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Mathematical modeling of memory effects on Gang dynamics

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Abstract

Crime, a complex and dynamic phenomenon influenced by various factors such as social, economic, and environmental variables, has been the focus of numerous studies aiming to model and mitigate delinquent behavior while fostering prosocial growth. However, existing mathematical models, predominantly based on ordinary and fractional differential equations, often overlook the intricate dynamics of gang warfare and competition effects among different gangs, crucial for accurately representing criminal behavioral changes. Hence, the proposed model incorporates competition effects among different gang members and categorizes the total population into five clusters using an epidemiological population-based approach: non-criminals (N), criminals (C), gang-1 (G_1), gang-2 (G_2), and prisoners (P). The article establishes the well-posedness and stability of the proposed model. Additionally, the criminal generation number, a key metric for eradicating crime transmission, is calculated using the next-generation matrix technique, offering insights into optimal crime-control strategies. Equilibrium points, including criminal-free, gangs-free, gang-1 free, gang-2 free, and an endemic equilibrium, are examined. The criminal-free equilibrium is globally asymptotically stable if the criminal generation number is less than one, while the gangs-free, gang-1 free, gang-2 free, and endemic equilibrium are locally asymptotically stable if the criminal generation number exceeds one. A sensitivity analysis is employed to assess the impact of various model parameters on crime transmission control. Theoretical conclusions are validated through numerical simulations, concluding with a discussion on the societal consequences inferred from the model's findings.

Keywords. Mathematical modeling, Fractional differential equations, Criminal gangs, Stability analysis.
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1. Introduction

Crime remains a focal concern for governments worldwide, prompting extensive research attention in the literature [8–10, 26]. The substantial financial resources allocated annually underscore the global commitment to crime control, with dedicated imprisonment and restoration centers addressing the challenges posed by addicted individuals [38]. In the historical context of crime, the pervasive influence of gangs cannot be overlooked. Gangs, both in urban and rural settings, contribute significantly to criminal activities, particularly violent ones, accounting for 48 percent of global violent crimes. Overall, they are responsible for 66 percent of general crimes in large cities, compared to 27 percent in smaller urban areas [22].

Internationally, criminal organizations such as the Italian-American Mafia, Cosa Nostra of Sicily, Apulian Sacra Corona Unita, Neapolitan Camorra, and Calabrian Ndrangheta, as well as the Russian Bratva, Chinese Triads, Japanese Yakuza, and the Irish Mob, exemplify the diversity and global prevalence of criminal gangs [6, 21, 47]. Closer to home, the Chaddi Baniyan Gang in India, including the notorious Kala Kachcha Gang (also known as Kale-Kachchewale or Kale Kachche Gang) in Punjab, further exemplify the expansive influence of organized criminal activities [16, 42]. Notable figures, such as Chhota Rajan, underscore the presence of influential crime syndicates, adding layers to the complex landscape of criminal gangs [18]. Effectively addressing and managing criminal gangs

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necessitates a multifaceted approach, incorporating strategies from the community, social, and law enforcement realms. Techniques such as law enforcement initiatives, legislative measures, community outreach programs, intervention and rehabilitation efforts, socioeconomic issue resolution, and cooperation strategies underscore the multifaceted nature of combating criminal gangs [47]. This paper delves into the complexities of gang-related dynamics, proposing a novel fractional-order model that contributes to the broader understanding of crime transmission within the context of gang warfare.

In India, rising unemployment, which surged from 7 % to 27.11 % in April 2020, has driven many individuals into unlawful activities [17, 46]. Criminologists have explored various factors to understand the motivations behind criminal behavior in the twenty-first century. Traditionally, explanations for criminal actions have been drawn from biological, psychological, social, and economic perspectives. Mathematical models have been developed to aid judicial systems in functioning more effectively. Previous research has utilized compartmental crime models, typically employing ordinary differential equations (integer-order) for initial analyses [23, 25, 44]. Research shows that children of criminal parents are twice as likely to obtain criminal convictions compared to children of non-criminal parents [28]. Standard mathematical models, however, lack the precision needed to fully describe the transmission process of criminal behavior.

In 2020, a mathematical model was introduced to investigate the transmission dynamics of financial crime and to assess the efficacy of different intervention strategies. The study concluded that keeping the reproduction number below one is critical for controlling financial crime, with integrated approaches combining public awareness campaigns and corrective measures proving most cost-efficient, especially when resources are constrained [5]. In 2021, researchers investigated the impact of COVID-19 social distancing mandates on gang-related crime in Los Angeles and found that crime levels and hot spots remained largely unchanged. These results indicate that gang activities may persist irrespective of broader public health restrictions and societal disruptions [14]. In 2022, an age-structured model was developed to examine the influence of correctional interventions on the dynamics of criminal gangs in Nigeria. The findings highlight that timely prosecution and rehabilitation of at least 60 percent of adolescent offenders (ages 8–17) can significantly reduce youth delinquency and overall gang proliferation. The model also demonstrates the presence of backward and Hopf bifurcations, revealing the complex and potentially cyclical nature of gang population behavior [29]. Mathematical modeling has gained prominence as a powerful tool to conceptualize criminal behavior as a socially transmitted phenomenon. Research in this area highlights the critical role of early intervention measures, particularly education and awareness programs, in effectively mitigating the propagation of criminal activities within communities [43]. In 2023, a nonlinear optimal control framework was formulated to explore the dynamics of criminal gangs within resource-constrained environments. The analysis identified that an integrated approach—merging preventive interventions with targeted prosecution—offers the most cost-effective means of suppressing gang activity[30]. Recently, a deterministic modeling approach was introduced to examine the complex interactions between competing criminal gangs within an actively criminal population. The analysis reveals that initiation rates critically drive the escalation of gang rivalries, particularly in environments characterized by limited law enforcement presence. The findings highlight that targeting and reducing initiation rates among at-risk individuals can effectively diminish rivalry intensity and promote greater public safety [31].

Recently, Gebrezabher and Eroglu [24] introduced a nonlinear differential equation model capturing crime dynamics driven by imitation behavior. Their framework delineates the thresholds for both crime-free and endemic scenarios based on the basic reproduction number. The study underscores that curbing the relapse rate among former offenders is pivotal for achieving long-term crime reduction. Fractional calculus, in contrast, offers a more accurate representation of the retention and transmission dynamics than integer-order models, making fractional differential equations more suitable for addressing this issue. These equations have been applied to model biological phenomena for several decades [4, 7, 33, 35]. More recently, fractional-order differential equations have been employed to construct crime propagation models that account for the role of crime history [8, 13, 48, 49]. Nevertheless, these models frequently fail to consider complex factors like gang rivalry and competition, which are essential to understanding shifts in criminal behavior. This study aims to fill this gap by proposing a new fractional-order crime transmission model, incorporating competition effects among different gangs.

In the realm of societal dynamics, understanding the intricate interactions between non-criminals, criminals, gangs, and prisoners is a complex challenge that has far-reaching implications for social science, criminology, and public policy.

This manuscript introduces a comprehensive mathematical model designed to capture the dynamic relationships among these distinct entities and shed light on the evolving nature of social structures. The proposed model, expressed through a system of differential equations, provides a formalized framework for examining the temporal evolution of non-criminals ($N(t)$), criminals ($C(t)$), gang-1 ($G_1(t)$), gang-2 ($G_2(t)$), and prisoners ($P(t)$). The significance of this model lies in its potential to unravel the underlying mechanisms that drive the emergence and transformation of criminal and social structures. By investigating equilibrium points and dynamic trends, researchers can gain valuable insights into the stability, resilience, and potential tipping points within these complex systems. This paper aims to contribute to the growing body of interdisciplinary research at the intersection of mathematics and the social sciences. The utilization of mathematical modeling enables us to bridge the gap between theoretical concepts and empirical observations, fostering a deeper comprehension of the intricate dynamics that characterize human societies.

This paper is structured as follows: Section 2 introduces the formulation of the mathematical model. Section 3 establishes the existence and uniqueness of solutions, ensuring the model is well-posed. Section 4 focuses on the computation of the crime generation number and the determination of equilibrium points. Section 5 analyzes the stability of these equilibrium points. Finally, Section 6 presents sensitivity analyses and numerical simulations, leading to the conclusions.

