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Isolated toughness and fractional (k, m)-covered graph

Wei Gao[1,*], **Carlo Cattani**[2], **Armando Ciancio**[3], **and Haci Mehmet Baskonus**[4]

[1]School of Mathematics, Hohai University, Nanjing 211100, China.

[2]Engineering School (DEIM), University of Tuscia, Viterbo 01100, Italy.

[3]Universitá degli Studi di Messina, Dipartimento di Scienze Biomediche, Odontoiatriche e delle Immagini Morfologiche e Funzionali, Messina, 98122, Italy.

[4]Department of Mathematics and Science Education, Faculty of Education, Harran University, Sanliurfa 63159, Turkey.

Abstract

The fractional k-factor is a spanning subgraph determined by the support set of the fractional indicator function, which requires that the sum of the fractional indicator function values of the edges incident with each vertex is k, where k is a positive integer. A graph G is fractional (k, m)-covered if for any $H \subseteq E(G)$ with $|H| = m$, there is a fractional factor with fractional indicator function h satisfying $h(e) = 1$ for any $e \in H$. Isolated toughness is a pivotal indicator in network security and a prominent performance indicator that engineers considering during the network design phase. In this work, we present the isolated toughness and its variant bounds for fractional (k, m)-covered graphs, and the sharpness of given bounds are delineated by counterexamples.

Keywords. Fractional factor, Isolated toughness, Fractional (k, m)-covered graph.

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1. INTRODUCTION

As a paradigm of transformation mechanism from integers to real numbers, the fractional theory has penetrated into various branches of mathematics, such as fractional derivatives, fractional equations, etc (see Ata and Onur Kıymaz [1] and Öztürk et al. [10] for instance). It extends classical integer-order operators to continuous or arbitrary orders, providing a more refined tool for modeling systems with memory and hereditary properties. This theory has found significant applications in physics, engineering, and biological sciences due to its ability to describe complex non-local and chaotic phenomena. Furthermore, fractional calculus enriches mathematical analysis by introducing new function spaces and generalized integral-differential operators. Ongoing research continues to explore its theoretical foundations and practical implementations across interdisciplinary fields.

As a rational approach to graph theory, fractional graph theory is the application of fractional ideas in graph theory, which converts integer-valued graph variables into fractional-valued versions and then obtains the corresponding relaxed forms. This work focuses on the topic of fractional graph factor which is closely related to data transform and network flow. The subject, notations, problems and contributions are elaborated in the subsequent subsections successively.

1.1. Fractional covered graph.

This article only discusses simple graphs. Let G be a graph, $\delta(G)$ be the minimum degree, and $d_G(x)$ be the degree of vertex x. Denote $i(G)$ by the number of isolated vertices (i.e., vertices with degree zero) in G. For $V_1, V_2 \subseteq V(G)$ and $V_1 \cap V_2 = \emptyset$, let $E_G(V_1, V_2)$ be the set of edges connecting V_1 and V_2 and $e_G(V_1, V_2) = |E_G(V_1, V_2)|$.

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* Corresponding author. Email: hmbaskonus@harran.edu.tr.

Let $h : E(G) \to [0,1]$ be the fractional indicator function, and $k \in \mathbb{N}$. If there exists a fractional indicator function h that satisfies $\sum_{x' \in N(x)} h(xx') = k$ for any $x \in V(G)$, then we say G admits a fractional k-factor.

If for any $H \subseteq E(G)$ with $|H| = m$ ($m \in \mathbb{N}$), a graph G admits a fractional k-factor such that $h(e) = 1$ for every $e \in H$, then we say G is a fractional (k,m)-covered graph. Especially, fractional k-covered graph is a particular case for $m = 1$. The fractional covered graph is the opposite of the fractional deleted graph, the latter forces the values of the fractional indicator function of a given number of edges to be 0, while the former is just the opposite which forces the values of the fractional indicator function of the fixed number of edges to be 1. In network applications, the fractional deleted graph describes the ability of the remaining subgraphs to transmit data after a fixed number of channels are damaged, while the fractional covered graph characterizes the feasibility of transmission when there are channels that must be passed during data flow transmission.

From the perspective of information theory, each piece of data (or signal) has its own vibration frequency, while each channel (corresponding to each edge in the graph) possesses its own transmission frequency band. Generally, signals or data require modulation and demodulation processes also known as frequency shifting to align the frequency of the transmitted content with the channel's inherent frequency. The underlying implication of fractional covering graphs is that each channel has a distinct transmission frequency, resulting in varying energy consumption for frequency adaptation. However, it has been observed that certain channels in the network naturally exhibit compatibility between their inherent frequency and the frequency of the data (or signal). In such cases, algorithms naturally prioritize these channels during path selection, thereby minimizing the total energy consumption required for frequency shifting across the network. From a graph-theoretic standpoint, these channels with inherently compatible frequencies correspond to the edges that need to be covered.

