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Operational discrete Petrov-Galerkin method for solving distributed order FDEs

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Abstract

This study presents a novel discrete Petrov-Galerkin method for solving distributed-order fractional differential equations, where fractional derivatives are expressed in the Caputo sense. The proposed approach begins by reducing the distributed-order equation to a multi-term fractional equation using the Newton-Cotes quadrature rule, thereby simplifying the numerical approximation process. The discrete Petrov-Galerkin framework utilizes generalized Jacobi polynomials as basis functions and fractional Legendre functions as test functions, chosen for their robust approximation capabilities. By exploiting these mathematical properties, the method transforms the problem into a system of algebraic equations. A comprehensive convergence analysis is conducted to validate the accuracy and reliability of the method. The effectiveness of the proposed approach is further illustrated through three numerical examples, demonstrating its superior performance compared to existing techniques in terms of accuracy and computational efficiency.

Keywords. Distributed-order fractional differential equation (DOFDE), Discrete Petrov-Galerkin method, Fractional Legendre functions.
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1. Introduction

Fractional calculus extends the traditional concepts of integrals and derivatives to non-integer orders, offering a robust and versatile tool for modeling complex phenomena across disciplines such as hydrology [10, 15], viscoelasticity [40, 51], finance [29, 81], fractional kinetics [41, 80], and mass transfer processes with chemical reactions [4]. By accommodating memory and hereditary properties, fractional calculus enables accurate modeling of dynamic, multi-scale phenomena.

Despite the utility of fixed-order fractional operators, their rigidity limits their effectiveness in modeling systems influenced by varying dynamics or multiple orders. Distributed-order operators, which integrate fractional kernels over a continuous range of orders, overcome this limitation. As a result, DOFDEs have emerged as potent tools for modeling phenomena such as diffusion and wave propagation [45]. The difficulty of deriving analytical solutions for DOFDEs has spurred the development of approximate numerical methods. Initial contributions by Diethelm and Ford [19] established fundamental numerical frameworks for addressing these equations. In the first step, the distributed-order derivative is approximated through a numerical integration technique, ensuring the accurate representation of the distributed nature of the derivative. The second step involves applying a suitable numerical method to solve the resulting equation, which typically contains multiple fractional derivative terms. This framework has since been widely adopted in subsequent studies, with variations primarily arising from the specific discretization techniques employed in the second step. These advancements have paved the way for more precise and flexible numerical solutions to DOFDEs, some of which are discussed in the remainder of this paper.

A notable investigation into the diffusion equation incorporating a distributed-order time fractional derivative was conducted in [33]. Further refinements of this work were carried out in [74], where the stability and accuracy of the proposed numerical scheme were rigorously analyzed. Building on this, the authors in [52] extended the

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approach by introducing a nonlinear component to the distributed-order reaction-diffusion equation, demonstrating the adaptability of the numerical methods employed. Another significant contribution to solving equations with distributed-order derivatives was made by Katsikadelis [37], who proposed a numerical method based on the trapezoidal rule to approximate the distributed integral. This approach was combined with the analog equation method to handle multiple fractional derivatives within the equations. Atanackovic et al. [5] further advanced the field by investigating the properties of fractional distributed-order systems, such as oscillators and wave equations [6, 7]. In a complementary development, the authors of [14] introduced a diffusion-like equation with distributed-order time-fractional derivatives aimed at modeling anomalous diffusion and relaxation phenomena. They provided mathematical proofs demonstrating the positivity of the solutions. Moreover, the study of boundary value problems within the context of DOFDEs was explored in [46], which focused on the existence and uniqueness of solutions through the application of the maximum principle. Furthermore, a notable investigation into the diffusion equation incorporating a distributed-order time fractional derivative was conducted in [64]. More recently, optimization-based numerical frameworks have also been considered. In particular, Sabermahani and Ordokhani [65] proposed fractional-order Fibonacci functions together with pseudo-operational matrices to solve multi-dimensional DOFDEs. They further developed a Fibonacci wavelet method for solving distributed-order fractional optimal control problems by transforming the original system into a set of algebraic equations [66]. In addition to these deterministic approaches, the study of stochastic analogs of DOFDEs was undertaken in [50], where explicit strong solutions were developed. Eab et al. [23] expanded the scope of applications by investigating equations analogous to the Langevin equation, highlighting the potential of DOFDEs in modeling random processes. Finally, Mashayekhi and Razzaghi [47] introduced hybrid functions, including block-pulse functions, to solve DOFDEs.

Spectral methods are prominent tools for addressing various functional equations. Unlike finite difference or finite element methods, which approximate solutions at discrete points or within finite elements, spectral methods express the solution as a series expansion using basis functions, often orthogonal polynomials. This unique approach harnesses the rapid convergence of these basis functions, enabling spectral methods to achieve exponential accuracy, particularly when the basis functions share features with the asymptotic behavior of the exact solution. One of the most effective approaches within spectral methods is the discrete Petrov-Galerkin approach, which generalizes the Galerkin approach by allowing the test functions (used for minimizing the residual) to differ from the basis functions while employing sufficient quadrature rules to approximate the resulting inner products. The flexibility to choose different basis and test functions is particularly valuable in complex scenarios.

Despite recent advances, spectral solutions for DOFDEs remain relatively unexplored. The inherently global nature of fractional derivatives aligns well with spectral methods, which are known for their high accuracy in solving functional equations. Applying spectral methods to discretize fractional differential equations can also substantially reduce data storage needs by requiring fewer time steps to achieve accurate approximate solutions compared to low-order methods. For a deeper discussion on the application of spectral methods to DOFDEs, the following relevant literature is referenced.