1.1. **Preliminaries for Fractional Calculus.** Fractional calculus provides an ideal framework for describing the memory and hereditary characteristics of various processes [13, 48, 49]. Differentiation and integration of arbitrary order can be defined in multiple ways, including the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative, and the Caputo derivative, among others. In this study, Caputo fractional-order operators are utilized due to their practical applicability.

Caputo Fractional Derivative. Suppose that $\eta > 0$, $t > 0$, and $n \in \mathbb{N}$. Then

$$
{}^{C}D^{\eta} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\eta)} \displaystyle\int_0^t \dfrac{f^{(n)}(x)}{(t-x)^{\eta+1-n}}\, dx, & n-1 < \eta < n, \\ \dfrac{d^n}{dt^n} f(t), & \eta = n, \end{cases} \tag{1.1}
$$

is called the Caputo fractional derivative of order η [15]. Goyal and Mathur [27, 37] implement this definition in generalized fractional diffusion equations. The main advantage of this approach is that the fractional differential equation in Caputo form retains classical derivative-type initial conditions.

Mittag-Leffler Function. The Mittag-Leffler is denoted by [36]

$$
\mathbb{E}_{\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\eta k + 1)}.
$$

The two-parameter function of the Mittag-Leffler is defined by the series expansion

$$
\mathbb{E}_{\eta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\eta k + \gamma)}, \text{ where } \eta > 0 \text{ and } \gamma > 0.
$$

Theorem 1.1. (Matignon Criteria)
Consider our model described by the following fractional differential equation:

$$
\frac{d^{\eta} X(t)}{dt^{\eta}} = F(X(t)),
$$

where $X(t) = (x_1(t); x_2(t); \ldots; x_n(t))^T$ *and* $F(X(t)) = (f_1; f_2; \ldots; f_n)^T$. *The equilibrium point is denoted by* $E^* = (x_1^*; x_2^*; \ldots; x_n^*)^T$. *For this fractional-order system, the equilibrium points are considered asymptotically stable if all eigenvalues* (λ_j) *of the Jacobian matrix* J *evaluated at equilibrium* E^* *satisfy the criterion:*

$$
|arg(eig(J))| = |arg(\lambda_j)| > \frac{\pi}{2}\eta,
$$

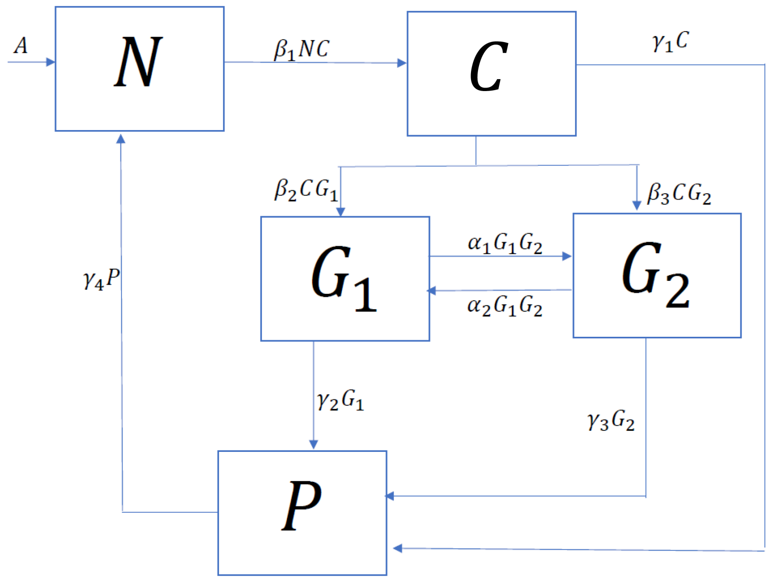

FIGURE 1. A diagram depicting the suggested model for crime propagation.

where $j = 1, 2, \ldots, n$, J *is the Jacobian matrix of the system at the equilibrium point* E^* , *and* λ_j *is eigenvalues of the Jacobian matrix* [3].

2. Formulation of Mathematical Model

This section introduces a fractional-order mathematical model viewed through the lens of a social epidemic, accounting for substantial interactions among non-criminals, offenders, prisoners, and various other individuals not examined in prior research. The model categorizes the population into five distinct clusters based on criminal behavior: non-criminals $N(t)$, criminals $C(t)$, gang-1 $G_1(t)$, gang-2 $G_2(t)$, and prisoners $P(t)$ (see Figure 1).

The model parameters have specific functions in influencing the system's behavior. Non-criminals are integrated into society at a rate denoted by A. The transition of non-criminals to criminal status occurs through interactions with criminals at a rate represented by β_1. The death rate, denoted as δ_0, influences the overall decrease in the population. Parameter β_1 is associated with the transition of individuals between the non-criminal and criminal populations, capturing the flux between these two groups. Similarly, β_2 represents the rate at which criminals transform into gang-1, while β_3 signifies the transformation of criminals into gang-2. Parameters α_1 and α_2 govern the inter-conversion between gang-1 and gang-2, adding complexity to the gang dynamics. The rates at which gang-1 and gang-2 transform into prisoners are denoted by γ_2 and γ_3, respectively, reflecting the law enforcement processes. Additionally, γ_1 represents the rate at which criminals transform into prisoners. On the other hand, γ_4 captures the transformation of prisoners back into the non-criminal population.

It's important to note that while some individuals may revert to criminal behavior immediately after being released from prison, the proposed model assumes that all released individuals initially refrain from illegal activity, even if only for a brief period. Similarly, individuals typically engage in criminal behavior before becoming gang members, and every gang member is first apprehended by law enforcement and incarcerated before any non-criminals.

The mathematical model, comprising fractional differential equations and subject to non-negative initial conditions, encapsulates these intricate interactions within the system as follows:

$$
\begin{aligned}
D^\eta N(t) =& A^\eta - \beta_1^\eta N(t)C(t) + \gamma_4^\eta P(t) - \delta_0^\eta N(t),\\
D^\eta C(t) =& \beta_1^\eta N(t)C(t) - \gamma_1^\eta C(t) - \beta_2^\eta C(t)G_1(t) - \beta_3^\eta C(t)G_2(t) - \delta_0^\eta C(t),
\end{aligned}
$$

$$
\begin{aligned}
D^\eta G_1(t) =&\beta_2^\eta C(t)G_1(t) - \alpha_1^\eta G_1(t)G_2(t) + \alpha_2^\eta G_1(t)G_2(t) - \delta_0^\eta G_1(t) - \gamma_2^\eta G_1(t), \quad (2.1)\\
D^\eta G_2(t) =&\beta_3^\eta C(t)G_1(t) + \alpha_1^\eta G_1(t)G_2(t) - \alpha_2^\eta G_1(t)G_2(t) - \delta_0^\eta G_2(t) - \gamma_3^\eta G_2(t),\\
D^\eta P(t) =&\gamma_2^\eta G_1(t) + \gamma_3^\eta G_2(t) + \gamma_1^\eta C(t) - \gamma_4^\eta P(t) - \delta_0^\eta P(t),\\
N(0) =&N_0,\ C(0) = C_0,\ G_1(0) = G_{10},\ G_2(0) = G_{20},\ P(0) = P_0. \quad (2.2)
\end{aligned}
$$

Here, $X(t) = N(t) + C(t) + G_1(t) + G_2(t) + P(t)$, with $\eta \in (0, 1]$ denoting the order of the derivative. To ensure dimensional consistency, each parameter is elevated to the power of η.

3. Model Analysis

This section demonstrates that the solution to the proposed model is not only existent but also uniquely determined, bounded, and positive.

