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Oscillatory behavior of Ψ-Hilfer generalized proportional fractional differential equations with mixed nonlinearities

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Abstract

This paper deals with the forced oscillation of FDEs (Fractional Differential Equations) with mixed nonlinearities via Ψ-Hilfer GPFD (Generalized Proportional Fractional Derivative). Volterra integral equation and Young's inequality are used to specify the appropriate conditions for each solution to oscillate. The effects of this study enhance and generalize earlier results in the literature. Examples are presented to illustrate the practicality of the outcomes.

Keywords. Oscillation theory, Fractional differential equations, Ψ-Hilfer generalized proportional fractional derivative, Mixed nonlinearity.
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1. Introduction

Fractional calculus handles integrals and derivatives of arbitrary order. Many branches of science and engineering depend on it for their further development, see [14, 22, 31]. In recent years, different types of fractional derivatives have drawn much attention. A growing variety of definitions and applications are made possible by the appearance of new fractional integrals and derivatives.

In 2014, Khalil et al.[21] initiated a novel fractional differential operator so called "the conformable derivative". However it does not prone to original function when the order is zero. To resolve this difficulty, Anderson et al. [6] defined another kind of conformable derivative and it gives the original function as the order tends to 0. Additionally, Anderson in [5] describes a few more generalizations about these operators.

In 2015, Abdeljawad in [1] created more general types of fractional operators by developing basic concepts of conformable derivatives and explaining how to use them. In addition, PFOs (Proportional Fractional Operators) of general type were discussed in [18]. The authors in [17] proposed the idea of PFOs of a function concerning another function.

In 2017, Jarad et al. [16] presented GPFD of the Caputo-type and Riemann-Liouville-type with exponential functions as their kernels. In addition to the classical features, it confirms nonlocal behavior and semi group property. In 2020, Jarad et al. [18] generalized the previous results done in [16], based on the idea of PFOs of a function concerning another function. After that, Sousa and Oliveira [34] proposed the Ψ-HFD (Hilfer Fractional Derivative) which unifies HFD and Riemann-Liouville derivative concerning another function. Moreover, Ahmed et al. [3] introduced the Hilfer GPFD which combines the operators described in [16].

Motivated by [3, 17], Mallah et al. [25] presented a new derivative called the Ψ-Hilfer GPFD of a function concerning another function. This new derivative merges the proportional fractional operators of Riemann-Liouville-type and

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Caputo-type described in [18] and also generalize the fractional operators provided in [3]. The main advantage of this Ψ-Hilfer GPFD is the freedom to select the kernel Ψ, and it allows us to combine and recover most of the earlier results on FDEs.

Classical fractional derivatives such as the Riemann–Liouville, Caputo, and Hilfer operators are widely used to model systems with memory and hereditary properties. To incorporate proportional effects, the proportional fractional derivative was later introduced. However, most of these operators are defined with respect to the standard time variable, which restricts their applicability when modelling processes that evolve on non-uniform or transformed time scales.

To address this limitation, the Ψ-Hilfer GPFD operator introduces a kernel function Ψ into its formulation. This function is assumed to be continuously differentiable and monotonically increasing on the considered interval, with $\Psi'(t) \neq 0$. The presence of Ψ allows a transformation of the independent variable, thereby providing greater analytical flexibility in modelling complex dynamical behaviors. By choosing different forms of Ψ, several well-known fractional derivatives can be obtained as special cases. In particular, when $\Psi(t) = t$, the operator reduces to the classical fractional derivatives, including the Riemann–Liouville and Caputo types. Hence, the Ψ-Hilfer GPFD operator serves as a unified framework for a wide class of fractional models.

Furthermore, the choice of the function Ψ influences the memory structure and the weighting of past states, which affects the qualitative behavior of solutions, especially their oscillatory properties. Therefore, it becomes important to investigate the oscillatory behavior of differential equations involving this generalized operator. Oscillation theory is an important branch of the qualitative theory of differential equations. Over the past few decades, the qualitative analysis of differential equations has attracted considerable interest, especially with respect to existence, uniqueness, stability, and oscillation properties [8, 26, 37].

Oscillation is an essential aspect of the qualitative behavior of solutions to differential equations. Its theory plays a crucial role in understanding oscillatory phenomena in technology as well as in the natural and social sciences. One of the central problems in oscillation theory is to establish the existence or nonexistence of oscillatory solutions for a given equation or system. In addition, the theory also investigates how other solutions behave in relation to a particular oscillatory or nonoscillatory solution.

Each year, numerous studies are published on the theoretical foundations of oscillation theory. However, this important field is not purely theoretical, as it has significant applications in areas such as physics, biology, ecology, and physiology [10, 15]. The study of oscillations helps in gaining deeper insight into the dynamics of solutions of equations that arise in engineering, technology, and scientific models. In particular, the study of oscillatory behavior of integer-order differential equations has attracted significant attention over the past several decades; see the monographs [2, 23] and the references therein. Despite its wide-ranging importance, the available literature reveals that relatively few works address the oscillatory behavior of fractional differential equations.

In 2007, Sun and Wong [35] considered the forced oscillation of the nonlinear differential equations of the form

$$(p(\kappa)x')' + q(\kappa)x + \sum_{j=1}^{n} q_j(\kappa)\Phi_{\alpha_j}(x) = f(\kappa); \qquad p(\kappa) > 0,$$

where $\Phi_*(s) := |s|^{*-1}s$, and that $p, q, f, q_j \in \mathrm{C}[0,\infty)$, $j = 1, \ldots, n$, $p(\kappa)x'$ is differentiable, and the nonlinearities satisfy

$$\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0. \tag{1.1}$$

In 2011, Hassan et al. [12] interested in the oscillatory behavior of forced differential equations

$$(a(\kappa)[x']^{\gamma})' + p_0(\kappa)x^{\gamma}(g_0) + \sum_{j=1}^{n} p_j(\kappa)\Phi_{\alpha_j}(x(g_j)) = f(\kappa),$$

where $\gamma = c_1/c_2$ ($c_1, c_2 \in \mathbb{Z}^+$ are odd integers), α_j's satisfy (1.1), and $a, f, p_j \in \mathrm{C}([\kappa_0,\infty),\mathbb{R})$, $a(\kappa) > 0$, $g_j \in \mathrm{C}(\mathbb{R},\mathbb{R}^+)$ with

$$\lim_{\kappa\to\infty} g_j(\kappa) = \infty; \qquad i = 0, 1, \ldots, n.$$

As far as impulsive equations are considered, in 2011, Muthulakshmi and Thandapani [27] examined the oscillaton of the nonlinear forced IDEs (Impulsive Differential Equations)

$$(r(\kappa)[x']^{\gamma})' + p(\kappa)x' + q(\kappa)x + \sum_{j=1}^{n} q_j(\kappa)\Phi_{\alpha_j}(x) = f(\kappa); \qquad \kappa \neq \theta_i,$$
$$x(\theta_i^+) = a_i x(\theta_i), \quad x'(\theta_i^+) = b_i x'(\theta_i); \qquad b_i \geq a_i > 0, \quad (i \in \mathbb{N})$$

where $r \in \mathrm{C}^1([\kappa_0,\infty),\mathbb{R}^+)$, $p, q_j, q, f \in \mathrm{C}([\kappa_0,\infty),\mathbb{R})$ $(j = 1,\dots,n)$, and the nonlinearities satisfy (1.1).