The fractional covered graph is a salient study context recently. Zhou et al. [14], [12] and [13] proposed degree, binding number and independence number conditions for a graph to be fractional (a,b,k)-critical covered (the variable k refers to the number of removed vertices), respectively. However, the foregoing contributions only discussed settings covering only one edge.

To our delight, there has been an important breakthrough recently on the fractional covered graph by Gao et al. [3] which covered m edges. In what follows, $e_H(T,S) = |\{e = xy \mid x \in S, y \in T, e \in H\}|$.

Lemma 1.1. (Gao et al. [3]) *Let G be a graph, $k \in \mathbb{N}$. Let $m \in \mathbb{N} \cup \{0\}$ and $m \le k$. Then G is a fractional (k,m)-covered graph if and only if for any $S \subseteq V(G)$,*

$$k|S| - k|T| + d_{G-S}(T) \ge \max_{H \subseteq E(G), |H| = m} \{\sum_{x \in S} d_H(x) - e_H(T,S) + \Theta(S,T)\}, \tag{1.1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le k-1\}$ and

$$\Theta(S,T) = \sum_{1 \le d_{G\setminus H-S}(x) - k + d_H(x) \le e_H(x,S) - 1,\, e_H(x,S) \ge 2} \{d_{G\setminus H-S}(x) - k + d_H(x)\}.$$

It is imperative to instantiate the following points: (1) The prerequisites $m \le k$ is necessary in Lemma 1.1. Suppose $m > k$ and H is selected as m edges incident with the same vertex x. Then the fractional degree of vertex x is larger than k if it is mandatory to cover H, which contradicts to the definition of fractional k-factor. (2) The item $\Theta(S,T)$ is unique to the fractional covered graph (compare the other settings of fractional graphs) which is limited to $e_H(x,S) \ge 2$. Hence, $\Theta(S,T) = 0$ if $m = 1$ (see Li et al. [6]).

1.2. **Isolated toughness and its variant.** As a prominent graph-based network parameter which delineates the vulnerability of the network, isolated toughness is defined by [11]

$$I(G) = \begin{cases} \min\left\{\frac{|S|}{i(G-S)} \middle| S \subset V(G), i(G-S) \ge 2\right\}, & \text{if } G \text{ is not a complete graph,} \\ +\infty, & \text{otherwise.} \end{cases}$$

The isolated toughness variant, also a salient graph-based network parameter, is stated by [9]

$$I'(G) = \begin{cases} \min\left\{\frac{|S|}{i(G-S)-1} \middle| S \subset V(G), i(G-S) \ge 2\right\}, & \text{if } G \text{ is not a complete graph,} \\ +\infty, & \text{Otherwise.} \end{cases}$$

Here, the isolated toughness and its variant of a complete graph are forced to assign to infinity since there is no subset S satisfies $i(G-S)\geq 2$.

The isolated toughness characterizes the robustness of the network and is a crux indicator for measuring network quality in network security. The vertex subset S that achieves the value of $I(G)$ (or $I'(G)$) characterizes the target site of the network attack, i.e., attacking the site corresponding to S can achieve the maximum attack effect at the lowest cost. The pertinent investigation on fractional factors and isolated toughness has pivotal guiding significance for both graph theory and network security.

Although there have been some advances in isolated toughness bounds for fractional factors, its fundamental properties are still open to fractional covered graph settings. It motivates us to explore isolated toughness conditions on fractional covered graphs in various settings.

1.3. Main contributions.

Set

$$\lfloor\frac{2m}{k}\rfloor^{*}=\begin{cases}\lfloor\frac{2m}{k}\rfloor, & \text{if } \frac{2m}{k} \text{ is not an integer,}\\ \lfloor\frac{2m}{k}\rfloor-1, & \text{if } \frac{2m}{k} \text{ is an integer.}\end{cases}$$

In the following theorem, we present the tight $I(G)$ bound for a graph to be fractional (k,m)-covered, which extends the result on isolated toughness bound for fractional k-covered graphs in Li et al. [7].

Theorem 1.2. *Let G be a graph, $k\geq 2$, $m\geq 1$ be integers, and $m\leq k$. Then, G is a fractional (k,m)-covered graph if $\delta(G)\geq k+m$ and*

$$I(G)>k+\frac{\lfloor\frac{2m}{k}\rfloor^{*}}{m+1-\lfloor\frac{2m}{k}\rfloor^{*}}.$$

Theorem 1.2 provides sufficient conditions for a graph to be fractional (k,m)-covered, requiring the minimum degree $\delta(G)\geq k+m$ and the isolated toughness $I(G)>k+\frac{\lfloor\frac{2m}{k}\rfloor^{*}}{m+1-\lfloor\frac{2m}{k}\rfloor^{*}}$. These conditions ensure the graph maintains high connectivity and structural resilience against vertex appointment. The result connects key graph parameters to fractional covering properties, highlighting how global robustness enables specific local covering behaviors. This theorem offers a practical criterion for identifying fractional (k,m)-covered graphs in structural graph theory.