3.1. Invariant region.

The behavior of the proposed model is examined within a feasible region $\Omega \in \mathbb{R}_+^5$, defined as

$$\Omega = \left\{(N, C, G_1, G_2, P) \in \mathbb{R}_+^5 : N + C + G_1 + G_2 + P \leq \frac{A^\eta}{\delta_0^\eta}\right\}.$$

Theorem 3.1. *The region $\Omega \in \mathbb{R}_+^5$ represents a positively invariant set with non-negative initial conditions.*

Proof. Let $X(t) = N(t) + C(t) + G_1(t) + G_2(t) + P(t)$ denote the total population at time t. Then,

$$
\begin{aligned}
D^\eta X(t) &= D^\eta N(t) + D^\eta C(t) + D^\eta G_1(t) + D^\eta G_2(t) + D^\eta P(t)\\
&= A^\eta - \delta_0^\eta X(t). \quad (3.1)
\end{aligned}
$$

By taking the Laplace transformation[1]:

$$X(s) = \frac{\dfrac{A^\eta}{s} + s^{\eta-1}X(0)}{s^\eta + \delta_0^\eta}.$$

Next, by applying the inverse Laplace transformation[2]:

$$X(t) = \frac{A^\eta}{\delta_0^\eta}[1 - \mathbb{E}_\eta(-\delta_0^\eta t^\eta)] + X(0)\mathbb{E}_\eta(-\delta_0^\eta t^\eta).$$

as $0 \leq \mathbb{E}_\eta(-\delta_0^\eta t^\eta) \leq 1$, $\mathbb{E}_\eta(-x)$ is completely monotonic for $x > 0$ if $0 < \eta \leq 1$ [34, 36].

$$\implies X(t) \leq \frac{A^\eta}{\delta_0^\eta}. \quad (3.2)$$

[1] $\mathcal{L}(D^\eta f(t)) = s^\eta F(s) - \sum_{k=1}^{n} s^{\eta-k} f^{(k-1)}(0)$ where $n - 1 < \eta \leq n$.

[2] $\mathcal{L}(t^{\gamma-1}\mathbb{E}_{\eta,\gamma}(at^\eta) = \dfrac{s^{\eta-\gamma}}{s^\eta - a}$.

We have the following to show that the proposed model 2.1 has a non-negative solution:

$$\left.\frac{d^\eta N(t)}{dt^\eta}\right|_{N(t_0)=0} = A^\eta + \gamma_1^\eta P(t) \geq 0,$$

$$\left.\frac{d^\eta C(t)}{dt^\eta}\right|_{C(t_0)=0} = 0 \geq 0,$$

$$\left.\frac{d^\eta G_1(t)}{dt^\eta}\right|_{G_1(t_0)=0} = 0 \geq 0,$$

$$\left.\frac{d^\eta G_2(t)}{dt^\eta}\right|_{G_2(t_0)=0} = 0 \geq 0,$$

$$\left.\frac{d^\eta P(t)}{dt^\eta}\right|_{P(t_0)=0} = \gamma_2^\eta G_1(t) + \gamma_3^\eta G_2(t) + \gamma_1^\eta C(t) \geq 0.$$

Now, using the generalized mean value theorem[3] , the solution of the proposed model 2.1 is non-negative.

It is established that the solution to the proposed model remains non-negative. Consequently, with non-negative initial conditions, the solution stays positive and is confined within the region Ω. This indicates that Ω is a positively invariant region, attracting all solutions of the model into it. □

3.2. Existence and uniqueness.

Theorem 3.2. *The proposed model (2.1) has a unique solution for non-negative initial conditions with $t \geq 0$. Additionally, all solutions are bounded.*

Proof. The right-hand side of the model (2.1) is continuous and bounded, expressed in polynomial form, allowing us to conclude that the Lipschitz condition is satisfied. Polynomials are continuous functions and are naturally Lipschitz continuous on any closed interval. Furthermore, the boundedness of the right-hand side guarantees a finite Lipschitz constant, which can be determined as the maximum absolute value of the polynomial's derivative within the relevant interval. Therefore, based on Lemma 2[4], the solution to the model (2.1), given suitable initial conditions, is not only guaranteed to exist but also to be unique and bounded for all $t \geq 0$. □

4. Equilibrium Points and Criminal Generation Number

To find equilibrium points, we equate the right-hand side of the proposed model 2.1 to zero. Hence, for the proposed model, there are five equilibrium points.

- **Criminal-free equilibrium point:** This equilibrium point occurs when the populations of criminals (C), Gang-1 (G_1), and Gang-2 (G_2) are all reduced to zero, essentially, the criminal activity has been completely eradicated i.e.

$$E_0 = \left(\frac{A^\eta}{\delta_0^\eta}, 0, 0, 0, 0\right).$$

  In practical terms, this represents an ideal scenario where crime and gang influence are completely eradicated from the system; however, achieving this outcome is unrealistic.

[3]**Generalized Mean Value Theorem** Consider $f(t) \in \mathbf{C}[0,b]$ and $\frac{d^\eta f(t)}{dt^\eta} \in \mathbf{C}[0,b]$ for $0 < \eta \leq 1$. Then, if

- $\frac{d^\eta f(t)}{dt^\eta} \geq 0 \ \forall\ t \in (0,b]$, then $f(t)$ is non-decreasing,
- $\frac{d^\eta f(t)}{dt^\eta} \leq 0 \ \forall\ t \in (0,b]$, then $f(t)$ is non-increasing [39].

[4]**Lemma 2** Consider the fractional differential equation with the Caputo derivative $\frac{d^\eta f(t)}{dt^\eta} = y(t,x), t > t_0$, $0 < \eta \leq 1$ and $y : [t_0, \infty] \times \Omega \to R^n, \Omega \in R^n$. Then there exists a unique solution on $[t_0, \infty] \times \Omega$ if $y(t,x)$ is continuous and bounded and assumes that it satisfies a Lipschitz condition with respect to the second variable x [41].

- **Criminal Generation Number:** The criminal generation number is defined as the estimated number of secondary cases generated by an individual who is actively engaged in criminal behavior within a population that is otherwise free from crime [20]. This metric indicates the potential for crime to spread: if, on average, each criminal produces fewer than one new criminal during their period of offending, the crime is unlikely to propagate. Conversely, if each criminal generates, on average, more than one new offender, crime is likely to diffuse throughout the population. To determine this criminal generation number, we utilize the next-generation matrix approach [20] within a criminal-free equilibrium framework. Let

$$D^\eta X = f(X) - v(X),$$

$$X = [N, C, G_1, G_2, P].$$

The terms for entering and exiting a class are represented by $v(X)$, while the terms for describing newly created criminals are represented by $f(X)$.
The criminal generation number is the greatest eigenvalue of FV^{-1}, where

$$F = \left(\frac{\partial f}{\partial X}\right)_{E_0}, \; V = \left(\frac{\partial v}{\partial X}\right)_{E_0}.$$

The Jacobian matrix of F at E_0

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_1^\eta A^\eta}{\delta_0^\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Jacobian matrix of V at E_0

$$V = \begin{bmatrix} \delta_0^\eta & \frac{\beta_1^\eta A^\eta}{\delta_0^\eta} & 0 & 0 & -\gamma_4^\eta \\ 0 & \gamma_1^\eta + \delta_0^\eta & 0 & 0 & 0 \\ 0 & 0 & \gamma_2^\eta + \delta_0^\eta & 0 & 0 \\ 0 & 0 & 0 & \gamma_3^\eta + \delta_0^\eta & 0 \\ 0 & -\gamma_1^\eta & -\gamma_2^\eta & -\gamma_3^\eta & \gamma_4^\eta + \delta_0^\eta \end{bmatrix}.$$

and

$$V^{-1} = \begin{bmatrix} \frac{1}{\delta_0^\eta} & \frac{A^\eta \beta_1^\eta \gamma_4^\eta + A^\eta \beta_1^\eta \delta_0^\eta + \gamma_1^\eta \gamma_4^\eta \delta_0^\eta}{\delta_0^{2\eta}(\delta_0^\eta + \gamma_4^\eta)(\delta_0^\eta + \gamma_1^\eta)} & \frac{\gamma_4^\eta \gamma_2^\eta}{\delta_0^\eta(\delta_0^\eta + \gamma_4^\eta)(\delta_0^\eta + \gamma_2^\eta)} & \frac{\gamma_4^\eta \gamma_3^\eta}{\delta_0^\eta(\delta_0^\eta + \gamma_4^\eta)(\delta_0^\eta + \gamma_3^\eta)} & \frac{\gamma_4^\eta}{\delta_0^\eta(\delta_0^\eta + \gamma_4^\eta)} \\ 0 & \frac{1}{\delta_0^\eta + \gamma_1^\eta} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\delta_0^\eta + \gamma_2^\eta} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\delta_0^\eta + \gamma_3^\eta} & 0 \\ 0 & \frac{\gamma_1^\eta}{(\delta_0^\eta + \gamma_4^\eta)(\delta_0^\eta + \gamma_1^\eta)} & \frac{\gamma_2^\eta}{(\delta_0^\eta + \gamma_4^\eta)(\delta_0^\eta + \gamma_2^\eta)} & \frac{\gamma_3^\eta}{(\delta_0^\eta + \gamma_4^\eta)(\delta_0^\eta + \gamma_3^\eta)} & \frac{1}{\delta_0^\eta + \gamma_4^\eta} \end{bmatrix}.$$

