Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-20
DOI:10.22034/cmde.2026.61736.2682

A novel generalized type of the Caputo fractional derivative: integral transforms, illustrative examples, and solution of fractional-order generalized differential equations

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Abstract

In this article, we introduce a novel generalized Caputo fractional derivative, using a special type of function known as the Wright function in the definition of the classical Caputo fractional derivative. We also apply the Fourier, Laplace, and Mellin integral transform methods, which are very useful popular mathematical tools in various scientific fields, to the new generalized fractional derivative. Moreover, as illustrative examples, we calculate the new generalized fractional derivative of constant, power, exponential, sine, and cosine functions. Furthermore, we obtain the solutions of the generalized motion, harmonic vibration, and Bessel differential equations defined by the new generalized fractional derivative using the Fourier, Laplace, and Mellin integral transform methods. Finally, we obtain approximate behavior graphs for both the classical Caputo fractional derivative and the generalized Caputo fractional derivative using some specific data and present these graphs comparatively.

Keywords. Caputo fractional derivative, Fourier integral transform, Laplace integral transform, Mellin integral transform, Wright function.
2010 Mathematics Subject Classification. 26A33, 42A38, 44A10, 44A15, 33E20.

1. Introduction

Fractional calculus has become a rapidly developing branch of mathematics in recent years, thanks to the discovery of applications in many scientific fields. One of the main goals of researchers has been to find the most appropriate and useful fractional operators to better describe the complexity of various problems that exist in nature. The articles by Liouville [35] and Caputo [18], which introduced the fractional operator models known as the Riemann-Liouville and Caputo definitions, both of which are the most standard in the field of fractional calculus, have motivated much of the work that now exists in fractional calculus. In this way, researchers have defined new fractional derivatives that can be used to model real-life problems to achieve successful results in both theory and practice. The readers can refer to various generalized Riemann-Liouville fractional operators in [2, 5, 10–12, 15, 25, 41, 42, 46, 49] and various generalized Caputo fractional operators in [3, 6, 10–12, 19, 25, 30, 31, 37]. The same process continues to the present day, with works appearing every year and introducing new models of fractional calculus every year [13, 17, 36, 47]. The final criterion for which fractional derivative is more appropriate for a given real-life process has to be determined by means of relevant experimental (real) data. Only by comparing real data can one determine which of the fractional operators used provides more reliable results or whether the classical derivative gives better results. Therefore, fractional calculus is a field of mathematical calculus that is worth thinking about and studying due to the wide range of open problems it still contains and the wide range of its applicability to various fields.

Integral transforms are the most effective mathematical tools for solving ordinary, partial, and fractional differential equations. These integral transforms convert differential equations into algebraic equations and provide great simplicity in computations. There are many integral transforms available in the literature. The most popular of these integral transforms are the Fourier, Laplace, and Mellin integral transforms, which have strong applications in many disciplines

Received: 21 May 2024; Accepted: 25 May 2026.
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such as mathematics, physics, chemistry, biology, engineering, and astronomy. Sometimes, these integral transforms are insufficient to solve the problems under consideration. For this reason, researchers have defined many new integral transforms in order to overcome the inadequacies of these integral transforms in related fields of study and to expand their application areas. The readers may refer to various generalized Fourier integral transforms in [9, 28, 33, 38, 44], various generalized Laplace integral transforms in [1, 7, 22, 27, 39, 48, 50] and various generalized Mellin integral transforms in [8, 14, 21, 23, 24, 26].

In the following, we list the purpose of this article and the innovations it will present.

- In this study, a **generalized Caputo fractional derivative** involving the **Wright function**, a multiparameter special function, is introduced to broaden the applicability of fractional calculus in modeling complex real-world phenomena. The proposed operator provides a flexible framework capable of capturing diverse dynamical behaviors.
- The **analytical evaluation** of the newly proposed fractional derivative on various **elementary functions** yields novel expressions characterized by the Wright function. The obtained results exhibit rich structural diversity depending on the specific choices of the Wright function parameters.
- To explore the analytical properties of the new operator, we employ the **Fourier**, **Laplace**, and **Mellin** integral transform techniques. Through these classical transformation methods, several new relations are established, which in turn provide new insights and solution forms for fractional differential equations.
- Based on the derived theoretical results, **comparative graphical analyses** are performed to illustrate the approximate behaviors of both the **classical Caputo fractional derivative** and the **proposed generalized Caputo fractional derivative**, demonstrating the enhanced modeling capability and flexibility of the new formulation.

We organize the balance of this article as follows:
In section 2, we provide the basic concepts that we will need throughout the article. In section 3, we describe the generalized Caputo fractional derivative and give some properties such as boundness, linearity, and interpolation. In section 4, we take the Fourier, Laplace, and Mellin integral transforms of the new generalized Caputo fractional derivative. In section 5, as illustrative examples, we compute the new generalized Caputo fractional derivative of the constant, power, exponential, sine, and cosine functions. In section 6, we obtain the solution of the fractional order generalized motion, harmonic vibration, and Bessel differential equations. In section 7, we present some numerical results with specific data and approximate behavior graphs. In section 8, we give the conclusion of the article.

2. Preliminaries

This section presents the basic mathematical materials required for the following sections. These are the definitions of special functions, fractional derivatives, integral transformations and their convolutions, and some theorems. Namely, the definitions of the gamma function, Wright function, and Mittag-Leffler function as special functions; Caputo fractional derivatives as fractional derivatives; Fourier, inverse Fourier, Laplace, inverse Laplace, Mellin, inverse Mellin integral transforms as integral transforms and convolution of these integral transforms are presented and then the convolution and derivative theorems for these integral transforms are also presented.

Definition 2.1 (See Ref. [4])**.** The gamma function is introduced by

$$\Gamma(\sigma) = \int_0^\infty \omega^{\sigma-1} \exp(-\omega)\, d\omega, \quad (\Re(\sigma) > 0).$$

Definition 2.2 (See Ref. [32])**.** The Wright function is denoted by the series

$$\phi(\varpi, \tau; z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\varpi k + \tau)} \frac{z^k}{k!}, \quad (z, \varpi, \tau \in \mathbb{C}; \Re(\varpi) > -1). \tag{2.1}$$

Definition 2.3 (See Ref. [32]). The Mittag-Leffler function with two parameters is denoted by the series

$$E_{\varpi,\tau}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\varpi k+\tau)},\quad (z,\varpi,\tau\in\mathbb{C};\Re(\varpi)>0).$$

Definition 2.4 (See Ref. [32]). The Caputo fractional derivatives of order ψ are defined by

$$\left({}^cD_{x+}^{\psi}f\right)(t)=\frac{1}{\Gamma(\ell-\psi)}\int_x^t(t-\nu)^{\ell-\psi-1}f^{(\ell)}(\nu)d\nu,\quad (t>x), \tag{2.2}$$

$$\left({}^cD_{+}^{\psi}f\right)(t)=\frac{1}{\Gamma(\ell-\psi)}\int_{-\infty}^t(t-\nu)^{\ell-\psi-1}f^{(\ell)}(\nu)d\nu,\quad (t\in\mathbb{R}), \tag{2.3}$$

and

$$\left({}^cD_{0+}^{\psi}f\right)(t)=\frac{1}{\Gamma(\ell-\psi)}\int_0^t(t-\nu)^{\ell-\psi-1}f^{(\ell)}(\nu)d\nu,\quad (t\in\mathbb{R}^+), \tag{2.4}$$

where $\ell-1<\Re(\psi)\le\ell,\ \ell\in\mathbb{N}$.