For the isolated toughness variant, we have the following conclusion.

Theorem 1.3. *Let G be a graph, $k\geq 2$ and $m\geq 1$ be integers, and $m\leq k$. Then, G is a fractional (k,m)-covered graph if $\delta(G)\geq k+m$ and*

$$I'(G)>k-1+\frac{m+k}{m-\lfloor\frac{2m}{k}\rfloor^{*}}.$$

Like Theorem 1.2, Theorem 1.3 extends structural graph theory by introducing a refined toughness parameter to guarantee the existence of fractional covers under stronger combinatorial constraints.

All the conditions (including minimum degree bound and isolated toughness (variant) bound) in Theorems 1.2 and 1.3 are tight, and counterexamples are presented in Section 3 to illustrate the optimality of these bounds.

The $\delta(G)$ and $I(G)$ bounds for fractional (k,m)-covered graphs are similar to corresponding conditions for fractional (k,m)-deleted graphs which are presented in Gao et al. [5] very recently. Therefore, we must emphasize the following prominent notes: (1) Due to the different conditions of establishment, there are discrepancies between details of the proofs, although the overall framework of the proofs are similar; (2) The extreme graphs (the counterexamples to interpret the sharpness of $\delta(G)$ and $I(G)$ (resp. $I'(G)$) bounds) are different for fractional (k,m)-covered graph and fractional (k,m)-deleted graph.

The next section aims to prove Theorems 1.2 and 1.3 together, and the skeleton in the proofing follows from Gao et al. [5].

2. PROOF OF MAIN THEOREMS

Theorem 1.2 and Theorem 1.3 can be directly verified by using the minimum degree condition if G is a complete graph. In what follows, G is not complete, $\Lambda(S,T,H)=\sum_{x\in S}d_H(x)-e_H(T,S)+\Theta(S,T)$ which is bounded by $2m$, and

$$\lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^* = \begin{cases} \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor, & \text{if } \frac{\Lambda(S,T,H)}{k} \text{ is not an integer or equal to zero,} \\ \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor - 1, & \text{if } \frac{\Lambda(S,T,H)}{k} \text{ is a positive integer.} \end{cases}$$

Using reduction to absurdity, we assume that G satisfies conditions of Theorem 1.2 (resp. Theorem 1.3), but is not fractional (k,m)-covered.

In view of Lemma 1.1, there exist $H \subseteq E(G)$, $|H| = m$, $S \subseteq V(G)$ satisfying

$$k|S| - k|T| + \sum_{x \in T} d_{G-S}(x) \leq \Lambda(S,T,H) - 1 \leq 2m - 1, \tag{2.1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k-1\}$.

If $T = \emptyset$, then by $m \leq k$, we get $k|S| \leq 2m-1 \leq 2k-1$ and thus $|S| = 1$. In this case, $k|S| - k|T| + \sum_{x\in T} d_{G-S}(x) = k$, $\sum_{x\in S} d_H(x) - e_H(T,S) \leq m$, $\Theta(S,T) = 0$, a contradiction. Hence, $T \neq \emptyset$, and we further yield $|S| \geq m+1$ in terms of $\delta(G) \geq k+m$.

Let l and T_0 be the number of K_k components and K_1 components in $G[T]$, respectively. Let G' be the subgraph obtained from $G[T]$ by deleting those K_k components and K_1 components, and S' be a set of vertices that contains exactly $k-1$ vertices in each K_k component in $G[T]$.

Claim 1. $|V(G')| > 0$.

Proof of Claim 1. If $|V(G')| = 0$, then from (2.1), we get $m+1 \leq |S| \leq |T_0| + l + \lfloor \frac{2m}{k} \rfloor^*$. We verify $i(G - S \cup S') = |T_0| + l \geq m + 1 - \lfloor \frac{2m}{k} \rfloor^* \geq 2$.

- If G satisfies the $I(G)$ condition in Theorem 1.2, then

$$\begin{aligned} I(G) &\leq \frac{|T_0| + l + \lfloor \frac{2m}{k} \rfloor^* + l(k-1)}{|T_0| + l} \\ &\leq k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{|T_0| + l} \leq k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m + 1 - \lfloor \frac{2m}{k} \rfloor^*}, \end{aligned}$$

a contradiction.