In 2011, motivated with the work [35], Özbekler and Zafer [29] considered the oscillation of mixed nonlinear forced IDEs

$$(r(\kappa)\Phi_\alpha(x'))' + q(\kappa)\Phi_\alpha(x) + \sum_{j=1}^{n} q_j(\kappa)\Phi_{\beta_j}(x) = f(\kappa), \qquad \kappa \neq \theta_i$$
$$x(\theta_i^+) = a_i x(\theta_i), \quad x'(\theta_i^+) = b_i x'(\theta_i),$$

where $z(\kappa^{\pm}) = \lim_{\tau\to\kappa^{\pm}} z(\tau)$;

(i) $\beta_1 > \beta_2 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$;

(ii) $\{\theta_i\}$, $\{a_i\}$ and $\{b_i\}$ are real sequences with

$$\theta_1 < \theta_2 < \cdots < \theta_n \to \infty \quad \text{and} \quad b_i/a_i \geq 1, \qquad i \in \mathbb{N};$$

(iii) $r, q, q_j, f \in \mathrm{PLC}[\kappa_0,\infty) := \big\{\mathrm{h} : [\kappa_0,\infty) \to \mathbb{R}\text{: } h \text{ is continuous on each interval } (\theta_i,\theta_{i+1}),\ h(\theta_i^{\pm}) \text{ exist, } h(\theta_i) = h(\theta_i^-) \text{ for } i \in \mathbb{N}\big\}$, $j = 1, 2, \dots, n$; $r(\kappa) > 0$ is a nondecreasing function.

The obtained results in [29] generalize and improve the results by Sun and Wong [35].

In 2020, Santra and Dix [32] obtained some oscillation criteria for neutral IDEs

$$(r(\kappa)[z']^{\gamma})' + \sum_{j=1}^{m} q_j(\kappa) f_j(x(\sigma_j)) = 0; \qquad \kappa \neq \lambda_k$$
$$\Delta r(\kappa)[z']^{\gamma} + \sum_{j=1}^{m} \tilde{q}_j(\kappa) f_j(x(\sigma_j)) = 0; \qquad \kappa = \lambda_k$$

for $\kappa \geq \kappa_0$, $k \in \mathbb{N}$, where $z = x + p(\kappa)x(\tau)$, $\Delta x(\nu) = x(\nu^+) - x(\nu^-)$, the functions $f_j, p, q_j, \tilde{q}_j, r, \sigma_j$ and τ are continuous functions satisfying the conditions will be stated below. $\{\lambda_k\}$ is the real-valued sequence with

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \quad \text{as} \quad n \to \infty,$$

γ is defined as before.

Many academics are now interested in researching the oscillation of noninteger-order equations. To address this, researchers have been analyzing the oscillatory behavior of FDEs for last twenty years.

In 2012, Grace et al. [9] introduced an oscillation result for the fractional initial value problem

$$\begin{cases} {}_a\mathcal{D}^{\alpha} x + f_1(\kappa, x) = v(\kappa) + f_2(\kappa, x); & \kappa > a, \\ \lim\limits_{\kappa\to a^+} {}_aI^{1-\alpha} x(\kappa) = b, & (b \in \mathbb{R}) \end{cases}$$

where ${}_a\mathcal{D}^{\alpha}$ is the RLF (Riemann-Liouville Fractional) derivative of order α, $I^{1-\alpha}$ is the RLF integral of order $1-\alpha$; $f_j \in \mathrm{C}([a,\infty)\times\mathbb{R},\mathbb{R})$, $j = 1, 2$.

In 2013, Shao et al. [33] examined the forced oscillation of fractional equation

$$\begin{cases} {}_a\mathcal{D}_{\kappa}^{\alpha} x - p(\kappa)x + \sum\limits_{j=1}^{m} q_j(\kappa)\Phi_{\lambda_j}(x) = v(\kappa); & \kappa \geq a, \\ \lim\limits_{\kappa\to a^+} \mathcal{J}_a^{1-\alpha} x(\kappa) = a_1, & \end{cases}$$

where ${}_a\mathcal{D}^{\alpha}$ indicates the RLF derivative of order α and $\mathcal{J}_a^{1-\alpha}$ defines the RLF integral of order $1-\alpha$. The functions p, v and q_j, $j = 1, \ldots, m$, are continuous on $[a, +\infty)$. $\lambda_j = c_1/c_2$ ($c_1, c_2 \in \mathbb{Z}^+$ are odd integers), $j = 1, \ldots, m$ with

$$\lambda_1 > \cdots > \lambda_l > 1 > \lambda_{l+1} > \cdots > \lambda_m. \tag{1.2}$$

In 2020, Jehad et al. [4] obtained some criteria for forced oscillation of GPF initial value problem with damping term in the Riemann-Liouville setting

$$\begin{cases} {}_a\mathcal{D}^{1+\alpha,\rho}y + p(l)\,{}_a\mathcal{D}^{\alpha,\rho}y + q(l)f(y) = g(l); \qquad l > a \geq 0, \\ \lim\limits_{l \to a^+} {}_aI^{j-\alpha,\rho}y(l) = b_j \qquad (j = 1, 2, \ldots, m) \end{cases}$$

for $\rho \in (0, 1]$ and $m = \lceil \alpha \rceil$, where ${}_a\mathcal{D}^{\alpha,\rho}$ represents the left GPF derivative of order $\alpha \in \mathbb{C}$ of y with $\mathrm{Re}(\alpha) \geq 0$, and that ${}_aI^{j-\alpha,\rho}$ is the left GPF integral of order $j - \alpha \in \mathbb{C}$, $b_j \in \mathbb{R}$ ($j = 1, 2, \ldots, m$), and $p, g \in \mathrm{C}(\mathbb{R}^+, \mathbb{R})$, $q \in \mathrm{C}(\mathbb{R}^+, \mathbb{R}^+)$, $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$. They also established forced oscillation of GPF initial value problem with damping term in the Caputo setting

$$\begin{cases} {}_a^C\mathcal{D}^{1+\alpha,\rho}y + p(l)\,{}_a^C\mathcal{D}^{\alpha,\rho}y + q(l)f(y) = g(l); \qquad l > a \geq 0, \\ \mathcal{D}^{k,\rho}y(a) = b_k, \qquad (k = 0, 1, \ldots, n-1) \end{cases}$$

for $\rho \in (0, 1]$ and $n = \lceil \alpha \rceil$, where ${}_a^C\mathcal{D}^{\alpha,\rho}$ is the left GPF derivative of order $\alpha \in \mathbb{C}$ of y with $\mathrm{Re}(\alpha) \geq 0$ in the Caputo setting, and that

$$\mathcal{D}^{k,\rho} := \underbrace{\mathcal{D}^{\rho}\mathcal{D}^{\rho}\cdots\mathcal{D}^{\rho}}_{\text{k-times}},$$

where $\mathcal{D}^{\rho}$ is called the proportional differential operator.