Definition 2.5 (See Refs. [20, 32]). The Fourier, Laplace, and Mellin integral transforms respectively are described by

$$\mathfrak{F}\Big[f(t)\Big](\omega)=\int_{-\infty}^{\infty}\exp(-i\omega t)f(t)dt,\quad (\omega\in\mathbb{R}),$$

$$\mathfrak{L}\Big[f(t)\Big](s)=\int_0^{\infty}\exp(-st)f(t)dt,\quad (s\in\mathbb{C}),$$

and

$$\mathfrak{M}\Big[f(t)\Big](p)=\int_0^{\infty}t^{p-1}f(t)dt,\quad (p\in\mathbb{C}).$$

Definition 2.6 (See Refs. [20, 32]). The inverse Fourier, Laplace, and Mellin integral transforms respectively are defined by

$$\mathfrak{F}^{-1}\Big[\mathfrak{F}\Big[f(t)\Big](\omega)\Big](t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\omega t)\ \mathfrak{F}\Big[f(t)\Big](\omega)d\omega,\quad (t\in\mathbb{R}),$$

$$\mathfrak{L}^{-1}\Big[\mathfrak{L}\Big[f(t)\Big](s)\Big](t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp(st)\ \mathfrak{L}\Big[f(t)\Big](s)ds,\quad (t\in\mathbb{R}_0^+),$$

and

$$\mathfrak{M}^{-1}\Big[\mathfrak{M}\Big[f(t)\Big](p)\Big](t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}t^{-p}\ \mathfrak{M}\Big[f(t)\Big](p)dp,\quad (t\in\mathbb{R}^+).$$

Definition 2.7 (See Ref. [20]). The convolutions of Fourier, Laplace, and Mellin integral transforms respectively are presented by

$$(f\star g)(t)=\int_{-\infty}^{\infty}f(t-\nu)g(\nu)d\nu,\quad (t\in\mathbb{R}),$$

$$(f*g)(t)=\int_0^{t}f(t-\nu)g(\nu)d\nu,\quad (t\in\mathbb{R}^+),$$

and

$$(f\circ g)(t)=\int_0^{\infty}f(t\nu)g(\nu)d\nu,\quad (t\in\mathbb{R}^+).$$

Theorem 2.8 (See Ref. [20]). *The convolution theorems of Fourier, Laplace, and Mellin integral transforms respectively are obtained by*

$$\mathfrak{F}\Big[(f \star g)(t)\Big](\omega) = \mathfrak{F}\Big[f(t)\Big](\omega)\ \mathfrak{F}\Big[g(t)\Big](\omega),$$
$$\mathfrak{L}\Big[(f * g)(t)\Big](s) = \mathfrak{L}\Big[f(t)\Big](s)\ \mathfrak{L}\Big[g(t)\Big](s),$$

and

$$\mathfrak{M}\Big[(f \circ g)(t)\Big](p) = \mathfrak{M}\Big[f(t)\Big](p)\ \mathfrak{M}\Big[g(t)\Big](1-p).$$

Theorem 2.9 (See Ref. [20]). *The derivative theorems of Fourier, Laplace, and Mellin integral transforms respectively are given by*

$$\mathfrak{F}\Big[f^{(\ell)}(t)\Big](\omega) = (i\omega)^{\ell}\ \mathfrak{F}\Big[f(t)\Big](\omega),$$
$$\mathfrak{L}\Big[f^{(\ell)}(t)\Big](s) = s^{\ell}\mathfrak{L}\Big[f(t)\Big](s) - \sum_{r=0}^{\ell-1} s^{\ell-r-1} f^{(r)}(0),$$

and

$$\mathfrak{M}\Big[f^{(\ell)}(t)\Big](p) = \frac{\Gamma(1-p+\ell)}{\Gamma(1-p)}\mathfrak{M}\Big[f(t)\Big](p-\ell),$$

where $\ell \in \mathbb{N}$.

3. A novel generalized type of the Caputo fractional derivative

This section introduces a new generalized Caputo fractional derivative defined by the Wright function. Moreover, it provides some properties such as boundedness, linearity, and interpolation.

Obviously, as shown in (3.1) below, we use the Wright function. Since the Wright function in the generalized Caputo fractional derivative is multiparameter, we believe that this fractional derivative will give better results than the classical Caputo fractional derivative in applications to real-life problems. For this reason, in this research article, we present a new generalization of the Caputo fractional derivative using this special function. Note also that the Wright function given in (2.1) must satisfy the conditions in the case $\Re(\varpi) > -1$, (2.1) is absolutely convergent for every $z \in \mathbb{C}$ and in the case $\Re(\varpi) = -1$, (2.1) is absolutely convergent on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 3.1. Let $\varpi, \Delta, \psi \in \mathbb{C}$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \leq \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$, $-\infty \leqq x < t < y \leqq +\infty$ and $f \in C^{\ell}[x, y]$. Then the generalized Caputo fractional derivative of order ψ is defined by

$$\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} f\right)(t) := \int_x^t (t-\nu)^{\ell-\psi-1} f^{(\ell)}(\nu)\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) d\nu, \tag{3.1}$$

where denotes $\phi(\cdot)$ is the Wright function and its definition is given in (2.1).

If we take $x = -\infty$ in (3.1), we have

$$\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} f\right)(t) := \int_{-\infty}^t (t-\nu)^{\ell-\psi-1} f^{(\ell)}(\nu)\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) d\nu, \tag{3.2}$$

where $\varpi, \Delta, \psi \in \mathbb{C}$, $t \in \mathbb{R}$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \leq \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$ and $f \in C^{\ell}[x, y]$.

If we take $x = 0$ in (3.1), we obtain

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t) := \int_0^t (t-\nu)^{\ell-\psi-1} f^{(\ell)}(\nu)\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) d\nu, \tag{3.3}$$

where $\varpi, \Delta, \psi \in \mathbb{C}$, $t \in \mathbb{R}^{+}$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \leq \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$ and $f \in C^{\ell}[x, y]$.

Remark 3.2. Notice that when we take $\varpi = 0$ in (3.1)-(3.3), we get (2.2)-(2.4).

Theorem 3.3 (Boundedness)**.** *Let $\varpi, \Delta, \psi \in \mathbb{C}$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \leq \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$ and $y > x$. Then,*

$$\|\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} f\right)(t)\|_C \leq B\|f\|_{C^\ell},$$

where

$$B = (y-x)^{\ell-\Re(\psi)} \sum_{k=0}^{\infty} \frac{|\varpi\Delta(\ell-\psi-1)(\ell-\psi)(y-x)^{\Re(\varpi)}|^k}{|\Gamma(\varpi k + \ell - \psi)|\,[\Re(\varpi)k + \ell - \Re(\psi)]\,k!}.$$

Proof. We consider the norms $\|\cdot\|_C : C[x,y] \to \mathbb{R}$ and $\|\cdot\|_{C^\ell} : C^\ell[x,y] \to \mathbb{R}$ given by

$$\|f\|_C := \max_{t\in[x,y]} |f(t)| \quad \text{and} \quad \|f\|_{C^\ell} := \sum_{k=0}^{\ell} \|f^{(k)}\|_C.$$

Since $|f^{(\ell)}(\nu)| \leq \|f\|_{C^\ell}$ for all $\nu \in [x,y]$, we have

$$\begin{aligned}
\|\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} f\right)\|_C &= \max_{t\in[x,y]} \left| \int_x^t (t-\nu)^{\ell-\psi-1}\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) f^{(\ell)}(\nu)d\nu \right| \\
&\leq \|f\|_{C^\ell} \max_{t\in[x,y]} \left| \int_x^t (t-\nu)^{\ell-\psi-1} \sum_{k=0}^{\infty} \frac{\Big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big)^k}{\Gamma(\varpi k + \ell - \psi)k!} d\nu \right| \\
&\leq B\|f\|_{C^\ell}.
\end{aligned}$$

□

Theorem 3.4 (Linearity)**.** *Let $\varpi, \Delta, \psi \in \mathbb{C}$, $\kappa, \mu \in \mathbb{R}$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \leq \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$. Then,*

$$\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} \kappa f + \mu g\right)(t) = \kappa\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} f\right)(t) + \mu\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} g\right)(t), \tag{3.4}$$

where $t > x$.

Proof. Using (3.1), we have

$$\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} \kappa f + \mu g\right)(t) = \int_x^t (t-\nu)^{\ell-\psi-1}\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) \times \frac{d^\ell}{d\nu^\ell}\Big(\kappa f(\nu) + \mu g(\nu)\Big) d\nu.$$

Then,

$$\begin{aligned}
\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} \kappa f + \mu g\right)(t) &= \kappa \int_x^t (t-\nu)^{\ell-\psi-1} f^{(\ell)}(\nu)\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) d\nu \\
&\quad + \mu \int_x^t (t-\nu)^{\ell-\psi-1} g^{(\ell)}(\nu)\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) d\nu.
\end{aligned}$$

Hence,

$$\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} \kappa f + \mu g\right)(t) = \kappa\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} f\right)(t) + \mu\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)} g\right)(t).$$

□

Corollary 3.5. *If we take $x = -\infty$ and $x = 0$ in (3.4) respectively, we get*

$$\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} \kappa f + \mu g\right)(t) = \kappa\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} f\right)(t) + \mu\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} g\right)(t), \quad t \in \mathbb{R}$$

and

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \kappa f + \mu g\right)(t) = \kappa\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t) + \mu\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} g\right)(t), \quad t \in \mathbb{R}^+.$$