- If G satisfies the $I'(G)$ assumption in Theorem 1.3, then

$$\begin{aligned} I'(G) &\leq \frac{|T_0| + l + \lfloor \frac{2m}{k} \rfloor^* + l(k-1)}{|T_0| + l - 1} \leq k + \frac{k + \lfloor \frac{2m}{k} \rfloor^*}{|T_0| + l - 1} \\ &\leq k + \frac{k + \lfloor \frac{2m}{k} \rfloor^*}{m + 1 - \lfloor \frac{2m}{k} \rfloor^* - 1} = k - 1 + \frac{m+k}{m - \lfloor \frac{2m}{k} \rfloor^*}, \end{aligned}$$

which contradicts to the assumption of Theorem 1.3. □

Let $G' = G_1 \cup G_2$ where G_1 is the union of components of G' which satisfies that $d_{G-S}(v) = k-1$ for each vertex $v \in V(G_1)$ and $G_2 = G' - G_1$. In terms of Lemma 2.2 in [8], G_1 has a maximum independent set I_1 and the covering set $C_1 = V(G_1) - I_1$ such that

$$|V(G_1)| \leq \sum_{i=1}^{k} (k - i + 1)|I^{(i)}| - \frac{|I^{(1)}|}{2} \leq (k - \frac{1}{2})|I_1|. \tag{2.2}$$

where $I^{(i)} = \{v \in I_1, d_{G_1}(v) = k - i\}$ for $1 \leq i \leq k$. By the definition of G_2 and $\delta(G) \geq k+m$, we confirm that if $G_2 \neq \emptyset$, then $k \geq 3$ and $|S| \geq m+2$.

Denote I_2 by an independent set of G_2. Set $W = V(G) - S - T$ and $U = S \cup S' \cup N_{G-S}(I_1) \cup N_{G-S}(I_2)$.

Case 1. $|T_0| + l \geq 1$.

Claim 2. *If* $|T_0| + l \geq 1$, *then* $|I_2| \neq 0$.

Proof of Claim 2. Suppose $|I_2|=0$, then $|I_1|\neq 0$ by $|V(G')|>0$.

By virtue of (2.1) and (2.2), we get $|T|=|V(G_1)|+|T_0|+lk$,

$$k|S|\leq (k-\frac{1}{2})|I_1|+k|T_0|+lk+\Lambda(S,T,H)-1,$$

and

$$|S|\leq |I_1|+|T_0|+l+\lfloor\frac{2m}{k}\rfloor^*.$$

Then, we infer

$$m+1\leq |S|\leq |I_1|+|T_0|+l+\lfloor\frac{2m}{k}\rfloor^*,$$

which implies $|T_0|+l+|I_1|\geq m+1-\lfloor\frac{2m}{k}\rfloor^*$.

- If G meets the $I(G)$ assumption of Theorem 1.2, then

$$\begin{aligned}
I(G)&\leq \frac{|S\cup S'\cup N_{G-S}(I_1)|}{i(G-S\cup S'\cup N_{G-S}(I_1))}\\
&\leq \frac{|I_1|+\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*+|T_0|+l+l(k-1)+|I_1|(k-1)}{|I_1|+l+|T_0|}\\
&\leq k+\frac{\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*}{|I_1|+l+|T_0|}\\
&\leq k+\frac{\lfloor\frac{2m}{k}\rfloor^*}{m+1-\lfloor\frac{2m}{k}\rfloor^*},
\end{aligned}$$

a contradiction.

- If G meets the $I'(G)$ assumption of Theorem 1.3, then

$$\begin{aligned}
I'(G)&\leq \frac{|S\cup S'\cup N_{G-S}(I_1)|}{i(G-S\cup S'\cup N_{G-S}(I_1))-1}\\
&\leq \frac{|I_1|+\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*+|T_0|+l+l(k-1)+|I_1|(k-1)}{|I_1|+l+|T_0|-1}\\
&\leq k+\frac{\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*+k}{|I_1|+l+|T_0|-1}\\
&\leq k-1+\frac{m+k}{m-\lfloor\frac{2m}{k}\rfloor^*},
\end{aligned}$$

a contradiction. □

Claim 3. *If* $|T_0|+l\geq 1$, *then* $|I_1|\neq 0$.

Proof of Claim 3. Suppose $|I_1|=0$. We yield $|I_2|\neq 0$ and $k\geq 3$.