Over the last few years, many researchers have focused on generalized fractional operators and their qualitative properties. New classes of weighted and generalized fractional operators have been introduced, unifying several existing definitions and extending their analytical features [7, 13, 24, 38–40]. In parallel, several authors have developed new oscillation criteria for fractional differential and partial differential equations, highlighting the importance of qualitative analysis in fractional models [19, 20, 28, 30, 36]. These developments indicate a growing research interest in generalized operators and their oscillatory behavior.

As far as we know, the proposed Ψ-Hilfer GPF derivative has not yielded any results for the oscillatory behaviour of fractional diferential equations.

Inspired by the introductory literature, in this work, we investigate the oscillatory behavior of Ψ-Hilfer GPF initial value problem of the form

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha,\beta,\eta,\Psi}x - p(\kappa)x + \sum\limits_{j=1}^{m} q_j(\kappa)\Phi_{\lambda_j}(x) = v(\kappa), \\ \mathcal{J}_{a^+}^{1-\gamma,\eta,\Psi}x(a) = \sum\limits_{j=1}^{m} \mu_j \; x(\tau_j) \end{cases} \tag{1.3}$$

for $\kappa \geq a$, $\tau_j \in (a, \kappa)$ and $\mu_j \in \mathbb{R}$, $j = 1, \ldots, m$, where $\mathcal{D}_{a^+}^{\alpha,\beta,\eta,\Psi}$ denotes the left sided Ψ-Hilfer GPF derivative of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$ of x concerning another function $\Psi, \eta \in (0, 1]$. $\mathcal{J}_{a^+}^{1-\gamma,\eta,\Psi}$ denotes the left sided GPF integral operator of order $1 - \gamma$ (with $\gamma = \alpha + \beta(1-\alpha)$) of x concerning another function Ψ. $p, v, q_j \in \mathrm{C}[a, \infty)$ ($j = 1, \ldots, m$), and that λ_j's are defined as before. Furthermore $\tau_j \in (a, \kappa)$ satisfy $a < \tau_1 < \tau_2 < \cdots < \tau_m < \kappa$ for $j = 1, \ldots, m$.

The condition in (1.3) represents a nonlocal weighted initial condition. Unlike classical initial conditions, which depend solely on the solution's value at a single initial point, this condition incorporates a weighted sum of the solution evaluated at multiple interior points, specifically $\sum\limits_{j=1}^{m} \mu_j \; x(\tau_j)$ for $j = 1, \ldots, m$. The parameters τ_j show where the sampling points are within the interval (a, κ). The coefficients μ_j represent the weights that measure the extent to which each point influences the initial state.

From both physical and analytical perspectives, nonlocal conditions are employed to model systems exhibiting memory effects or distributed initial data, where the current state depends on past values across a range of points rather than a single instant. This approach suits fractional-order models, where derivatives naturally capture memory and hereditary effects. The weighted terms $\mu_j\ x(\tau_j)$ provide a realistic framework for modelling physical phenomena with distributed initial conditions.

Solution $x \in \mathrm{C}([a,\infty),\mathbb{R})$ $(x \not\equiv 0)$ of (1.3) is defined to be a function satisfying (1.3) for $\kappa \geq a$ and $\mathcal{D}_{a^+}^{\alpha,\beta,\eta,\Psi}x \in \mathrm{C}([a,\infty),\mathbb{R})$. The solution x of problem (1.3) is said to be oscillatory if it has arbitrarily large zeros on $(0,\infty)$; otherwise, it is said to be nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

The structure of the paper is as follows: In section 2 we provide a few introductory information that are essential to demonstrate the main results of the paper. Using the recently proposed Ψ-Hilfer GPFD in section 3, we create a new oscillation result for problem (1.3). We discuss three examples in section 4 to explain the importance of results given in section 3. We conclude the importance of the results of the paper for the particular choice of parameters in section 5 and discuss how they will coincide with the former results by Shao et al. [33].

2. Preliminaries

This section deals with providing some basic definitions and lemmas which are necessary will be used throughout this work.

Definition 2.1 ([18])**.** Let $\varphi_0, \varphi_1 \in \mathrm{C}([0,1]\times\mathbb{R},[0,\infty))$ such that $\varphi_0(\eta,*) \neq 0$ for $\eta \in (0,1]$, $\varphi_1(\eta,*) \neq 0$ for $\eta \in [0,1)$, and that

$$\lim_{\eta\to 0^+}\varphi_0(\eta,*) = \lim_{\eta\to 1^-}\varphi_1(\eta,*) = 0,$$
$$\lim_{\eta\to 1^-}\varphi_0(\eta,*) = \lim_{\eta\to 0^+}\varphi_1(\eta,*) = 1$$

for all $\eta \in [0,1]$.

Let Ψ be a positive, strictly increasing continuous function. Then,

$$\mathcal{D}^{\eta,\Psi}f(\kappa) = \varphi_1(\eta,\kappa)f(\kappa) + \varphi_0(\eta,\kappa)\frac{f'(\kappa)}{\Psi'(\kappa)}, \tag{2.1}$$

gives the proportional differential operator of order η with respect to the function Ψ of a function f. In particular, when $\varphi_0(\eta,\kappa) = \eta$ and $\varphi_1(\eta,\kappa) = 1-\eta$, the operator $\mathcal{D}^{\eta,\Psi}$ in (2.1) becomes

$$\mathcal{D}^{\eta,\Psi}f(\kappa) = (1-\eta)f(\kappa) + \eta\frac{f'(\kappa)}{\Psi'(\kappa)}, \tag{2.2}$$

and the integral corresponding to proportional derivative (2.2) is given as

$$\mathcal{J}_a^{1,\eta,\Psi}f(\kappa) = \frac{1}{\eta}\int_a^\kappa e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}\Psi'(s)f(s)\mathrm{d}s, \tag{2.3}$$

where it is assumed that $\mathcal{J}_a^{0,\eta,\Psi}f(\kappa) = f(\kappa)$.

The generalized proportional integral of order n corresponding to proportional derivative $\mathcal{D}^{n,\eta,\Psi}f(\kappa)$ is given as follows

$$(\mathcal{J}_a^{n,\eta,\Psi}f)(\kappa) = \frac{1}{\eta^n\Gamma(n)}\int_a^\kappa e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{n-1}\Psi'(s)f(s)\mathrm{d}s, \tag{2.4}$$

where $\Gamma(\cdot)$ is the *gamma function* and

$$\mathcal{D}^{n,\eta,\Psi} = \underbrace{\mathcal{D}^{\eta,\Psi}\mathcal{D}^{\eta,\Psi}\mathcal{D}^{\eta,\Psi}\cdots\mathcal{D}^{\eta,\Psi}}_{n-\text{times}}.$$

Definition 2.2 ([18])**.** Let $\eta \in (0,1]$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Then the fractional integral

$$(\mathcal{J}_{a^+}^{\alpha,\eta,\Psi}f)(\kappa) = \frac{1}{\eta^\alpha\Gamma(\alpha)}\int_{a^+}^\kappa e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)f(s)\mathrm{d}s, \tag{2.5}$$

is called the left-sided GPF integral of order α of the function f with respect to another function Ψ for $\kappa > a$.

