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Flexibility conjugate gradient methods for image processing

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Abstract

The Conjugate Gradient approach is a technique for numerical optimization that identifies the optimal solution by focusing on the coefficient conjugate. This helps the approach find the optimal answer more quickly. This technique bears the same name as its namesake. In this paper, a novel approach utilizing conjugate gradients and coefficients is proposed for the purpose of removing impulsive noise from photographs. It has been demonstrated that the approach, which is based on a quadratic function, converges to the same value everywhere in the world. The research indicates that the method for recovering photos using this methodology is highly successful.

Keywords. Conjugate gradient method, Coefficient conjugate gradient, Image restoration.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

An edge-preserving regularization (EPR) functional is used as the objective function in this research's proposed set of iterative techniques to addressing optimization issues. These approaches are proposed as part of this study. Finding the optimal solution to the problem at hand is the current task that has to be completed. The elimination of impulsive noise may be accomplished by the use of an adaptive median filter (AMF) equation [24].

The letter $A = \{1, 2, 3, ...M\} \times \{1, 2, 3, ...N\}$ stands in for the index set that is associated with the displayed picture and is indicated by the letter X. The indices of noisy pixels discovered in the first phase are represented by the sets $N \subset A$ and $P_{i,j}$, which represent the pixel's four closest neighbors and are placed at the $(i, j) \in A$ location, respectively. These indices were located using the $N \subset A$ and PIL sets. The resultant vector, denoted by the initials $u_{i,j} = [u_{i,j}]_{(i,j)\in N}$, has a length of c and is organized in lexicographic order when a pixel value, denoted by the initials $y_{i,j}$, is placed at position (i, j), the amount of elements that make up a set, denoted by the letter c and symbolized by the number N. Here, c denotes the total number of components that make up N. A procedure consisting of two steps for recovering noise pixels. In the first stage, we will minimize a functional, and in the second step, we will minimize a another functional in order to recover the noisy pixels. Both steps require minimization:

$$f_\alpha(u) = \sum_{(i,j)\in N} [|u_{i,j} - y_{i,j}| + \frac{\beta}{2}(2 \times S^1_{i,j} + S^2_{i,j})], \tag{1.1}$$

where β is the parameter used to regularize the data, and: $S^1_{i,j} = 2\sum_{(m,n)\in P_{i,j}} \bigcap N^c \varphi_\alpha(u_{i,j} - y_{m,n})$, $S^2_{i,j} = \sum_{(m,n)\in P_{i,j}} \bigcap N \varphi_\alpha(u_{i,j} - y_{m,n})$. One example of a potential function that keeps the image's edges intact is the $\varphi_\alpha = \sqrt{\alpha + x^2}, \alpha > 0$ edge-preserving potential function. It is applied to a picture in order to lessen the amount of noise present in it and to improve the edges of the image, hence making it simpler to recognize individual features. Minimizing a smooth edge-preserving regularization (EPR) functional may be accomplished via the application of

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optimization techniques. In the second phase, when just the noisy pixels are being restored, it should be noticed that the non-smooth data-fitting term is not required:

$$f_\alpha(u) = \sum_{(i,j)\in N} [(2 \times S^1_{i,j} + S^2_{i,j})]. \tag{1.2}$$

Regarding additional information found at [23] and [24], respectively.

In image restoration, the conjugate gradient approach is often used, and its primary purpose is to reduce the magnitude of a specified goal function:

$$Minf(x), u \in R^n, \tag{1.3}$$

where a function, such as in [4], may be differentiated in a continuous manner. The Conjugate Gradient Method repeatedly improves the answer by using a starting point and a gradient as a guide:

$$u_{k+1} = u_k + \alpha_k d_k. \tag{1.4}$$

Utilizing a search direction d_k and an exact step size α_k can be used to optimize the quadratic case up until the point where it converges on the best possible solution. These two components are very necessary in order to accomplish the most successful outcomes possible:

$$\alpha_k = \frac{-g_k^T d_k}{d_k^T Q d_k}. \tag{1.5}$$

It is necessary to use an iterative approach in order to get a solution for generic nonlinear functions [21]. The Wolfe conditions are a collection of criteria that are used to determine the step length in a certain task. These requirements are as follows:

$$f(u_k + \alpha_k d_k) \le f(u_k) + \delta \alpha_k g_k^T d_k, \tag{1.6}$$

$$d_k^T g(u_k + \alpha_k d_k) \ge \sigma d_k^T g_k, \tag{1.7}$$

where $0 < \delta < \sigma < 1$. Additional details are available in [1]. The search direction equation is as follows:

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \tag{1.8}$$

where and are required to meet the conditions of the conjugacy property, and the coefficient β_k is selected in such a manner as to guarantee that. Fletcher and Reeves (FR) [5] and Dai and Yuan (DY) [3] are two notable β_k that are believed to be the most effective approaches, as stated by them:

$$\beta_k^{FR} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}, \beta_k^{DY} = \frac{g_{k+1}^T g_{k+1}}{d_k^T y_k}. \tag{1.9}$$

Alternately, by utilizing different equations (like [6, 17, 20, 22]). When it comes to numerical computing, certain methods for conjugating gradients in computer-generated images are more effective than others. Each of these methods is equivalent if the objective function is strictly convex and quadratic. When used with general non-quadratic functions, as shown in [2], they behave in a particular way. After some time, the conjugate gradient approach underwent modification to enable the application of the formula to solve increasingly challenging and unrestricted optimization problems [5]. Conjugate gradient techniques are currently acknowledged as being especially useful for handling large-scale unconstrained optimization issues as they do not need the storage of matrices. Numerous conjugate gradient techniques with strong global convergence properties and excellent numerical performance have recently been proposed. These approaches are based on the requirement of conjugacy of the many conjugate gradient parameters. see, [8, 10, 14, 15]. In previous conjugate gradient algorithms, the conjugacy criteria was applied:

$$d_{k+1}^T Q d_k = 0. \tag{1.10}$$

This is of utmost significance for the mathematical experiment as well as the investigation of convergence [15]. Fairly effective conjugate gradient methods of optimization were described and still draw significant attention [9, 11–13, 16, 18, 19].

The quadratic function was used as the foundation for the revolutionary conjugate gradient approach, which, following its conception, underwent additional rationalization and investigation in order to improve upon its effectiveness. Studies that made use of numerical techniques demonstrated that the suggested method performed more effectively than other conjugate gradient methods. This was discovered after the studies were carried out.

2. Modification of Coefficients Conjugate Gradient:

We utilizes the quadratic model and Taylor series to derivation the new coefficients conjugate gradient:

$$f_{k+1} = f_k + s_k^T g_k + \frac{1}{2} s_k^T Q s_k, \tag{2.1}$$

and

$$f_k = f_{k+1} - g_{k+1}^T s_k + \frac{1}{2} s_k^T Q s_k, \tag{2.2}$$

where $Q(u_k)$ is the Hessian matrix. Taking the derivative of (2.1) and (2.2) on both sides for s_k, we get:

$$\nabla f_{k+1} = g_k + Q(u_k) s_k = 0. \tag{2.3}$$

There are easy to see that the following secant equation:

$$s_k^T Q s_k = 2r(f_k - f_{k+1} + g_{k+1}^T s_k) + 2(1-r)(f_{k+1} - f_k - s_k^T g_k) \tag{2.4}$$

where $0 \leq r \leq 1$. By combining (2.1), (2.2), and (2.3), we get:

$$s_k^T Q(u_k) s_k = s_k^T y_k + (2r-1)(2(f_k - f_{k+1}) - s_k^T y_k). \tag{2.5}$$

The result of applying the conjugacy criteria is:

$$d_{k+1}^T Q s_k = 0. \tag{2.6}$$

After algebra, (2.5) - (2.6) yield:

$$\beta_k = \frac{\left[1 + \frac{(2r-1)(2(f_k - f_{k+1}) - s_k^T y_k)}{s_k^T y_k}\right] g_{k+1}^T y_k}{d_k^T y_k}. \tag{2.7}$$

Exact line search in (2.6) would result in BBB being as follows:

$$\beta_k^{BBB1} = \frac{\left[1 + \frac{(2r-1)(2(f_k - f_{k+1}) - s_k^T y_k)}{s_k^T y_k}\right] \|g_{k+1}\|^2}{d_k^T y_k}, \tag{2.8}$$

and

$$\beta_k^{BBB2} = \frac{\left[1 + \frac{(2r-1)(2(f_k - f_{k+1}) + s_k^T g_k)}{-s_k^T g_k}\right] \|g_{k+1}\|^2}{-d_k^T g_k}. \tag{2.9}$$