Therefore, the criminal generation number can be defined as follows:

$$C_g = \frac{\beta_1^\eta A^\eta}{\delta_0^\eta} \frac{1}{(\delta_0^\eta + \gamma_1^\eta)}. \tag{4.1}$$

- **Gangs-free equilibrium point:** This equilibrium point exists when the populations of Gang-1 (G_1) and Gang-2 (G_2) are both reduced to zero, i.e.

$$E_1 = (N^*, C^*, 0, 0, P^*),$$

where

$$N^* = \frac{\gamma_1^\eta + \delta_0^\eta}{\beta_1^\eta}, \; P^* = \frac{\gamma_1^\eta C^*}{\delta_0^\eta + \gamma_4^\eta}, \; C^* = \frac{(\delta_0^\eta + \gamma_4^\eta)(\gamma_1^\eta + \delta_0^\eta)(C_g - 1)}{\beta_1^\eta(\gamma_1^\eta + \delta_0^\eta + \gamma_4^\eta)}.$$

Gangs-free equilibrium point exists iff $C_g > 1$. In practical terms, this reflects a scenario where individual criminal activity continues, but organized crime or gang activity has been effectively dismantled. This equilibrium occurs only if the threshold parameter $C_g > 1$, meaning specific conditions must be met for gangs to vanish while individual criminals persist.

- **Gang-1 free equilibrium point:** This equilibrium point exists when gang-1 (G_1) populations become zero, i.e.

$$E_2 = (\tilde{N}, \tilde{C_2}, 0, \tilde{G_2}, \tilde{P}),$$

where

$$\tilde{C} = \frac{\delta_0^\eta + \gamma_3^\eta}{\beta_3^\eta}, \; \tilde{P} = \frac{\gamma_3^\eta \tilde{G_2} + \gamma_1^\eta \tilde{C}}{\delta_0^\eta + \gamma_4^\eta}, \; \tilde{N} = \frac{\gamma_1^\eta + \beta_3^\eta \tilde{G_2} + \delta_0^\eta}{\beta_1^\eta},$$

and

$$\tilde{G_2} = \frac{\beta_3^\eta(\delta_0^\eta + \gamma_4^\eta)(\gamma_1^\eta + \delta_0^\eta)(C_g - 1) - \beta_1^\eta(\delta_0^\eta + \gamma_3^\eta)(\gamma_1^\eta + \delta_0^\eta + \gamma_4^\eta)}{\beta_3^\eta(\beta_1^\eta(\delta_0^\eta + \gamma_4^\eta + \gamma_3^\eta) + \beta_3^\eta(\delta_0^\eta + \gamma_4^\eta))}.$$

Gang-1 free equilibrium point exists iff $C_g > \dfrac{\beta_1^\eta(\delta_0^\eta + \gamma_3^\eta)(\gamma_1^\eta + \delta_0^\eta + \gamma_4^\eta)}{\beta_3^\eta(\delta_0^\eta + \gamma_4^\eta)(\gamma_1^\eta + \delta_0^\eta)} + 1 = C_{g_1} > 0.$

From a practical perspective, this signifies a situation where efforts to eliminate Gang-1 have been successful, but Gang-2 continues its operations, alongside individual criminal activities. Physically, this equilibrium represents a system in which environmental and social factors allow Gang-2 to survive, while Gang-1 is unable to persist. This state is sustained only if specific conditions, involving the threshold parameter C_g and other model factors, are met, indicating that the environment still supports the existence of Gang-2 but not Gang-1.

- **Gang-2 free equilibrium point:** This equilibrium point exists when gang-2 (G_2) populations become zero, i.e.

$$E_3 = (\grave{N}, \grave{C}, \grave{G}_1, 0, \grave{P}).$$

Where

$$\grave{C} = \frac{\delta_0^\eta + \gamma_2^\eta}{\beta_2^\eta}, \; \grave{P} = \frac{\gamma_1^\eta \grave{C} + \gamma_2^\eta \grave{G}_1}{\delta_0^\eta + \gamma_4^\eta}, \; \grave{N} = \frac{\gamma_1^\eta + \beta_2^\eta \grave{G}_1 + \delta_0^\eta}{\beta_1^\eta},$$

and

$$\grave{G}_1 = \frac{\beta_2^\eta(\delta_0^\eta + \gamma_4^\eta)(C_g - 1)(\gamma_1^\eta + \delta_0^\eta) - \beta_1^\eta(\delta_0^\eta + \gamma_2^\eta)(\gamma_1^\eta + \delta_0^\eta + \gamma_4^\eta)}{\beta_2^\eta(\beta_1^\eta(\delta_0^\eta + \gamma_4^\eta + \gamma_2^\eta) + \beta_2^\eta(\delta_0^\eta + \gamma_4^\eta))}.$$

Gang-2 free equilibrium point exists iff $C_g > \dfrac{\beta_1^\eta(\delta_0^\eta + \gamma_2^\eta)(\gamma_1^\eta + \delta_0^\eta + \gamma_4^\eta)}{\beta_2^\eta(\delta_0^\eta + \gamma_4^\eta)(\gamma_1^\eta + \delta_0^\eta)} + 1 = C_{g_2} > 0.$

Practically, this reflects a situation where efforts have successfully dismantled Gang-2, yet Gang-1 and individual criminal behavior persist. Physically, this implies that the environment supports the survival of Gang-1 but not Gang-2, indicating that conditions, such as resources or social factors, favor one gang's presence over the other. This equilibrium is sustained if a specific threshold condition involving C_g is satisfied, enabling Gang-1 persistence while Gang-2 vanishes.

- **Endemic equilibrium point:** This equilibrium point exists when both criminal gangs are non-zero, i.e.

$$E_4 = (N^e, C^e, G_1^e, G_2^e, P^e).$$

Where

$$P^e = \frac{\gamma_1^\eta C^e + \gamma_2^\eta G_1^e + \gamma_3^\eta G_2^e}{\delta_0^\eta + \gamma_4^\eta}, \; S^e = \frac{\gamma_1^\eta + \beta_2^\eta G_1^e + \beta_3^\eta G_2^e + \delta_0^\eta}{\beta_1^\eta},$$

Let $\alpha_1^\eta > \alpha_2^\eta$. Then,

$$G_1^e = \frac{(\delta_0^\eta + \gamma_3^\eta) - \beta_3^\eta C^e}{\alpha_1^\eta - \alpha_2^\eta}, \text{ exists if } C^e < \frac{\delta_0^\eta + \gamma_3^\eta}{\beta_3^\eta},$$

$$G_2^e = \frac{(\delta_0^\eta + \gamma_2^\eta) - \beta_2^\eta C^e}{\alpha_2^\eta - \alpha_1^\eta}, \text{ exists if } C^e > \frac{\delta_0^\eta + \gamma_2^\eta}{\beta_2^\eta},$$

$$C^e = \frac{\gamma_4^\eta \beta_1^\eta (\gamma_2^\eta G_1^* + \gamma_3^\eta G_2^e) - \delta_0^\eta(\delta_0^\eta + \gamma_4^\eta)(\beta_2^\eta G_1^e + \beta_3^\eta G_2^e) + (\delta_0^\eta + \gamma_4^\eta)\delta_0^\eta(\gamma_1^\eta + \delta_0^\eta(\gamma_1^\eta + \delta_0^\eta)(C_g - 1)}{\beta_1^\eta((\delta_0^\eta + \gamma_4^\eta)(\beta_2^\eta G_1^e + \beta_3^\eta G_2^e + \delta_0^\eta) + \delta_0^\eta \gamma_1^\eta)}.$$

For $\alpha_1^\eta > \alpha_2^\eta$, endemic equilibrium point exists if $\frac{\delta_0^\eta + \gamma_2^\eta}{\beta_2^\eta} < C^e < \frac{\delta_0^\eta + \gamma_3^\eta}{\beta_3^\eta}$ and $C_g > 1$ and for $\alpha_1^\eta < \alpha_2^\eta$, endemic equilibrium point exists if $\frac{\delta_0^\eta + \gamma_2^\eta}{\beta_2^\eta} > C^e > \frac{\delta_0^\eta + \gamma_3^\eta}{\beta_3^\eta}$ and $C_g > 1$.