Definition 2.3 ([18]). Let $\eta \in (0,1]$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \geq 0$ and let $\Psi \in \mathrm{C}[a,b]$ with $\Psi'(s) > 0$. Then the left GPF derivative of order α of the function f with respect to Ψ is defined as

$$(\mathcal{D}_{a^+}^{\alpha,\eta,\Psi} f)(\kappa) = \frac{1}{\eta^{n-\alpha}\Gamma(n-\alpha)} \mathcal{D}_{\kappa}^{n,\eta,\Psi} \left(\int_{a^+}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta} [\Psi(\kappa)-\Psi(s)]^{n-\alpha-1} \Psi'(s) f(s) \mathrm{d}s \right),$$

where $n = \mathrm{Re}(\alpha) + 1$.

Definition 2.4 ([25]). Let $I = [a,b]$, $-\infty \leq a < b \leq \infty$, and let $f, \Psi \in \mathrm{C}^n[a,b]$ such that Ψ is positive, strictly increasing and $\Psi' \neq 0$ for all $\kappa \in I$. The Ψ-Hilfer GPFD (left-sided/right-sided) of order α and type β of f with respect to another function Ψ are defined by

$$(\mathcal{D}_{a^\pm}^{\alpha,\beta,\eta,\Psi} f)(\kappa) = \left(\mathcal{J}_{a^\pm}^{\beta(n-\alpha),\eta,\Psi} (\mathcal{D}^{n,\eta,\Psi}) \mathcal{J}_{a^\pm}^{(1-\beta)(n-\alpha),\eta,\Psi} f \right)(\kappa), \tag{2.6}$$

where $n-1 < \alpha < n,\ 0 \leq \beta \leq 1$ with $n \in \mathbb{N}$ and $\eta \in (0,1]$. Also,

$$\mathcal{D}^{\eta,\Psi} f(\kappa) = (1-\eta) f(\kappa) + \eta \frac{f'(\kappa)}{\Psi'(\kappa)},$$

and $\mathcal{J}$ is the GPF integral defined in (2.5).

When $n = 1$, (2.6) becomes

$$(\mathcal{D}_{a^\pm}^{\alpha,\beta,\eta,\Psi} f)(\kappa) = \left(\mathcal{J}_{a^\pm}^{\beta(1-\alpha),\eta,\Psi} (\mathcal{D}^{1,\eta,\Psi}) \mathcal{J}_{a^\pm}^{(1-\beta)(1-\alpha),\eta,\Psi} f \right)(\kappa), \tag{2.7}$$

for $\alpha \in (0,1)$ and $\beta \in [0,1]$.

Theorem 2.5 ([25]). *Let $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, $\eta \in (0,1]$, and $\gamma = \alpha + \beta(n-\alpha)$, $n \in \mathbb{N}$. If $\xi > n$ for $\xi \in \mathbb{R}$, then the function*

$$f(\kappa) = e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta} [\Psi(\kappa)-\Psi(a)]^{\xi-1},$$

under the operator $\mathcal{D}_{a^+}^{\alpha,\beta,\eta,\Psi}$ is given as

$$\mathcal{D}_{a^+}^{\alpha,\beta,\eta,\Psi} f(\kappa) = \frac{\eta^{\alpha}\Gamma(\xi)}{\Gamma(\xi-\alpha)} e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta} [\Psi(\kappa)-\Psi(a)]^{\xi-\alpha-1}. \tag{2.8}$$

Lemma 2.6 ([25]). *Let $n-1 < \alpha < n$, $\eta \in (0,1]$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta(n-\alpha)$ such that $n-1 < \gamma < n$, $n \in \mathbb{N}$. If $f \in \mathrm{C}_\gamma[a,b]$ and $\mathcal{J}_{a^+}^{n-\gamma,\eta,\Psi} f \in \mathrm{C}_{\gamma,\Psi}^n[a,b]$, then*

$$\mathcal{J}_{a^+}^{\alpha,\eta,\Psi} \mathcal{D}_{a^+}^{\alpha,\beta,\eta,\Psi} f(\kappa) = f(\kappa) - \sum_{k=1}^{n} \frac{e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta} [\Psi(\kappa)-\Psi(a)]^{\gamma-k}}{\eta^{\gamma-k}\Gamma(\gamma-k+1)} \times (\mathcal{J}_{a^+}^{k-\gamma,\eta,\Psi} f)(a).$$

Lemma 2.7 ([11]). *Let X, Y, U and V be nonnegative constants. Then the inequalities*

(i) $\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda-1) Y^{\lambda}$; $\quad \lambda > 1$,
(ii) $\lambda U V^{\lambda-1} - U^{\lambda} \geq (\lambda-1) V^{\lambda}$; $\quad 0 < \lambda < 1$

hold. Equality holds in both of the inequalities if and only if $X = Y$ and $U = V$, respectively.

Lemma 2.8 ([35]). *Let $(\alpha_1, \alpha_2, \dots, \alpha_m)$ be an m-tuple satisfying*

$$\alpha_1 > \cdots > \alpha_l > 1 > \alpha_{l+1} > \cdots > \alpha_m > 0.$$

Then there exists an m-tuple $(\eta_1, \eta_2, \dots, \eta_m)$ satisfying

$$\sum_{k=1}^{l} \alpha_k \eta_k = \sum_{k=l+1}^{m} \alpha_k \eta_k,$$

with $\sum_{k=1}^{m} \eta_k = 1$ and $0 < \eta_k < 1$ for $k = 1, 2, \dots, m$.

3. Main results

This section deals with the forced oscillation of Ψ-Hilfer GPF, i.e. problem (1.3). By using Lemma 2.6, the solution representation of problem (1.3) is defined by

$$\begin{aligned} x(\kappa) = &\frac{\Lambda}{\eta^{\alpha}\Gamma(\alpha)} e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta}[\Psi(\kappa)-\Psi(a)]^{\gamma-1} \\ &\times \sum_{k=1}^{m}\mu_k \int_{a^+}^{\tau_i} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s \\ &+ \frac{1}{\eta^{\alpha}\Gamma(\alpha)} \int_{a^+}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s, \end{aligned} \tag{3.1}$$

where

$$\Lambda^{-1} = \eta^{\gamma-1}\Gamma(\gamma) - \sum_{i=1}^{m}\mu_i e^{(\eta-1)[\Psi(\tau_i)-\Psi(a)]/\eta}[\Psi(\tau_i)-\Psi(a)]^{\gamma-1},$$

and

$$F(s) = v(s) + p(s)x(s) - \sum_{k=1}^{m} q_k(s)\Phi_{\lambda_k}(x(s)).$$

Theorem 3.1. *Assume that* $p(\kappa) > 0$ *and*

$$q_k(\kappa)\begin{cases} \geq 0 & \text{for } 1 \leq k \leq l, \\ \leq 0 & \text{for } l+1 \leq k \leq m. \end{cases} \tag{3.2}$$

