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Unifrom convergence of a higher order finite element method on an exponentially graded Bakhvalov mesh for convection-diffusion problems possessing boundary layers

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Abstract

In this article, we established the convergence of a pth order ($p \geq 1$) finite element method on an exponentially graded Bakhvalov mesh for a convection-diffusion problem which posses boundary layer. Optimal uniform convergence order is obtained by a careful selection of the interpolation operator, considering the characteristics of the layers, allows the finite element method. Numerical results are presented to support the theoretical findings.

Keywords. Singularly perturbed, Finite element method, Exponentially graded Bakhvlov mesh, Uniform convergence.
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1. Introduction

Consider the singularly perturbed convection-diffusion problem:

$$-\varepsilon u'' - b(x)u' + c(x)u = f \ in \ \Omega := (0,1), \ u(0) = u(1) = 0, \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a positive perturbation parameter, b, c and f are sufficiently smooth functions. Furthermore, we assume two positive constants β and γ such that

$$b(x) \geq \beta \geq 1 \ on \ \overline{\Omega}, \tag{1.2}$$

and

$$c(x) + \frac{1}{2}b'(x) \geq \gamma > 0 \ on \ \overline{\Omega}. \tag{1.3}$$

With these conditions (1.2) and (1.3), the problem (1.1) posses a unique solution with presence of an exponential boundary layer close to $x = 0$ as $\varepsilon \to 0$. Solution of this type of problem typically exhibits a boundary layer of width $\mathcal{O}(\varepsilon ln(1/\varepsilon))$ at $x = 0$ (see[11]).

It is well known that for the perturbation parameter $\varepsilon \to 0$, classic numerical techniques do not perform appropriately on a uniform mesh. Many researchers have adapted multiple techniques using different ideas to resolve the boundary layers formed by the problem (1.1). For cost efficiency it is preferable to resolve the layer behavior by use of layer adapted meshes. Two of such famous meshes are Bakhvalov-type meshes (B-type) and Shishkin-type meshes (S-type) (see [6]). The idea of S-type meshes was very acceptable due to their easy construction as it is a piecewise uniform mesh, which was easy to construct and facilitates analysis. The drawback of S-type meshes is the production of errors that include lnN factors for N mesh steps, which is not the case in B-type meshes. B-type meshes are graded in the inner region and have a smoother transition to the outer region at a point. Although the numerical results are better on B-type meshes, their convergence is difficult to prove near the transition point. Therefore, optimal order convergence for higher-dimensional problems has remained an open area of research, as mentioned in [12]. So naturally, quite a few studies have been done on the finite element method applied to Bakhvalov-type meshes (see [8, 15–17]). In [3, 10] a quasi interpolation method was derived for optimal order convergence, but it can not be extended to higher

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dimensions or higher-order elements. This issue was resolved in [16], which introduced a new interpolation technique for a pth-order finite element method ($p \geq 1$) for B-type meshes, with some constraints on the mesh parameter (see (6) and (7) in [16]) for problem (1.1).

. In [9], the authors introduced a new B-type mesh called the exponentially graded Bakhvalov mesh, which is a Bakhvalov-type modification of the exponentially graded mesh (see [5]). As far as the knowledge of the authors is concerned, the finite element analysis for the exponentially graded Bakhvalov mesh (EGBM) has not yet been done. The article [16] derives the analysis of optimal order convergence for a Bakhvalov-type mesh of the specific structure considered therein. However, the currently available B-type meshes represent different approximations of the original Bakhvalov mesh idea and differ in their structure. EGBM has a slightly different structure than the Bakhvalov-type meshes considered in [16]. The specialty of this mesh is that it is a Bakhvalov-type mesh, but it also retains the simplicity of its predecessor, the eXp-mesh, and hence keeps one of its points as the transition point.

. In this article, we derive some preliminary results in the form of lemmas for EGBM in the next section, which are used to prove the optimal order convergence for the proposed problem using a slightly different approach than the analysis performed in [16]. Although the article aims to achieve the optimal order convergence of the finite element method on EGBM, we also compare it with the results of the Vulanović Bakhvalov (V-B) mesh , which is one of the approximations of Bakhvalov-type meshes [7], to understand its numerical effectiveness. It can be observed that EGBM performs better than the V-B mesh.