Practically, it signifies the ongoing presence of both gang activities and individual criminal behavior within the population.

5. Stability Analysis

The stability of these equilibrium points illustrates the dynamic interplay between various forms of criminal behavior and the effectiveness of interventions. A shift from one equilibrium to another indicates changes in societal conditions, law enforcement efficacy, and the underlying factors influencing crime and gang activity. Understanding these points' stability provides insight into designing targeted prevention and intervention strategies to mitigate crime effectively.

Theorem 5.1. *In the system described in (2.1), the criminal-free equilibrium point $E_0 = (N, 0, 0, 0, 0)$ is globally asymptotically stable if $C_g < 1$. Conversely, it becomes unstable when $C_g > 1$.*

Proof. The Jacobian matrix evaluated at E_0 is

$$J_{E_0} = \begin{bmatrix} -\delta_0^\eta & \frac{-\beta_1^\eta A^\eta}{\delta_0^\eta} & 0 & 0 & \gamma_4^\eta \\ 0 & \frac{\beta_1^\eta A^\eta}{\delta_0^\eta} - \gamma_1^\eta - \delta_0^\eta & 0 & 0 & 0 \\ 0 & 0 & -\delta_0^\eta - \gamma_2^\eta & 0 & 0 \\ 0 & 0 & 0 & -\delta_0^\eta - \gamma_3^\eta & 0 \\ 0 & \gamma_1^\eta & \gamma_2^\eta & \gamma_3^\eta & -\delta_0^\eta - \gamma_4^\eta \end{bmatrix}.$$

It is evident that all eigenvalues occur in the second quadrant, especially $-\delta_0^\eta, -\delta_0^\eta - \gamma_2^\eta, -\delta_0^\eta - \gamma_3^\eta, -\delta_0^\eta - \gamma_4^\eta$, and $\frac{\beta_1^\eta A^\eta}{\delta_0^\eta} - \gamma_1^\eta - \delta_0^\eta$ if $C_g < 1$. By using Matignon-criteria [3], E_0 is stable. □

Theorem 5.2. *The gang-free equilibrium point $E_1 = (N^*, C^*, 0, 0, P^*)$ is locally asymptotically stable when $C_g > 1$; otherwise, it becomes unstable.*

Proof. The Jacobian matrix evaluated at E_1 is

$$J_{E_1} = \begin{bmatrix} -\beta_1^\eta C^* - \delta_0^\eta & -\beta_1^\eta N^* & 0 & 0 & \gamma_4^\eta \\ \beta_1^\eta C^* & \beta_1^\eta N^* - \gamma_1^\eta - \delta_0^\eta & -\beta_2^\eta C^* & -\beta_3^\eta C^* & 0 \\ 0 & 0 & \beta_3^\eta C^* - \delta_0^\eta - \gamma_2^\eta & 0 & 0 \\ 0 & 0 & 0 & \beta_3^\eta C^* - \delta_0^\eta - \gamma_3^\eta & 0 \\ 0 & \gamma_1^\eta & \gamma_2^\eta & \gamma_3^\eta & -\delta_0^\eta - \gamma_4^\eta \end{bmatrix}.$$

In this case, the two eigenvalues are $\beta_3^\eta C^* - \delta_0^\eta - \gamma_2^\eta$, $\beta_3^\eta C^* - \delta_0^\eta - \gamma_3^\eta$. Now, for the remaining eigenvalues:

$$J_{E_1}^* = \begin{bmatrix} -\beta_1^\eta C^* - \delta_0^\eta & -\beta_1^\eta N^* & \gamma_4^\eta \\ \beta_1^\eta C^* & \beta_1^\eta N^* - \gamma_1^\eta - \delta_0^\eta & 0 \\ 0 & \gamma_1^\eta & -\delta_0^\eta - \gamma_4^\eta \end{bmatrix}.$$

The equation of characteristic for $J_{E_1}^*$ is

$$\Lambda^3 + a_1\Lambda^2 + a_2\Lambda + a_3 = 0, \tag{5.1}$$

where
a_1 = -trace($J_{E_1}^*$), a_2 = sum of co-factor of diagonal elements of $J_{E_1}^*$, a_3 = -determinant($J_{E_1}^*$).

For the Routh-Hurwitz conditions to be satisfied and the equilibrium point E_1 to be locally asymptotically stable, the Hurwitz determinants, H_i, where $i = 1, 2, 3$, must all be positive [2]. Therefore, the Gangs-free equilibrium point is stable iff $a_1 > 0,\ a_3 > 0$, and $a_1a_2 > a_3$. □

Theorem 5.3. *In the system (2.1), the Gangs-1 free equilibrium point $E_2 = (\tilde{N}, \tilde{C}, 0, \tilde{G}_2, \tilde{P})$ is locally asymptotically stable when $C_g > C_{g1}$; otherwise, it becomes unstable.*

Proof. The Jacobian matrix evaluated at E_2 is

$$J_{E_2} = \begin{bmatrix} -\beta_1^\eta\tilde{C} - \delta_0^\eta & -\beta_1^\eta\tilde{N} & 0 & 0 & \gamma_4^\eta \\ \beta_1^\eta\tilde{C} & \beta_1^\eta\tilde{N} - \gamma_1^\eta - \delta_0^\eta - \beta_3^\eta\tilde{G}_2 & -\beta_2^\eta\tilde{C} & -\beta_3^\eta\tilde{C} & 0 \\ 0 & 0 & \begin{matrix}\beta_3^\eta\tilde{C} - \delta_0^\eta - \gamma_2^\eta\tilde{G}_2 \\ +(-\alpha_1^\eta + \alpha_2^\eta)\tilde{G}_2\end{matrix} & 0 & 0 \\ 0 & \beta_3^\eta\tilde{G}_2 & \alpha_1^\eta\tilde{G}_2 - \alpha_2^\eta\tilde{G}_2 & \beta_3^\eta\tilde{C} - \delta_0^\eta - \gamma_3^\eta & 0 \\ 0 & \gamma_1^\eta & \gamma_2^\eta & \gamma_3^\eta & -\delta_0^\eta - \gamma_4^\eta \end{bmatrix}.$$

The equation of characteristic for J_{E_2} is

$$\Lambda^5 + a_1\Lambda^4 + a_2\Lambda^3 + a_3\Lambda^2 + a_4\Lambda + a_5 = 0. \tag{5.2}$$

To satisfy the Routh-Hurwitz criteria and ensure that the equilibrium point E_2 is locally asymptotically stable, all Hurwitz determinants H_i (for $i = 1, 2, \ldots, 5$) must be positive [2]. Therefore, the endemic equilibrium point is stable if the following conditions are satisfied: $a_i > 0$ for $i = 1, 2, 3, 4, 5$; $a_1a_2a_3 > a_3^2 + a_1^2a_4$; and $(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_4(a_1a_2 - a_3)^2$. □

Theorem 5.4. *In the system (2.1), the Gangs-2 free equilibrium point $E_3 = (\grave{N}, \grave{C}, \grave{G}_1, 0, \grave{P})$ is locally asymptotically stable when $C_g > C_{g2}$; otherwise, it becomes unstable.*