Let $v_1,v_2,\cdots,v_{|I_2|}$ be vertices in I_2 such that $d_{G-S}(v_1)\leq k-2$ and $d_{G-S}(v_1)\leq d_{G-S}(v_2)\leq\cdots\leq d_{G-S}(v_{|I_2|})$. Then $i(G-U)\geq 2$ where $U=S\cup S'\cup N_{G-S}(I_2)$ and

$$\begin{aligned}
|U|&\leq |S|+|S'|+|N_{G-S}(I_2)|\\
&\leq |T_0|+l+\frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(k-d_{G-S}(v_i))}{k}+\frac{\Lambda(S,T,H)}{k}-\frac{1}{k}+l(k-1)+\sum_{i=1}^{|I_2|}d_{G-S}(v_i)\\
&=|T_0|+lk+\sum_{i=1}^{|I_2|}(-\frac{d_{G-S}^2(v_i)}{k}+(2-\frac{1}{k})d_{G-S}(v_i)+1)+\frac{\Lambda(S,T,H)}{k}-\frac{1}{k}\\
&\leq |T_0|+lk+k|I_2|+\frac{\Lambda(S,T,H)}{k}-\frac{3}{k}.
\end{aligned}$$

Hence, $|U| \leq |T_0| + lk + k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*$.

By maximizing $|U|$, we infer,

$$\begin{aligned}|S| &\leq |T_0| + l + \frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(k-d_{G-S}(v_i))}{k} + \frac{\Lambda(S,T,H)}{k} - \frac{1}{k}\\ &\leq |T_0| + l + |I_2| + \frac{\Lambda(S,T,H)}{k} + 1 - \frac{3}{k},\end{aligned}$$

which reveals $m+2 \leq |S| \leq |T_0| + l + |I_2| + 1 + \lfloor \frac{\Lambda(S,T,H)-3}{k} \rfloor$ and

$$|T_0| + l + |I_2| \geq m + 1 - \lfloor \frac{\Lambda(S,T,H)-3}{k} \rfloor \geq m + 1 - \lfloor \frac{2m}{k} \rfloor^*.$$

- If G satisfies the $I(G)$ assumption of Theorem 1.2, then

$$\begin{aligned}I(G) &\leq \frac{|U|}{i(G-U)} \leq \frac{|T_0| + lk + k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_2| + |T_0| + l}\\ &\leq k + \frac{\lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_2| + |T_0| + l}\\ &\leq k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m + 1 - \lfloor \frac{2m}{k} \rfloor^*},\end{aligned}$$

a contradiction.

- If G satisfies the $I'(G)$ assumption of Theorem 1.3, then

$$\begin{aligned}I'(G) &\leq \frac{|T_0| + lk + k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_2| + |T_0| + l - 1}\\ &\leq k + \frac{\lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^* + k}{|I_2| + |T_0| + l - 1}\\ &\leq k - 1 + \frac{m+k}{m - \lfloor \frac{2m}{k} \rfloor^*},\end{aligned}$$

a contradiction. □

From above discussions, we have $|I_1| \geq 1$, $|I_2| \geq 1$ and $k \geq 3$.

Denote $v_1, v_2, \cdots, v_{|I_2|}$ as vertices in I_2. We obtain

$$k|S| \leq k(|T_0| + l) + (k - \frac{1}{2})|I_1| + \sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(k-d_{G-S}(v_i)) + (\Lambda(S,T,H)) - 1.$$

We confirm that $i(G-U) \geq 3$ where $U = S \cup S' \cup C_1 \cup (N_G(I_1) \cap W) \cup N_{G-S}(I_2)$ and $|U| \leq |T_0| + lk + k|I_1| + k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*$.

By maximizing $|U|$, we deduce,

$$|S| \leq |T_0| + l + |I_1| + |I_2| + \frac{\Lambda(S,T,H)}{k} + 1 - \frac{3}{k},$$

which reveals $m+2 \leq |S| \leq |T_0| + l + |I_1| + |I_2| + 1 + \lfloor \frac{\Lambda(S,T,H)-3}{k} \rfloor$. Hence, we verify $|T_0| + l + |I_1| + |I_2| \geq m + 1 - \lfloor \frac{2m}{k} \rfloor^*$.

- If G meets the $I(G)$ assumption of Theorem 1.2, then

$$\begin{aligned}I(G) &\leq \frac{|T_0| + lk + k|I_1| + k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_1| + |I_2| + |T_0| + l}\\ &\leq k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m + 1 - \lfloor \frac{2m}{k} \rfloor^*},\end{aligned}$$

a contradiction.
- If G meets the $I'(G)$ assumption of Theorem 1.3, then

$$\begin{aligned}
I'(G) &\leq \frac{|U|}{i(G-U)-1} \leq \frac{|T_0|+lk+k|I_1|+k|I_2|+\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*}{|I_1|+|I_2|+|T_0|+l-1}\\
&\leq k+\frac{\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*+k}{|I_1|+|I_2|+|T_0|+l-1}\\
&\leq k-1+\frac{m+k}{m-\lfloor\frac{2m}{k}\rfloor^*},
\end{aligned}$$

a contradiction.