Notations. Throughout this article, C is defined as a generic positive constant that is independent of the perturbation parameter, whereas C_i designates specific constant values. The usual Sobolev norm $||.||_{p,I}$ and semi-norm $|.|_{p,I}$ are used. For $p = 2$, the subscript p will be dropped to write $||.||_{L_2(I)} = ||.||_I$. When $I = \Omega$, we can also drop I and use the usual norm and semi-norm symbols.

2. Decomposition, Mesh and finite element method

2.1. **Decomposition of solution.** Higher-order derivatives of the solution u of (1.1) are necessary for uniform convergence of finite element methods. In accordance with [11], the solution u of (1.1) can be decomposed as

$$u = u_S + u_L, \tag{2.1}$$

where u_S and u_L are smooth and layer components that satisfy $Lu_S = f$ and $Lu_L = 0$, respectively. Further, the bounds of higher-order derivatives of u_S and u_L are as below:

Lemma 2.1. *Assume that (1.2) and (1.3) holds true and $b,c,f \in C^l[0,1]$ for any fixed integer $l \geq 1$. Then*

$$|u_S^{(j)}(x)| \leq C, \text{ and } |u_L^{(j)}(x)| \leq C\varepsilon^{-j} \exp\left(-\frac{\beta x}{\varepsilon}\right), 0 \leq j \leq l+1. \tag{2.2}$$

2.2. **Exponentially graded Bakhvalov Mesh.** We divide $\Omega = [0,1]$ into N sub-intervals such that x_i , $i = 0,1,...,N$ are the discretization points of the mesh such that $0 \leq x_0 \leq x_1 \leq ... \leq x_N = 1$. Let p be the order of the finite element and $q \in (0,1)$ be a user defined parameter and $\sigma \geq p+1$. Then for $t_i = \frac{i}{N}$, the mesh point x_i of eXp-graded Bakhvalov mesh can been seen as

$$x_i = \begin{cases} \psi(t_i), & i = 1,...,k-1, \\ x_{k-1} + \psi'(t_{k-1})(t_i - t_{k-1}), & i = k,..,N, \end{cases} \tag{2.3}$$

with

$$\psi(t) = -\frac{\sigma\varepsilon}{\beta}\phi_{EB}(t) \text{ for } t \in \left[0, \frac{q}{M_\varepsilon}\right), \tag{2.4}$$

where

$$\phi_{EB} = \log\left(1 - \frac{M_\epsilon t}{q}\right) \text{ and } M_\epsilon = 1 - \exp\left(\frac{-\beta}{\sigma\varepsilon}\right). \tag{2.5}$$

Here k is the index chosen such that

$$t_{k-1} < \tau \leq t_k, \tag{2.6}$$

where τ is solution of nonlinear equation

$$\psi(\tau)+\psi'(\tau)(1-\tau)=1. \tag{2.7}$$

We also consider

$$\phi=1-\frac{M_\varepsilon}{q}t. \tag{2.8}$$

From (2.8), we have $\phi'=-\frac{M_\varepsilon}{q}\in R^-$, which means that this mesh is of optimal type [13].
From (2.4), we have $t=e^{\frac{\beta x}{\sigma\varepsilon}}$.
Now from (2.3), we have

$$\begin{aligned} h_i =&\psi\left(\frac{i+1}{N}\right)-\psi\left(\frac{i}{N}\right)\le \sigma\varepsilon N^{-1}\max_{I_i}\phi'_{EB}\\ \le&\sigma\varepsilon N^{-1}\max_{I_i}|\phi|e^{\frac{x_{i+1}}{(p+1)\varepsilon}}\le \sigma\varepsilon N^{-1}\max_{I_i}|\phi|e^{\frac{x_{i+1}}{(p+1)\varepsilon}}. \end{aligned} \tag{2.9}$$