Proof. The Jacobian matrix evaluated at E_3 is

$$J_{E_3} = \begin{bmatrix} -\beta_1^\eta\grave{C} - \delta_0^\eta & -\beta_1^\eta\grave{N} & 0 & 0 & \gamma_4^\eta \\ \beta_1^\eta\grave{C} & \beta_1^\eta\grave{N} - \gamma_1^\eta - \delta_0^\eta - \beta_2^\eta\grave{G}_1 & -\beta_2^\eta\grave{C} & -\beta_3^\eta\grave{C} & 0 \\ 0 & \beta_2^\eta\grave{G}_1 & \beta_3^\eta\grave{C} - \delta_0^\eta - \gamma_2^\eta & -\alpha_1^\eta\grave{G}_1 + \alpha_2^\eta\grave{G}_1 & 0 \\ 0 & 0 & 0 & \begin{matrix}\beta_3^\eta\grave{C} - \delta_0^\eta - \gamma_3^\eta \\ +(\alpha_1^\eta - \alpha_2^\eta)\grave{G}_1\end{matrix} & 0 \\ 0 & \gamma_1^\eta & \gamma_2^\eta & \gamma_3^\eta & -\delta_0^\eta - \gamma_4^\eta \end{bmatrix}.$$

The equation of characteristic for J_{E_3} is

$$\Lambda^5 + a_1\Lambda^4 + a_2\Lambda^3 + a_3\Lambda^2 + a_4\Lambda + a_5 = 0. \tag{5.3}$$

To satisfy the Routh-Hurwitz criteria and ensure that the equilibrium point E_3 is locally asymptotically stable, all Hurwitz determinants H_i (for $i = 1, 2, \ldots, 5$) must be positive [2]. Therefore, the endemic equilibrium point is stable if the following conditions are satisfied: $a_i > 0$ for $i = 1, 2, 3, 4, 5$; $a_1a_2a_3 > a_3^2 + a_1^2a_4$; and $(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_4(a_1a_2 - a_3)^2$. □

Theorem 5.5. *In the system (2.1), the endemic equilibrium point $E_4 = (N^e, C^e, G_1^e, G_2^e, P^e)$ is locally asymptotically stable when $C_g > 1$; otherwise, it becomes unstable.*

Proof. The Jacobian matrix evaluated at E_4 is

$$J_{E_4} = \begin{bmatrix} -\beta_1^\eta C^e - \delta_0^\eta & -\beta_1^\eta N^e & 0 & 0 & \gamma_4^\eta \\ \beta_1^\eta C^e & \beta_1^\eta N^e - \gamma_1^\eta - \delta_0^\eta & -\beta_2^\eta C^e & -\beta_3^\eta C^e & 0 \\ & -\beta_2^\eta G_1^e - \beta_3^\eta G_2^e & & & \\ 0 & \beta_2^\eta G_1^e & \beta_3^\eta C^e - \delta_0^\eta - \gamma_2^\eta & (-\alpha_1^\eta + \alpha_2^\eta) G_1^e & 0 \\ & & +(-\alpha_1^\eta + \alpha_2^\eta) G_2^e & & \\ 0 & \beta_3^\eta G_2^e & (\alpha_1^\eta - \alpha_2^\eta) G_2^e & \beta_3^\eta C^e - \delta_0^\eta - \gamma_3^\eta & 0 \\ & & & +(\alpha_1^\eta - \alpha_2^\eta) G_1^e & \\ 0 & \gamma_1^\eta & \gamma_2^\eta & \gamma_3^\eta & -\delta_0^\eta - \gamma_4^\eta \end{bmatrix}.$$

The equation of characteristic for J_{E_4} is

$$\Lambda^5 + a_1\Lambda^4 + a_2\Lambda^3 + a_3\Lambda^2 + a_4\Lambda + a_5 = 0. \tag{5.4}$$

To satisfy the Routh-Hurwitz criteria and ensure that the equilibrium point E_4 is locally asymptotically stable, all Hurwitz determinants H_i (for $i = 1, 2, \ldots, 5$) must be positive [2]. Therefore, the endemic equilibrium point is stable if the following conditions are satisfied: $a_i > 0$ for $i = 1, 2, 3, 4, 5$; $a_1a_2a_3 > a_3^2 + a_1^2a_4$; and $(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_4(a_1a_2 - a_3)^2$. □

6. Sensitivity Evaluation

This study investigates the effects of various parameters in the proposed model (2.1) on individual behavior, with a particular focus on the threshold parameter, C_g, which significantly influences criminal behavior. The partial derivatives of C_g, known as elasticity, are calculated for each parameter to assess their impact. Positive elasticity values indicate that an increase in the parameter leads to a rise in C_g, while negative values suggest a decrease in C_g as the parameter increases. The findings of this sensitivity analysis for the threshold parameter C_g are summarized in Table 1.

$$\begin{aligned}
\zeta_q &= \frac{q}{C_g} \cdot \frac{\partial C_g}{\partial q}, \\
\zeta_A &= \frac{A}{C_g} \cdot \frac{\partial C_g}{\partial A} = \frac{A}{C_g} \cdot \frac{\beta_1^\eta \eta A^{\eta-1}}{\delta_0^\eta(\delta_0^\eta + \gamma_1^\eta)} = \eta, \\
\zeta_{\beta_1} &= \frac{\beta_1}{C_g} \cdot \frac{\partial C_g}{\partial \beta_1} = \frac{\beta_1}{C_g} \cdot \frac{\beta_1^{\eta-1} \eta A^\eta}{\delta_0^\eta(\delta_0^\eta + \gamma_1^\eta)} = \eta, \\
\zeta_{\gamma_1} &= \frac{\gamma_1}{C_g} \cdot \frac{\partial C_g}{\partial \gamma_1} = -\frac{\gamma_1}{C_g} \cdot \frac{\beta_1^\eta A^\eta \eta \gamma_1^{\eta-1}}{\delta_0^\eta(\delta_0^\eta + \gamma_1^\eta)^2} = -\frac{\eta\gamma_1^\eta}{(\delta_0^\eta + \gamma_1^\eta)}, \\
\zeta_{\delta_0} &= \frac{\delta_0}{C_g} \cdot \frac{\partial C_g}{\partial \delta_0} = -\frac{\delta_0}{C_g} \cdot \frac{\beta_1^\eta A^\eta (2\delta_0^\eta + \gamma_1^\eta)}{\delta_0^{\eta+1}(\delta_0^\eta + \gamma_1^\eta)^2} = -\frac{2\delta^\eta + \gamma_1^\eta}{(\delta_0^\eta + \gamma_1^\eta)}, \\
\zeta_\eta &= \frac{\eta}{C_g} \cdot \frac{\partial C_g}{\partial \eta} = \frac{\eta(\delta_0^\eta log(A) + \gamma_1^\eta log(A) + \delta_0^\eta log(\beta_1) - 2\delta_0^\eta log(\delta_0) + \gamma_1^\eta log(\beta_1) - \gamma_1^\eta log(\delta_0) - \gamma_1^\eta log(\gamma_1))}{(\delta_0^\eta + \gamma_1^\eta)}.
\end{aligned}$$

Table 1. Sensitivity analysis results for the threshold parameter C_g

Metric	$\frac{\partial C_g}{\partial A}$	$\frac{\partial C_g}{\partial \beta_1}$	$\frac{\partial C_g}{\partial \gamma_1}$	$\frac{\partial C_g}{\partial \delta_0}$	$\frac{\partial C_g}{\partial \eta}$
Sensitivity Index Sign	Positive	Positive	Negative	Negative	Positive

We aim to show that $\zeta_\eta > 0$.

