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C^1–conforming finite element method for fourth-order ordinary differential equations

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Abstract

This paper presents a C^1-conforming finite element method (FEM) for solving fourth-order ordinary differential equations (ODEs) using cubic Hermite basis functions. The proposed scheme inherently enforces C^1-continuity and accommodates both clamped and simply supported boundary conditions. A rigorous theoretical analysis demonstrates the method's stability and optimal convergence rates, achieving $\mathcal{O}(h^2)$ in the H^2-norm and $\mathcal{O}(h^4)$ in the L^2-norm. Numerical experiments validate the theoretical results, showing excellent agreement with exact solutions. The study underscores the efficacy of Hermite FEM for high-order BVPs, providing a robust and accurate computational framework.

Keywords. Hermite FEM, Fourth-order ODEs, C^1-conforming, Boundary value problems, Optimal convergences.
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1. Introduction

Boundary Value Problems (BVPs) commonly emerge in engineering and physics, particularly in applications such as beam deflection, plate bending, and fluid dynamics modeling [2]. The complexity of these problems often makes obtaining exact analytical solutions difficult or infeasible, especially for irregular domains or variable coefficient cases [7]. As a result, numerical approaches have become essential for approximating solutions to such high-order differential equations.

The Finite Element Method (FEM) distinguishes itself among numerical techniques due to its capability to manage complex geometries, non-uniform meshes, and high-order derivatives effectively [8]. Compared to finite difference and spectral methods, FEM offers superior flexibility in implementing boundary conditions while delivering higher accuracy with reduced computational expense for smooth solutions [28]. EM's inherent meshing provides excellent local approximations, crucial for detailed stress/deformation analysis [4, 28], while meshless methods bypass complex meshing [5, 20] but sometimes sacrifice accuracy or require complex kernel functions [3, 19].

Given the limitations of analytical methods, numerical techniques play a crucial role in solving these problems. Recent developments encompass finite difference and collocation methods [15, 17], Runge-Kutta and block methods [21, 23], and wavelet and spectral approaches [6, 25]. Additionally, machine learning techniques, including neural networks [18], and hybrid strategies [13, 16], have improved accuracy and efficiency for stiff and nonlinear BVPs. This study examines these methods, highlighting their comparative advantages.

Although Lagrange polynomials are widely employed in FEM, they prove inadequate for fourth-order problems due to their inability to ensure derivative continuity [14]. Hermite interpolation, on the other hand, guarantees C^1 continuity, making it particularly suitable for approximating solutions to fourth-order BVPs [24]. Cubic Hermite basis functions maintain both function values and their first derivatives at nodal points, resulting in enhanced solution stability and accuracy [22]. For solving high-order differential equations, cubic Hermite basis functions generally provide a smoother and more accurate approximation than Lagrange polynomials due to the incorporation of derivative

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information, leading to higher-order continuity [9, 11]. However, Lagrange polynomials are better suited for treating discontinuous solutions in standard finite element methods [14].

Despite these benefits, Hermite interpolation has received limited attention in FEM applications for fourth-order ODEs, with most research concentrating on lower-order problems or alternative approaches like B-splines and mixed finite elements [26]. This research gap inspires the current work, which introduces an efficient FEM framework utilizing cubic Hermite interpolation to solve fourth-order BVPs.

The rest of this paper is organized as follows. In Section 2, we present the derivation of the proposed numerical scheme. Section 3 is devoted to the rigorous error analysis and stability estimates. In Section 4, we perform comprehensive numerical experiments to validate the theoretical findings and demonstrate the scheme's efficiency. Finally, concluding remarks are given in Section 5.

2. Derivation of the Numerical Scheme

Consider the fourth-order boundary value problem (BVP):

$$\frac{d^4u(x)}{dx^4} + c(x)\frac{d^2u(x)}{dx^2} + d(x)\frac{du(x)}{dx} + e(x)u(x) = f(x), \quad x \in \Omega = (a,b), \tag{2.1}$$

with two possible sets of boundary conditions:

$$u(a) = \gamma_1, \quad u(b) = \gamma_2, \quad u'(a) = \eta_1, \quad u'(b) = \eta_2, \tag{2.2}$$

or

$$u(a) = \gamma_1, \quad u(b) = \gamma_2, \quad u''(a) = \mu_1, \quad u''(b) = \mu_2. \tag{2.3}$$

To derive the weak form, we introduce a test function $v(x)$ belonging to the Sobolev space $H_0^2(\Omega)$, which satisfies the homogeneous essential boundary conditions. Multiplying (2.1) by $v(x)$ and integrating over Ω yields:

$$\int_a^b \frac{d^4u}{dx^4} v\,dx + \int_a^b \left[c(x)\frac{d^2u}{dx^2} + d(x)\frac{du}{dx} + e(x)u\right] v\,dx = \int_a^b f(x)v\,dx. \tag{2.4}$$