Lemma 2.2. *From the mesh generated by (2.3),(2.9), and (2.7), we have*

$$C_1\varepsilon\le\frac{q}{M_\varepsilon}-\tau\le C_2\varepsilon, \tag{2.10}$$

$$C_3N^{-1}\le h_i\le C_4N^{-1}\ for\ i=0,1,2,...,k-2, \tag{2.11}$$

Proof. By following the steps given in [9], we can get (2.10) and (2.11). □

Lemma 2.3. *For exponentially graded Bakhvalov mesh (2.3), we have*

$$\max_{t\in[0,t_{k-1}]}\phi'_{EB}\le N, \tag{2.12}$$

$$h_0\le h_1\le ...\le h_{k-2}, \tag{2.13}$$

$$h_i\le C\varepsilon\ for\ i=0,1,2,...,k-3, \tag{2.14}$$

$$h_i\le CN^{-1},\ for\ i=0,1,2,...,N-1, \tag{2.15}$$

$$e^{\beta x_{k-1}/\varepsilon}\le C\left(\varepsilon+N^{-1}\right)^\sigma, \tag{2.16}$$

$$e^{-\beta x_k/\varepsilon}\le C\varepsilon^\sigma. \tag{2.17}$$

Proof. Using (2.10) in (2.5), we have

$$\begin{aligned} \max_{t\in[0,t_{k-1}]}\phi'_{EB}=&\max_{t\in[0,t_{k-1}]}\frac{1}{q/M_\varepsilon-t}\le\frac{1}{q/M_\varepsilon-t_{k-1}}\\ &\le\frac{1}{q/M_\varepsilon-(t_k-1/N)}\le\frac{1}{q/M_\varepsilon-t_k+N^{-1}}\\ &\le\frac{1}{C_2\varepsilon+N^{-1}}\le N. \end{aligned}$$

This proves (2.12). Next using the mesh (2.3) for $0\le i\le k-2$, we have

$$h_i=x_{i+1}-x_i=\psi(t_{i+1})-\psi(t_i)=\int_{t_i}^{t_{i+1}}\psi'(s)ds=\int_{t_i}^{t_{i+1}}\frac{\sigma\varepsilon M_\varepsilon}{q-M_\varepsilon t}ds,$$

and

$$\frac{\sigma\varepsilon M_\varepsilon}{q-M_\varepsilon t_i}N^{-1}\le h_i\le\frac{\sigma\varepsilon M_\varepsilon}{q-M_\varepsilon t_{i+1}}N^{-1}. \tag{2.18}$$

By using this, we can verify (2.13) easily.

Using $t_{k-2} < t_{k-1} < \tau < q/M_\varepsilon$ and (2.3), we get

$$h_i \leq \int_{t_i}^{t_{i+1}} \psi'(s)ds \leq N^{-1}\psi'(t_{i+1}) = \frac{\sigma\varepsilon}{N}\frac{1}{q/M_\varepsilon - t_{i+1}}.$$

For $i = k-3$, we have

$$h_{k-3} \leq \frac{\sigma\varepsilon}{N}\frac{1}{q/M_\varepsilon - t_{k-2}} \leq \frac{\sigma\varepsilon}{N}\frac{1}{t_{k-1} - t_{k-2}} \leq \sigma\varepsilon.$$

Now by using (2.13), one has $h_i \leq C\varepsilon$ for $i = 0, 1, 2, ..., k-3$.

From (2.5) and (2.11), we get (2.15). Using (2.9) and (2.10) one gets (2.16) by the steps below

$$\begin{aligned} e^{\beta x_{k-1}/\varepsilon} &\leq e^{\beta\psi(t_{k-1})/\varepsilon} \leq \left(1 - \frac{M_\epsilon t_{k-1}}{q}\right)^\sigma \\ &\leq M_\varepsilon^\sigma \left(\frac{\frac{q}{M_\varepsilon} - t_{k-1}}{q}\right)^\sigma \leq C\left((q/M_\varepsilon - \tau) + (\tau - t_{k-1})\right)^\sigma \\ &\leq C\left(\varepsilon + N^{-1}\right). \end{aligned}$$

Using same calculations of (2.16) and (2.17) can be proven. □

2.3. **The finite element method.** The variation formulation of (1.1) is given by: Find $u \in H_0^1$ such that

$$A(u, v) = F(v) \ for\ all\ v \in H_0^1, \tag{2.19}$$

where

$$A(u, v) = \varepsilon \int_\Omega u'v'dx + \int_\Omega (-bu' + cu)v\ dx \text{ and } F(v) = \int_\Omega fv\ dx,\ for\ all\ u, v \in H_0^1.$$

By using the conditions (1.2), (1.3), and Lax-Milgramm Lemma, the variational form has a unique solution.