If

$$\begin{aligned} \liminf_{\kappa\to\infty} &\Psi^{1-\alpha}(\kappa) \int_{a^+}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \\ &\times \left[v(s) + K\sum_{k=1}^{m}[p(s)]^{\lambda_k/(\lambda_k-1)}|q_k(s)|^{1/(1-\lambda_k)}\right]\mathrm{d}s = -\infty, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \limsup_{\kappa\to\infty} &\Psi^{1-\alpha}(\kappa) \int_{a^+}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \\ &\times \left[v(s) + K\sum_{k=1}^{m}[p(s)]^{\lambda_k/(\lambda_k-1)}|q_k(s)|^{1/(1-\lambda_k)}\right]\mathrm{d}s = \infty, \end{aligned} \tag{3.4}$$

for some constant $K > 0$, *then every solution of problem* (1.3) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of problem (1.3). Without loss of generality, let $T > a$ be large enough such that $x(\kappa) > 0$ for all $\kappa \geq T$. From Equation (3.1), we have

$$\begin{aligned} x(\kappa) \leq &\frac{\Lambda}{\eta^{\alpha}\Gamma(\alpha)} e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta}[\Psi(\kappa)-\Psi(a)]^{\gamma-1} \\ &\times \sum_{k=1}^{m}\mu_k \int_{a^+}^{\tau_k} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s \\ &+ \frac{1}{\eta^{\alpha}\Gamma(\alpha)} \int_{a^+}^{T} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s \\ &+ \frac{1}{\eta^{\alpha}\Gamma(\alpha)} \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \times \sum_{k=1}^{l}\left[\lambda_k p(s)x(s) - q_k(s)x^{\lambda_k}(s)\right]\mathrm{d}s \end{aligned}$$

$$+\frac{1}{\eta^{\alpha}\Gamma(\alpha)}\int_{T}^{\kappa}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\times\sum_{k=l+1}^{m}\Big[-Ap(s)x(s)+|q_k(s)|x^{\lambda_k}(s)\Big]\mathrm{d}s, \tag{3.5}$$

where

$$F(s)=v(s)+p(s)x(s)-\sum_{k=1}^{m}q_k(s)x^{\lambda_k}(s),$$

and

$$A=\frac{1}{m-l}\sum_{k=1}^{l}(\lambda_k-1)>0.$$

Multiplying (3.5) by $\eta^{\alpha}\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)$, we have

$$\begin{aligned}
\eta^{\alpha}\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)x(\kappa) &\le \Psi^{1-\alpha}(\kappa)\Lambda e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta}[\Psi(\kappa)-\Psi(a)]^{\gamma-1}\\
&\times\sum_{k=1}^{m}\mu_k\int_{a^+}^{\tau_k}e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s\\
&+\Psi^{1-\alpha}(\kappa)\int_{a^+}^{T}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s\\
&+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)v(s)\mathrm{d}s\\
&+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\times\sum_{k=1}^{l}\Big[\lambda_k p(s)x(s)-q_k(s)x^{\lambda_k}(s)\Big]\mathrm{d}s\\
&+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\times\sum_{k=l+1}^{m}\Big[-Ap(s)x(s)+|q_k(s)|x^{\lambda_k}(s)\Big]\mathrm{d}s.
\end{aligned} \tag{3.6}$$

Set

$$X=[q_k(s)]^{1/\lambda_k}x(s),\quad Y=\Big[p(s)[q_k(s)]^{-1/\lambda_k}\Big]^{1/(\lambda_k-1)},\qquad 1\le k\le l,$$

and

$$U=|q_k(s)|^{1/\lambda_k}x(s),\quad V=\Big[A\lambda_k^{-1}p(s)|q_k(s)|^{-1/\lambda_k}\Big]^{1/(\lambda_k-1)},\qquad l+1\le k\le m.$$

Then by using Lemma 2.7 in (3.6), we get

$$\begin{aligned}
\eta^{\alpha}\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)x(\kappa) &\le \Psi^{1-\alpha}(\kappa)\Phi(\kappa)+\Psi^{1-\alpha}(\kappa)\Omega(\kappa,T)\\
&+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)v(s)\mathrm{d}s\\
&+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\\
&\times\sum_{k=1}^{l}(\lambda_k-1)[p(s)]^{\lambda_k/(\lambda_k-1)}[q_k(s)]^{1/(1-\lambda_k)}\mathrm{d}s\\
&+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa}e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\\
&\times\sum_{k=l+1}^{m}(1-\lambda_k)\lambda_k^{\lambda_k/(1-\lambda_k)}A^{-\lambda_k/(1-\lambda_k)}[p(s)]^{\lambda_k/(\lambda_k-1)}|q_k(s)|^{1/(1-\lambda_k)}\mathrm{d}s,
\end{aligned}$$

where

$$\Phi(\kappa)=\Lambda e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta}[\Psi(\kappa)-\Psi(a)]^{\gamma-1}$$

$$\times \sum_{k=1}^{m} \mu_k \int_{a^+}^{\tau_k} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s,$$

and

$$\Omega(\kappa,T) = \int_{a^+}^{T} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s,$$

for $\kappa \geq T$. Therefore,

$$\begin{aligned} &0 < \eta^\alpha \Gamma(\alpha)\Psi^{1-\alpha}(\kappa)x(\kappa) \\ &\leq \Psi^{1-\alpha}(\kappa)\Phi(\kappa) + \Psi^{1-\alpha}(\kappa)\Omega(\kappa,T) \\ &+ \Psi^{1-\alpha}(\kappa)\int_T^\kappa e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\times\left[v(s)+K\sum_{k=1}^{m}[p(s)]^{\lambda_k/(\lambda_k-1)}|q_k(s)|^{1/(1-\lambda_k)}\right]\mathrm{d}s, \end{aligned} \tag{3.7}$$

where

$$K = \max\left\{\lambda_1 - 1, \max_{l+1\leq k\leq m}(1-\lambda_k)\lambda_k^{\lambda_k/(1-\lambda_k)}A^{-\lambda_k/(1-\lambda_k)}\right\}.$$

Take $T_1 \geq T$. Since $|e^{(\eta-1)\Psi(\kappa)/\eta}| \leq 1$ and the function

$$h_1(\kappa) = \Psi^{1-\alpha}(\kappa)[\Psi(\kappa)-\Psi(a)]^{\gamma(\alpha-1)},$$

is decreasing for $0<\alpha<1$, we get

$$\begin{aligned} \Psi^{1-\alpha}(\kappa)\Phi(\kappa) &\leq \Psi^{1-\alpha}(\kappa)|\Lambda|\sum_{k=1}^{m}\mu_k\int_{a^+}^{\tau_k}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}[\Psi(s)-\Psi(a)]^{\gamma-1}\Psi'(s)|F(s)|\mathrm{d}s \\ &= \mathcal{B}(\gamma,\alpha)|\Lambda||F(s)|\sum_{k=1}^{m}\mu_k h_1(\tau_k), \end{aligned}$$

for $\kappa \geq T_1$, where

$$\mathcal{B}(\gamma,\alpha) = \int_0^1 u^{\gamma-1}(1-u)^{\alpha-1}\mathrm{d}u,$$

is the beta function with $\mathrm{Re}(\gamma), \mathrm{Re}(\alpha) > 0$.