Integrating the first term by parts twice:

$$\begin{aligned}\int_a^b \frac{d^4u}{dx^4} v\,dx &= \left[\frac{d^3u}{dx^3}v\right]_a^b - \int_a^b \frac{d^3u}{dx^3}\frac{dv}{dx}\,dx \\ &= \left[\frac{d^3u}{dx^3}v\right]_a^b - \left[\frac{d^2u}{dx^2}\frac{dv}{dx}\right]_a^b + \int_a^b \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx.\end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4) gives:

$$\int_a^b \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_a^b \left[c(x)\frac{d^2u}{dx^2} + d(x)\frac{du}{dx} + e(x)u\right] v\,dx + \left[\frac{d^3u}{dx^3}v\right]_a^b - \left[\frac{d^2u}{dx^2}\frac{dv}{dx}\right]_a^b = \int_a^b f(x)v\,dx. \tag{2.6}$$

For boundary conditions (2.2) and (2.3) (u and u' specified), we have essential boundary conditions on both u and u'. Therefore, we require $v(a) = v(b) = 0$ and $v'(a) = v'(b) = 0$, which eliminates both boundary terms in (2.6). The weak form becomes:

$$\int_a^b \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_a^b \left[c(x)\frac{d^2u}{dx^2} + d(x)\frac{du}{dx} + e(x)u\right] v\,dx = \int_a^b f(x)v\,dx. \tag{2.7}$$

For boundary conditions (2.2) and (2.3)(u and u'' specified), we have essential boundary conditions on u only. We require $v(a) = v(b) = 0$, which eliminates the $\left[\frac{d^3u}{dx^3}v\right]_a^b$ term. The weak form becomes:

$$\int_a^b \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_a^b \left[c(x)\frac{d^2u}{dx^2} + d(x)\frac{du}{dx} + e(x)u\right] v\,dx - \left[\frac{d^2u}{dx^2}\frac{dv}{dx}\right]_a^b = \int_a^b f(x)v\,dx. \tag{2.8}$$

Using the specified values $u''(a) = \mu_1$ and $u''(b) = \mu_2$, we obtain:

$$\left[\frac{d^2u}{dx^2}\frac{dv}{dx}\right]_a^b = \mu_2 v'(b) - \mu_1 v'(a). \tag{2.9}$$

Thus, (2.8) can be written as:

$$\int_a^b \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_a^b \left[c(x)\frac{d^2u}{dx^2} + d(x)\frac{du}{dx} + e(x)u\right] v\,dx = \int_a^b f(x)v\,dx + \mu_2 v'(b) - \mu_1 v'(a). \tag{2.10}$$

In selecting basis functions, Hermite interpolation is often preferred in cases involving derivative data or requiring smoother approximations. While Lagrange interpolation remains an option, Hermite methods provide enhanced continuity.

We partition the domain $\Omega = [a, b]$ into N elements with nodes $a = x_0 < x_1 < \cdots < x_N = b$. Let $h_i = x_{i+1} - x_i$ be the length of element i. We approximate $u(x)$ using cubic Hermite polynomials, which ensure C^1 continuity and are suitable for fourth-order problems. On each element $[x_i, x_{i+1}]$, the approximation is:

$$u_h(x) = \sum_{j=1}^{4} u_j^e \phi_j^e(\xi), \tag{2.11}$$

where $\xi = (x - x_i)/h_i$ is the local coordinate, $u_1^e = u(x_i)$, $u_2^e = h_i u'(x_i)$, $u_3^e = u(x_{i+1})$, $u_4^e = h_i u'(x_{i+1})$, and the local basis functions are:

$$\phi_1^e(\xi) = 1 - 3\xi^2 + 2\xi^3, \tag{2.12}$$

$$\phi_2^e(\xi) = \xi - 2\xi^2 + \xi^3, \tag{2.13}$$

$$\phi_3^e(\xi) = 3\xi^2 - 2\xi^3, \tag{2.14}$$

$$\phi_4^e(\xi) = -\xi^2 + \xi^3. \tag{2.15}$$

These satisfy:

$$\phi_1^e(0) = 1,\ \phi_1^e(1) = 0,\ \frac{d\phi_1^e}{d\xi}(0) = 0,\ \frac{d\phi_1^e}{d\xi}(1) = 0,$$

$$\phi_2^e(0) = 0,\ \phi_2^e(1) = 0,\ \frac{d\phi_2^e}{d\xi}(0) = 1,\ \frac{d\phi_2^e}{d\xi}(1) = 0,$$

$$\phi_3^e(0) = 0,\ \phi_3^e(1) = 1,\ \frac{d\phi_3^e}{d\xi}(0) = 0,\ \frac{d\phi_3^e}{d\xi}(1) = 0,$$

$$\phi_4^e(0) = 0,\ \phi_4^e(1) = 0,\ \frac{d\phi_4^e}{d\xi}(0) = 0,\ \frac{d\phi_4^e}{d\xi}(1) = 1.$$