It's clear from the above modification:

$$\beta_k^{BBB3} = \frac{\left[1 + \frac{(2r-1)(2(f_k - f_{k+1}) - \alpha_k \|g_k\|^2)}{\alpha_k \|g_k\|^2}\right] \|g_{k+1}\|^2}{\|g_k\|^2}. \tag{2.10}$$

From (2.8), we obtain BBB1, BBB2, and BBB3 respectively. Moreover, when $r = 1/2$, i.e., the formula (2.8) turns out to be the standard other formula ignoring whether the objective function is quadratic or not, while for other cases the function value information is exploited. The algorithm that was proposed, which was driven by the direction given above, is referred to as:

Algorithm 1 BBB1, BBB2, and BBB3 conjugate gradient algorithms.

Choose an initial point $x_0 \in \mathbb{R}^n$.
Set $k = 0$.
Compute $g_0 = \nabla f(x_0)$.
Set the initial search direction $d_0 = -g_0$.
while $\|g_k\| > \varepsilon$ **do**
    Determine the step size α_k such that it satisfies conditions (1.6) and (1.7).
    Compute the new iterate
    $$x_{k+1} = x_k + \alpha_k d_k.$$
    Compute the new gradient
    $$g_{k+1} = \nabla f(x_{k+1}).$$
    Compute $\beta_k^{BBB_i}$, for $i = 1, 2, 3$, using the formula given in (2.8).
    Define the new search direction by
    $$d_{k+1} = -g_{k+1} + \beta_k^{BBB_i} d_k.$$
    Set $k = k + 1$.
end while
return x_k as the approximate solution.

3. Convergence analysis

In order to get the convergence result, the following set of assumptions is used. Assumptions:

The f is differentiable and its gradient is Lipschitz continuous in an open convex set Ψ that includes the level set $\Psi = \{x \in R^n : f(x) \leq f(x_0)\}$ and where x_0 is a constant. More exactly, a constant $L > 0$ occurs in the following way:

$$\|g(z) - g(u)\| \leq L \|z - u\|, \forall z, u \in R^n. \tag{3.1}$$

With a constant ¥ ≥ 0 such that $\|\nabla f(x)\| \leq$ ¥ , see [14].

Zoutendijk [26] came to the following significant conclusion throughout his research.

Lemma 3.1. *Any iteration method where α_k is found using the Wolfe line search and the premises are regarded as true. Then:*

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty. \tag{3.2}$$

Theorem 3.2. *Let the new algorithms produce the direction d_{k+1}. Then, we discovered:*

$$d_{k+1}^T g_{k+1} < 0 \text{ and } d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k. \tag{3.3}$$

Proof. It may be deduced using (1.8) that:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1}. \tag{3.4}$$

On (2.7) and after algebra:

$$\|g_{k+1}\|^2 = \frac{\beta_k^{BB1} d_k^T y_k}{\left[1 + \frac{(2r-1)(2(f_k - f_{k+1}) - s_k^T y_k)}{s_k^T y_k}\right]}. \tag{3.5}$$

The (3.5) on (3.4) yields:

$$d_{k+1}^T g_{k+1} = \beta_k^{BBB1}[d_k^T g_{k+1} - d_k^T y_k] = \beta_k^{BBB1} d_k^T g_k < 0. \tag{3.6}$$

It follows from this inequality that $k + 1$ is met by (3.3) □

Theorem 3.3. *Accept the presumptions. Let the new algorithm generate* $\{u_k\}$*. One has to:*

$$\lim_{k\to\infty} \inf \|g_k\| = 0. \tag{3.7}$$

Proof. By contradiction, equation (3.7) is false. For any k, we can find a r > 0 such that:

$$\|g_{k+1}\| > r. \tag{3.8}$$

By squaring the search duration as $d_{k+1} + g_{k+1} = \beta_k d_k$on both sides, one can obtain the following result:

$$\|d_{k+1}\|^2 + \|g_{k+1}\|^2 + 2d_{k+1}^T g_{k+1} = (\beta_k)^2 \|d_k\|^2 . \tag{3.9}$$