Using the monotonicity of Ψ on (a,κ), we get

$$\begin{aligned} \Psi^{1-\alpha}(\kappa)\Phi(\kappa) &< \mathcal{B}(\gamma,\alpha)|\Lambda||F(s)|\sum_{k=1}^{m}\mu_k h_1(\kappa) \\ &\leq \mathcal{B}(\gamma,\alpha)|\Lambda||F(s)|\sum_{k=1}^{m}\mu_k h_1(T_1) \\ &:= C(T_1). \end{aligned} \tag{3.8}$$

Since $|e^{(\eta-1)\Psi(\kappa)/\eta}| \leq 1$ and the function $h_2(\kappa) = [1-\Psi(s)/\Psi(\kappa)]^{\alpha-1}$ is decreasing for $0<\alpha<1$, we get

$$\Psi^{1-\alpha}(\kappa)\Omega(\kappa,T) \leq \int_{a^+}^{T} h_2(T_1)\Psi'(s)|F(s)|\mathrm{d}s := C(T,T_1). \tag{3.9}$$

for $\kappa \geq T_1$.

Substituting (3.8) and (3.9) in (**??**), we have

$$\begin{aligned} &0 < \eta^\alpha\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)x(\kappa) \\ &\leq C(T_1) + C(T,T_1) + \Psi^{1-\alpha}(\kappa)\int_T^\kappa e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \end{aligned}$$

$$\times \Big[v(s) + K\sum_{k=1}^{m}[p(s)]^{\lambda_k/(\lambda_k-1)}|q_k(s)|^{1/(1-\lambda_k)}\Big]\mathrm{d}s,$$

which implies that

$$\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\times\Big[v(s) + K\sum_{k=1}^{m}[p(s)]^{\lambda_k/(\lambda_k-1)}|q_k(s)|^{1/(1-\lambda_k)}\Big]\mathrm{d}s \geq -[C(T_1)+C(T,T_1)]. \tag{3.10}$$

Choosing the limit infimum of both sides of inequality (3.10) as $\kappa\to\infty$, we obtain a contradiction to condition (3.7). Similarly, when x is eventually negative, we get a contradiction to condition (3.4). Hence, the proof is completed. □

Corollary 3.2. *Let $l = m$ in (1.3). Then we have $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 1$. Suppose further that $p(\kappa) > 0$, $q_k(\kappa) \geq 0$, $k = 1,\dots,m$. If (3.7) and (3.4) hold for some constant $K_1 > 0$, then problem (1.3) is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of problem (1.3). Without loss of generality, let $T > a$ be large enough such that $x(\kappa) > 0$ for all $\kappa \geq T$. From equation (3.1), we have

$$\begin{aligned}\eta^{\alpha}\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)x(\kappa) &= \Psi^{1-\alpha}(\kappa)\Lambda e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta}[\Psi(\kappa)-\Psi(a)]^{\gamma-1}\\ &\times\sum_{k=1}^{m}\mu_k\int_{a^+}^{\tau_k} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s\\ &+\Psi^{1-\alpha}(\kappa)\int_{a^+}^{T} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s\\ &+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)v(s)\mathrm{d}s\\ &+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\\ &\times\sum_{k=1}^{m}\Big[\frac{1}{m}p(s)x(s) - q_k(s)x^{\lambda_k}(s)\Big]\mathrm{d}s.\end{aligned} \tag{3.11}$$

Set

$$X = [q_k(s)]^{1/\lambda_k}x(s), \quad \text{and} \quad Y = \Big[\frac{1}{m\lambda_k}p(s)[q_k(s)]^{-1/\lambda_k}\Big]^{1/(\lambda_k-1)}, \qquad (1\leq k\leq m),$$

for $\kappa \geq T$. Then by using Lemma 2.7 (i) in the above equation (3.11), we get

$$\begin{aligned}\eta^{\alpha}\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)x(\kappa) &\leq \Psi^{1-\alpha}(\kappa)\Lambda e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta}[\Psi(\kappa)-\Psi(a)]^{\gamma-1}\\ &\times\sum_{k=1}^{m}\mu_k\int_{a^+}^{\tau_k} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s\\ &+\Psi^{1-\alpha}(\kappa)\int_{a^+}^{T} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s\\ &+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)v(s)\mathrm{d}s\\ &+\Psi^{1-\alpha}(\kappa)\int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\\ &\times K_1\sum_{k=1}^{m}[p(s)]^{\lambda_k/(\lambda_k-1)}|q_k(s)|^{1/(1-\lambda_k)}\Big]\mathrm{d}s,\end{aligned}$$

where $K_1 \geq (\lambda_1 - 1)/m$. As the remaining proof is similar to that of Theorem 3.1, we ignore it. □

Similarly, we obtain the next corollary.

Corollary 3.3. *Let $l = 0$ in (1.3), then we have $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_m$. Further suppose that $p(\kappa) < 0$, $q_k(\kappa) \le 0$, $k = 1, \dots, m$. If (3.7) and (3.4) hold for some constant $K_2 > 0$, then problem (1.3) is oscillatory.*

Corollary 3.4. *When $p(s) \equiv 0$ problem (1.3), we have $1 < l < m$. Assume that*

$$q_k(\kappa) \begin{cases} \ge 0, & \text{for } \ 1 \le k \le l, \\ \le 0, & \text{for } \ l+1 \le k \le m. \end{cases} \tag{3.12}$$

If there exists a positive function $r(\kappa)$ on $[a, \infty)$ such that

$$\liminf_{\kappa\to\infty} \Psi^{1-\alpha}(\kappa) \int_{a^+}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta} [\Psi(\kappa) - \Psi(s)]^{\alpha-1} \Psi'(s) \times \Big[v(s) + K_3 \sum_{k=1}^{m} [r(s)]^{\lambda_k/(\lambda_k-1)} |q_k(s)|^{1/(1-\lambda_k)} \Big] \mathrm{d}s = -\infty \tag{3.13}$$

and

$$\limsup_{\kappa\to\infty} \Psi^{1-\alpha}(\kappa) \int_{a^+}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta} [\Psi(\kappa) - \Psi(s)]^{\alpha-1} \Psi'(s) \times \Big[v(s) + K_3 \sum_{k=1}^{m} [r(s)]^{\lambda_k/(\lambda_k-1)} |q_k(s)|^{1/(1-\lambda_k)} \Big] \mathrm{d}s = \infty \tag{3.14}$$

for some constant $K_3 > 0$, then every solution of problem (1.3) is oscillatory.