The derivatives transform as:

$$\frac{du_h}{dx} = \frac{1}{h_i}\frac{du_h}{d\xi}, \quad \frac{d^2u_h}{dx^2} = \frac{1}{h_i^2}\frac{d^2u_h}{d\xi^2}. \tag{2.16}$$

For each element, we compute the element stiffness matrix $\mathbf{K}^e$ and load vector $\mathbf{F}^e$. Let $\mathbf{u}^e = [u_1^e, u_2^e, u_3^e, u_4^e]^\top$. The weak form contributions are::

$$K_{ij}^{e,1} = \int_0^1 \frac{1}{h_i^3}\frac{d^2\phi_i^e}{d\xi^2}\frac{d^2\phi_j^e}{d\xi^2}\,d\xi. \tag{2.17}$$

$$K_{ij}^{e,2} = \int_0^1 c(x(\xi))\frac{1}{h_i}\frac{d^2\phi_i^e}{d\xi^2}\phi_j^e\,d\xi. \tag{2.18}$$

$$K_{ij}^{e,3} = \int_0^1 d(x(\xi))\frac{d\phi_i^e}{d\xi}\phi_j^e\,d\xi. \tag{2.19}$$

$$K_{ij}^{e,4} = \int_0^1 e(x(\xi))h_i\phi_i^e\phi_j^e\, d\xi. \tag{2.20}$$

The total element stiffness matrix is:

$$\mathbf{K}^e = \mathbf{K}^{e,1} + \mathbf{K}^{e,2} + \mathbf{K}^{e,3} + \mathbf{K}^{e,4}. \tag{2.21}$$

The element load vector is:

$$F_i^e = \int_0^1 f(x(\xi))h_i\phi_i^e\, d\xi. \tag{2.22}$$

For boundary condition set (2.3), additional contributions arise:

$$F_2^e \leftarrow F_2^e - \mu_1 \quad \text{(if element includes node } x = a\text{)}, \quad F_4^e \leftarrow F_4^e + \mu_2 \quad \text{(if element includes node } x = b\text{)}. \tag{2.23}$$

We employ three-point Gauss-Legendre quadrature for numerical integration:

$$\int_0^1 g(\xi)\, d\xi \approx \sum_{k=1}^{3} w_k g(\xi_k), \tag{2.24}$$

Assembling all element contributions yields the global linear system:

$$\mathbf{KU} = \mathbf{F}, \tag{2.25}$$

where $\mathbf{U} = [u_0, u_0', u_1, u_1', \ldots, u_N, u_N']^\top$ is the global vector of unknowns.

- Dirichlet conditions: Set $u_0 = \gamma_1$, $u_N = \gamma_2$.
- Neumann conditions: Set $u_0' = \eta_1$, $u_N' = \eta_2$.

These are enforced by modifying the corresponding rows and columns of $\mathbf{K}$ and $\mathbf{F}$.

- Dirichlet conditions: Set $u_0 = \gamma_1$, $u_N = \gamma_2$.
- The conditions $u''(a) = \mu_1$ and $u''(b) = \mu_2$ are natural and have been incorporated into the weak form.

After solving the global system, the approximate solution is reconstructed as:

$$u_h(x) = \sum_{i=0}^{N} \left[u_i\phi_i(x) + u_i'\psi_i(x)\right], \tag{2.26}$$

where:

- $\phi_i(x)$ = shape functions for u (nodal values),
- $\psi_i(x)$ = shape functions for u' (derivative values).

u_i and u_i' are the unknown nodal values of the function and its first derivative, respectively. $\phi_i(x)$ is the global shape function associated with the value u_i. It satisfies $\phi_i(x_j) = \delta_{ij}$ and $\phi_i'(x_j) = 0$. $\psi_i(x)$ is the global shape function associated with the derivative u_i'. It satisfies $\psi_i(x_j) = 0$ and $\psi_i'(x_j) = \delta_{ij}$.

3. Error estimates

This section addresses two principal elements of the analysis, which are stability analysis and convergence analysis.