Theorem 3.6 (Interpolation)**.** *Let $\varpi, \Delta, \psi \in \mathbb{C}$, $t \in \mathbb{R}^+$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \leq \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$. Then,*

$$\lim_{\psi\to\ell} \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t) = f^{(\ell)}(t),$$

$$\lim_{\psi\to\ell-1} \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t) = f^{(\ell-1)}(t) - f^{(\ell-1)}(0).$$

Proof. Rewriting (3.3), we have

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t) = \int_0^t (t-\nu)^{\ell-\psi-1} f^{(\ell)}(\nu)\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) d\nu$$
$$= \sum_{k=0}^{\infty} \frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^k}{\Gamma(\varpi k+\ell-\psi)k!} \int_0^t (t-\nu)^{\varpi k+\ell-\psi-1} f^{(\ell)}(\nu) d\nu.$$

Using integration by parts, then

$$u = f^{(\ell)}(\nu), \qquad dv = (t-\nu)^{\varpi k+\ell-\psi-1}\, d\nu$$
$$du = f^{(\ell+1)}(\nu)\, d\nu, \qquad v = -\frac{(t-\nu)^{\varpi k+\ell-\psi}}{\varpi k+\ell-\psi}$$

Thus,

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t) = t^{\ell-\psi} f^{(\ell)}(0) \sum_{k=0}^{\infty} \frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)t^{\varpi}\big)^k}{\Gamma(\varpi k+\ell-\psi+1)k!}$$
$$+ \int_0^t (t-\nu)^{\ell-\psi} \sum_{k=0}^{\infty} \frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\big)^k}{\Gamma(\varpi k+\ell-\psi+1)k!} f^{(\ell+1)}(\nu) d\nu.$$

For $\psi \to \ell$ and $\psi \to \ell - 1$ respectively, we get

$$\lim_{\psi\to\ell} \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t) = f^{(\ell)}(0) + \int_0^t f^{(\ell+1)}(\nu) d\nu = f^{(\ell)}(t)$$

and

$$\lim_{\psi\to\ell-1} \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t) = t f^{(\ell)}(0) + \int_0^t (t-\nu) f^{(\ell+1)}(\nu) d\nu$$
$$= t f^{(\ell)}(0) + \Big[(t-\nu) f^{(\ell)}(\nu)\Big]_0^t + \int_0^t f^{(\ell)}(\nu) d\nu$$
$$= f^{(\ell-1)}(t) - f^{(\ell-1)}(0).$$

□

4. Integral transforms

This section presents some results obtained by applying Fourier, Laplace, and Mellin integral transforms to the generalized Caputo fractional derivative. These results are of great importance for obtaining solutions of fractional differential equations of various orders involving the generalized Caputo fractional derivative.

4.1. Fourier integral transform.

Theorem 4.1. *Let $\varpi, \Delta, \psi \in \mathbb{C}$, $t \in \mathbb{R}$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \le \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$. Then,*

$$\mathfrak{F}\left[\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](\omega) = \exp\left(\frac{\varpi\Delta(\ell-\psi-1)(\ell-\psi)}{(i\omega)^{\varpi}}\right)(i\omega)^{\psi}\ \mathfrak{F}\Big[f(t)\Big](\omega).$$

Proof. Using the Fourier integral transform and (3.2), we have

$$\mathfrak{F}\left[\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](\omega) = \mathfrak{F}\left[\int_{-\infty}^t (t-\nu)^{\ell-\psi-1}\ \phi\Big(\varpi, \ell-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big) f^{(\ell)}(\nu) d\nu\right](\omega)$$
$$= \sum_{k=0}^{\infty} \frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^k}{\Gamma(\varpi k+\ell-\psi)k!} \mathfrak{F}\left[\int_{-\infty}^t (t-\nu)^{\varpi k+\ell-\psi-1} f^{(\ell)}(\nu) d\nu\right](\omega).$$

Considering the convolution of Fourier integral transform, we get

$$\mathfrak{F}\left[\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} f\right)(t)\right](\omega) = \sum_{k=0}^{\infty} \frac{\left(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\right)^k}{k!} \mathfrak{F}\left[\frac{t^{\varpi k+\ell-\psi-1}}{\Gamma(\varpi k+\ell-\psi)} \star f^{(\ell)}(t)\right](\omega).$$

Using the convolution theorem of Fourier integral transform, we find

$$\mathfrak{F}\left[\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} f\right)(t)\right](\omega) = \sum_{k=0}^{\infty} \frac{\left(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\right)^k}{k!} \mathfrak{F}\Big[g_+(t)\Big](\omega)\ \mathfrak{F}\left[f^{(\ell)}(t)\right](\omega), \tag{4.1}$$

where

$$g_+(t) = \begin{cases} \frac{t^{\varpi k+\ell-\psi-1}}{\Gamma(\varpi k+\ell-\psi)}, & \text{for } \ t>0, \\ 0, & \text{for } \ t\le 0. \end{cases}$$

Then,

$$\mathfrak{F}\Big[g_+(t)\Big](\omega) = |\omega|^{-(\varpi k+\ell-\psi)} \exp\left(-\left(\frac{i\pi}{2}\right)(\varpi k+\ell-\psi)\,\mathrm{sgn}(\omega)\right).$$

Here using the formula (see Ref. [45]) $(\mp i\omega)^\lambda = |\omega|^\lambda \exp\left(\mp\frac{i\lambda\pi}{2}\mathrm{sgn}(\omega)\right)$ for $\lambda\in\mathbb{R}$, we obtain

$$\mathfrak{F}\Big[g_+(t)\Big](\omega) = (i\omega)^{-(\varpi k+\ell-\psi)}. \tag{4.2}$$

Finally, using the derivative theorem of Fourier integral transform and (4.2) in (4.1), we get

$$\mathfrak{F}\left[\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} f\right)(t)\right](\omega) = \exp\left(\frac{\varpi\Delta(\ell-\psi-1)(\ell-\psi)}{(i\omega)^\varpi}\right)(i\omega)^\psi\ \mathfrak{F}\Big[f(t)\Big](\omega).$$

□

4.2. **Laplace integral transform.**

Theorem 4.2. *Let $\varpi,\Delta,\psi\in\mathbb{C}$, $t\in\mathbb{R}^+$, $\ell\in\mathbb{N}$, $\ell-1<\Re(\psi)\le\ell$, $\Re(\psi)>0$, $\Re(\varpi)>-1$. Then,*

$$\mathfrak{L}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t)\right](s) = \exp\left(\frac{\varpi\Delta(\ell-\psi-1)(\ell-\psi)}{s^\varpi}\right)\left(s^\psi\mathfrak{L}\Big[f(t)\Big](s) - \sum_{r=0}^{\ell-1} s^{\psi-r-1} f^{(r)}(0)\right).$$

Proof. Using the Laplace integral transform and (3.3), we have

$$\begin{aligned}\mathfrak{L}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t)\right](s) &= \mathfrak{L}\left[\int_0^t (t-\nu)^{\ell-\psi-1} f^{(\ell)}(\nu)\ \phi\Big(\varpi,\ell-\psi;\varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^\varpi\Big)d\nu\right](s)\\ &= \sum_{k=0}^{\infty} \frac{\left(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\right)^k}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{L}\left[\int_0^t (t-\nu)^{\varpi k+\ell-\psi-1} f^{(\ell)}(\nu)d\nu\right](s).\end{aligned}$$

Considering the convolution of Laplace integral transform, we get

$$\mathfrak{L}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t)\right](s) = \sum_{k=0}^{\infty} \frac{\left(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\right)^k}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{L}\Big[t^{\varpi k+\ell-\psi-1} * f^{(\ell)}(t)\Big](s).$$

Using the convolution theorem of Laplace integral transform, we find

$$\mathfrak{L}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} f\right)(t)\right](s) = \sum_{k=0}^{\infty} \frac{\left(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\right)^k}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{L}\Big[t^{\varpi k+\ell-\psi-1}\Big](s)\ \mathfrak{L}\left[f^{(\ell)}(t)\right](s). \tag{4.3}$$

By calculating the first Laplace integral transform, we obtain

$$\mathfrak{L}\Big[t^{\varpi k+\ell-\psi-1}\Big](s) = \frac{\Gamma(\varpi k+\ell-\psi)}{s^{\varpi k+\ell-\psi}}. \tag{4.4}$$

Finally, using the derivative theorem of Laplace integral transform and (4.4) in (4.3), we get

$$\mathfrak{L}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](s)=\exp\left(\frac{\varpi\Delta(\ell-\psi-1)(\ell-\psi)}{s^{\varpi}}\right)\left(s^{\psi}\mathfrak{L}\Big[f(t)\Big](s)-\sum_{r=0}^{\ell-1}s^{\psi-r-1}f^{(r)}(0)\right).\qquad\square$$

4.3. **Mellin integral transform.**

Theorem 4.3. *Let* $\varpi,\Delta,\psi\in\mathbb{C}$, $t\in\mathbb{R}^{+}$, $\ell\in\mathbb{N}$, $\ell-1<\Re(\psi)\leq\ell$, $\Re(\psi)>0$, $\Re(\varpi)>-1$. *Then,*

$$\mathfrak{M}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](p)=\sum_{k=0}^{\infty}\frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^{k}}{k!}\frac{\Gamma(1-p-\varpi k+\psi)}{\Gamma(1-p)}\,\mathfrak{M}\Big[f(t)\Big](p+\varpi k-\psi),$$

where $\Re(1-p)>0$ *and* $\Re(1-p-\varpi k+\psi)>0$ *for* $k=0,1,2,\cdots$.