Then for the exponentially graded Bakhvalov mesh $\{x_i\}_{i=0}^N$, the finite element method for (2.19) can be read by

$$A(u^N, v^N) = (f, v^N)\ \forall v^N \in V^N, \tag{2.20}$$

where $V^N := \{y \in C(\overline{\Omega}) : y(0) = y(1) = 0, y|_{I_i} \in P_k(I_i), i = 0, 1, ..., N-1\}$ and $P_k(I_i)$ is the Lagrange polynomial space of degree k on the interval I_i. Based on (1.2) and (1.3), we have coercivity

$$A(v^N, v^N) \geq \|v\|_\epsilon^2\ \forall v^N \in V^h, \tag{2.21}$$

where $\|v\|_\varepsilon^2 = \varepsilon|v|_1^2 + \|v\|^2\ \forall v \in H^1(\Omega)$.

The condition (2.21) demonstrate the well-posedness of finite element solution u^N defined by (2.19)(see [2]).

3. Error Analysis

Consider $x_i^j = x_i + (\frac{j}{h})h_i$ for $i = 0, 1, ..., N-1$ and $j = 0, 1, ..., p-1$. Let θ_i^j be the nodal basis functions corresponding to nodes x_i^j in the finite element space V^N. For the notation consistency, we set $x_N^0 = x_N$ *and* $\theta_i^0 = \theta_i$. For any $v \in C^0(\Omega)$ its Lagrange interpolant $v^I \in V^N$ on each mesh point is defined as

$$v^I(x) = \sum_{i=0}^N v(x_i^0)\theta_i^0(x) + \sum_{i=0}^{N-1}\sum_{j=0}^{p-1} v(x_i^j)\theta_i^j(x).$$

Using the interpolation theory of Sobolev spaces [Theorem 3.1.4, [4]] , for any $v \in W^{k+1,p}(I_i)$, we have

$$||v - v^I||_{W^{l,q}(I_i)} \leq Ch_i^{k+1-l+\frac{1}{q}-\frac{1}{p}}|v|_{W^{k+1,p}(I_i)}, \tag{3.1}$$

where $i = 0, 1, ..., N-1,\ l = 0, 1$ *and* $1 \leq p, q \leq \infty$.

. Next by utilization of the properties of exponentially graded Bakhvalov mesh from Lemma 2.3, we derive the following error estimates :

Lemma 3.1. *For exponentially graded Bakhvalov mesh (2.3), we have*

$$||u_L - u_L^I||_\infty + ||u_S - u_S^I||_\infty + ||u - u^I||_\infty \le CN^{-(p+1)}, \tag{3.2}$$

$$||u_L - u_L^I|| + ||u_S - u_S^I|| + ||u - u^I|| \le CN^{-(p+1)}, \tag{3.3}$$

$$||u_L - u_L^I||_\varepsilon + ||u_S - u_S^I||_\varepsilon + ||u - u^I||_\varepsilon \le CN^{-p}. \tag{3.4}$$

Proof. Using (2.2) and (3.1), one has

$$||u_L - u_L^I||_{L^\infty(I_i)} \le C{h_i}^{p+1}|u_L|_{W^{p+1,\infty}(I_i)}. \tag{3.5}$$