Proof. Suppose that x is a nonoscillatory solution of the problem (1.3). Without loss of generality, let $T > a$ be large enough such that $x(\kappa) > 0$ for all $\kappa \ge T$. From equation (3.1), we have

$$\begin{aligned} x(\kappa) \le {} & \frac{\Lambda}{\eta^\alpha \Gamma(\alpha)} e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta} [\Psi(\kappa) - \Psi(a)]^{\gamma-1} \\ & \times \sum_{k=1}^{m} \mu_k \int_{a^+}^{\tau_k} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta} [\Psi(\tau_k) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \mathrm{d}s \\ & + \frac{1}{\eta^\alpha \Gamma(\alpha)} \int_{a^+}^{T} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta} [\Psi(\kappa) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \mathrm{d}s \\ & + \frac{1}{\eta^\alpha \Gamma(\alpha)} \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta} [\Psi(\kappa) - \Psi(s)]^{\alpha-1} \Psi'(s) v(s) \mathrm{d}s \\ & - \frac{1}{\eta^\alpha \Gamma(\alpha)} \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta} [\Psi(\kappa) - \Psi(s)]^{\alpha-1} \Psi'(s) \sum_{k=1}^{l} q_k(s) x^{\lambda_k}(s) \mathrm{d}s \\ & + \frac{1}{\eta^\alpha \Gamma(\alpha)} \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta} [\Psi(\kappa) - \Psi(s)]^{\alpha-1} \Psi'(s) \sum_{k=l+1}^{m} |q_k(s)| x^{\lambda_k}(s) \mathrm{d}s. \end{aligned}$$

Since $\lambda_1 > \cdots > \lambda_l > 1 > \lambda_{l+1} > \cdots > \lambda_m$, by using Lemma 2.8, there exists an m-tuple $(\eta_1, \dots, \eta_m)$ satisfying

$$\sum_{k=1}^{l} \lambda_k \eta_k = \sum_{k=l+1}^{m} \lambda_k \eta_k,$$

and hence we have

$$x(\kappa) \le \frac{\Lambda}{\eta^\alpha \Gamma(\alpha)} e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta} [\Psi(\kappa) - \Psi(a)]^{\gamma-1}$$

$$\begin{aligned}
&\times \sum_{k=1}^{m} \mu_k \int_{a^+}^{\tau_k} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s \\
&+ \frac{1}{\eta^\alpha\Gamma(\alpha)} \int_{a^+}^{T} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s \\
&+ \frac{1}{\eta^\alpha\Gamma(\alpha)} \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)v(s)\mathrm{d}s \\
&+ \frac{1}{\eta^\alpha\Gamma(\alpha)} \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \times \sum_{k=1}^{l}\Big[\lambda_k\eta_k r(s)x(s) - q_k(s)x^{\lambda_k}(s)\Big]\mathrm{d}s \\
&+ \frac{1}{\eta^\alpha\Gamma(\alpha)} \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \times \sum_{k=l+1}^{m}\Big[-\lambda_k\eta_k r(s)x(s) + |q_k(s)|x^{\lambda_k}(s)\Big]\mathrm{d}s.
\end{aligned} \tag{3.15}$$

Multiplying (3.15) by $\eta^\alpha\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)$, we have

$$\begin{aligned}
\eta^\alpha\Gamma(\alpha)\Psi^{1-\alpha}(\kappa)x(\kappa) \le{}& \Psi^{1-\alpha}(\kappa)\Lambda e^{(\eta-1)[\Psi(\kappa)-\Psi(a)]/\eta}[\Psi(\kappa)-\Psi(a)]^{\gamma-1} \\
&\times \sum_{k=1}^{m} \mu_k \int_{a^+}^{\tau_k} e^{(\eta-1)[\Psi(\tau_k)-\Psi(s)]/\eta}[\Psi(\tau_k)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s \\
&+ \Psi^{1-\alpha}(\kappa) \int_{a^+}^{T} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\mathrm{d}s \\
&+ \Psi^{1-\alpha}(\kappa) \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)v(s)\mathrm{d}s \\
&+ \Psi^{1-\alpha}(\kappa) \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \\
&\times \sum_{k=1}^{l}\Big[\lambda_k\eta_k r(s)x(s) - q_k(s)x^{\lambda_k}(s)\Big]\mathrm{d}s \\
&+ \Psi^{1-\alpha}(\kappa) \int_{T}^{\kappa} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \\
&\times \sum_{k=l+1}^{m}\Big[-\lambda_k\eta_k r(s)x(s) + |q_k(s)|x^{\lambda_k}(s)\Big]\mathrm{d}s.
\end{aligned}$$

Since the rest of the proof is similar, we skip the remaining details. □

4. Examples

In this section, three examples are provided to highlight our main results.

Example 4.1. Given the Ψ-HGPF problem as follows

$$\begin{cases} \mathcal{D}_{0^+}^{1/4,1/2,1,\Psi}x(\kappa) - \kappa^3 x(\kappa) + \kappa^5|x(\kappa)|^{2/3}x(\kappa) - \dfrac{\kappa}{4}|x(\kappa)|^{2/3}x(\kappa) = \cos(\kappa), \\ \mathcal{J}_{0^+}^{1-\gamma,1,\Psi}x(0) = 4x(1/3) + 6x(5/7). \end{cases} \tag{4.1}$$

for $\kappa \ge 0$. Here, $a = 0$, $\alpha = 1/4$, $\beta = 1/2$, $\eta = 1$, $l = 1$, $m = 2$, $p(\kappa) = \kappa^3$, $q_1(\kappa) = \kappa^5$, $q_2(\kappa) = -\kappa/4$, $\lambda_1 = 5/3$, $\lambda_2 = 1/3$, $v(\kappa) = \cos\kappa$, $\mu_1 = 4$, $\mu_2 = 6$, $\tau_1 = 1/3$, $\tau_2 = 5/7$.

In addition, let $\Psi(\kappa) = \kappa$. By choosing $K = 2/3$, we get

$$\Psi^{1-\alpha}(\kappa) \int_{a^+}^{t} e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s) \times \Big[v(s) + K\sum_{k=1}^{m} p^{\lambda_k/(\lambda_k-1)}(s)|q_k(s)|^{1/(1-\lambda_k)}\Big]\mathrm{d}s$$

$$= \kappa^{3/4}\Big(\int_0^\kappa (\kappa-s)^{-3/4}\cos s\mathrm{d}s + 3\kappa^{1/4}\Big).$$

Therefore,

$$\liminf_{\kappa\to\infty} \kappa^{3/4}\left(\int_0^\kappa (\kappa-s)^{-3/4}\cos s\mathrm{d}s + 3\kappa^{1/4}\right)\mathrm{d}s = -\infty,$$

and

$$\limsup_{\kappa\to\infty} \kappa^{3/4}\left(\int_0^\kappa (\kappa-s)^{-3/4}\cos s\mathrm{d}s + 3\kappa^{1/4}\right)\mathrm{d}s = \infty.$$

Hence, according to Theorem 3.1, every solution of problem (4.1) is oscillatory.