Kharazmi et al. [39] proposed a highly accurate spectral collocation method for nonlinear DOFDEs. In their approach, the distributed operator was approximated using Gauss-Legendre quadrature, while the resulting multi-term FDE was solved with a spectral collocation method. To address linear DOFDEs, Morgado et al. [53] introduced a Chebyshev collocation method for the distributed-order time-fractional diffusion equation, where Gauss-Legendre quadrature was employed to approximate the distributed operator. In a related contribution, a collocation method based on Chebyshev polynomials was also developed for solving distributed-order fractional differential equations, demonstrating accurate numerical performance [57]. In a separate development, Zhang et al. [35] presented a numerical approach for solving the two-dimensional linear Riesz space distributed-order advection-diffusion equation. Their method also relied on Gauss-Legendre quadrature for approximating both the distributed order and the resulting multi-term FDE. Furthermore, operational matrix techniques based on orthonormal Bernoulli polynomials were developed for solving distributed-order time-fractional partial differential equations, whereby the original problems were transformed into systems of algebraic equations with satisfactory numerical accuracy [59].

The aforementioned studies demonstrate the potential of spectral methods for solving DOFDEs, yet gaps remain, particularly in practical implementation approaches and error analysis. Most spectral methods applied to DOFDEs in

current research rely on direct, non-operational approaches. These approaches often lead to increased computational costs and complex implementation requirements, demanding substantial storage capacity. Although some studies have introduced operational approaches to improve efficiency [56, 58, 77], these methods typically result in dense matrices with computationally complex elements. In light of these limitations, our work aims to propose a discrete Petrov-Galerkin scheme for solving the following DOFDEs

$$\begin{cases} \int_0^1 \varphi(\alpha) D^\alpha u(x) d\alpha = f(x), \\ u(0) = 0, \end{cases} \tag{1.1}$$

by addressing these challenges. Here f is absolutely integrable on $[0, \infty)$, and D^α denotes a bounded Caputo-type fractional derivative of order $0 < \alpha \leq 1$, defined by

$$D^\alpha u(x) = I^{1-\alpha} Du(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha} u'(\tau) d\tau, \quad x > 0,$$

where $I^\alpha u(x)$ is the fractional integral of order α [17]. Furthermore, the weight function $\varphi(\alpha)$ is absolutely integrable on $\Lambda = [0, 1]$ and satisfies

$$\int_0^1 \varphi(\alpha) s^\alpha d\alpha \neq 0, \quad \mathrm{Re}(s) > 0.$$

In [19], the fundamental analytical results of (1.1) were investigated under the above assumptions, demonstrating that the unique solution is given by

$$u(x) = f * \mathcal{L}^{-1} \left(\frac{1}{\int_0^1 \varphi(\alpha) s^\alpha d\alpha} \right),$$

and $u'(x)$ is bounded and measurable on Λ, where $\mathcal{L}^{-1}$ and $*$ denote the inverse Laplace operator and convolution product, respectively. This conclusion indicates that the discrete Petrov-Galerkin method can serve as a powerful tool for obtaining an approximate solution to (1.1), owing to its superior accuracy in handling equations with smooth solutions. Accordingly, our approach is devised based on the following key steps:

- The distributed operator is typically approximated through midpoint or composite numerical integration methods, such as the trapezoidal or Simpson's rule. In this paper, we implement high-order Newton–Cotes quadrature, specifically Milne's and Weddle's rules, to efficiently approximate the distributed operator.
- An easily implementable discrete Petrov-Galerkin scheme is developed, employing generalized Jacobi polynomials and fractional Legendre functions as basis and test functions, respectively, and fractional Jacobi quadrature to discretize the resulting inner products. This approach captures the approximate solution with sparse and computationally efficient operational matrices.
- Finally, a rigorous theoretical analysis is performed to evaluate the convergence properties of the proposed scheme. Additionally, all theoretical predictions are validated through illustrative numerical examples.

The proposed approach offers several significant advantages. First, the flexibility in choosing distinct basis and test functions, tailored to the asymptotic behavior of the underlying function spaces, enables the method to achieve very high accuracy for the resulting multi-term fractional differential equation, without compromising the precision of the initial quadrature discretization. Second, the method is implemented in a fully operational (matrix-based) fashion, which leads to sparse system matrices and significantly reduces computational cost and storage requirements compared to conventional non-operational spectral methods. Moreover, the proposed scheme is parameter-free and straightforward to implement, making it an attractive choice for practical applications.

In addition to these advantages, the proposed method has certain limitations. However, their impact can be minimized by employing a robust quadrature rule. Indeed, the convergence rate of the proposed method is determined not only by the spectral accuracy of the Petrov-Galerkin discretization (governed by the polynomial degree N) but also by the accuracy of the numerical integration rule used to approximate the distributed-order integral in (1.1). Since the distributed-order operator is discretized using a composite Newton-Cotes quadrature prior to applying the spectral scheme, the overall convergence behavior is inherently influenced by the quadrature order p. In particular, if a

low-order quadrature rule (e.g., the trapezoidal rule) is employed, the quadrature error may dominate and degrade the overall convergence rate. Nevertheless, this limitation can be effectively mitigated by using high-order Newton-Cotes rules, such as Milne's rule ($p = 6$) or Weddle's rule ($p = 8$), which ensure that the quadrature error remains negligible compared to the spectral error, thereby preserving the overall high accuracy of the method.

The remainder of the paper is organized as follows: Section 2 introduces some preliminaries, including Newton–Cotes quadrature rules and Jacobi polynomials, along with their approximation properties. This section also defines the fractional Jacobi functions and associated quadrature formulas. Section 3 outlines an efficient discrete Petrov-Galerkin approach for the numerical solution of (1.1). In Section 4, the convergence of the numerical solution is analyzed and verified. Section 5 provides numerical examples to demonstrate the effectiveness of our method. Finally, Section 6 offers concluding remarks.

2. Preliminaries

In this section, we present some required formulations and lemmas for the sequel.