Applying (3.8) to (3.7) yields the following outcomes:

$$\|d_{k+1}\|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2} \|d_k\|^2 - 2d_{k+1}^T g_{k+1} - \|g_{k+1}\|^2 . \tag{3.10}$$

The result of dividing (3.8) by $(d_{k+1}^T g_{k+1})^2$ is:

$$\begin{aligned} \frac{\|d_{k+1}\|^2}{\left(d_{k+1}^T g_{k+1}\right)^2} &= \frac{\|d_k\|^2}{\left(d_k^T g_k\right)^2} - \frac{\|g_{k+1}\|^2}{\left(d_{k+1}^T g_{k+1}\right)^2} - \frac{2}{d_{k+1}^T g_{k+1}} \\ &\le \frac{\|d_k\|^2}{\left(d_k^T g_k\right)^2} - \left(\frac{\|g_{k+1}\|}{d_{k+1}^T g_{k+1}} + \frac{1}{\|g_{k+1}\|^2}\right) + \frac{1}{\|g_{k+1}\|^2} \\ &\le \frac{\|d_k\|^2}{\left(d_k^T g_k\right)^2} + \frac{1}{\|g_{k+1}\|^2}. \end{aligned} \tag{3.11}$$

Thus, we discovered:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} \le \sum_{i=1}^{k+1} \frac{1}{\|g_i\|^2}. \tag{3.12}$$

Then:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{r_1^2}. \tag{3.13}$$

Finally, we have:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \infty. \tag{3.14}$$

Similarly, Lemma 3.1 states that $\lim_{k\to\infty} \inf \|g_k\| = 0$ holds. There are several formulae that may be used to get different results. □

4. Computational Experience

The performance of the FR approach is then evaluated numerically, and its effectiveness is contrasted with that of salt-and-pepper impulse noise reduction. Table 1 lists the methods for judging photos, including Lena, Home, Cameraman, and Elaine. The PSNR (peak signal to noise ratio) is used to qualitatively evaluate the effectiveness of restoration:

$$PSNR = 10 \log_{10} \frac{255^2}{\frac{1}{MN} \sum_{i,j} (u_{i,j}^r - u_{i,j}^*)^2}, \tag{4.1}$$

where $u^r_{i,j}$ and $u^*_{i,j}$ denote the restored and original picture pixels. Both methods stop at:

$$\frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \text{ and } \|f(u_k)\| \leq 10^{-4}(1 + |f(u_k)|). \tag{4.2}$$

For more details see [7], [25].

Table 1 displays the PSNR, total number of iterates (NI), and function evaluation (NF).

Table 1. Numerical results of FR, BBB1, BBB2 and BBB3 algorithms.

Image	**Noise level r (%)**	**FR-Method**			**BBB1-Method**			**BBB2-Method**			**BBB3-Method**		
		NI	**NF**	**PSNR (dB)**	**NI**	**NF**	**PSNR (dB)**	**NI**	**NF**	**PSNR (dB)**	**NI**	**NF**	**PSNR (dB)**
Le	50	82	153	30.5529	69.0	138.0	30.5656	64.0	128.0	30.6883	67.0	133.0	30.5187
	70	81	155	27.4824	67.0	133.0	27.4031	35.0	71.0	27.5016	42.0	83.0	27.3399
	90	108	211	22.8583	53.0	109.0	23.0148	52.0	106.0	22.785	49.0	99.0	23.0567
Ho	50	52	53	30.6845	39.0	77.0	34.6713	40.0	78.0	35.0561	23.0	45.0	34.9347
	70	63	116	31.2564	48.0	96.0	31.0653	52.0	103.0	31.2084	28.0	55.0	31.1998
	90	111	214	25.287	42.0	84.0	25.3282	52.0	105.0	25.1151	40.0	83.0	25.1078
El	50	35	36	33.9129	31.0	59.0	33.9044	32.0	61.0	33.9167	31.0	59.0	33.897
	70	38	39	31.864	23.0	42.0	32.0146	23.0	42.0	31.9748	23.0	42.0	31.9005
	90	65	114	28.2019	30.0	58.0	28.2423	46.0	90.0	28.3144	33.0	65.0	28.4865
c512	50	59	87	35.5359	46.0	92.0	35.5935	42.0	84.0	35.4469	41.0	82.0	35.6108
	70	78	142	30.6259	30.0	61.0	30.7481	29.0	60.0	30.7247	30.0	62.0	31.0196
	90	121	236	24.3962	71.0	144.0	25.1506	50.0	103.0	25.119	58.0	118.0	24.8045