Case 2. $|T_0|+l=0$.

Claim 4. *If $|T_0|+l=0$, then $|I_2|\neq 0$.*

Proof of Claim 4. Suppose $|I_2|=0$, then we infer $|I_1|\neq 0$, $|T|=|V(G_1)|$ and $k|S|\leq k|T|-d_{G-S}(T)+\Lambda(S,T,H)-1=|T|+\Lambda(S,T,H)-1$.

If $|I_1|=1$, then $|T|\leq k-1$ and $|S|\leq\frac{|T|+\Lambda(S,T,H)-1}{k}\leq\frac{\Lambda(S,T,H)-2}{k}+1$. Thus, $k+m\leq\delta(G)\leq|S|+(k-1)\leq k+\frac{\Lambda(S,T,H)-2}{k}\leq k+\frac{2m-2}{k}$, a contradiction. Hence, $|I_1|\geq 2$.

By virtue of the discussion in Claim 2, we acquire

$$\begin{aligned}
|I_1| &\geq m+1-\lfloor\frac{2m}{k}\rfloor^*,\\
|U| &\leq k|I_1|+\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*,
\end{aligned}$$

where $U=S\cup C_1\cup(N_G(I_1)\cap W)$, and thus $i(G-U)=|I_1|\geq 2$.
- If G satisfies the $I(G)$ condition in Theorem 1.2, then

$$\begin{aligned}
I(G) &\leq \frac{|U|}{i(G-U)} \leq \frac{k|I_1|+\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*}{|I_1|}\\
&= k+\frac{\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*}{|I_1|}\\
&\leq k+\frac{\lfloor\frac{2m}{k}\rfloor^*}{m+1-\lfloor\frac{2m}{k}\rfloor^*},
\end{aligned}$$

a contradiction.
- If G satisfies the $I'(G)$ condition in Theorem 1.3, then

$$\begin{aligned}
I'(G) &\leq \frac{k|I_1|+\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*}{|I_1|-1}\\
&= k+\frac{\lfloor\frac{\Lambda(S,T,H)}{k}\rfloor^*+k}{|I_1|-1}\\
&\leq k-1+\frac{m+k}{m-\lfloor\frac{2m}{k}\rfloor^*},
\end{aligned}$$

a contradiction. □

Claim 5. *If $|T_0|+l=0$, then $|I_1|\neq 0$.*

Proof of Claim 5. Suppose $|I_1|=0$, then $|I_2|\neq 0$ and $k\geq 3$.

If $|I_2| = 1$, then we set $d_{min} = \min\{d_{G-S}(v)|v \in G_2\}$ and $z \in V(G_2)$ such that $d_{G-S}(z) = d_{min}$, thus $d_{min} \in \{1, \cdots, k-2\}$. Hence,

$$\begin{aligned}|S| &\le \frac{|T|(k-d_{min}) + (\Lambda(S,T,H) - 1}{k}\\ &\le \frac{(k-1)(k-d_{min}) - 1}{k} + \frac{\Lambda(S,T,H)}{k},\end{aligned}$$

and

$$\begin{aligned}k + m \le \delta(G) &\le d_{min} + |S|\\ &\le d_{min} + \frac{(k-1)(k-d_{min}) - 1}{k} + \frac{\Lambda(S,T,H)}{k}\\ &\le k + \frac{2m-3}{k},\end{aligned}$$

a contradiction.

Thus, $i(G-U) = |I_2| \ge 2$ where $U = S \cup N_{G-S}(I_2)$. We verify $|U| \le k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*$ and $|I_2| \ge m + 1 - \lfloor \frac{2m}{k} \rfloor^*$.

- If G meets the $I(G)$ assumption of Theorem 1.2, then

$$I(G) \le \frac{k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_2|} = k + \frac{\lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_2|} \le k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m + 1 - \lfloor \frac{2m}{k} \rfloor^*},$$

a contradiction.

- If G meets the $I'(G)$ assumption of Theorem 1.3, then

$$I'(G) \le \frac{k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_2| - 1} = k + \frac{\lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^* + k}{|I_2| - 1} \le k - 1 + \frac{m + k}{m - \lfloor \frac{2m}{k} \rfloor^*},$$

a contradiction. □

From above discussions, we ensure $|I_1| \ge 1$, $|I_2| \ge 1$, and $k \ge 3$.