2.1. **Composite Newton-Cotes quadrature.** Let $h = \frac{1}{M}$, where M is a positive integer and $\alpha_i = ih$ for $i = 0, 1, \ldots, M$ define a partition of Λ. For numerical integration over the interval Λ, the composite Newton–Cotes quadrature is employed:

$$\int_0^1 f(\alpha)d\alpha \simeq h\sum_{i=0}^{M} f(\alpha_i)w_i, \tag{2.1}$$

where for the second-order trapezoidal rule we have $f \in C^2(\Lambda)$, and

$$\begin{aligned} w_0 &= w_M = \frac{1}{2}, \\ w_i &= 1, \quad i = 1, 2, \ldots, M-1. \end{aligned}$$

Also, for the fourth-order Simpson's rule we have $M = 2k$, $k \in \mathbb{N}$, $f \in C^4(\Lambda)$, and

$$\begin{aligned} w_0 &= w_M = \frac{1}{3}, \\ w_{2i-1} &= \frac{4}{3}, \quad i = 1, 2, \ldots, \frac{M}{2}, \\ w_{2i} &= \frac{2}{3}, \quad i = 1, 2, \ldots, \frac{M}{2} - 1. \end{aligned}$$

Moreover, for the sixth-order Milne's rule we have $M = 4k$, $k \in \mathbb{N}$, $f \in C^6(\Lambda)$, and

$$\begin{aligned} w_0 &= w_M = \frac{14}{45}, \\ w_{2i-1} &= \frac{64}{45}, \quad i = 1, 2, \ldots, \frac{M}{2}, \\ w_{2i} &= \frac{24}{45}, \quad i = 1, 2, \ldots, \frac{M}{2} - 1, \ i \neq 2k, \\ w_{4i} &= \frac{28}{45}, \quad i = 1, 2, \ldots, \frac{M}{4} - 1. \end{aligned}$$

Finally, for the eighth-order Weddle's rule we have $M = 6k$, $k \in \mathbb{N}$, $f \in C^8(\Lambda)$, and

$$\begin{aligned} w_0 &= w_M = \frac{123}{420}, \\ w_{2i-1} &= \frac{648}{420}, \quad i = 1, 2, \ldots, \frac{M}{2}, \ i \neq 3k-1, \\ w_{2i} &= \frac{81}{420}, \quad i = 1, 2, \ldots, \frac{M}{2} - 1, \ i \neq 3k, \end{aligned}$$

$$w_{6i} = \frac{246}{420}, \quad i = 1, 2, \ldots, \frac{M}{6} - 1,$$
$$w_{6i-3} = \frac{816}{420}, \quad i = 1, 2, \ldots, \frac{M}{6}.$$

Note that all the aforementioned weights are obtained by extending the Newton–Cotes quadrature formulas [69] to approximate (2.1) throughout the integration interval Λ. Employing established error estimation theorems [69], we can observe that, for $f \in C^p(\Lambda)$ with $p \in \mathbb{N}$, we have

$$\|\mathrm{ENC}(f(\alpha))\|_\infty \leq ch^p \|\partial_\alpha^p f(\alpha)\|_\infty, \tag{2.2}$$

where c is a positive constant and $\mathrm{ENC}(f(\alpha))$ represents the error function derived from applying one of the aforementioned composite Newton–Cotes methods to approximate the integral of $f(\alpha)$ on Λ.

2.2. **Generalized Jacobi polynomials.** For all $\theta, \vartheta \in \mathbb{Z}$, the generalized Jacobi polynomials are defined by

$$\hat{J}_n^{\theta,\vartheta}(x) = 2^{\tilde{\theta}+\tilde{\vartheta}}(1-x)^{\tilde{\theta}} x^{\tilde{\vartheta}} J_{\hat{n}}^{\hat{\theta},\hat{\vartheta}}(x), \quad \hat{n} = n - (\tilde{\theta} + \tilde{\vartheta}), \tag{2.3}$$

where

$$\tilde{\theta} = \begin{cases} -\theta, & \theta \leq -1, \\ 0, & \theta > -1, \end{cases} \quad \hat{\theta} = \begin{cases} -\theta, & \theta \leq -1, \\ \theta, & \theta > -1, \end{cases}$$

and similarly for $\tilde{\vartheta}$ and $\hat{\vartheta}$. $J_{\hat{n}}^{\hat{\theta},\hat{\vartheta}}(x)$ is the classical shifted Jacobi polynomial on Λ. These polynomials extend classical shifted Jacobi polynomials to eliminate the constraints $\theta, \vartheta > -1$ and are mutually orthogonal over Λ with respect to the $L^2_{w_{\theta,\vartheta}}$ inner product, where $w^{\theta,\vartheta} = (1-x)^\theta x^\vartheta$ [26]. Owing to the following distinctive properties of the shifted generalized Jacobi polynomials for $\theta, \vartheta \leq 0$,

$$\partial_x^j \hat{J}_n^{\theta,\vartheta}(0) = 0, \quad j = 0, 1, \ldots, \tilde{\vartheta} - 1, \tag{2.4}$$
$$\partial_x^j \hat{J}_n^{\theta,\vartheta}(1) = 0, \quad j = 0, 1, \ldots, \hat{\theta} - 1, \tag{2.5}$$

they serve as highly suitable basis functions for Galerkin and Petrov-Galerkin approximations of functional equations with homogeneous boundary conditions.