Step 1: Positivity of the Denominator

$$\delta_0^\eta + \gamma_1^\eta > 0, \quad \text{for} \quad \delta_0 > 0, \gamma_1 > 0, \eta > 0.$$

Step 2: Simplify the Numerator

$$\delta_0^\eta(\log A + \log \beta_1 - 2\log \delta_0) + \gamma_1^\eta(\log A + \log \beta_1 - \log \delta_0 - \log \gamma_1).$$

Let:

$$L_1 = \log A + \log \beta_1 - 2\log \delta_0,$$

$$L_2 = \log A + \log \beta_1 - \log \delta_0 - \log \gamma_1.$$

Then the expression becomes:

$$\zeta_\eta = \frac{\eta\,(\delta_0^\eta L_1 + \gamma_1^\eta L_2)}{\delta_0^\eta + \gamma_1^\eta}.$$

Since the denominator is always positive, we need the numerator to be positive:

$$\delta_0^\eta L_1 + \gamma_1^\eta L_2 > 0.$$

This will hold if both $L_1 > 0$ and $L_2 > 0$, i.e.,

$$\log A + \log \beta_1 > 2\log \delta_0 \quad \Rightarrow \quad A \cdot \beta_1 > \delta_0^2,$$

$$\log A + \log \beta_1 > \log \delta_0 + \log \gamma_1 \quad \Rightarrow \quad A \cdot \beta_1 > \delta_0 \cdot \gamma_1,$$

Thus,

$$\zeta_\eta > 0 \quad \text{if} \quad A \cdot \beta_1 > \max(\delta_0^2, \delta_0 \cdot \gamma_1).$$

Since the parameter values are chosen from previously published articles and they satisfy the required conditions, it follows that $\zeta_\eta > 0$.
Moreover, sensitivity analysis reveals that crime prevention is the most effective strategy for curbing the spread of criminal behavior. It underscores the significance of reducing interactions between vulnerable individuals and criminals to prevent crime transmission. These results emphasize the role of advanced mathematical techniques in developing targeted prevention and intervention measures, ultimately leading to more effective strategies for reducing the negative effects of crime.

7. Numerical simulation

Numerical simulations validate the theoretical results and underscore the significance of the threshold value C_g. These simulations reveal the interactions between various population groups and the overall dynamics of crime. The predictor-corrector method from the Adams-Bashforth-Moulton scheme, implemented in MATLAB [19], is utilized for the simulations. Details of the variables used in the analysis are provided in Table 2 [1, 10, 11, 44]. Using the suggested model, we examine the dynamics of various populations $N(t)$, $C(t)$, $G_1(t)$, $G_2(t)$, and $P(t)$. The trajectories of these populations, regardless of the order, validate the stability of the crime transmission model. Figure 2 presents a numerical simulation where the criminal population, $C(t)$, initially rises and then begins to decline, independent of the derivative order. The population of gang-1 decreases, while gang-2 population increases, given the assumption that $\alpha_1 > \alpha_2$. Meanwhile, Figure 3 illustrates the effect of the derivative order on various population groups. As depicted in Figure 3, all populations decrease as the derivative order decreases. The populations likely evolve according to

TABLE 2. Description of variables and parameters

Parameter	Description	Value	Source
A	Rate of recruitment	0.127	[1]
β_1	Rate at which individuals move from non-criminals to criminals.	0.12	[1]
β_2	Rate from which criminals become gang-1 members	0.001	[1]
β_3	Rate from which criminals become gang-2 members	0.0001	[44]
γ_1	Rate from which criminals become Prisoners	0.0001	[44]
γ_2	Rate from which gang-1 members become Prisoners	0.0001	[1]
γ_3	Rate from which gang-2 members become Prisoners	0.0001	[1]
γ_4	Rate of prisoners become non-criminals	0.5	[44]
α_1	Rate of gang-1 members become gang-2	0.00013	Assumed
α_2	Rate of gang-2 members become gang-1	0.000013	Assumed
δ_0	Natural death rate	0.012	[1]

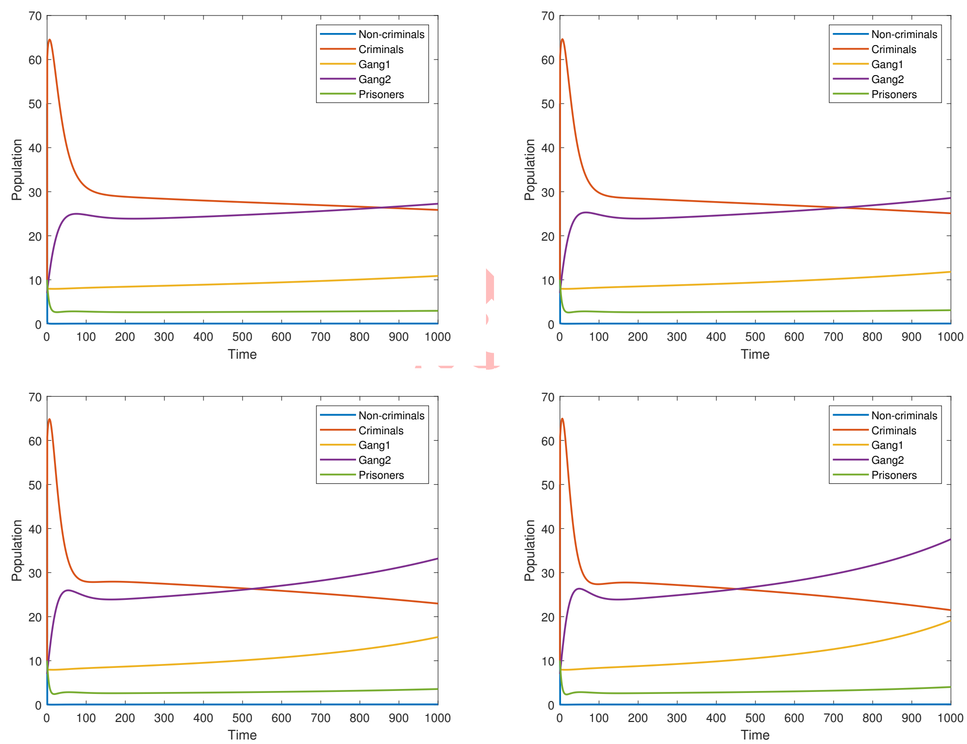

FIGURE 2. Variations in the non-criminal population (N), criminals (C), gang-1 (G_1), gang-2 (G_2), and prisoners (P) corresponding to various values of the order of derivative (η): 0.9, 0.925, 0.95, and 1.

a mathematical model representing the interactions between non-criminals, criminals, gang members, and prisoners. The fractional order η could be affecting the rate of change in the population dynamics. In Figure 3:

(1) Non-criminals seems to start at a higher value but gradually decreases or stabilizes over time. Non-criminal generally follows a downward trend, represented by curves that taper off, indicating a decrease or stabilization in the non-criminal population.