Proof. Using the Mellin integral transform and (3.3), we have

$$\begin{aligned}\mathfrak{M}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](p)&=\mathfrak{M}\left[\int_0^t(t-\nu)^{\ell-\psi-1}f^{(\ell)}(\nu)\;\phi\Big(\varpi,\ell-\psi;\varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big)d\nu\right](p)\\&=\sum_{k=0}^{\infty}\frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^{k}}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{M}\left[\int_0^t(t-\nu)^{\varpi k+\ell-\psi-1}f^{(\ell)}(\nu)d\nu\right](p).\end{aligned}$$

Then, taking $g(t)=f^{(\ell)}(t)$, we get

$$\mathfrak{M}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](p)=\sum_{k=0}^{\infty}\frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^{k}}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{M}\left[\int_0^t(t-\nu)^{\varpi k+\ell-\psi-1}g(\nu)d\nu\right](p).\tag{4.5}$$

Considering the derivative theorem of Mellin integral transform, we find

$$\mathfrak{M}\Big[g(t)\Big](p)=\mathfrak{M}\Big[f^{(\ell)}(t)\Big](p)=\frac{\Gamma(1-p+\ell)}{\Gamma(1-p)}\mathfrak{M}\Big[f(t)\Big](p-\ell).\tag{4.6}$$

Taking $\nu=t\xi$ in (4.5), we obtain

$$\begin{aligned}\mathfrak{M}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](p)&=\sum_{k=0}^{\infty}\frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^{k}}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{M}\left[t^{\varpi k+\ell-\psi}\int_0^1(1-\xi)^{\varpi k+\ell-\psi-1}g(t\xi)d\xi\right](p)\\&=\sum_{k=0}^{\infty}\frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^{k}}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{M}\left[t^{\varpi k+\ell-\psi}\int_0^{\infty}h(\xi)g(t\xi)d\xi\right](p),\end{aligned}\tag{4.7}$$

where

$$h(t)=\begin{cases}(1-t)^{\varpi k+\ell-\psi-1}, & \text{for}\quad 0\leq t<1,\\ 0, & \text{for}\quad t\geq 1.\end{cases}$$

Then,

$$\mathfrak{M}\Big[h(t)\Big](p)=\frac{\Gamma(p)\Gamma(\varpi k+\ell-\psi)}{\Gamma(\varpi k+\ell+p-\psi)}.\tag{4.8}$$

Using the formula (see Ref. [43]) $\mathfrak{M}\left[t^{\lambda}\int_0^{\infty}\nu^{\mu}f(t\nu)g(\nu)d\nu\right](p)=\mathfrak{M}\Big[f(t)\Big](p+\lambda)\;\mathfrak{M}\Big[g(t)\Big](1-p-\lambda+\mu)$ in (4.7), we have

$$\mathfrak{M}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](p)=\sum_{k=0}^{\infty}\frac{\big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\big)^{k}}{\Gamma(\varpi k+\ell-\psi)k!}\mathfrak{M}\Big[g(t)\Big](p+\varpi k+\ell-\psi)\times\mathfrak{M}\Big[h(t)\Big](1-p-\varpi k-\ell+\psi).$$

Finally, using the (4.6) and (4.8), we get

$$\mathfrak{M}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}f\right)(t)\right](p)=\sum_{k=0}^{\infty}\frac{\Big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\Big)^{k}}{k!}\frac{\Gamma(1-p-\varpi k+\psi)}{\Gamma(1-p)}\times\mathfrak{M}\Big[f(t)\Big](p+\varpi k-\psi). \qquad \square$$

Corollary 4.4. *The following formula holds true:*

$$\mathfrak{M}\left[t^{\psi}\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,t^{-\varpi};(\psi)}f\right)(t)\right](p)=\Gamma(1-p)\ \phi\Big(\varpi,1-\psi-p;\varpi(\ell-\psi-1)(\ell-\psi)\Big)\ \mathfrak{M}\Big[f(t)\Big](p).$$

5. Illustrative examples

This section calculates the generalized Caputo fractional derivative of constant, power, exponential, sine, and cosine functions and obtains various relations. In Section 7, numerical results are presented for these relations given specific data, and Figure 1 shows the graphs of their approximate behavior in comparison to the classical Caputo fractional derivative.

Example 5.1 (Constant function)**.** Let $\varpi,\Delta,\psi\in\mathbb{C}$, $\ell\in\mathbb{N}$, $\ell-1<\Re(\psi)\le\ell$, $\Re(\psi)>0$, $\Re(\varpi)>-1$. Then,

$$\left(\widehat{\mathfrak{D}}_{x+}^{\varpi,\Delta;(\psi)}c\right)(t)=0,\quad t>x,$$

where c is constant.

Using the formula $\frac{d^{\ell}c}{d\nu^{\ell}}=0$ in (3.1), the desired result is achieved.

Example 5.2 (Power function)**.** Let $\varpi,\Delta,\psi\in\mathbb{C}$, $t\in\mathbb{R}^{+}$, $\ell\in\mathbb{N}$, $\ell-1<\Re(\psi)\le\ell$, $\Re(\psi)>0$, $\Re(\varpi)>-1$. Then,

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}t^{r}\right)(t)=\Gamma(r+1)t^{r-\psi}\ \phi\Big(\varpi,r+1-\psi;\varpi\Delta(\ell-\psi-1)(\ell-\psi)t^{\varpi}\Big), \tag{5.1}$$

where $r>-1$.

Using (3.3), we have

$$\begin{aligned}\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}t^{r}\right)(t)&=\int_{0}^{t}(t-\nu)^{\ell-\psi-1}\left(\frac{d^{\ell}\nu^{r}}{d\nu^{\ell}}\right)\ \phi\Big(\varpi,\ell-\psi;\varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big)d\nu\\&=\sum_{k=0}^{\infty}\frac{\Big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\Big)^{k}}{\Gamma(\varpi k+\ell-\psi)k!}\int_{0}^{t}(t-\nu)^{\varpi k+\ell-\psi-1}\frac{d^{\ell}\nu^{r}}{d\nu^{\ell}}d\nu.\end{aligned}$$

Here using the formula $\frac{d^{\ell}\nu^{r}}{d\nu^{\ell}}=\frac{\Gamma(r+1)}{\Gamma(r+1-\ell)}\nu^{r-\ell}$, we get

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}t^{r}\right)(t)=\sum_{k=0}^{\infty}\frac{\Big(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\Big)^{k}}{\Gamma(\varpi k+\ell-\psi)k!}\frac{\Gamma(r+1)}{\Gamma(r+1-\ell)}\int_{0}^{t}(t-\nu)^{\varpi k+\ell-\psi-1}\nu^{r-\ell}d\nu.$$

Taking $\nu=ut$, we obtain

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}t^{r}\right)(t)=\Gamma(r+1)t^{r-\psi}\ \phi\Big(\varpi,r+1-\psi;\varpi\Delta(\ell-\psi-1)(\ell-\psi)t^{\varpi}\Big).$$

Example 5.3 (Exponential function)**.** Let $\varpi,\Delta,\psi,\lambda\in\mathbb{C}$, $t\in\mathbb{R}^{+}$, $\ell\in\mathbb{N}$, $\ell-1<\Re(\psi)\le\ell$, $\Re(\psi)>0$, $\Re(\varpi)>-1$. Then,