Definition 3.1. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ be a multi-index, where:

- Each α_i is a non-negative integer,
- The order of the derivative is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

For a function u, the derivative $D^\alpha u$ is defined as:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}.$$

Definition 3.2. [1] The Sobolev space $H^m(\Omega)$ consists of functions whose weak derivatives up to order m are square-integrable:

$$H^m(\Omega) = \left\{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega), \forall|\alpha| \leq m\right\},$$

with norm:

$$\|u\|_{H^m} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^2}^2\right)^{1/2}.$$

Definition 3.3 (Weak Formulation). Multiply (2.1) by a test function $v \in H_0^1(\Omega)$ and integrate by parts:

$$\int_\Omega u''v''\,dx + \int_\Omega c(x)u''v\,dx + \int_\Omega d(x)u'v\,dx + \int_\Omega e(x)uv\,dx = \int_\Omega fv\,dx.$$

The bilinear form $a(u, v)$ and linear form $L(v)$ are:

$$a(u,v) = \int_\Omega u''v''\,dx + \int_\Omega c(x)u''v\,dx + \int_\Omega d(x)u'v\,dx + \int_\Omega e(x)uv\,dx,$$

$$L(v) = \int_\Omega fv\,dx.$$

Definition 3.4. [10] The cubic Hermite basis functions ϕ_i on an element $[x_i, x_{i+1}]$ satisfy:

$$\phi_i(x_j) = \delta_{ij}, \tag{3.1}$$

$$\phi_i'(x_j) = \delta_{ij}, \tag{3.2}$$

ensuring C^1-continuity.

Theorem 3.5. *(Coercivity) If $c(x), d(x), e(x)$ are bounded and $e(x) \geq e_0 > 0$) then exists $\alpha > 0$ such that:* $a(u,u) \geq \alpha\|u\|_{H^2}^2 \quad \forall u \in H_0^2(\Omega)$.

Proof. For $u, v \in H_0^2(\Omega)$, the bilinear form is defined as:

$$a(u,v) = \int_\Omega u''v''\,dx + \int_\Omega c(x)u''v\,dx + \int_\Omega d(x)u'v\,dx + \int_\Omega e(x)uv\,dx.$$

For $u = v$, we have:

$$a(u,u) = \|u''\|_{L^2}^2 + \int_\Omega c(x)u''u\,dx + \int_\Omega d(x)u'u\,dx + \int_\Omega e(x)u^2\,dx. \tag{3.3}$$

Since c is bounded, let $M_c = \|c\|_{L^\infty}$. Using Young's inequality with $\epsilon_1 > 0$:

$$\left|\int_\Omega c(x)u''u\,dx\right| \leq M_c \int_\Omega |u''||u|\,dx \leq M_c\left(\frac{\epsilon_1}{2}\|u''\|_{L^2}^2 + \frac{1}{2\epsilon_1}\|u\|_{L^2}^2\right). \tag{3.4}$$

Let $M_d = \|d\|_{L^\infty}$. Using Young's inequality with $\epsilon_2 > 0$:

$$\left|\int_\Omega d(x)u'u\,dx\right| \leq M_d \int_\Omega |u'||u|\,dx \leq M_d\left(\frac{\epsilon_2}{2}\|u'\|_{L^2}^2 + \frac{1}{2\epsilon_2}\|u\|_{L^2}^2\right). \tag{3.5}$$

From the assumption on e, we have:

$$\int_\Omega e(x)u^2\,dx \geq e_0\|u\|_{L^2}^2. \tag{3.6}$$

Substituting (3.4), (3.5), and (3.6) into (3.3):

$$\begin{aligned} a(u,u) &\geq \|u''\|_{L^2}^2 - M_c\left(\frac{\epsilon_1}{2}\|u''\|_{L^2}^2 + \frac{1}{2\epsilon_1}\|u\|_{L^2}^2\right) - M_d\left(\frac{\epsilon_2}{2}\|u'\|_{L^2}^2 + \frac{1}{2\epsilon_2}\|u\|_{L^2}^2\right) + e_0\|u\|_{L^2}^2 \\ &= \left(1 - \frac{M_c\epsilon_1}{2}\right)\|u''\|_{L^2}^2 - \frac{M_d\epsilon_2}{2}\|u'\|_{L^2}^2 + \left(e_0 - \frac{M_c}{2\epsilon_1} - \frac{M_d}{2\epsilon_2}\right)\|u\|_{L^2}^2. \end{aligned} \tag{3.7}$$

Since $u \in H_0^2(\Omega)$, we have the following Poincaré inequalities:

$$\|u\|_{L^2} \le C_p \|u'\|_{L^2}, \quad \|u'\|_{L^2} \le C_p \|u''\|_{L^2}.$$

Combining these: $\|u\|_{L^2} \le C_p^2 \|u''\|_{L^2}$ and $\|u'\|_{L^2} \le C_p \|u''\|_{L^2}$. Substituting these into (3.7):

$$\begin{aligned} a(u,u) &\ge \left(1 - \frac{M_c \epsilon_1}{2} - \frac{M_d \epsilon_2 C_p^2}{2}\right) \|u''\|_{L^2}^2 + \left(e_0 - \frac{M_c}{2\epsilon_1} - \frac{M_d}{2\epsilon_2}\right) C_p^4 \|u''\|_{L^2}^2 \\ &= \left[1 - \frac{M_c \epsilon_1}{2} - \frac{M_d \epsilon_2 C_p^2}{2} + \left(e_0 - \frac{M_c}{2\epsilon_1} - \frac{M_d}{2\epsilon_2}\right) C_p^4\right] \|u''\|_{L^2}^2. \end{aligned} \tag{3.8}$$