2.3. **Fractional Jacobi functions.** The fractional Jacobi functions are denoted by $J_n^{\rho,\sigma,\lambda}(x)$, for $\rho, \sigma > -1$, $\lambda \in (0, 1]$, and are defined as

$$J_n^{\rho,\sigma,\lambda}(x) = J_n^{\rho,\sigma}(t^\lambda).$$

These functions exhibit orthogonality with respect to the weight function $w^{\rho,\sigma,\lambda}(x) = \lambda x^{\lambda\sigma+\lambda-1}(1-x^\lambda)^\rho$, and are used in the fractional Jacobi interpolation operator $I_N^{\rho,\sigma,\lambda}$ [25]:

$$I_N^{\rho,\sigma,\lambda} f(x) = \sum_{j=0}^{N} f_j J_j^{\rho,\sigma,\lambda}(x), \quad f_j = \frac{\langle f, J_j^{\rho,\sigma,\lambda}\rangle_{N,w^{\rho,\sigma,\lambda}}}{\|J_j^{\rho,\sigma,\lambda}\|^2_{L^2_{w^{\rho,\sigma,\lambda}}(\Lambda)}}, \tag{2.6}$$

where $\|J_j^{\rho,\sigma,\lambda}\|_{L^2_{w^{\rho,\sigma,\lambda}}(\Lambda)}$ denotes the weighted L^2 norm of the basis function and $\langle\cdot,\cdot\rangle_{N,w^{\rho,\sigma,\lambda}}$ is the following fractional Jacobi-Gauss discrete inner product formula:

$$\langle U, V\rangle_{N,w^{\rho,\sigma,\lambda}} = \sum_{j=0}^{N} U\big((s_j^{\rho,\sigma})^\lambda\big) V\big((s_j^{\rho,\sigma})^\lambda\big) w_j^{\rho,\sigma}, \tag{2.7}$$

such that $\{s_j^{\rho,\sigma}, w_j^{\rho,\sigma}\}_{j=0}^N$ are the nodes and weights of the Gauss-Jacobi quadrature on Λ [25]. In particular, for $\rho = \sigma = 0$, a proper upper bound for the truncation error of the fractional Legendre interpolation operator is provided in [25]:

$$\|e_{I_N^{0,0,\lambda} f}\|_2 \leq CN^{-k} \|\partial_s^k(f(s))\|_{w^{k,k}}, \tag{2.8}$$

where $s = x^{\lambda}$, $e_{I_N^{0,0,\lambda}} f = f - I_N^{0,0,\lambda} f$ is the fractional Legendre interpolation error, and $\|\cdot\|_2$ and $\|\cdot\|_{w^{k,k}}$ are the norms associated with the L^2 and weighted L^2 spaces on Λ, respectively. Moreover, k is the largest natural number such that the norm on the right-hand side of (2.8) remains finite.

3. Numerical approach

In this section, the proposed operational discrete Petrov-Galerkin method addresses the DOFDE (1.1). The distributed operator is discretized using high-order Newton-Cotes quadrature, yielding a multi-term fractional differential equation

$$\sum_{i=0}^{M} w_i \varphi\left(\frac{i}{M}\right) D^{\frac{i}{M}} u(x) \simeq f(x), \tag{3.1}$$

where $\{w_i\}_{i=0}^{M}$ are the Newton-Cotes weights given in subsection 2.1. This transformation enables efficient computation and facilitates the application of the Petrov-Galerkin approach in a discrete setting. Assume that the solution is approximated as:

$$u_N(x) = \sum_{j=0}^{\infty} a_j \hat{J}_j^{(0,-1)}(x) = \underline{a_N} J \underline{X}, \tag{3.2}$$

which denotes the approximate Petrov-Galerkin solution of degree N for the derived multi-term FDE (3.1). In this formulation, $\underline{a_N} = (a_0, a_1, \ldots, a_N, 0, \ldots)$, $\underline{X} = (x, x^2, \ldots, x^N, \ldots)^T$, and J is the following non-singular lower triangular coefficient matrix,

$$J = \begin{pmatrix} 1 & & & & \\ -2 & 3 & & & \\ 3 & -12 & 10 & & \\ -4 & 30 & -60 & 35 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Trivially from (2.4) we have $u_N(0) = 0$, and the initial condition is fulfilled. Now, substituting (3.2) into (3.1) yields the approximate solution in such a manner that the approximate relation becomes an exact equality. Namely, we obtain:

$$\underline{a_N} J \sum_{i=0}^{M} w_i \ \varphi\left(\frac{i}{M}\right) \ D^{\frac{i}{M}} \underline{X} = f(x). \tag{3.3}$$

Using the following relation [17]

$$D^{\alpha} x^j = \begin{cases} 0, & j < \lceil \alpha \rceil, \quad j \in \mathbb{N}, \\ \frac{j!}{\Gamma(j+1-\alpha)} x^{j-\alpha}, & j \geq \lceil \alpha \rceil, \end{cases}$$

we obtain

$$D^{\frac{i}{M}} x^j = \frac{j!}{\Gamma\left(j - \frac{i}{M} + 1\right)} x^{j - \frac{i}{M}} \simeq \Big[\underbrace{0, 0, \ldots, 0}_{jM-i}, \ \frac{j!}{\Gamma\left(j + 1 - \frac{i}{M}\right)}, 0, \ldots\Big] \underline{X_M}$$
$$= \underline{\xi}^{i,j} \underline{X_M}, \ j \in \mathbb{N},$$

where $\underline{X_M} = [1, x^{\frac{1}{M}}, x^{\frac{2}{M}}, \ldots, x^{\frac{jM-i}{M}}, \ldots]^T$ and

$$\underline{\xi}^{i,j} = [\vec{O}_{jM-i} \mid A_{i,j} e_1^T], \ i = 0, 1, \ldots, M, \ j \in \mathbb{N}$$

such that $\vec{O}_{jM-i}$ denotes the zero vector of order $jM-i$, $A_{i,j}=\frac{j!}{\Gamma\left(j+1-\frac{i}{M}\right)}$, and $e_1^T=[1,0,0,...]$. Thus we can conclude