Example 4.2. Given the Ψ-HGPF problem as follows

$$\begin{cases}\mathcal{D}_{0^+}^{3/4,2/3,1,\Psi}x(\kappa) - \dfrac{\kappa}{3}x(\kappa) + \kappa^3|x(\kappa)|^2x(\kappa) = \sin\kappa, \quad \kappa\ge 0,\\ \mathcal{J}_{0^+}^{1-\gamma,\eta,\Psi}x(0) = 5x(1/6).\end{cases} \tag{4.2}$$

Here, $a=0$, $\alpha=3/4$, $\beta=2/3$, $\eta=1$, $l=m=1$, $p(\kappa)=\kappa/3$, $q_1(\kappa)=\kappa^3$, $\lambda_1=3$, $\mu_1=5$, $\tau_1=1/6$, $v(\kappa)=\sin\kappa$.

In addition, let $\Psi(\kappa)=\kappa/2$. By choosing $K_1=2$, we get

$$\Psi^{1-\alpha}(\kappa)\int_{a^+}^t e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)\times\Big[v(s)+K_1\sum_{k=1}^m p^{\lambda_k/(\lambda_k-1)}(s)|q_k(s)|^{1/(1-\lambda_k)}\Big]\mathrm{d}s$$
$$= \kappa^{1/4}\Big(\frac{1}{2}\int_0^\kappa (\kappa-s)^{-1/4}\sin s\mathrm{d}s + \frac{4}{9}\kappa^{3/4}\Big).$$

Therefore,

$$\liminf_{\kappa\to\infty} \kappa^{1/4}\left(\frac{1}{2}\int_0^\kappa (\kappa-s)^{-1/4}\sin s\mathrm{d}s + \frac{4}{9}\kappa^{3/4}\right)\mathrm{d}s = -\infty,$$

and

$$\limsup_{\kappa\to\infty} \kappa^{1/4}\left(\frac{1}{2}\int_0^\kappa (\kappa-s)^{-1/4}\sin s\mathrm{d}s + \frac{4}{9}\kappa^{3/4}\right)\mathrm{d}s = \infty.$$

Hence, according to Corollary 3.2, every solution to problem (4.2) is oscillatory.

Example 4.3. Given the Ψ-HGPF problem as follows

$$\begin{cases}\mathcal{D}_{0^+}^{1/2,1,1,\Psi}x(\kappa) - \kappa^5x(\kappa) + \kappa^7|x(\kappa)|^{2/5}x(\kappa) = \dfrac{2\kappa}{\sqrt{\pi}} + \kappa^{49/5} - \kappa^7, \quad \kappa\ge 0,\\ \mathcal{J}_{0^+}^{1-\gamma,\eta,\Psi}x(0) = 0.\end{cases} \tag{4.3}$$

Here, $a=0$, $\alpha=1/2$, $\beta=1$, $\eta=1$, $l=m=1$, $p(\kappa)=\kappa^5$, $q_1(\kappa)=\kappa^7$, $\lambda_1=7/5$, $\mu_1=0$, $\tau_1=5/2$ and

$$v(\kappa) = \frac{2\kappa}{\sqrt{\pi}} + \kappa^{49/5} - \kappa^7.$$

In addition, let $\Psi(\kappa)=\kappa^2$. It is easy to obtain $K_1=2/5$. Since $2\kappa/\sqrt{\pi}+\kappa^{49/5}-\kappa^7\ge 0$ and one can verify that

$$\Psi^{1-\alpha}(\kappa)\int_{a^+}^\kappa e^{(\eta-1)[\Psi(\kappa)-\Psi(s)]/\eta}[\Psi(\kappa)-\Psi(s)]^{\alpha-1}\Psi'(s)$$
$$\times\Big[v(s)+K_1\sum_{k=1}^m p^{\lambda_k/(\lambda_k-1)}(s)|q_k(s)|^{1/(1-\lambda_k)}\Big]\mathrm{d}s$$

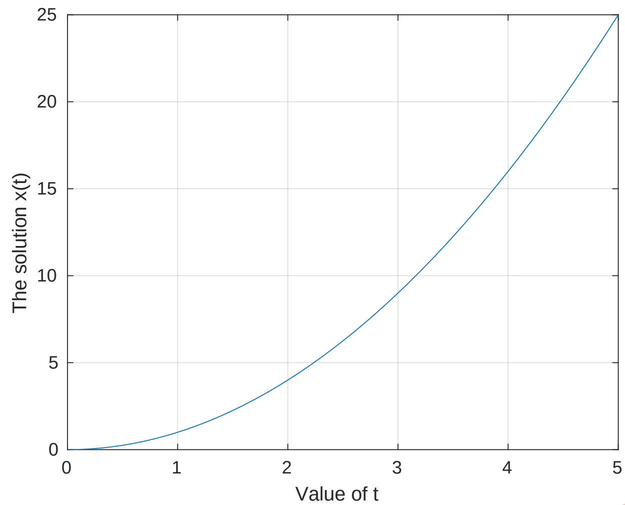

FIGURE 1. Nonoscillatory behavior of $x(\kappa) = \kappa^2$.

$$= (\kappa^2)^{1/2}\left(\int_0^\kappa (\kappa^2 - s^2)^{-1/2}(2s)\left[\left(\frac{2s}{\sqrt{\pi}} + s^{49/5} - s^7\right) + \frac{2}{5}\right]\mathrm{d}s\right)$$
$$\geq \kappa \int_0^\kappa (\kappa^2 - s^2)^{-1/2}\left(\frac{4s}{5}\right)\mathrm{d}s$$
$$= \frac{4}{5}\kappa^2.$$

This confirms that neither (3.7) nor (3.4) is satisfied. Infact, using Theorem 2.5 with $\xi = 2$, it is simple to verify that the nonoscillatory solution of the problem (4.3) is $x(\kappa) = \kappa^2$ and its nonoscillatory behavior is demonstrated in Figure 1.

5. Conclusion

In this paper, we studied the oscillatory behavior of the Ψ-Hilfer GPF(Ψ-Hilfer generalized proportional fractional) initial value problem. By employing the Ψ-Hilfer GPFD operator within the framework of the GPFD (generalized proportional fractional derivative), new sufficient conditions for forced oscillation were established. The obtained results extend and generalize several existing oscillation criteria available in the literature. In particular, for the special case $\eta = 1$ and $\Psi(\kappa) = \kappa$, our results reduce to those derived under the classical Riemann–Liouville and Caputo fractional settings in [33]. Moreover, illustrative examples were provided to demonstrate the applicability and effectiveness of the theoretical findings.

This study provides a foundation for several future research directions. The proposed approach can be extended to analyse the oscillatory behavior of impulsive fractional differential equations, neutral-type systems, and delay fractional models within the Ψ-Hilfer generalized proportional framework. Furthermore, examining boundary value problems involving the Ψ-Hilfer GPFD may provide deeper insights and broaden the applicability of these findings.

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