Table 1 illustrates that the suggested algorithms are better to the FR technique in terms of the peak signal-to-noise ratio, the number of iterations, and the function evaluations. Additionally, the number of iterations is much less in the proposed algorithms.

5. Conclusions

The problem of picture restoration is one that we investigate and attempt to solve in this research project by introducing a parameter conjugate gradient approach that is founded on the quadratic function. In addition, we analyze the extent to which this tactic converges globally under a number of other constraints that are lax. The findings of numerical tests carried out on the problem of restricted picture restoration have shown that our method is both promising and capable of being used in real-world situations.

Acknowledgment

This section should come before the References and should be unnumbered. Funding information may also be included here.

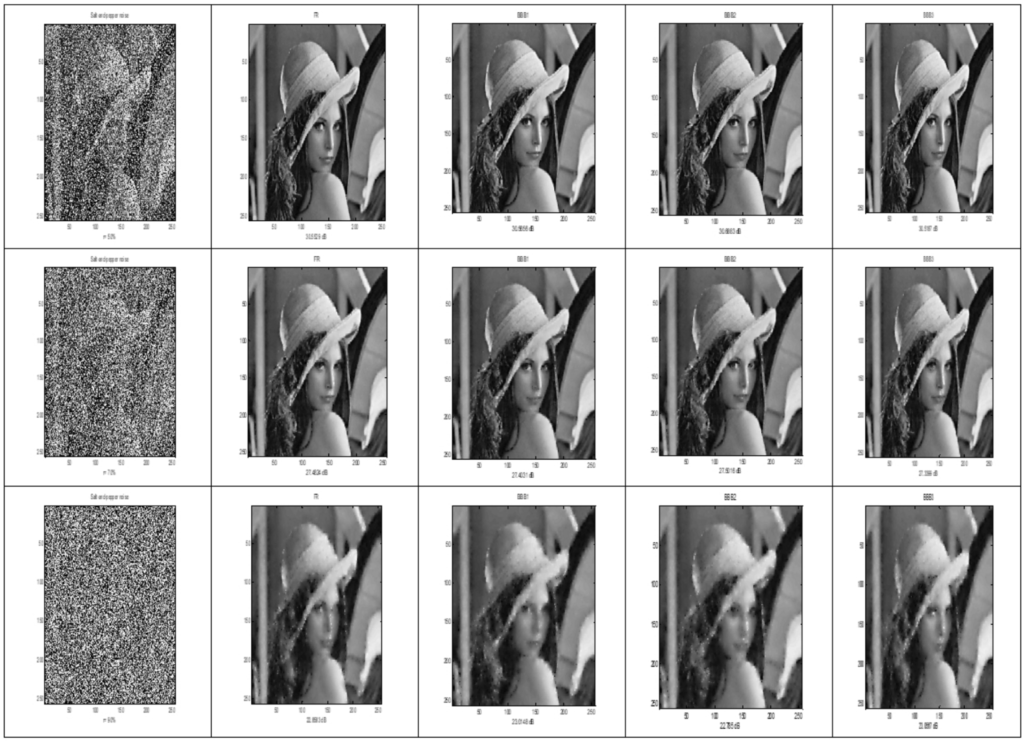

FIGURE 1. Demonstrates the results of algorithms FR, BBB1, BBB2, and BBB3 of 256 × 256 Lena image.

FIGURE 2. Demonstrates the results of algorithms FR, BBB1, BBB2, and BBB3 of 256 × 256 House image.

FIGURE 3. Demonstrates the results of algorithms FR, BBB1, BBB2, and BBB3 of 256×256 Elaine image.

FIGURE 4. Demonstrates the results of algorithms FR, BBB1, BBB2, and BBB3 of 256×256 Cameraman image.

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