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}\exp(\lambda t)\right)(t)=\lambda^{\ell}t^{\ell-\psi}\sum_{r=0}^{\infty}(\lambda t)^{r}\ \phi\Big(\varpi,\ell+r+1-\psi;\varpi\Delta(\ell-\psi-1)(\ell-\psi)t^{\varpi}\Big). \tag{5.2}$$

Using (3.3), we have

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}\exp(\lambda t)\right)(t)=\int_{0}^{t}(t-\nu)^{\ell-\psi-1}\left(\frac{d^{\ell}\exp(\lambda\nu)}{d\nu^{\ell}}\right)\times\phi\Big(\varpi,\ell-\psi;\varpi\Delta(\ell-\psi-1)(\ell-\psi)(t-\nu)^{\varpi}\Big)d\nu$$

$$= \sum_{k=0}^{\infty} \frac{\left(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\right)^k}{\Gamma(\varpi k+\ell-\psi)k!} \int_0^t (t-\nu)^{\varpi k+\ell-\psi-1} \frac{d^\ell \exp(\lambda\nu)}{d\nu^\ell} d\nu.$$

Here using the formula $\frac{d^\ell \exp(\lambda\nu)}{d\nu^\ell} = \lambda^\ell \sum_{r=0}^{\infty} \frac{(\lambda\nu)^r}{r!}$, we get

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \exp(\lambda t)\right)(t) = \lambda^\ell \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\varpi\Delta(\ell-\psi-1)(\ell-\psi)\right)^k}{\Gamma(\varpi k+\ell-\psi)k!} \frac{\lambda^r}{r!} \int_0^t (t-\nu)^{\varpi k+\ell-\psi-1} \nu^r d\nu.$$

Taking $\nu = ut$, we obtain

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \exp(\lambda t)\right)(t) = \lambda^\ell t^{\ell-\psi} \sum_{r=0}^{\infty} (\lambda t)^r \ \phi\Big(\varpi, \ell+r+1-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)t^\varpi\Big).$$

Example 5.4 (Trigonometric functions)**.** Let $\varpi, \Delta, \psi, \lambda \in \mathbb{C}$, $t \in \mathbb{R}^+$, $\ell \in \mathbb{N}$, $\ell - 1 < \Re(\psi) \leq \ell$, $\Re(\psi) > 0$, $\Re(\varpi) > -1$. Then,

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \sin(\lambda t)\right)(t) = \frac{(i\lambda)^\ell t^{\ell-\psi}}{2i} \Bigg(\sum_{r=0}^{\infty} (i\lambda t)^r \ \phi\Big(\varpi, \ell+r+1-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)t^\varpi\Big) - (-1)^\ell \sum_{r=0}^{\infty} (-i\lambda t)^r \ \phi\Big(\varpi, \ell+r+1-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)t^\varpi\Big) \Bigg) \tag{5.3}$$

and

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \cos(\lambda t)\right)(t) = \frac{(i\lambda)^\ell t^{\ell-\psi}}{2} \Bigg(\sum_{r=0}^{\infty} (i\lambda t)^r \ \phi\Big(\varpi, \ell+r+1-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)t^\varpi\Big) + (-1)^\ell \sum_{r=0}^{\infty} (-i\lambda t)^r \ \phi\Big(\varpi, \ell+r+1-\psi; \varpi\Delta(\ell-\psi-1)(\ell-\psi)t^\varpi\Big) \Bigg). \tag{5.4}$$

Using the formulas $\sin(\lambda t) = \frac{\exp(i\lambda t)-\exp(-i\lambda t)}{2i}$ and $\cos(\lambda t) = \frac{\exp(i\lambda t)+\exp(-i\lambda t)}{2}$ respectively, we have

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \sin(\lambda t)\right)(t) = \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \ \frac{\exp(i\lambda t)-\exp(-i\lambda t)}{2i}\right)(t)$$

and

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \cos(\lambda t)\right)(t) = \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \ \frac{\exp(i\lambda t)+\exp(-i\lambda t)}{2}\right)(t).$$

Then from linearity properties, respectively we get

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \sin(\lambda t)\right)(t) = \frac{1}{2i}\left(\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \exp(i\lambda t)\right)(t) - \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \exp(-i\lambda t)\right)(t) \right)$$

and

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \cos(\lambda t)\right)(t) = \frac{1}{2}\left(\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \exp(i\lambda t)\right)(t) + \left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)} \exp(-i\lambda t)\right)(t) \right).$$

Finally, considering (5.2) for both expressions, the desired results are obtained.

6. Solution of fractional-order generalized differential equations

This section obtains the solution of fractional-order generalized motion, harmonic vibration, and Bessel differential equations via Fourier, Laplace, and Mellin integral transform methods. Since these differential equations contain the generalized Caputo fractional derivative, we call them fractional-order generalized differential equations.

Application 6.1. We consider the fractional-order generalized motion differential equation

$$y''(t) + \kappa \left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} y\right)(t) + \mu y(t) = f(t),$$

where $\kappa, \mu \in \mathbb{R}$ and $0 < \Re(\psi) \leq 1$.

Applying the Fourier integral transform to the fractional-order generalized motion differential equation, we have

$$\begin{aligned}\mathfrak{F}\Big[f(t)\Big](\omega) &= \mathfrak{F}\Big[y''(t)\Big](\omega) + \kappa\, \mathfrak{F}\Big[\left(\widehat{\mathfrak{D}}_{+}^{\varpi,\Delta;(\psi)} y\right)(t)\Big](\omega) + \mu\, \mathfrak{F}\Big[y(t)\Big](\omega) \\ &= (i\omega)^2\, \mathfrak{F}\Big[y(t)\Big](\omega) + \kappa (i\omega)^{\psi} \exp\left(-\frac{\varpi\Delta\psi(1-\psi)}{(i\omega)^{\varpi}}\right) \mathfrak{F}\Big[y(t)\Big](\omega) + \mu \mathfrak{F}\Big[y(t)\Big](\omega).\end{aligned}$$

That is,

$$\mathfrak{F}\Big[y(t)\Big](\omega) = \frac{\mathfrak{F}\Big[f(t)\Big](\omega)}{(i\omega)^2 + \kappa(i\omega)^{\psi} \exp\left(-\frac{\varpi\Delta\psi(1-\psi)}{(i\omega)^{\varpi}}\right) + \mu}.$$

If we choose

$$\mathfrak{F}\Big[g(t)\Big](\omega) = \frac{1}{(i\omega)^2 + \kappa(i\omega)^{\psi} \exp\left(-\frac{\varpi\Delta\psi(1-\psi)}{(i\omega)^{\varpi}}\right) + \mu}. \tag{6.1}$$

Then,

$$\mathfrak{F}\Big[y(t)\Big](\omega) = \mathfrak{F}\Big[f(t)\Big](\omega)\mathfrak{F}\Big[g(t)\Big](\omega).$$

Thus,

$$\mathfrak{F}\Big[y(t)\Big](\omega) = \mathfrak{F}\Big[(f \star g)(t)\Big](\omega).$$

From the inverse Fourier integral transform and convolution, we get

$$y(t) = \int_{-\infty}^{\infty} f(t-\nu) g(\nu) d\nu. \tag{6.2}$$

Applying the inverse Fourier integral transform to (6.1), we obtain

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{(i\omega)^2 + \kappa(i\omega)^{\psi} \exp\left(-\frac{\varpi\Delta\psi(1-\psi)}{(i\omega)^{\varpi}}\right) + \mu} d\omega. \tag{6.3}$$

Finally, using (6.3) in (6.2), we have

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(t-\nu)\exp(i\omega\nu)}{(i\omega)^2 + \kappa(i\omega)^{\psi} \exp\left(-\frac{\varpi\Delta\psi(1-\psi)}{(i\omega)^{\varpi}}\right) + \mu} d\omega d\nu. \tag{6.4}$$

Corollary 6.2. *If we choose $\varpi = 0$ in (6.4), we get the following formula:*

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(t-\nu)\exp(i\omega\nu)}{(i\omega)^2 + \kappa(i\omega)^{\psi} + \mu} d\omega d\nu,$$

which is the solution of the fractional-order motion differential equation (see Refs. [16, 29]).

Application 6.3. We consider the fractional-order generalized harmonic vibration differential equation

$$\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}y\right)(t)+\kappa^2 y(t)=0,$$

with the initial conditions

$$y(0)=A \quad \text{and} \quad y'(0)=B,$$

where $\kappa\in\mathbb{R}$ and $1<\Re(\psi)\leq 2$.