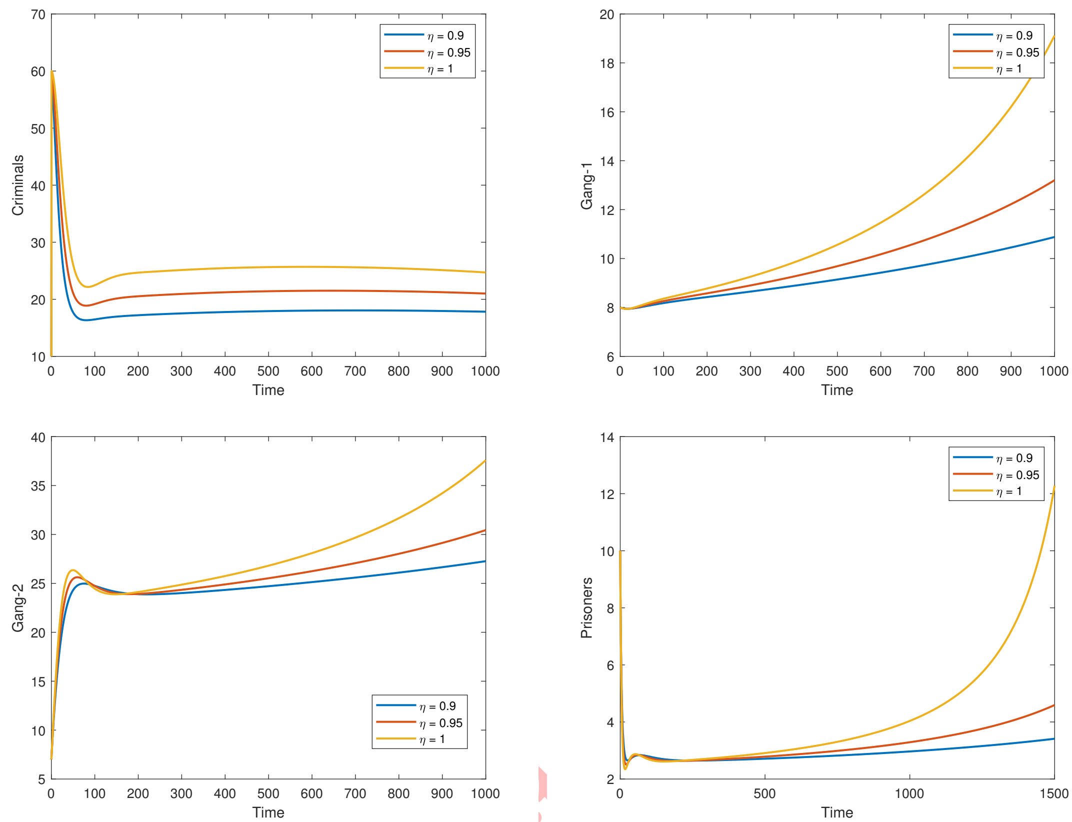

FIGURE 3. Variations in criminals (C), gang-1 (G_1), and gang-2 (G_2) for different orders of the derivative (η) indicate that populations decrease as the order of the derivative decreases.

(2) The criminal population spikes early on but decreases as time progresses, possibly indicating early growth in criminal activity, followed by a decline due to interventions like arrests or rehabilitation. Criminals exhibits a steep peak initially, followed by a decay toward a stable value, indicating that criminal activity surges and then decreases as external forces (like law enforcement) become more effective.

(3) Gang-1 and Gang-2 exhibit a similar pattern with lower initial values and varying growth rates, depending on η. The distinction between these two gangs could represent different levels of criminal organization or influence. The curves for both gangs show gradual increases, with higher values of η potentially leading to a slower rate of convergence toward stability.

(4) The prisoner population shows an increase early on, reaching a steady value later, possibly reflecting the balance between arrests and rehabilitation. The prisoner population tends to increase and then level off, representing the arrest and incarceration rates balancing out over time.

As η increases (from 0.9 to 1), the model becomes more deterministic (closer to an integer-order model), leading to more rapid stabilization of the populations. For smaller η values (fractional-order cases), the system dynamics evolve more gradually.

Figure 3 shows that as the order of the derivative (η) increases from 0.9 to 1, the populations of criminals, gang-1 members, gang-2 members, and prisoners grow more rapidly over time. For $\eta = 1$, the system exhibits sharper increases in these populations, indicating that higher η values correspond to more dynamic or aggressive growth. In contrast, lower η values lead to more gradual changes, reflecting slower progression in the system's behavior. Thus, increasing η accelerates the evolution of crime and incarceration in the modeled system.

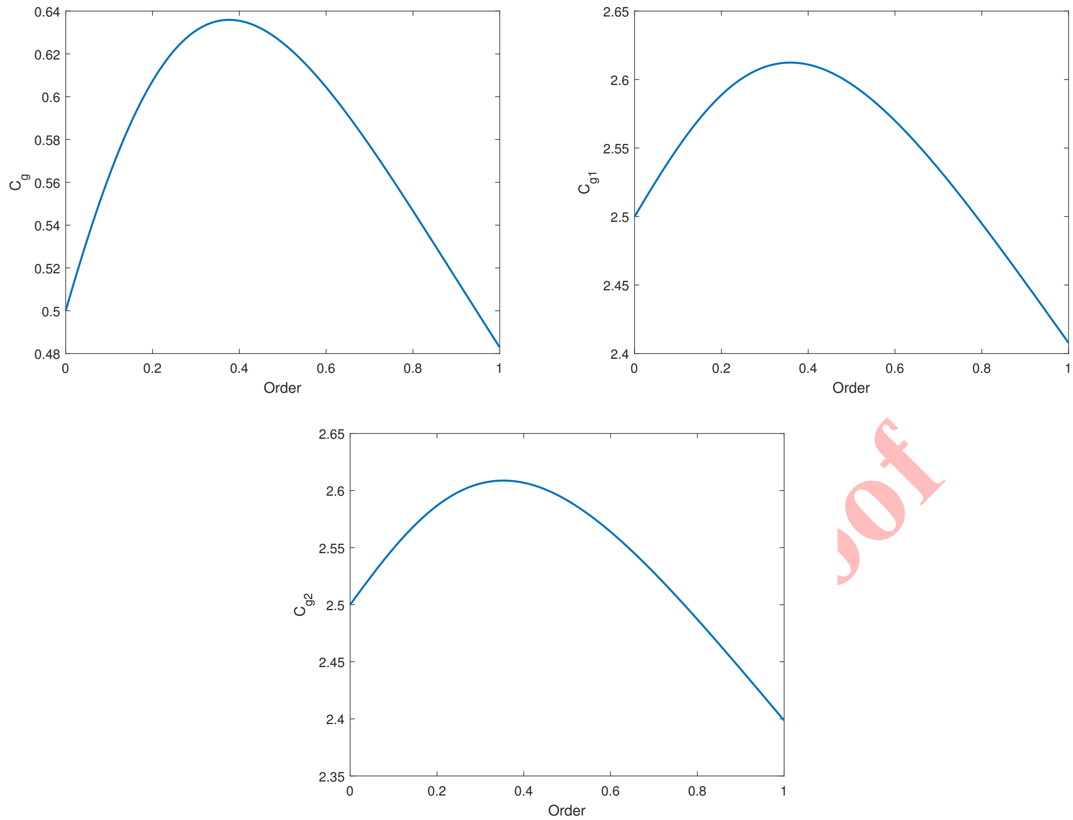

FIGURE 4. Relation of C_g, C_{g1}, C_{g2} with the order of derivative η.

Figure 4 illustrates the relationship between the order of the derivative (η) and the criminal generation numbers (C_g, C_{g1}, and C_{g2}). In this context, C_g represents the overall criminal generation, while C_{g1} and C_{g2} correspond to gang-1 and gang-2 affiliations, respectively. The figure demonstrates that as the order of the derivative increases from 0 to 0.5, the values of C_g, C_{g1}, and C_{g2} rise, reaching their maximum before declining as η approaches 1. This behavior indicates that memory effects, captured by the derivative order, significantly influence both the formation of criminal groups and gang affiliations. Initially, greater memory effects result in more variability and heightened criminal activity across all groups, but this influence diminishes as the memory factor intensifies further.

8. Conclusion

This study introduces a fractional-order model for crime transmission that incorporates the influences of gang rivalry and competition among different gangs, while considering important social interactions and behavioral changes within the community. It shows that when the criminal reproduction number, C_g, is less than one, criminal activity ultimately declines. In contrast, crime persists when C_g exceeds one. Theoretical results were supported by numerical simulations. Furthermore, sensitivity analysis highlights that crime prevention is the most effective strategy for reducing crime transmission, emphasizing the importance of minimizing contact between vulnerable individuals and criminals to prevent criminal behavior from spreading. Some key findings of this study are:

- For the first time, a nonlinear fractional-order model for criminal gang behaviour analyzes is proposed.
- Memory effects, indicated by the derivative order (η), significantly influence criminal generation and gang affiliations.

- The most effective strategy for reducing the crime transmission is through crime prevention and law enforcement, focusing on minimizing interactions between vulnerable individuals and criminals to prevent the transmission of criminal behavior.

These findings highlight the value of advanced mathematical approaches in designing targeted prevention and intervention measures, ultimately contributing to more effective strategies for mitigating the adverse impacts of crime.

Future Scope

This model can be expanded to higher dimensions in the future by including more variables. Various algorithms can be utilized to identify the optimal fractional order of derivatives. Additionally, simulation results can be verified with real-world data across different regions and age demographics.

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Ethics declarations

We (The authors) undertake to adopt ethical principles and practices and abide by good ethical conduct in scientific research in accordance with the DEDS Ethical Statement for Ethics in Publication.

Conflict of interest. The authors declare that there is no any competing of interest regarding the publication of this research work.

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