$$D^{\frac{i}{M}}\underline{X}\simeq\begin{pmatrix}\xi^{i,1}\\ \xi^{i,2}\\ \xi^{i,3}\\ \vdots\end{pmatrix}\underline{X}^M=\begin{pmatrix}\underbrace{0,0,...,0}_{M-i} & A_{i,1} & 0 & 0 & \dots\\ \underbrace{0,0,...,0}_{2M-i} & A_{i,2} & 0 & 0 & \dots\\ \underbrace{0,0,...,0}_{3M-i} & A_{i,3} & 0 & 0 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\underline{X}^M=\eta^i\underline{X}^M, \tag{3.4}$$

where

$$\eta^i=\Big[\mathbf{O}_{\infty\times(M-i)}\mid B_i\Big],\ i=0,1,...,M,$$

such that $\mathbf{O}_{\infty\times(M-i)}$ is an infinite zero matrix of order $\infty\times(M-i)$ and B_i is an infinite matrix such that its j-th row is given by

$$(B_i)_j=[\vec{O}_{(j-1)M}\mid A_{i,j}e_1^T],\ i=0,1,...,M,\ j\in\mathbb{N}.$$

Substituting (3.4) into (3.3) we obtain approximately

$$\underline{a_N}J\sum_{i=0}^{M}\tilde{\eta}^i\underline{X}^M=f(x), \tag{3.5}$$

where $\tilde{\eta}^i=w_i\varphi\left(\frac{i}{M}\right)\eta^i$. Clearly, defining $\underline{J}^M=(J_0^{0,0,\frac{1}{M}},J_1^{0,0,\frac{1}{M}},...,J_N^{0,0,\frac{1}{M}},...)^T$ as a vector of fractional Legendre basis functions, we can write

$$J^M\underline{X}^M=\underline{J}^M,$$

where J^M is the following infinite lower triangular coefficient matrix

$$J^M=\begin{pmatrix}1 & & & & \\ -1 & 2 & & & \\ 1 & -6 & 6 & & \\ -1 & 12 & -30 & 20 & \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Therefore, the approximate relation (3.5) can be rewritten as

$$\underline{a_N}\ \Pi^{J,M}\ \underline{J}^M=f(x), \tag{3.6}$$

with $\Pi^{J,M}=J\Big(\sum_{i=0}^{M}\tilde{\eta}^i\Big)\big(J^M\big)^{-1}$. After multiplying both sides by $\{J_k^{0,0,\frac{1}{M}}\}_{k=0}^N$ and applying the fractional Legendre-Gauss discrete inner product (2.7), we deduce that

$$\underline{a_N}\ \Pi^{J,M}\ \langle\underline{J}^M,J_k^{0,0,\frac{1}{M}}\rangle_{N,w^{0,0,\frac{1}{M}}}=\langle f,J_k^{0,0,\frac{1}{M}}\rangle_{N,w^{0,0,\frac{1}{M}}},\ k=0,1,...,N. \tag{3.7}$$

Given the orthogonality of $\{J_k^{0,0,\frac{1}{M}}\}_{k\geq0}$, (3.7) implies

$$\underline{a_N}\ \Pi_k^{J,M}=\frac{1}{\|J_k^{0,0,\frac{1}{M}}\|^2_{L^2_{w^{0,0,\frac{1}{M}}}(\Lambda)}}\langle f,J_k^{0,0,\frac{1}{M}}\rangle_{N,w^{0,0,\frac{1}{M}}},\ k=0,1,...,N,$$

and consequently,

$$\underline{a_N}\ \left(\Pi_0^{J,M},\Pi_1^{J,M},...,\Pi_N^{J,M}\right)=(f_0^M,f_1^M,...,f_N^M), \tag{3.8}$$

where $\Pi_k^{J,M}$ is the $k-$th column of $\Pi^{J,M}$ and

$$f_k^M = \frac{1}{\|J_k^{0,0,\frac{1}{M}}\|^2_{L^2_{w^{0,0,\frac{1}{M}}}(\Lambda)}} \langle f, J_k^{0,0,\frac{1}{M}} \rangle_{N,w^{0,0,\frac{1}{M}}}, \ k = 0, 1, ..., N.$$

Finally, the unknown vector $\underline{a_N}$ is determined by solving the linear system of algebraic equations of order $(N+1)$ (3.8).

4. Convergence analysis

This section focuses on analyzing the convergence properties of the proposed method, with particular emphasis on deriving rigorous error bounds. To this end, we establish the following theorem, which formalizes the error bound for the proposed Petrov-Galerkin approximation.

Theorem 4.1. *Let $u_N(x)$ represent the Petrov-Galerkin solution of (1.1), as defined by (3.2). Then the following estimate is established:*

$$\|e_N\|_2 \leq C_1 N^{-k} \|\partial_s^k f(s)\|_{w^{k,k}} + C_2 h^p \|\varphi(\alpha) D^\alpha u\|_\infty + C_3 \sum_{i=0}^{M} N^{-\nu_i} \|\partial_s^{\nu_i} D^{\frac{i}{M}} u(s)\|_{w^{\nu_i,\nu_i}}. \tag{4.1}$$

Here $e_N(x) = u(x) - u_N(x)$ represents the error, while the positive constants k and ν_i denote the largest integers for which the weighted L^2 norms on the right-hand side of (4.1) remain bounded. Furthermore, h represents the quadrature step size, and p denotes the quadrature order described in subsection 2.1.