Applying the Laplace integral transform to the fractional-order generalized harmonic vibration differential equation, we have

$$\mathfrak{L}\left[\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}y\right)(t)\right](s)+\kappa^2\,\mathfrak{L}\Big[y(t)\Big](s)=0,$$

and

$$\exp\left(\frac{\varpi\Delta(1-\psi)(2-\psi)}{s^{\varpi}}\right)\left(s^{\psi}\mathfrak{L}\Big[y(t)\Big](s)-As^{\psi-1}-Bs^{\psi-2}\right)+\kappa^2\mathfrak{L}\Big[y(t)\Big](s)=0.$$

That is,

$$\mathfrak{L}\Big[y(t)\Big](s)=\frac{\left(As^{\psi-1}+Bs^{\psi-2}\right)\exp\left(\frac{\varpi\Delta(1-\psi)(2-\psi)}{s^{\varpi}}\right)}{s^{\psi}\exp\left(\frac{\varpi\Delta(1-\psi)(2-\psi)}{s^{\varpi}}\right)+\kappa^2},$$

and

$$\mathfrak{L}\Big[y(t)\Big](s)=\frac{As^{-1}}{1+\kappa^2 s^{-\psi}\exp\left(-\frac{\varpi\Delta(1-\psi)(2-\psi)}{s^{\varpi}}\right)}+\frac{Bs^{-2}}{1+\kappa^2 s^{-\psi}\exp\left(-\frac{\varpi\Delta(1-\psi)(2-\psi)}{s^{\varpi}}\right)}.$$

Using the formula $\frac{1}{1+t}=\sum\limits_{m=0}^{\infty}(-t)^m$, we have

$$\begin{aligned}\mathfrak{L}\Big[y(t)\Big](s)=&\,A\sum_{m=0}^{\infty}(-1)^m\kappa^{2m}s^{-\psi m-1}\exp\left(-\frac{m\varpi\Delta(1-\psi)(2-\psi)}{s^{\varpi}}\right)\\&+B\sum_{m=0}^{\infty}(-1)^m\kappa^{2m}s^{-\psi m-2}\exp\left(-\frac{m\varpi\Delta(1-\psi)(2-\psi)}{s^{\varpi}}\right).\end{aligned}$$

Applying the inverse Laplace integral transform, we obtain

$$\begin{aligned}y(t)=&\,A\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\kappa^{2m}\frac{\left(-m\varpi\Delta(1-\psi)(2-\psi)\right)^n t^{\psi m+\varpi n}}{n!(\psi m+\varpi n)!}\\&+Bt\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\kappa^{2m}\frac{\left(-m\varpi\Delta(1-\psi)(2-\psi)\right)^n t^{\psi m+\varpi n}}{n!(\psi m+\varpi n+1)!}.\end{aligned}\tag{6.5}$$

Corollary 6.4. *If we choose $\varpi=0$ in (6.5), we get the following formula:*

$$y(t)=AE_{\psi,1}\left(-\kappa^2t^{\psi}\right)+BtE_{\psi,2}\left(-\kappa^2t^{\psi}\right)\tag{6.6}$$

which is the solution of the fractional-order harmonic vibration differential equation (see Ref. [34]).

Application 6.5. We consider the fractional-order generalized Bessel differential equation

$$t^{\psi+1}\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,t^{-\varpi};(\psi+1)}y\right)(t)+t^{\psi}\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,t^{-\varpi};(\psi)}y\right)(t)=f(t),$$

with the initial conditions

$$y(0)=y'(0)=0 \quad \text{and} \quad y(\infty)=y'(\infty)=0,$$

where $0<\Re(\psi)\leq 1$.

Applying the Mellin integral transform to the fractional-order generalized Bessel differential equation, we have

$$\mathfrak{M}\Big[t^{\psi+1}\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,t^{-\varpi};(\psi+1)}y\right)(t)\Big](p)+\mathfrak{M}\Big[t^{\psi}\left(\widehat{\mathfrak{D}}_{0+}^{\varpi,t^{-\varpi};(\psi)}y\right)(t)\Big](p)=\mathfrak{M}\Big[f(t)\Big](p),$$

and

$$\Gamma(1-p)\phi\Big(\varpi,-p-\psi;\varpi\psi(1+\psi)\Big)\mathfrak{M}\Big[y(t)\Big](p)+\Gamma(1-p)\phi\Big(\varpi,1-p-\psi;-\varpi\psi(1-\psi)\Big)\times\mathfrak{M}\Big[y(t)\Big](p)=\mathfrak{M}\Big[f(t)\Big](p).$$

That is,

$$\mathfrak{M}\Big[y(t)\Big](p)=\frac{\mathfrak{M}\Big[f(t)\Big](p)}{\Gamma(1-p)}\left[\phi\Big(\varpi,-p-\psi;\varpi\psi(1+\psi)\Big)+\phi\Big(\varpi,1-p-\psi;-\varpi\psi(1-\psi)\Big)\right]^{-1}.$$

If we choose

$$\mathfrak{M}\Big[g(t)\Big](1-p)=\frac{1}{\Gamma(1-p)}\left[\phi\Big(\varpi,1-p-\psi-1;\varpi\psi(1+\psi)\Big)+\phi\Big(\varpi,1-p-\psi;-\varpi\psi(1-\psi)\Big)\right]^{-1}. \tag{6.7}$$

Then,

$$\mathfrak{M}\Big[y(t)\Big](p)=\mathfrak{M}\Big[f(t)\Big](p)\mathfrak{M}\Big[g(t)\Big](1-p).$$

Thus,

$$\mathfrak{M}\Big[y(t)\Big](p)=\mathfrak{M}\Big[(f\circ g)(t)\Big](p).$$

From the inverse Mellin integral transform and convolution, we get

$$y(t)=\int_0^\infty f(t\nu)g(\nu)d\nu. \tag{6.8}$$

Applying the inverse Mellin integral transform to (6.7), we obtain

$$g(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{t^{-p}}{\Gamma(p)}\left[\phi\Big(\varpi,p-\psi-1;\varpi\psi(1+\psi)\Big)+\phi\Big(\varpi,p-\psi;-\varpi\psi(1-\psi)\Big)\right]^{-1}dp. \tag{6.9}$$

Finally, using (6.9) in (6.8), we have

$$y(t)=\frac{1}{2\pi i}\int_0^\infty\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{f(t\nu)\nu^{-p}}{\Gamma(p)}\left[\phi\Big(\varpi,p-\psi-1;\varpi\psi(1+\psi)\Big)+\phi\Big(\varpi,p-\psi;-\varpi\psi(1-\psi)\Big)\right]^{-1}dpd\nu. \tag{6.10}$$

Corollary 6.6. *If we choose $\varpi=0$ in (6.10), we get the following formula:*

$$y(t)=\frac{1}{2\pi i}\int_0^\infty\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{f(t\nu)\nu^{-p}\Gamma(p-\psi)}{(p-\psi)\Gamma(p)}dpd\nu,$$

which is the solution of the fractional-order Bessel differential equation (see Ref. [40]).

7. Numerical results and approximate behavior graphs

This section obtains some results for the generalized Caputo fractional derivative ($\widehat{\mathfrak{D}}_{0+}^{\varpi,\Delta;(\psi)}$) and the classical Caputo fractional derivative (${}^{c}D_{0+}^{\psi}$) by giving specific data to (5.1)-(5.4), (6.5), and (6.6). It also presents Figures 1 and 2 to visualize these results. In order to visualize the equations we have obtained in terms of both the classical Caputo fractional derivative and the generalized Caputo fractional derivative, in Figures 1 and 2 we present comparative graphs of the approximate behavior of the results of these two fractional derivative operators via the Mathematica software system. In Figures 1 and 2, the first four and first three terms of the series are unfolded, respectively. This is because it is very difficult to plot all the terms of an unbounded series. It should be noted that since the numerical results obtained satisfy the convergence conditions, visually their approximate behavior graphs also converge.