Select $\epsilon_1 = \frac{1}{M_c C_p^2}$ and $\epsilon_2 = \frac{1}{M_d C_p^2}$ (assuming $M_c, M_d > 0$; if zero, the terms vanish). Then:

$$1 - \frac{M_c \epsilon_1}{2} - \frac{M_d \epsilon_2 C_p^2}{2} = 1 - \frac{1}{2C_p^2} - \frac{1}{2} = \frac{1}{2} - \frac{1}{2C_p^2},$$

$$e_0 - \frac{M_c}{2\epsilon_1} - \frac{M_d}{2\epsilon_2} = e_0 - \frac{M_c^2 C_p^2}{2} - \frac{M_d^2 C_p^2}{2}.$$

Substituting into (3.8):

$$a(u,u) \ge \left[\frac{1}{2} - \frac{1}{2C_p^2} + e_0 C_p^4 - \frac{M_c^2 C_p^6}{2} - \frac{M_d^2 C_p^6}{2}\right] \|u''\|_{L^2}^2. \tag{3.9}$$

For sufficiently small M_c and M_d relative to e_0, the bracket in (3.9) is positive. More precisely, we require:

$$e_0 > \frac{1}{2C_p^4} + \frac{M_c^2 C_p^2}{2} + \frac{M_d^2 C_p^2}{2}.$$

Since $e_0 > 0$ is given and C_p is fixed for the domain Ω, we can choose $\alpha_1 > 0$ such that:

$$a(u,u) \ge \alpha_1 \|u''\|_{L^2}^2.$$

By the norm equivalence in $H_0^2(\Omega)$ (via Poincaré inequalities), there exists $C_e > 0$ such that:

$$\|u\|_{H^2}^2 \le C_e \|u''\|_{L^2}^2.$$

Therefore, setting $\alpha = \alpha_1 / C_e$ gives:

$$a(u,u) \ge \alpha \|u\|_{H^2}^2.$$

This completes the coercivity proof. □

Theorem 3.6 (Continuity)**.** *If $c, d, e \in L^\infty(\Omega)$, then $\exists \beta > 0$ such that:*

$$|a(u,v)| \le \beta \|u\|_{H^2} \|v\|_{H^2}.$$

Proof. By the Cauchy-Schwarz inequality and boundedness of coefficients:

$$|a(u,v)| \le \|u''\|_{L^2} \|v''\|_{L^2} + \|\zeta\|_{L^\infty} \|u''\|_{L^2} \|v\|_{L^2}$$

The result follows from norm equivalence in H^2. □

Theorem 3.7 (Stability)**.** *If the bilinear form $a(u_h, v_h)$ is bounded and coercive on the finite element space V_h spanned by cubic Hermite elements, then the discrete solution u_h is stable in the energy norm.*

Proof. Let the bilinear form be given by: Let the bilinear form be given by:

$$a(u,v) = \int_\Omega \left(\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} + c\frac{d^2u}{dx^2}v + d\frac{du}{dx}v + euv\right) dx,$$

where $c, d, e \ge 0$. There exists $M > 0$ such that:

$$|a(u_h, v_h)| \le M \|u_h\|_{H^2} \|v_h\|_{H^2}, \quad \forall u_h, v_h \in V_h.$$

To show the Coercivity condition (from ellipticity of the ODE),

$$a(u_h, u_h) \geq \alpha \|u_h\|_{H^2}^2, \quad \alpha > 0.$$

By the Lax-Milgram theorem [9], boundedness and coercivity ensure existence, uniqueness, and stability:

$$\|u_h\|_{H^2} \leq \frac{1}{\alpha}\|f\|_{L^2}. \tag{3.10}$$

□

Theorem 3.8 (H^2-Error Estimate)**.** *Let $u \in H^4(\Omega)$ be the exact solution and u_h be the cubic Hermite FEM approximation. Then:*

$$\|u - u_h\|_{H^2} \leq Ch^2|u|_{H^4}.$$

Proof. Using Cea's Lemma [12], we have:

$$\|u - u_h\|_{H^2} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{H^2}. \tag{3.11}$$