Proof. Based on the implementation process outlined in the previous section, it can be concluded that the linear system of algebraic equations (3.8) corresponds to the following operator form:

$$\sum_{i=0}^{M} w_i \varphi\left(\frac{i}{M}\right) I_N^{0,0,\frac{1}{M}} D^{\frac{i}{M}} u_N(x) = I_N^{0,0,\frac{1}{M}} f(x). \tag{4.2}$$

Subtracting (4.2) from (1.1) yields

$$\int_0^1 \varphi(\alpha) D^\alpha u(x) d\alpha - \sum_{i=0}^{M} w_i \varphi\left(\frac{i}{M}\right) I_N^{0,0,\frac{1}{M}} D^{\frac{i}{M}} u_N(x) = e_{I_N^{0,0,\frac{1}{M}}} f. \tag{4.3}$$

Through a series of straightforward manipulations, the relation (4.3) can be expressed in terms of

$$\sum_{i=0}^{M} w_i \varphi\left(\frac{i}{M}\right) \left(D^{\frac{i}{M}} e_N + e_{I_N^{0,0,\frac{1}{M}}} D^{\frac{i}{M}} u_N(x)\right) = e_{I_N^{0,0,\frac{1}{M}}} f - \mathrm{ENC}\big(\varphi(\alpha) D^\alpha u\big). \tag{4.4}$$

Integrating (4.4) from 0 to x gives:

$$\begin{aligned} w_M \varphi(1) e_N(x) = &- \sum_{i=0}^{M-1} w_i \varphi\left(\frac{i}{M}\right) I^{1-\frac{i}{M}} e_N + I^1 \Bigg\{ e_{I_N^{0,0,\frac{1}{M}}} f - \mathrm{ENC}\big(\varphi(\alpha) D^\alpha u\big) \\ &- \sum_{i=0}^{M} w_i \varphi\left(\frac{i}{M}\right) e_{I_N^{0,0,\frac{1}{M}}} D^{\frac{i}{M}} u_N(x) \Bigg\}, \end{aligned} \tag{4.5}$$

and consequently,

$$|e_N(x)| \leq C \left[\sum_{i=0}^{M-1} I^{1-\frac{i}{M}} |e_N| + I^1 \left\{ |e_{I_N^{0,0,\frac{1}{M}}} f| + |\mathrm{ENC}\big(\varphi(\alpha) D^\alpha u\big)| + \sum_{i=0}^{M} |e_{I_N^{0,0,\frac{1}{M}}} D^{\frac{i}{M}} u_N(x)| \right\} \right], \tag{4.6}$$

where C is a positive constant. On the other hand, we have

$$\sum_{i=0}^{M-1} I^{1-\frac{i}{M}} |e_N| = \sum_{i=0}^{M-1} \frac{1}{\Gamma\left(1-\frac{i}{M}\right)} \int_0^x (x-\tau)^{-\frac{i}{M}} |e_N(\tau)| d\tau$$

$$= \int_0^x (x-\tau)^{\frac{1}{M}-1} K(x,\tau)|e_N(\tau)|d\tau,$$

where $K(x,t) = \sum_{i=0}^{M-1} \frac{(x-\tau)^{1-\frac{1+i}{M}}}{\Gamma(1-\frac{i}{M})}$ is a bounded operator. Accordingly, by incorporating the preceding result into (4.6), we can deduce

$$|e_N(x)| \leq C_K \int_0^x (x-\tau)^{\frac{1}{M}-1}|e_N(\tau)|d\tau + I^1\left\{|e_{I_N^{0,0,\frac{1}{M}}} f| + |\mathrm{ENC}(\varphi(\alpha)D^\alpha u)| + \sum_{i=0}^{M} |e_{I_N^{0,0,\frac{1}{M}}} D^{\frac{i}{M}} u_N(x)|\right\}, \tag{4.7}$$

where $C_K = C\|K\|_\infty$. Applying Gronwall's inequality, as established in [24], allows us to derive the following relation:

$$\|e_N\|_2 \leq \|e_{I_N^{0,0,\frac{1}{M}}} f\|_2 + \|\mathrm{ENC}(\varphi(\alpha)D^\alpha u)\|_\infty + \sum_{i=0}^{M} \|e_{I_N^{0,0,\frac{1}{M}}} D^{\frac{i}{M}} u_N(x)\|_2. \tag{4.8}$$

Applying the estimations (2.2) and (2.8) to the right-hand side of (4.8) yields:

$$\|e_N\|_2 \leq C_1 N^{-k}\|\partial_s^k f(s)\|_{w^{k,k}} + C_2 h^p \|\varphi(\alpha)D^\alpha u\|_\infty + C_3 \sum_{i=0}^{M} N^{-\nu_i}\|\partial_s^{\nu_i} D^{\frac{i}{M}} u_N(s)\|_{w^{\nu_i,\nu_i}}. \tag{4.9}$$

Finally, the desired result is achieved by using the first-order Taylor expansion of $\partial_s^{\nu_i} D^{\frac{i}{M}} u_N(s)$ around $u(s)$ and performing some simple manipulations in (4.9). □

5. Numerical results

In this section, we present a series of numerical experiments using the operational discrete Petrov-Galerkin method introduced in Section 3, applied to three test problems. To rigorously evaluate the accuracy and computational efficiency of the proposed method, we employ the L^2-norm error. Furthermore, we establish the convergence order (CO) of the method according to the following formulation:

$$CO = \left|\log_2 \frac{\|e_N^{2M}\|_2}{\|e_N^M\|_2}\right|.$$

Here, e_N^M denotes the error of the proposed method after two levels of approximation. First, the distributed-order integral is discretized using a Newton-Cotes quadrature rule with M subintervals, which yields an approximate multi-term fractional differential equation. Second, this multi-term equation is solved by a Petrov-Galerkin spectral method using basis polynomials of degree N. Consequently, the overall error depends on both discretization parameters: M controls the accuracy of the quadrature step, while N governs the spectral accuracy of the solution step. All computations were performed using Mathematica software on a personal computer equipped with a 13th-generation Intel(R) Core(TM) i3-1315U processor (1.20 GHz), 20 GB of RAM, and a 64-bit operating system, ensuring consistency and precision in results. Additionally, all linear systems are solved using the `LinearSolve` function in Wolfram Mathematica, which implements LU decomposition with partial pivoting to ensure numerical stability.