Using (2.9) and (2.12) , for $i = 0, 1, ..., k-2$, we have

$$\begin{aligned}||u_L - u_L^I||_{L^\infty(I_i)} &\le C\varepsilon^{-(p+1)}{h_i}^{p+1}e^{-\beta x_i/\varepsilon}\\ &\le C\left((p+1)\varepsilon N^{-1}e^{\frac{x_{i+1}}{(p+1)\varepsilon}}\right)^{(p+1)}\varepsilon^{-(p+1)}e^{-\beta x_i/\varepsilon}\\ &\le CN^{-(p+1)}\left(e^{x_{i+1}/\varepsilon} - e^{x_i/\varepsilon}\right) \le CN^{-(p+1)}e^{h_i/\varepsilon}\\ &\le CN^{-(p+1)}e^{h_i/\varepsilon} \le CN^{-(p+1)}e^{(p+1)N^{-1}\max_{I_i}\phi'_{EB}}\\ &\le CN^{-(p+1)}.\end{aligned}$$

Next for $I_{k-1} = [x_{k-1}, x_k]$, using (2.15) and (2.16) in (3.5), we get

$$||u_L - u_L^I||_{L^\infty(I_i)} \le CN^{-(p+1)}\varepsilon^{-(p+1)}(\varepsilon + N^{-1})^{-(p+1)}. \tag{3.6}$$

For $\varepsilon \le N^{-1}$ we directly get the desired result. Similarly for $\varepsilon > N^{-1}$, (3.6) becomes

$$||u_L - u_L^I||_{L^\infty(I_i)} \le CN^{-(p+1)}\varepsilon^{-(p+1)}\varepsilon^{(p+1)} \le CN^{-(p+1)}.$$

For $i = k, ..., N-1$, by substituting (2.15), (2.17) in (3.5) and direct calculations,one achieves

$$||u_L - u_L^I||_{L^\infty(I_i)} \le CN^{-(p+1)}. \tag{3.7}$$

Combining the results for $i = 0, 1, 2, .., N-1$ and (2.2) one can obtain (3.2).

Using Holder inequalities on (3.2), (3.3) is obtained. For $||u_L - u_L^I||_\varepsilon$, we first decompose $||(u_L - u_L^I)'||$ in to following 2 parts:

$$\|(u_L - u_L^I)'\|^2 = \sum_{i=0}^{k-2}\|(u_L - u_L^I)'\|_{I_i}^2 + \|(u_L - u_L^I)'\|_{[x_{k-1},x_N]}^2 = T_1 + T_2. \tag{3.8}$$

Using (2.9), (2.12), we have

$$\begin{aligned}T_1 &\le C\sum_{i=0}^{k-2}{h_i}^{2p}|u_L|_{p+1,I_i} \le C{h_i}^{2p}\int_{x_i}^{x_{i+1}}\varepsilon^{-2(p+1)}e^{-2x/\varepsilon}dx\\ &\le C\sum_{i=0}^{k-2}h^{2p}\varepsilon^{-2p-1}(e^{-2x_i/\varepsilon} - e^{2x_{i+1}/\varepsilon})\\ &\le \left(\sigma\varepsilon N^{-1}\max_{I_i}|\phi|e^{\frac{x_{i+1}}{(p+1)\varepsilon}}\right)^{2p}\varepsilon^{-2p-1}(e^{-2x_i/\varepsilon} - e^{2x_{i+1}/\varepsilon})\\ &\le C\varepsilon^{-1}N^{-2p}e^{2px_{i+1}/(p+1)\varepsilon}(e^{-2x_i/\varepsilon} - e^{2x_{i+1}/\varepsilon}).\end{aligned} \tag{3.9}$$

Now using $e^{2px_{i+1}/(p+1)\varepsilon}(e^{-2x_i/\varepsilon} - e^{2x_{i+1}/\varepsilon}) \le C$ in (3.9), one gets

$$T_1 \le C\varepsilon^{-1}N^{-2p}. \tag{3.10}$$

Next for the second term of (3.8), using (2.15),(2.16), (2.17) and (3.3), one has

$$T_2 \leq C \left(\|(u_L - u_L^I)'\|^2_{[x_{k-1},x_k]} + \|(u_L - u_L^I)'\|^2_{[x_k,x_N]} \right). \tag{3.11}$$

Using similar calculation as (3.9),one can get

$$\|(u_L - u_L^I)'\|^2_{[x_{k-1},x_k]} \leq C\varepsilon^{-1}N^{-2p}. \tag{3.12}$$

For $I_i \in [x_k, x_N]$ using (2.15),(2.17), and (3.1), one gets

$$\|(u_L - u_L^I)'\|^2_{I_i} \leq {h_i}^{2p}\varepsilon^{-2p-1}\|e^{-2x_i/\varepsilon}\|_{\infty,I_i} \leq C\varepsilon N^{-2p} \tag{3.13}$$

Combining (3.10), (3.12), and (3.13) with (3.3), we get (3.4).