Set $U = S \cup C_1 \cup (N_G(I_1) \cap W) \cup N_{G-S}(I_2)$. In light of the final discussion in Case 1, we get $i(G-U) \ge 2$,

$$|U| \le k|I_1| + k|I_2| + \lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*,$$

and

$$|I_1| + |I_2| \ge m + 1 - \lfloor \frac{2m}{k} \rfloor^*.$$

- If G meets the conditions in Theorem 1.2, then

$$I(G) \le k + \frac{\lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^*}{|I_1| + |I_2|} \le k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m + 1 - \lfloor \frac{2m}{k} \rfloor^*},$$

a contradiction.

- If G meets the conditions in Theorem 1.3, then

$$I'(G) \le k + \frac{\lfloor \frac{\Lambda(S,T,H)}{k} \rfloor^* + k}{|I_1| + |I_2| - 1} \le k - 1 + \frac{m + k}{m - \lfloor \frac{2m}{k} \rfloor^*},$$

a contradiction. □

3. SHARPNESS

To show that $\delta(G) \geq k+m$ is best possible, we consider $G_1 = ((m - \lfloor \frac{2m}{k} \rfloor^*)K_k \cup K_{k+2}) \vee K_m$, where $k \geq m$. We have $\delta(G_1) = k+m-1$,

$$\begin{aligned} I(G_1) &= \begin{cases} k + \frac{\lfloor \frac{2m}{k} \rfloor^* + 1}{m+1-\lfloor \frac{2m}{k} \rfloor^*}, & \text{if } m - \lfloor \frac{2m}{k} \rfloor^* = 1, \\ k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m-\lfloor \frac{2m}{k} \rfloor^*}, & \text{if } m - \lfloor \frac{2m}{k} \rfloor^* \geq 2, \end{cases} \\ &= \begin{cases} k + \frac{\lfloor \frac{2m}{k} \rfloor^* + 1}{m+1-\lfloor \frac{2m}{k} \rfloor^*}, & \text{if } m = 1 \text{ or } (m,k) = (2,2) \text{ or } (m,k) = (2,3), \\ k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m-\lfloor \frac{2m}{k} \rfloor^*}, & \text{otherwise,} \end{cases} \end{aligned}$$

which is strictly greater than $k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m+1-\lfloor \frac{2m}{k} \rfloor^*}$, and

$$I'(G_1) = k - 1 + \frac{m+k+1}{m - \lfloor \frac{2m}{k} \rfloor^*} > k - 1 + \frac{m+k}{m - \lfloor \frac{2m}{k} \rfloor^*}.$$

Indeed, to get at least two isolated vertices, K_m must be removed from G_1.

For $I(G_1)$, we ignore K_{k+2} part, and set $z \in \{1, 2, \cdots, m - \lfloor \frac{2m}{k} \rfloor^*\}$. We get

$$\min_z \{\frac{m + z(k-1)}{z}\} = k - 1 + \min_z \{\frac{m}{z}\} = k - 1 + \frac{m}{m - \lfloor \frac{2m}{k} \rfloor^*} = k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m - \lfloor \frac{2m}{k} \rfloor^*},$$

which implies that if we ignore the K_{k+2} part, then we get $m - \lfloor \frac{2m}{k} \rfloor^*$ isolated vertices can minimize the corresponding ratio of the isolated toughness. If we get one more isolated vertex from K_{k+2}, then we have to remove more $k+1$ vertices, and the corresponding ratio is

$$\frac{m + (m - \lfloor \frac{2m}{k} \rfloor^*)(k-1) + k + 1}{m + 1 - \lfloor \frac{2m}{k} \rfloor^*} = k + \frac{\lfloor \frac{2m}{k} \rfloor^* + 1}{m + 1 - \lfloor \frac{2m}{k} \rfloor^*}.$$

It can be verified that $k + \frac{\lfloor \frac{2m}{k} \rfloor^*}{m-\lfloor \frac{2m}{k} \rfloor^*} \leq k + \frac{\lfloor \frac{2m}{k} \rfloor^*+1}{m+1-\lfloor \frac{2m}{k} \rfloor^*}$ for combinations of (k, m). However, when $m - \lfloor \frac{2m}{k} \rfloor^* = 1$ (i.e., $m = 1$ or $m = 2$ and $k \in \{2, 3\}$), in order to yield at least two isolated vertices, we must remove $k+1$ vertices from K_{k+2}. Hence, we check the isolated toughness of G_1.