Taking $v_h = \Pi_h u$, the interpolation estimates yield:

$$\|u - \Pi_h u\|_{H^2} \leq Ch^2|u|_{H^4}, \tag{3.12}$$

which proves the result. □

Theorem 3.9 (L^2-Error Estimate)**.** *Under the same assumptions as Theorem 3.5, the L^2-error satisfies:*

$$\|u - u_h\|_{L^2} \leq Ch^4|u|_{H^4}.$$

Proof. We proceed via the Aubin–Nitsche duality argument.
For any $w \in L^2(\Omega)$, consider the dual problem: Find $\phi \in H_0^2(\Omega)$ such that

$$a(v, \phi) = (w, v)_{L^2} \quad \forall v \in H_0^2(\Omega),$$

where $a(\cdot,\cdot)$ is the bilinear form of the original fourth-order problem. Setting $w = u - u_h$, we obtain the specific dual problem:

$$a(v, \phi) = (u - u_h, v)_{L^2} \quad \forall v \in H_0^2(\Omega). \tag{3.13}$$

Assume the dual problem satisfies elliptic regularity: there exists a constant $C_R > 0$ such that

$$\phi \in H^4(\Omega) \quad \text{and} \quad \|\phi\|_{H^4(\Omega)} \leq C_R\|u - u_h\|_{L^2(\Omega)}. \tag{3.14}$$

From the original problem, the finite element solution satisfies Galerkin orthogonality:

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h. \tag{3.15}$$

Take $v = u - u_h$ in (3.13):

$$a(u - u_h, \phi) = \|u - u_h\|_{L^2(\Omega)}^2. \tag{3.16}$$

Let $\phi_h \in V_h$ be the finite element interpolant of ϕ. By Galerkin orthogonality (3.15) with $v_h = \phi_h$:

$$a(u - u_h, \phi_h) = 0.$$

Subtracting this from (3.16) gives:

$$\|u - u_h\|_{L^2(\Omega)}^2 = a(u - u_h, \phi - \phi_h). \tag{3.17}$$

Assume the bilinear form $a(\cdot,\cdot)$ is continuous on $H_0^2(\Omega)$: there exists $M > 0$ such that

$$|a(v, w)| \leq M\|v\|_{H^2(\Omega)}\|w\|_{H^2(\Omega)}.$$

Applying this to (3.17):

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq M\|u - u_h\|_{H^2(\Omega)}\|\phi - \phi_h\|_{H^2(\Omega)}. \tag{3.18}$$

(i) Interpolation error: For $\phi \in H^4(\Omega)$,

$$\|\phi - \phi_h\|_{H^2(\Omega)} \leq C_I h^2 |\phi|_{H^4(\Omega)}. \tag{3.19}$$

(ii) Energy-norm error (from Theorem 3.5):

$$\|u - u_h\|_{H^2(\Omega)} \leq C_E h^2 |u|_{H^4(\Omega)}. \tag{3.20}$$

Substitute (3.19) and (3.20) into (3.18):

$$\|u - u_h\|^2_{L^2(\Omega)} \leq M(C_E h^2 |u|_{H^4})(C_I h^2 |\phi|_{H^4}).$$

Using elliptic regularity (3.14) to bound $|\phi|_{H^4} \leq \|\phi\|_{H^4} \leq C_R \|u - u_h\|_{L^2}$:

$$\|u - u_h\|^2_{L^2} \leq M C_E C_I C_R\, h^4 |u|_{H^4} \|u - u_h\|_{L^2}.$$

Dividing both sides by $\|u - u_h\|_{L^2}$ (assuming nonzero error; otherwise the estimate holds trivially):

$$\|u - u_h\|_{L^2} \leq C h^4 |u|_{H^4},$$

where $C = M C_E C_I C_R$. □

4. Numerical Experiments

We verify the cubic Hermite FEM theoretical performance through stability and convergence tests. For exact solution u and FEM solution u_h, we compute:

$$\|u - u_h\|_{L^2} = \left(\int_\Omega |u - u_h|^2 dx \right)^{1/2}, \tag{4.1}$$

$$\|u - u_h\|_{H^2} = \left(\|u - u_h\|^2_{L^2} + \|u' - u_h'\|^2_{L^2} + \|u'' - u_h''\|^2_{L^2} \right)^{1/2}. \tag{4.2}$$

The convergence rate r is calculated as:

$$r = \frac{\log(\|e_{h_1}\| / \|e_{h_2}\|)}{\log(h_1/h_2)},$$

where e_{h_1} and e_{h_2} are the errors at h_1 and h_2, respectively.