□

4. Uniform Convergence

Consider $\vartheta := u^I - u^N$. Then using (2.12), (2.2),(3.3), and (3.6), one can have

$$\alpha||\vartheta||^2_\varepsilon \leq A(\vartheta,\vartheta) = A(u^I - u, \vartheta) = \sum_{r=1}^{5} \Upsilon_r, \tag{4.1}$$

where

$$\Upsilon_1 = \varepsilon \int_0^1 (u^I - u)'\vartheta' dx,\ \Upsilon_2 = \int_0^1 b(u_L^I - u_L)\vartheta' dx,$$

$$\Upsilon_3 = -\int_0^1 b({u_S}^I - u_S)'\vartheta dx,\ \Upsilon_4 = \int_0^1 b'(u_L^I - u_L)\vartheta dx,\ \Upsilon_5 = \varepsilon \int_0^1 c(u^I - u)\vartheta dx,$$

Using statements of Lemma 3.1 and Cauchy-Schwarz inequality on $\vartheta_1,\ \vartheta_3,\ \vartheta_4,\ \vartheta_5$, we get

$$(\Upsilon_1 + \Upsilon_5) \leq ||u - u^I||_\varepsilon ||\vartheta||_\varepsilon \leq CN^{-p}||\vartheta||_\varepsilon, \tag{4.2}$$

$$\Upsilon_3 + \Upsilon_4 \leq (||(u_S^I - u_S)'|| + ||u_L^I - u_L||)||\vartheta|| \leq CN^{-p}||\vartheta||. \tag{4.3}$$

Lemma 4.1. *Let u_L^I be defined in , then we have*

$$|\Upsilon_3| = \left| \int_0^1 b(u_L^I - u_L)\vartheta' dx \right| \leq CN^{-p}||\vartheta||_\epsilon. \tag{4.4}$$

Proof. The term in (4.4) can be divide in to 2 different parts follows:

$$\begin{aligned} \int_0^1 b(u_L^I - u_L)\vartheta' dx =& \int_0^{x_{k-1}} b(u_L^I - u_L)\vartheta' dx + \int_{x_{k-1}}^1 b(u_L^I - u_L)\vartheta' dx \\ =& \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{4.5}$$

Using Holder inequalities, (2.9), (2.13, (2.14), (3.1), and (3.2), one gets

$$\begin{aligned} |\mathcal{I}_1| \leq& C\sum_{i=0}^{k-2} \int_{x_i}^{x_{i+1}} |u_L - u_L^I||\vartheta'| dx \\ \leq& \sum_{i=0}^{k-2} \int_{x_i}^{x_{i+1}} \|u_L - u_L^I\|_{\infty,I_i} |\vartheta'|_{L^1(I_i)} dx \\ \leq& \sum_{i=0}^{k-2} CN^{-(p+1)}{h_i}^{1/2}|\vartheta'|_{I_i} \leq C\varepsilon^{1/2} \sum_{i=0}^{k-2} CN^{-(p+1)}|\vartheta'|_{I_i} \\ \leq& CN^{-(p+1)}\|\vartheta\|_\varepsilon, \end{aligned} \tag{4.6}$$

For the second integral using inverse inequality of [14], one has

$$\begin{aligned} |\mathcal{I}_2| \leq & \|u_L - u_L^I\|_{L_\infty,[x_{k-1},1]} \|\vartheta'\|_{L^1,[x_{k-1},1]} \\ \leq & \|u_L - u_L^I\|_{L_\infty,[x_{k-1},1]} \sum_{i=k-1}^{N-1} \|(u^I - u^N)'\|_{[x_{k-1},1]} \\ \leq & CN^{-(p+1)} \sum_{i=k-1}^{N-1} \frac{p}{h_i} \|(u^I - u^N)'\|_{[x_{k-1},1]} \\ \leq & CN^{-p} \|u_I - u_N\|_{\varepsilon}, \end{aligned} \tag{4.7}$$

where we have used (3.2). □

Now we derive the final result for uniform convergence.