Example 5.1. Consider (1.1) with $\varphi(\alpha) = \Gamma(4-\alpha)\sinh(\alpha)$ and

$$f(x) = \frac{6x(x^2 - \sinh(1)\ln(x) - \cosh(1))}{\ln(x^2) - 1}.$$

The equation under consideration possesses a unique solution, $u(x) = x^3$. To solve Example 5.1, we applied the technique outlined in section 3. Table 1 presents the L^2-norm errors for $N = 4$ over a range of M values, along with the corresponding convergence orders. Furthermore, Table 2 compares the maximum error of the proposed method ($\|e_N\|_\infty$) with the numerical results reported in [39]. It is noteworthy that μ in [39] is a free interpolation parameter introduced in the basis expansion to capture potential solution singularities and to enhance the convergence behavior of the spectral Petrov-Galerkin scheme. Consequently, the errors reported in that reference depend on the chosen values of μ, whereas the present method requires no additional tuning parameters. This comparison confirms that the proposed approach achieves competitive accuracy while offering a simpler, parameter-free implementation. Figure 1

TABLE 1. The L^2-norm errors for $N = 4$ versus M, alongside the order of convergence for Example 5.1.

	Trapezoidal rule		Simpson's rule		Milne's rule		Weddle's rule	
M	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO
2	$2e-2$	1.94	$9e-4$	3.89	-	-	-	-
4	$5e-3$	2.07	$6e-5$	3.93	$5e-6$	5.93	-	-
6	$2e-3$	1.98	$1e-5$	3.88	-	-	$4e-8$	6.57
8	$1e-3$	1.89	$4e-6$	4.02	$8e-8$	6.02	-	-
12	$6e-4$	2.04	$8e-7$	3.96	$7e-9$	5.98	$5e-10$	7.19
16	$4e-4$	1.99	$3e-7$	3.94	$1e-9$	5.98	-	-
24	$2e-4$	1.94	$5e-8$	3.98	$1e-10$	5.97	$3e-12$	7.64

TABLE 2. Comparison of the L_∞-errors for Example 5.1 with $N = 4$ and $M = 24$.

Method / Parameter		L_∞-error
Simpson	Proposed method	$6e-8$
Milne		$3e-10$
$\mu = 1 + 10^{-4}$	Ref.[39]	$6e-7$
$\mu = 1.1$		$6e-4$
$\mu = 1.5$		$4e-3$

also displays the L^2-norm errors on a logarithmic scale for varying M. The results demonstrate that increasing the quadrature order consistently reduces the error. Moreover, the observed convergence orders are in excellent agreement with the theoretical predictions established in Theorem 4.1, further validating the reliability of the method. The comparative analysis in Table 2 highlights the superior performance of our method relative to existing approaches [39].

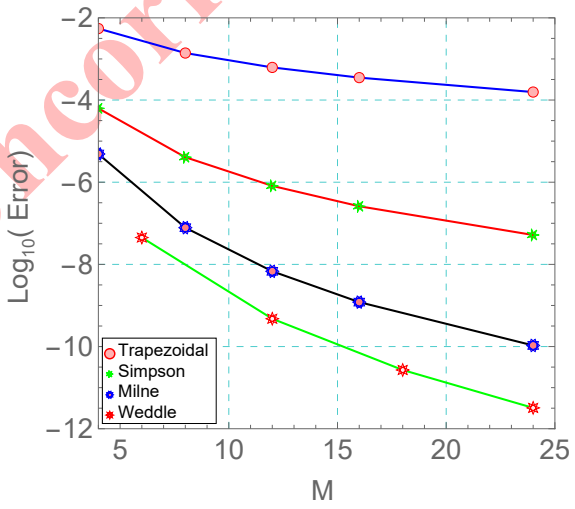

FIGURE 1. Semi-logarithmic plot of the numerical errors for Example 5.1 versus different M.

Example 5.2. Consider (1.1) with $\varphi(\alpha) = \Gamma(3-\alpha)$ and

$$f(x) = \frac{2(x^2 - x)}{\ln(x)}.$$

The equation under consieration admits a unique solution, $u(x) = x^2$. To solve Example 5.2, we applied the methodology outline in section 3. Table 3 reports the L^2-norm errors for $N = 4$ across various values of M, along with the corresponding convergence order.

In addition, Table 4 presents a comparison of the relative error at $t = 0.9$ with the corresponding results reported in [37]. In that reference, h and K denote the temporal step size and the number of subdivisions used for the distributed-order discretization, respectively. Also, Figure 2 illustrates the L^2-norm errors on a logarithmic scale as a function of M.

The results confirm that increasing the quadrature order systematically reduces the error. Moreover, the observed convergence order closely matches the theoretical predictions established in Theorem 4.1, further validating the robustness and accuracy of the proposed approach. Finally, the comparative data presented in Table 4 highlight the enhanced performance of the proposed scheme, confirming its superior accuracy and computational efficiency relative to the existing methods reported in [37].

TABLE 3. The L^2-norm errors for $N = 4$ versus M, alongside the order of convergence for Example 5.2.

	Trapezoidal rule		Simpson's rule		Milne's rule		Weddle's rule	
M	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO
2	$4e-3$	2.07	$2e-4$	2.76	-	-	-	-
4	$1e-3$	1.83	$2e-5$	1.92	$4e-6$	2.5	-	-
6	$5e-4$	1.58	$1e-5$	2.26	-	-	$3e-7$	4.09
8	$3e-4$	1.48	$6e-6$	2.68	$7e-7$	3.94	-	-
12	$2e-4$	1.44	$2e-6$	3.08	$2e-7$	4.57	$2e-8$	5.79
16	$1e-4$	1.55	$9e-7$	3.29	$4e-8$	4.91	-	-
24	$6e-5$	1.6	$2e-7$	3.5	$6e-9$	5.27	$3e-10$	6.93

Example 5.3. Consider (1.1) with $\varphi(\alpha) = \Gamma(5-\alpha)$ and

$$f(x) = \frac{24(x^4 - x^3)}{\ln(x)}.$$

The exact solution of the problem is $u(x) = x^4$. Similar to the previous examples, the proposed Petrov-Galerkin approach is implemented to obtain robust approximate solution. The results presented in Table 5 and Figure 3, further confirm the effectiveness and reliability of the proposed method.