Example 4.1. Consider the fourth order BVP with constant coefficients

$$u^{(4)}(x) - 2u^{(2)}(x) + u(x) = -\frac{e^{1-x}(3\cos(x) + 4\sin(x))}{\cos(1)}, \quad x \in (-1, 1). \tag{4.3}$$

The boundary conditions are given by:

$$u(-1) = e^2, u(1) = 1, u'(-1) = e^2(\tan 1 - 1), u'(1) = -(1 + \tan 1). \tag{4.4}$$

or

$$u(-1) = e^2, u(1) = 1, u''(-1) = -2e^2 \tan(1), u''(1) = 2\tan(1). \tag{4.5}$$

The exact solution of such equation is

$$u(x) = \frac{e^{1-x}\cos(x)}{\cos(1)}. \tag{4.6}$$

Example 4.2. [27] Consider fourth order the linear boundary value problem with constant coefficients

$$u^{(4)} - u = -4(2x\cos x + 3\sin x), \quad -1 < x < 1$$

subject to

$$u(-1) = u(1) = 0,\ u'(-1) = u'(1) = 2\sin(1). \tag{4.7}$$

The exact solution is

$$y = (x^2 - 1)\sin(x). \tag{4.8}$$

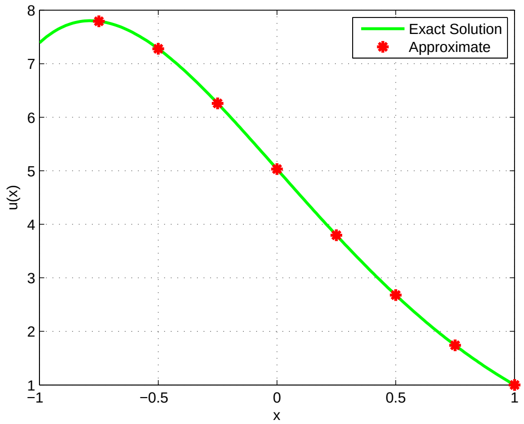

FIGURE 1. Comparison of exact and approximation solutions for Example 4.1.

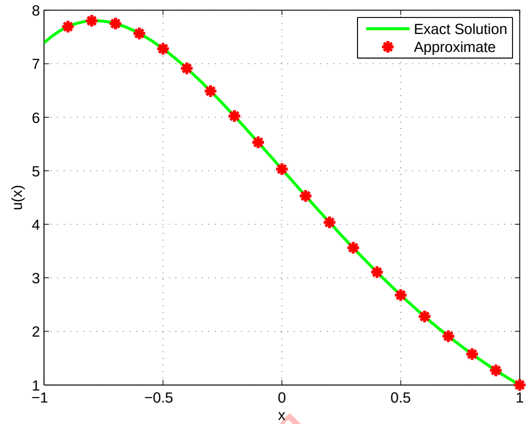

FIGURE 2. Comparison of exact and approximation solutions for Example 4.1.

TABLE 1. Error norms and convergence rates for Example 4.1 with boundary condition 4.4.

h	H^2	Order	L^2	Order
1/2	2.840×10^{-1}	–	3.000×10^{-3}	–
1/4	7.100×10^{-2}	2.000	1.941×10^{-4}	3.999
1/8	1.800×10^{-2}	1.980	1.212×10^{-5}	4.001
1/16	4.000×10^{-3}	2.170	7.572×10^{-7}	4.001
1/32	1.000×10^{-3}	2.000	4.722×10^{-8}	4.0031

TABLE 2. Error norms and convergence rates for Example 4.1 with boundary condition 4.5.

h	H^2	Order	L^2	Order
1/2	2.840×10^{-1}	–	2.400×10^{-3}	–
1/4	7.110×10^{-2}	2.002	1.530×10^{-4}	3.995
1/8	1.770×10^{-2}	2.000	9.571×10^{-6}	3.999
1/16	4.400×10^{-3}	2.000	5.982×10^{-7}	4.000
1/32	1.100×10^{-3}	2.000	3.707×10^{-8}	4.013

TABLE 3. Error norms and convergence rates for Example 4.2.

N	h	$\|e\|_{L^2}$	Order (for L^2)	$\|e\|_{H^2}$	Order (for H^2)
1	2.0	4.20×10^{-2}	–	1.89	–
2	1.0	4.01×10^{-3}	3.39	4.94×10^{-1}	1.94
4	0.5	2.76×10^{-4}	3.86	1.25×10^{-1}	1.98
8	0.25	1.78×10^{-5}	3.96	3.13×10^{-2}	2.00
16	0.125	1.12×10^{-6}	3.99	7.82×10^{-3}	2.00
32	0.0625	7.01×10^{-8}	4.00	1.96×10^{-3}	2.00

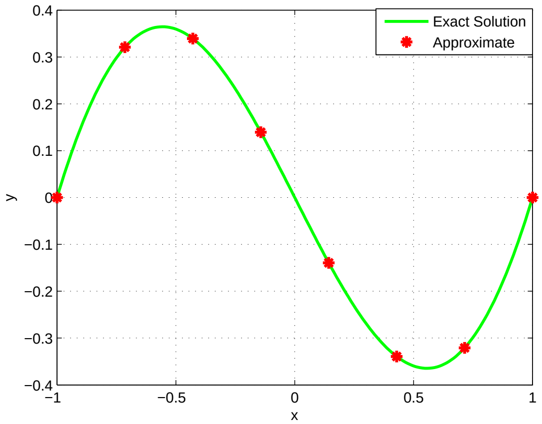

FIGURE 3. Comparison of exact and approximation solutions for Example 4.2.