Theorem 4.2. *Let u and u^N be the solutions of (1.1) and (2.1), respectively. Then the following estimate holds:*

$$||u - u^N||_{\varepsilon} \leq CN^{-p}.$$

Proof. Using triangle inequality, we have

$$||u - u^N||_{\varepsilon} \leq ||u - u^I||_{\varepsilon} + ||u^I - u^N||_{\varepsilon}. \tag{4.8}$$

Substituting (4.2), (4.3), and (4.4) in (4.1), one has

$$||u^I - u^N||_{\varepsilon} \leq CN^{-p}.$$

Combining this with (3.4), we have

$$||u - u^N||_{\varepsilon} \leq CN^{-p}.$$

This concludes the proof. □

5. Numerical Results

To verify the theoretical findings of the theorem, following test example is considered from [16] and [7]

$$\begin{cases} -\varepsilon u'' - (3 - x)u' + u = f(x), & in\ \Omega = (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{5.1}$$

where f can be determined by the use of exact solution of (5.1), which is

$$u(x) = cos\left(\frac{\pi}{2}x\right)\left(1 - \exp\left(\frac{-2x}{\varepsilon}\right)\right). \tag{5.2}$$

For computational purpose, we consider $q = 1/2$, $p = 1, 2, 3, 4$, $\varepsilon = 10^{-4}, ..., 10^{-9}$, and $N = 8, 16, ..., 1024$.

In the first table we discuss the errors in the norm $||.||_{\varepsilon}$ defined as $e^N = ||u - u^N||_{\varepsilon}$ and rate of convergence r^N, which can be calculated using the formula

$$r^N = \log_2\left(\frac{e_{\varepsilon}^N}{e_{\varepsilon}^{2N}}\right).$$

Here in Table 1 we have given a comparison of our current mesh to the Vulanović-Bakhvalov mesh given in [7] for first order finite element. It was observed that the exponentially graded Bakhvalov mesh performs better than the Vulanović-Bakhvalov mesh in the current framework. For simplicity of notation we have considered errors in exponentially graded Bakhvalov and Vulanović Bakhvalov mesh as e_{EB}^N and e_{VB}^N. Similarly, the rate of convergence for the meshes are given by r_{EB}^N and r_{VB}^N. Table 2 shows errors for different order of finite element methods on exponentially graded Bakhvalov mesh with respect to maximum error over a wide range of ε calculated as

$$E_{\varepsilon}^N = \max_{\varepsilon=10^{-4},...,10^{-9}} ||u - u^N||_{\varepsilon},$$

Table 1. Comparison of EB and VB meshes with errors and convergence rates ($p = 1$).

ε	Type	32	64	128	256	512	1024
Exponentially graded Bakhvalov mesh (EB)							
10^{-4}	e^N	8.336069e-02	4.167029e-02	2.083394e-02	1.041678e-02	5.208348e-03	2.604169e-03
	r^N	1.00	1.00	0.99	0.99	0.99	–
10^{-5}	e^N	8.335951e-02	4.166975e-02	2.083373e-02	1.041673e-02	5.208344e-03	2.604168e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-6}	e^N	8.335939e-02	4.166970e-02	2.083370e-02	1.041671e-02	5.208340e-03	2.604168e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-7}	e^N	8.335937e-02	4.166969e-02	2.083370e-02	1.041671e-02	5.208339e-03	2.604167e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-8}	e^N	8.335937e-02	4.166969e-02	2.083370e-02	1.041671e-02	5.208339e-03	2.604167e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-9}	e^N	8.335937e-02	4.166969e-02	2.083370e-02	1.041671e-02	5.208339e-03	2.604167e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
Vulanović–Bakhvalov mesh (VB)							
10^{-4}	e^N	9.538456e-02	4.762064e-02	2.380303e-02	1.190074e-02	5.950261e-03	2.975117e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-5}	e^N	9.540389e-02	4.762599e-02	2.380361e-02	1.190072e-02	5.950259e-03	2.975116e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-6}	e^N	9.540694e-02	4.762710e-02	2.380401e-02	1.190082e-02	5.950263e-03	2.975116e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-7}	e^N	9.540743e-02	4.762725e-02	2.380407e-02	1.190084e-02	5.950272e-03	2.975117e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-8}	e^N	9.540754e-02	4.762727e-02	2.380408e-02	1.190084e-02	5.950273e-03	2.975118e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–
10^{-9}	e^N	9.540757e-02	4.762728e-02	2.380408e-02	1.190085e-02	5.950273e-03	2.975118e-03
	r^N	1.00	1.00	1.00	1.00	1.00	–