Now we use some specific data in generalized and classical Caputo fractional derivatives and obtain the results listed below. Indeed:

⋆ **Power functions**

Taking $\Re(\psi)=0.5$, $\Re(\varpi)=2$, $\Re(\Delta)=1$ and $r=2$ in (5.1) yields

$$\left(\widehat{\mathfrak{D}}_{0+}^{2,1;(0.5)}t^2\right)(t)=\Gamma(3)t^{1.5}\phi\Big(2,2.5;(-0.5)t^2\Big), \tag{7.1}$$

and for $\Re(\varpi)=0$:

$$\left({}^{c}D_{0+}^{0.5}t^2\right)(t)=\frac{\Gamma(3)}{\Gamma(2.5)}t^{1.5}. \tag{7.2}$$

Taking $\Re(\psi)=0.5$, $\Re(\varpi)=2$, $\Re(\Delta)=1$ and $r=3$ in (5.1) yields

$$\left(\widehat{\mathfrak{D}}_{0+}^{2,1;(0.5)}t^3\right)(t)=\Gamma(4)t^{2.5}\phi\Big(2,3.5;(-0.5)t^2\Big), \tag{7.3}$$

and for $\Re(\varpi)=0$:

$$\left({}^{c}D_{0+}^{0.5}t^3\right)(t)=\frac{\Gamma(4)}{\Gamma(3.5)}t^{2.5}. \tag{7.4}$$

⋆ **Exponential functions**

Taking $\Re(\psi)=0.7$, $\Re(\varpi)=3$, $\Re(\Delta)=1$ and $\Re(\lambda)=1$ in (5.2), we have

$$\left(\widehat{\mathfrak{D}}_{0+}^{3,1;(0.7)}\exp(t)\right)(t)=t^{0.3}\sum_{r=0}^{\infty}\phi\Big(3,r+1.3;3(-0.7)(0.3)t^3\Big)t^r, \tag{7.5}$$

and for $\Re(\varpi)=0$:

$$\left({}^{c}D_{0+}^{0.7}\exp(t)\right)(t)=t^{0.3}\sum_{r=0}^{\infty}\frac{t^r}{\Gamma(r+1.3)}. \tag{7.6}$$

Taking $\Re(\psi)=0.7$, $\Re(\varpi)=3$, $\Re(\Delta)=1$ and $\Re(\lambda)=2$ in (5.2) yields

$$\left(\widehat{\mathfrak{D}}_{0+}^{3,1;(0.7)}\exp(2t)\right)(t)=2t^{0.3}\sum_{r=0}^{\infty}\phi\Big(3,r+1.3;3(-0.7)(0.3)t^3\Big)(2t)^r, \tag{7.7}$$

and for $\Re(\varpi)=0$:

$$\left({}^{c}D_{0+}^{0.7}\exp(2t)\right)(t)=2t^{0.3}\sum_{r=0}^{\infty}\frac{(2t)^r}{\Gamma(r+1.3)}. \tag{7.8}$$

⋆ **Trigonometric functions**

Taking $\Re(\psi)=0.9$, $\Re(\varpi)=4$, $\Re(\Delta)=1$ and $\Re(\lambda)=1$ in (5.3) yields

$$\left(\widehat{\mathfrak{D}}_{0+}^{4,1;(0.9)}\sin(t)\right)(t)=\frac{t^{0.1}}{2}\left(\sum_{r=0}^{\infty}\phi\Big(4,r+1.1;4(-0.9)(0.1)t^4\Big)(it)^r+\sum_{r=0}^{\infty}\phi\Big(4,r+1.1;4(-0.9)(0.1)t^4\Big)(-it)^r\right), \tag{7.9}$$

and for $\Re(\varpi)=0$:

$$\left({}^{c}D_{0+}^{0.9}\sin(t)\right)(t)=\frac{t^{0.1}}{2}\left(\sum_{r=0}^{\infty}\frac{(it)^r}{\Gamma(r+1.1)}+\sum_{r=0}^{\infty}\frac{(-it)^r}{\Gamma(r+1.1)}\right). \tag{7.10}$$

Taking $\Re(\psi)=0.9$, $\Re(\varpi)=4$, $\Re(\Delta)=1$ and $\Re(\lambda)=1$ in (5.4) yields

$$\left(\widehat{\mathfrak{D}}_{0+}^{4,1;(0.9)}\cos(t)\right)(t)=\frac{it^{0.1}}{2}\left(\sum_{r=0}^{\infty}\phi\Big(4,r+1.1;4(-0.9)(0.1)t^4\Big)(it)^r-\sum_{r=0}^{\infty}\phi\Big(4,r+1.1;4(-0.9)(0.1)t^4\Big)(-it)^r\right), \tag{7.11}$$

and for $\Re(\varpi) = 0$:

$$\left({}^{c}D_{0+}^{0.9} \cos(t)\right)(t) = \frac{it^{0.1}}{2}\left(\sum_{r=0}^{\infty} \frac{(it)^r}{\Gamma(r+1.1)} - \sum_{r=0}^{\infty} \frac{(-it)^r}{\Gamma(r+1.1)}\right). \tag{7.12}$$

☆ **Solving fractional differential equations**

Taking $\Re(\psi) = 1.2$, $\Re(\Delta) = \Re(\kappa) = 1$, $\Re(\varpi) = 2$ and $A = B = 1$ in (6.5) yields

$$\begin{aligned} y(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.2)(2-1.2)\right)^n}{n!(1.2m+2n)!} t^{1.2m+2n} \\ &+ t \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.2)(2-1.2)\right)^n}{n!(1.2m+2n+1)!} t^{1.2m+2n}, \end{aligned} \tag{7.13}$$

and for (6.6):

$$y(t) = E_{1.2,1}\left(-t^{1.2}\right) + tE_{1.2,2}\left(-t^{1.2}\right). \tag{7.14}$$

Taking $\Re(\psi) = 1.4$, $\Re(\Delta) = \Re(\kappa) = 1$, $\Re(\varpi) = 2$ and $A = B = 1$ in (6.5) yields

$$\begin{aligned} y(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.4)(2-1.4)\right)^n}{n!(1.4m+2n)!} t^{1.4m+2n} \\ &+ t \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.4)(2-1.4)\right)^n}{n!(1.4m+2n+1)!} t^{1.4m+2n}, \end{aligned} \tag{7.15}$$

and for (6.6):

$$y(t) = E_{1.4,1}\left(-t^{1.4}\right) + tE_{1.4,2}\left(-t^{1.4}\right). \tag{7.16}$$

Taking $\Re(\psi) = 1.6$, $\Re(\Delta) = \Re(\kappa) = 1$, $\Re(\varpi) = 2$ and $A = B = 1$ in (6.5) yields

$$\begin{aligned} y(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.6)(2-1.6)\right)^n}{n!(1.6m+2n)!} t^{1.6m+2n} \\ &+ t \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.6)(2-1.6)\right)^n}{n!(1.6m+2n+1)!} t^{1.6m+2n}, \end{aligned} \tag{7.17}$$

and for (6.6):

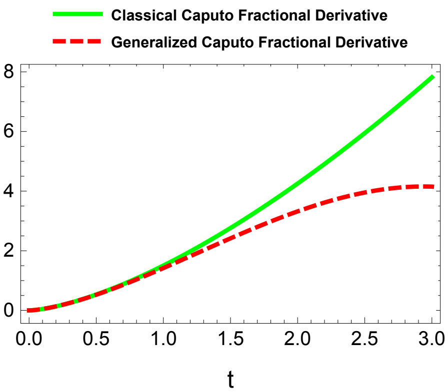
$$y(t) = E_{1.6,1}\left(-t^{1.6}\right) + tE_{1.6,2}\left(-t^{1.6}\right). \tag{7.18}$$

Taking $\Re(\psi) = 1.8$, $\Re(\Delta) = \Re(\kappa) = 1$, $\Re(\varpi) = 2$, and $A = B = 1$ in (6.5) yields

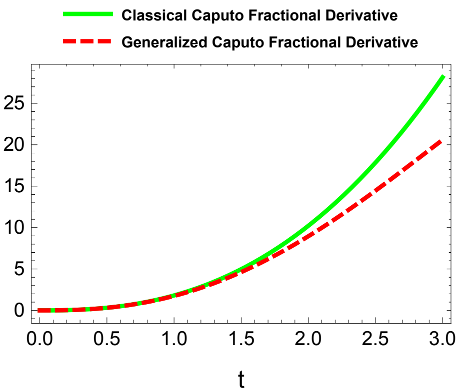
$$\begin{aligned} y(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.8)(2-1.8)\right)^n}{n!(1.8m+2n)!} t^{1.8m+2n} \\ &+ t \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m \left(-2m(1-1.8)(2-1.8)\right)^n}{n!(1.8m+2n+1)!} t^{1.8m+2n}, \end{aligned} \tag{7.19}$$

and for (6.6):

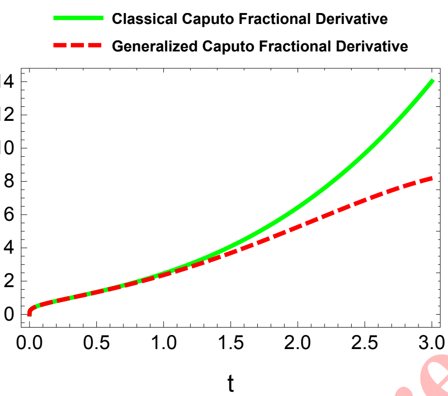
$$y(t) = E_{1.8,1}\left(-t^{1.8}\right) + tE_{1.8,2}\left(-t^{1.8}\right). \tag{7.20}$$



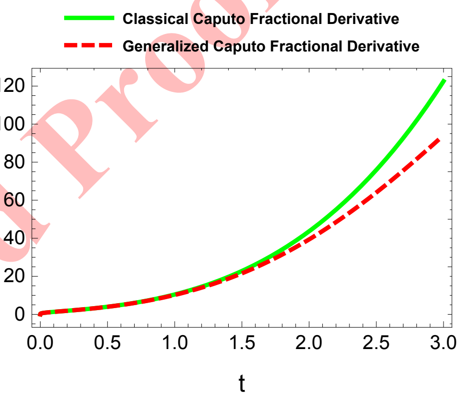
(A) The approximate graphs of (7.1) and (7.2) for the Wright function index $k = 0, 1, 2, 3$.