Example 5.4. Consider (1.1) with $\varphi(\alpha) = \delta(\alpha - \beta)$ and where $\delta(\cdot)$ denotes the Dirac delta function and $\beta \in (0, 1)$. The source term $f(x)$ is constructed in such a way that the exact solution is $u(x) = e^{\beta x} - 1$.

The numerical results generated by the proposed method are presented in Table 6. These findings confirm that the proposed technique effectively handles both nonlinear and non-integer β values, maintaining accuracy and stability

TABLE 4. Comparison of the relative error at $t = 0.9$ for Example 5.2 with $N = 4$ and $M = 24$.

Method / Parameter		Relative error
Simpson	Proposed method	$1e-7$
Milne		$2e-9$
$h = 10^{-4}, K = 64$	Ref. [37]	$6e-4$

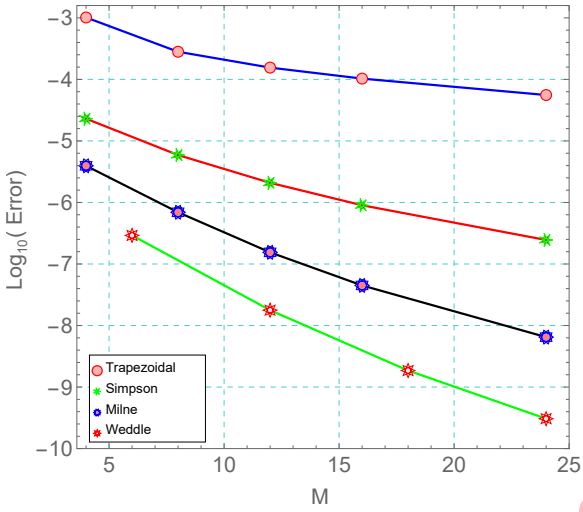

FIGURE 2. Semi-logarithmic plot of the numerical errors for Example 5.2 versus different M.

TABLE 5. The L^2-norm errors for $N = 4$ versus M, alongside the order of convergence for Example 5.3.

	Trapezoidal rule		Simpson's rule		Milne's rule		Weddle's rule	
M	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO	$\|e_N\|_2$	CO
2	$9e-4$	1.95	$9e-6$	3.90	-	-	-	-
4	$2e-4$	1.96	$6e-7$	3.84	$3e-8$	5.82	-	-
6	$1e-4$	1.97	$1e-7$	3.84	-	-	$2e-10$	7.19
8	$6e-5$	1.97	$4e-8$	3.85	$5e-10$	5.95	-	-
12	$3e-5$	1.98	$9e-9$	3.88	$4e-11$	5.98	$1e-12$	7.65
16	$1e-5$	1.99	$3e-9$	3.90	$7e-12$	5.98	-	-
24	$7e-6$	1.99	$6e-10$	3.93	$6e-13$	5.98	$7e-15$	7.86

TABLE 6. The L^2-norm errors for different values of N and β for Example 5.4.

N	$\beta = \frac{1}{4}$	$\beta = \frac{1}{2}$	$\beta = \frac{3}{4}$
4	$9e-10$	$7e-8$	$8e-7$
8	$4e-18$	$4e-15$	$3e-13$
12	$5e-27$	$9e-23$	$3e-20$
16	$2e-36$	$6e-31$	$1e-27$

in the numerical solutions. It is also noteworthy that the significant reduction in the reported errors compared to the previous examples is due to the fact that the weight function in the problem is the Dirac delta function, which eliminates the need to apply numerical integration methods to convert the problem into a classical fractional differential equation. Consequently, the convergence rate is determined solely by the adopted spectral approach, which, according to the theoretical results, exhibits exponential accuracy.

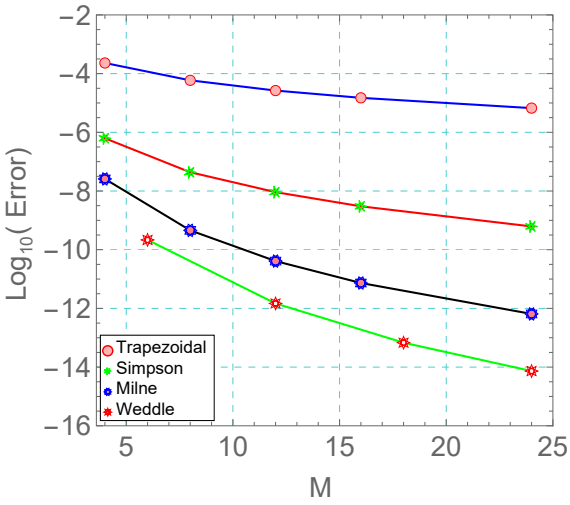

FIGURE 3. Semi-logarithmic plot of the numerical errors for Example 5.3 versus different M.

6. Conclusion

This paper presents an innovative discrete Petrov-Galerkin method for solving DOFDEs. By combining high-order Newton-Cotes quadrature with generalized Jacobi polynomials, the method achieves superior accuracy and computational efficiency. Theoretical convergence analysis and numerical experiments validate its effectiveness, positioning it as a significant advancement in the field.

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Conflicts of interest statement

Authors declare that they have no conflicts of interest.

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