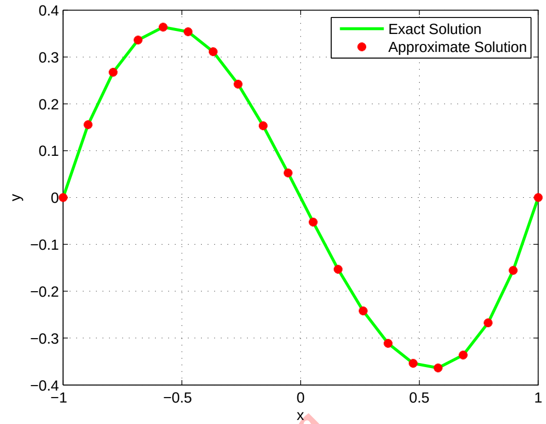

FIGURE 4. Comparison of exact and approximation solutions for Example 4.2.

Example 4.3. Consider fourth order BVP with variable coefficients

$$\frac{d^4u(x)}{dx^4} - (1+x)\frac{d^2u(x)}{dx^2} + \cos(x)u(x) = f(x), \quad x \in \Omega = (0, \pi), \tag{4.9}$$

with boundary conditions:

$$u(0) = 0, \quad u(\pi) = 0, \quad u'(0) = 1, \quad u'(\pi) = -1, \tag{4.10}$$

where $f(x) = \sin x(2 + x + \cos x)$ with the exact solution $u(x) = \sin(x)$.

TABLE 4. Error norms and convergence rates for Example 4.3.

N	h	$\|e\|_{L^2}$	Order	$\|e\|_{H^2}$	Order
2	$\frac{\pi}{2}$	$3.12E-03$	–	$4.51E-02$	–
4	$\frac{\pi}{4}$	$1.98E-04$	3.98	$1.12E-02$	2.01
8	$\frac{\pi}{8}$	$1.24E-05$	4.00	$2.79E-03$	2.01
16	$\frac{\pi}{16}$	$7.76E-07$	4.00	$6.96E-04$	2.00
32	$\frac{\pi}{32}$	$4.85E-08$	4.00	$1.74E-04$	2.00

The visual comparisons for Examples 4.1 and 4.2 (shown in Figures 1–4) demonstrate that the numerical and exact solutions align almost perfectly with mesh refinement, providing a clear visual confirmation of the method's stability and consistency. In Example 4.1, the numerical results produced using the two different boundary condition sets become identical on sufficiently refined meshes, demonstrating that the method is not sensitive to how the boundaries are enforced. Example 4.2 further demonstrates accurate capture of oscillatory behavior inherent in the exact solution, while Example 4.3 confirms that variable coefficients do not degrade qualitative accuracy. Across all graphs, refinement reduces phase and amplitude errors uniformly, corroborating the theoretical convergence mechanisms derived from the coercive and continuous bilinear form. The numerical results presented in Tables 1–4 demonstrate $O(h^2)$ convergence in the H^2-norm and nearly $O(h^4)$ convergence in the L^2-norm, in agreement with Theorems 3.8 and 3.9. Example 4.2 shows the same asymptotic rates, with L^2 orders approaching 4 and H^2 orders approaching 2 as N increases, validating the duality-based enhancement in L^2. The results for the variable-coefficient problem (Example 4.3) also achieve optimal convergence rates, confirming the method's robustness.

5. Conclusion

In this paper, we developed a cubic Hermite FEM for solving fourth-order BVPs. The proposed scheme effectively handles both clamped and simply supported boundary conditions while ensuring C^1-continuity through Hermite interpolation. Theoretical analysis confirmed the stability and optimal convergence rates, with $\mathcal{O}(h^2)$ in the H^2-norm and $\mathcal{O}(h^4)$ in the L^2-norm. Numerical experiments validated the method's accuracy and efficiency, demonstrating excellent agreement with exact solutions.

Future work includes developing a multidimensional implementation for fourth-order PDEs (e.g., biharmonic problems) using tensor-product Hermite finite elements. A natural extension of this work involves generalizing the cubic Hermite FEM to multidimensional fourth-order problems through tensor-product formulations, particularly for applications involving biharmonic operators in 2D/3D.

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