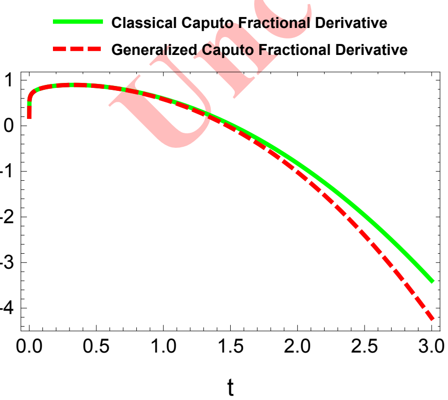
(B) The approximate graphs of (7.3) and (7.4) for the Wright function index $k = 0, 1, 2, 3$.



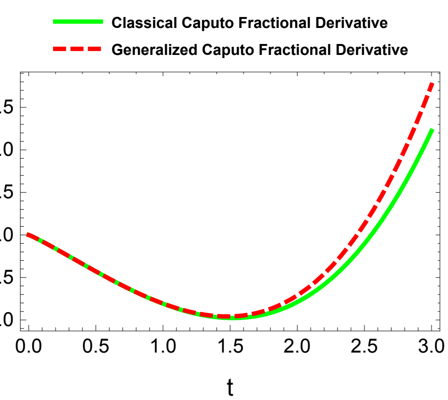
(C) The approximate graphs of (7.5) and (7.6) for $r = 0, 1, 2, 3$ and the Wright function index $k = 0, 1, 2, 3$.



(D) The approximate graphs of (7.7) and (7.8) for $r = 0, 1, 2, 3$ and the Wright function index $k = 0, 1, 2, 3$.

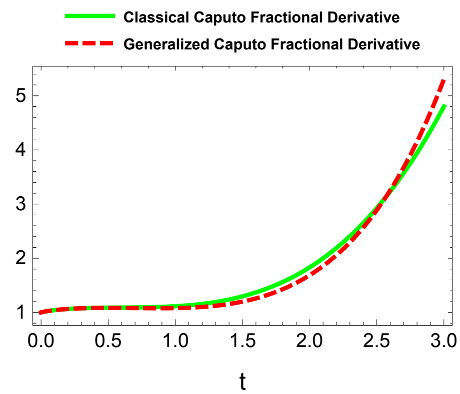


(E) The approximate graphs of (7.9) and (7.10) for $r = 0, 1, 2, 3$ and the Wright function index $k = 0, 1, 2, 3$.

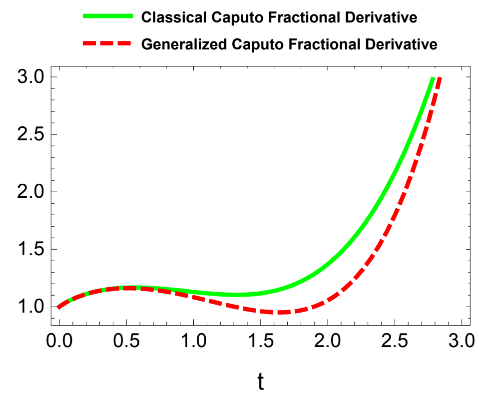


(F) The approximate graphs of (7.11) and (7.12) for $r = 0, 1, 2, 3$ and the Wright function index $k = 0, 1, 2, 3$.

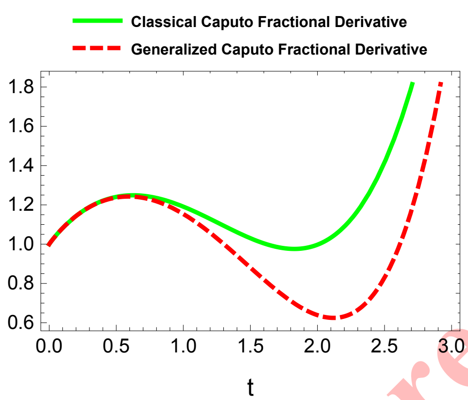
FIGURE 1. Comparison of the approximate behavior of the elementary functions in both the classical Caputo fractional derivative and the generalized Caputo fractional derivative.



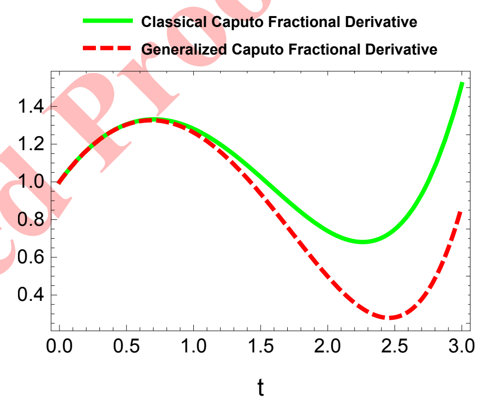
(A) The approximate graphs of (7.13) and (7.14) for the Mittag-Leffler functions and series indexes $m = n = k = 0, 1, 2$.



(B) The approximate graphs of (7.15) and (7.16) for the Mittag-Leffler functions and series indexes $m = n = k = 0, 1, 2$.



(C) The approximate graphs of (7.17) and (7.18) for the Mittag-Leffler functions and series indexes $m = n = k = 0, 1, 2$.



(D) The approximate graphs of (7.19) and (7.20) for the Mittag-Leffler functions and series indexes $m = n = k = 0, 1, 2$.

FIGURE 2. Comparison of the approximate behaviour for the solution functions of the harmonic vibration equation involving both the classical Caputo fractional derivative and the generalized Caputo fractional derivative.

8. Conclusion

In this research article, we introduce a new generalized type of this fractional derivative by using the Wright function, which belongs to the family of special functions, in the definition of a classical Caputo fractional derivative. We also apply the popular integral transforms Fourier, Laplace, and Mellin integral transforms to the generalized Caputo fractional derivative. We then give illustrative examples and present the approximate behavior graphs of their particular data results compared to the classical Caputo fractional derivative in Figure 1. Thus, we observe how both fractional derivatives behave. As applications of the article, we obtain the solutions of various differential equations involving the generalized fractional derivative by means of the popular integral transforms Fourier, Laplace and Mellin integral transform methods and present the approximate behavior graph for one of the results obtained in Figure 2 in comparison with the classical Caputo. Here again, we observe how both fractional derivatives behave. It should also be noted that in Figures 1 and 2, the red lines represent the approximate behavior of the classical Caputo fractional derivative and the green lines represent the approximate behavior of the generalized Caputo fractional derivative.

With regard to the results of the research article, it is possible to say the following. There are many works on fractional calculus in the literature and many of these works have been used by researchers to model real-life problems. We believe that the new generalized Caputo fractional derivative presented in this article can also be used to model various real-life problems and give better results. Because the fractional derivative we define contains a multiparameter special function, we believe that it can give more realistic results in applications to real-life problems. Thus, the application areas of fractional calculus will be expanded.

Future work will use the Wright function for various fractional operators in the literature, such as the Riemann-Liouville, Grünwald-Letnikov, Hadamard, and conformable derivatives. This will allow us to define new fractional operators, present their potential properties, and obtain solutions to some fractional differential equations using integral transformation methods.

Acknowledgment

The authors would like to note that a preprint version of this manuscript was previously made available on Research Square with the DOI: https://doi.org/10.21203/rs.3.rs-9172862/v1.

The authors also sincerely thank the editor and the anonymous reviewers for their valuable comments and constructive suggestions, which have significantly improved the quality and clarity of the manuscript.

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