For $I'(G_1)$, we ignore K_{k+2} part, and set $z \in \mathbb{N}$ and $z \leq m - \lfloor \frac{2m}{k} \rfloor^*$. We get

$$\min_z \{\frac{m + z(k-1)}{z-1}\} = k - 1 + \min_z \{\frac{m+k-1}{z-1}\} = k - 1 + \frac{m+k-1}{m - \lfloor \frac{2m}{k} \rfloor^* - 1},$$

which implies that if we ignore the K_{k+2} part, then we get $m - \lfloor \frac{2m}{k} \rfloor^*$ isolated vertices can minimize the corresponding ratio of the isolated toughness variant. If we get one more isolated vertex from K_{k+2}, then we have to remove more $k+1$ vertices, and the corresponding ratio is

$$\frac{m + (m - \lfloor \frac{2m}{k} \rfloor^*)(k-1) + k + 1}{m + 1 - \lfloor \frac{2m}{k} \rfloor^* - 1} = k - 1 + \frac{k+m+1}{m - \lfloor \frac{2m}{k} \rfloor^*}.$$

It can be verified that $k - 1 + \frac{m+k-1}{m-\lfloor \frac{2m}{k} \rfloor^*-1} > k - 1 + \frac{k+m+1}{m-\lfloor \frac{2m}{k} \rfloor^*}$ for all combinations of (k, m). Hence, we verify the isolated toughness variant of G_1.

Take $S = V(K_m)$, then $T = V((m - \lfloor \frac{2m}{k} \rfloor^*)K_k)$, $\max_{H \subseteq E(G_1), |H| = m}\{\Lambda(S, T, H)\} = 2m$ and

$$\begin{aligned} k|S| - k|T| + d_{G_1 - S}(T) &= km - k(m - \lfloor \frac{2m}{k} \rfloor^*) \\ &= k\lfloor \frac{2m}{k} \rfloor^* < 2m = \max_{H \subseteq E(G_1), |H| = m}\{\Lambda(S, T, H)\}. \end{aligned}$$

Hence, G_1 is not fractional (k, m)-covered, and the minimum degree bound $\delta(G) \geq k + m$ in both Theorem 1.2 and Theorem 1.3 is sharp.

To explain why the isolated toughness (variant) bounds are tight in Theorem 1.2 and Theorem 1.3, we consider $G_2 = ((m+1-\lfloor\frac{2m}{k}\rfloor^*)K_k)\vee K_{m+1}$. We obtain $\delta(G_2) = k+m$, $I(G_2) = k+\frac{\lfloor\frac{2m}{k}\rfloor^*}{m+1-\lfloor\frac{2m}{k}\rfloor^*}$ and $I'(G_2) = k-1+\frac{m+k}{m-\lfloor\frac{2m}{k}\rfloor^*}$. Set $S = V(K_{m+1})$, then $|S| = m+1$, $T = V((m+1-\lfloor\frac{2m}{k}\rfloor^*)K_k)$, $|T| = k(m+1-\lfloor\frac{2m}{k}\rfloor^*)$, $d_{G_2-S}(x) = k-1$ for each $x\in T$, $\max_{H\subseteq E(G_2),|H|=m}\{\Lambda(S,T,H)\} = 2m$, and

$$\begin{aligned} k|S| - k|T| + d_{G_2-S}(T) &= k(m+1) - k(m+1-\lfloor\frac{2m}{k}\rfloor^*) \\ &< 2m = \max_{H\subseteq E(G_2),|H|=m}\{\Lambda(S,T,H)\}. \end{aligned}$$

Hence, G_2 is not fractional (k,m)-covered. It reveals that $I(G) > k+\frac{\lfloor\frac{2m}{k}\rfloor^*}{m+1-\lfloor\frac{2m}{k}\rfloor^*}$ in Theorem 1.2 and $I'(G) > k-1+\frac{m+k}{m-\lfloor\frac{2m}{k}\rfloor^*}$ in Theorem 1.3 are sharp.

4. CONCLUSION AND DISCUSSION

In this paper, we present the $I(G)$ and $I'(G)$ bounds for a graph to be fractional (k,m)-covered, and the sharpness of bounds is instantiated by counterexamples. It is noted that $\lfloor\frac{2m}{k}\rfloor^*\in\{0,1\}$ since $k\geq m$ is an intrinsic requirement for fractional (k,m)-covered graphs, $I(G)$ and $I'(G)$ bounds in Theorem 1.2 and Theorem 1.3 can be replaced to

$$I(G) > \begin{cases} k+\frac{1}{m}, & \text{if } m\leq k<2m, \\ k, & \text{if } k\geq 2m, \end{cases}$$

and

$$I'(G) > \begin{cases} k+\frac{k+1}{m-1}, & \text{if } m\leq k<2m, \\ k+\frac{k}{m}, & \text{if } k\geq 2m, \end{cases}$$

respectively.

Finally, we end this article with the following open problem (for the concepts of toughness and toughness variant, which can be referred to Chvátal [2] and Enomoto [4]).

Problem 1. *What is the tight toughness (resp. toughness variant) bound for fractional (k,m)-covered graphs?*

DECLARATION

Conflict of Interests. The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

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