Table 2. Errors and convergence rates on exponentially graded Bakhvalov mesh.

	p=1		p=2		p=3		p=4	
N	E_ε^N	R_ε^N	E_ε^N	R_ε^N	E_ε^N	R_ε^N	E_ε^N	R_ε^N
8	3.895955e-01	1.00	1.027811e-01	1.99	3.010796e-02	2.93	8.977874e-03	3.88
16	1.920828e-01	1.20	2.570168e-02	2.00	3.930327e-03	2.99	6.085121e-04	3.97
32	8.341136e-02	1.00	6.412054e-03	1.99	4.963541e-04	2.99	3.881919e-05	3.99
64	4,168294e-02	1.00	1.604796e-03	1.99	6.219989e-05	2.99	2.438665e-06	3.99
128	2.083873e-02	0.99	4.013147e-04	1.99	7.779825e-06	2.99	1.526119e-07	3.99
256	1.041973e-02	0.99	1.003366e-04	2.00	9.726292e-07	2.99	9.541303e-09	3.99
512	5.211358e-03	1.00	2.508475e-05	2.00	1.215834e-07	3.00	5.963895e-10	3.91
1024	2.608615e-03	-	6.271236e-06	-	1.519807e-08	-	3.950656e-11	-

and rate of convergence is given by

$$R_\varepsilon^N = \log_2\left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}\right).$$

The errors and rate of convergence are in accordance with the theoretical results discussed previously.

In Figure 1, different errors $||u-u^N||$, $||u-u^N||_{L^\infty(\Omega)}$ and $||u-u^N||_\varepsilon$ are considered for studying exponentially graded Bakhvalov mesh for $\varepsilon = 10^{-3}$, $p = 1$, and $N = 2^3, ..., 2^{12}$. Data has been plotted in log-log graph. It was observed that the convergence rates in $||u-u^N||$ and $||u-u^N||_{L^\infty(\Omega)}$ are not optimal for $N^{-1} \geq \varepsilon$.

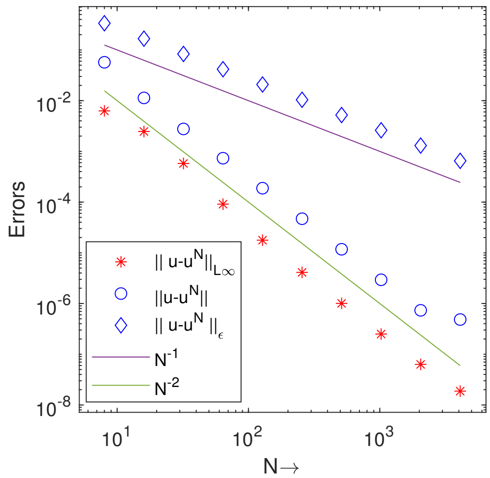

FIGURE 1. Errors for $\varepsilon = 10^{-3}$ on exponentially graded Bakhvalov mesh for $p = 1$.

6. Conclusions

In this article, we developed a higher order finite element analysis for a singularly perturbed convection-diffusion problem on an exponentially graded Bakhvalov mesh. Uniform error estimates are derived. Results are compared with the latest results [7] available in literature to show the better performance, which makes EGBM as one of the better approximations of origianl Bakhvalov mesh. We did not give any comparison with original Bakhvalov mesh as, for smaller values than $\epsilon = 10^{-1}$, the results of EGBM and original Bakhvalov mesh are identical.

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