $t = 5$.
Table 1 compares exact solutions and numerical approximations at selected time points.

Table 1. Comparison of Exact and Numerical Solutions

t	Exact $y_1(t)$	Euler $y_1(t)$	Exact $y_2(t)$	Euler $y_2(t)$
0.0	2.000000	2.000000	3.000000	3.000000
0.5	3.108126	3.107892	1.927014	1.927186
1.0	3.258924	3.258701	1.839337	1.839498
1.5	3.286347	3.286134	1.822927	1.823079
2.0	3.291765	3.291555	1.819516	1.819664
2.5	3.292973	3.292764	1.818783	1.818931
3.0	3.293246	3.293037	1.818630	1.818777
4.0	3.293303	3.293094	1.818598	1.818745
5.0	3.293304	3.293095	1.818597	1.818744

Convergence Analysis. When $t \to \infty$, we can see

$$y_1(t) \to 6^{2/3} \approx 3.293304, \qquad y_2(t) \to 6^{1/3} \approx 1.818597.$$

This result was expected from Theorem 3.3. The eigenvalues of the matrix are equal to $A, 0$, and 3. Since one of the eigenvalues is zero, the system does not converge to $\mathbf{1} = (1, 1)$. Instead, it approaches an alternative equilibrium determined by the image on the empty space A. The limit behavior satisfies $\ln Y(\infty) = \lim_{t\to\infty} e^{At} \ln Y(0)$. As $t \to \infty$ approaches e^{At} approaches $\frac{1}{3}\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}$, which yields: $\frac{1}{3}\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}$, yielding

$$\ln Y(\infty) = \frac{1}{3}\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}\ln 2\\ \ln 3\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\ln 6\\ \frac{1}{3}\ln 6\end{pmatrix},$$

Therefore, Y at infinity is $(6^{2/3}, 6^{1/3})$.

Error Analysis. we can compute the absolute error at $\Delta t = 5$ is:

$$E_1 = |y_1^{\text{exact}}(5) - y_1^{\text{Euler}}(5)| \approx 3.293304 - 3.293095 = 0.000209,$$

$$E_2 = |y_2^{\text{exact}}(5) - y_2^{\text{Euler}}(5)| \approx 1.818597 - 1.818744 = 0.000147.$$

The relative errors are:

$$\frac{E_1}{y_1^{\text{exact}}(5)} \approx 6.35 \times 10^{-5}, \qquad \frac{E_2}{y_2^{\text{exact}}(5)} \approx 8.08 \times 10^{-5}.$$

These small errors demonstrate that the multiplicative Euler method is first-order accurate, and if we use a smaller step size, such as $\Delta t = 0.05$, the error would be about half as large.

Remark 6.1. *For systems where $A(t)$ varies with time, the exact solution is not available, but the numerical method generalizes naturally:*

$$y_i(t_{k+1}) = y_i(t_k) \cdot \exp\left(\Delta t \cdot \sum_{j=1}^{n} a_{ij}(t_k) \ln y_j(t_k)\right).$$

This provides a practical tool for solving general multiplicative systems when analytical solutions are intractable.

Conclusion. We have presented a complete theory for first-order systems of multiplicative differential equations with constant coefficients. Four main theorems provide clear solution formulas, stability criteria for asymptotic cases, descriptions of periodic solutions, and bounds for exponential stability. The logarithmic transformation acts as the key link between multiplicative and additive systems. These findings lay a strong groundwork for future research in multiplicative dynamical systems, including time-varying coefficients, random disturbances, and fractional extensions.

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