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An innovative computational technique for a class of fractional BVPs with conformable derivative

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Abstract

This work will study linear conformable fractional boundary value problems with Dirichlet boundary conditions. First, the solution's existence and uniqueness will be verified using the Banach fixed point theorem. Then, an approximated solution to the problem will be obtained using a numerical method. Some numerical examples will be presented to show the efficiency of the result.

Keywords. Boundary value problem, Conformable fractional derivative, Fixed point theorem, Numerical solution.

2010 Mathematics Subject Classification. 34Axx, 34A08, 34Bxx.

1. Introduction

Fractional calculus is a branch of mathematical analysis that extends the concept of differentiation and integration to arbitrary orders (non-integer). This field, rooted in the 17th century, has gained significant attention in recent decades due to its potential applications in various scientific and engineering domains. Unlike traditional calculus, which deals with integer-order derivatives and integrals, fractional calculus allows for exploring fractional-order operators, offering a more nuanced perspective on physical and engineering phenomena [8, 13–15].

One of the most fascinating features of fractional calculus is the variety of definitions associated with fractional derivatives. Notable among these are the Riemann-Liouville, Caputo, and Grünwald-Letnikov definitions, each possessing distinct characteristics and uses. Recently, the conformable fractional derivative has gained attention as a viable alternative, owing to its straightforward nature and alignment with the principles of classical calculus.

The conformable fractional derivative, presented by Khalil et al. [12] in 2014, provides a novel viewpoint on fractional calculus. Its definition is relatively simple, similar to classical derivatives, yet incorporates a fractional order parameter. This ease of understanding has made it appealing to researchers and practitioners, facilitating more straightforward analysis and application.

While conformable fractional calculus offers a more intuitive approach to fractional calculus, the exploration of conformable fractional boundary value problems (CFBVPs) is still in its early stages compared to other definitions of fractional derivatives. There are some works in the literature, such as [2–4, 6, 11, 18, 20], that have investigated the existence of solutions for certain CFBVPs, and only a few work exists on finding numerical solutions for these problems [7, 9, 10, 17, 19]. This situation can be attributed to several factors, including the recent introduction of the conformable derivative, its perceived ease of use, and the limited availability of analytical tools specifically designed for CFBVPs. As a result, a significant gap exists in numerical methods and practical applications for addressing CFBVPs. Given the promising potential of CFBVPs in modeling complex real-world phenomena, further research in this area is urgently needed. By developing innovative numerical techniques and exploring additional applications, researchers

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can significantly contribute to the advancement of fractional calculus and its relevance across various scientific and engineering fields.

In this work we will focus to study the conformable fractional boundary value problem

$$\mathcal{T}_{\upsilon}\zeta(\tau) + \theta(\tau)\zeta(\tau) = \varrho(\tau), \quad 1 < \upsilon < 2, 0 \le \tau \le \wp, \tag{1.1}$$

$$\zeta(0) = \alpha, \quad \zeta(\wp) = \beta, \tag{1.2}$$

where $\mathcal{T}_{\upsilon}\zeta$ is the Conformable fractional derivative defined by Khalil et al. in [12] and the functions $\theta(\tau)$ and $\varrho(\tau)$ have some properties that will be specified in next sections. The rest of the paper is organized as below: In section 2, some preliminary facts and necessary tools will be introduced.In section 3, the existence and uniqueness of the solution will be demonstrated by use of the general nonlinear case and Banach contraction principle. In section 4, the numerical method which will be used will be explained and finally in section 5, some numerical examples will be presented to show our result.

2. Preliminaries

To enhance reader comprehension, we introduce essential notation and lemmas employed in our subsequent proofs.

Definition 2.1. [12] Let $\nu \in (0, 1]$. The conformable derivative of $\varphi : \mathbb{R}^+ \to \mathbb{R}$ of order ν is formulated as

$$\mathcal{T}_{\nu}\varphi(\tau) = \lim_{\epsilon \to 0} \frac{\varphi(\tau + \epsilon\tau^{1-\nu}) - \varphi(\tau)}{\epsilon}. \tag{2.1}$$

If $\mathcal{T}_{\nu}\varphi(\tau)$ exists on $(0, b)$, then $\mathcal{T}_{\nu}\varphi(0) = \lim_{\tau \to 0^+} \mathcal{T}_{\nu}\varphi(\tau)$.

From Definition 2.2 to Lemma 2.6, we set $\nu \in (k, k+1]$, $k \in \mathbb{N}$.

Definition 2.2. [12] The conformable derivative of $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is formulated as

$$\mathcal{T}_{\nu}\varphi(\tau) = \mathcal{T}_{\varsigma}\varphi^{(k)}(t), \tag{2.2}$$

where $\varsigma = \nu - k$.

Definition 2.3. [12] The conformable integral of $\varphi : \mathbb{R}^+ \to \mathbb{R}$ of order ν is given by

$$I^{\nu}\varphi(\tau) = \frac{1}{k!}\int_0^{\tau} (\tau - \varsigma)^k \varsigma^{\nu-k-1}\varphi(\varsigma)d\varsigma. \tag{2.3}$$

Lemma 2.4. *[12] For each $\tau > 0, \mathcal{T}_{\nu}I^{\nu}\varphi(\tau) = \varphi(\tau)$ whenever φ is continuous on $\mathbb{R}^+$.*

Lemma 2.5. *[12] For all $\mathcal{T}_{\nu}\, t^l = l\, t^{l-\nu}$, $\forall\, l \in \mathbb{R}$*

Lemma 2.6. *(See [1, 12]) If $\mathcal{T}_{\nu}\varphi(\tau)$ is continuous on $\mathbb{R}^+$, then*

$$I^{\nu}\mathcal{T}_{\nu}\varphi(\tau) = \varphi(\tau) + C_0 + C_1\tau + C_2\tau^2 + \cdots + C_n\tau^n, \tag{2.4}$$

for some real numbers $C_0, C_1, \ldots C_n$.

Definition 2.7. [16] Let $C([0, p])$ denote the space of continuous real- (or complex-) valued functions on the interval $[0, p]$. The *supremum norm* (or *uniform norm*) of a function $f \in C([0, p])$ is defined by

$$\|f\|_{\infty} := \sup_{t \in [0,p]} |f(t)|.$$

Definition 2.8. [16] The space $L^1[0, p]$ consists of all Lebesgue measurable functions $f : [0, p] \to \mathbb{R}$ (or $\mathbb{C}$) such that the L^1-*norm* is finite:

$$\|f\|_1 := \int_0^p |f(t)|\, dt < \infty.$$

Functions in $L^1[0, p]$ are identified if they differ only on a set of Lebesgue measure zero.

Theorem 2.9 (Banach Fixed Point Theorem)**.** *[5] Let (X, d) be a non-empty complete metric space and let $T : X \to X$ be a contraction mapping; that is, there exists a constant $0 \le k < 1$ such that*

$$d(Tx, Ty) \le k\, d(x, y) \quad \text{for all } x, y \in X.$$

Then T has a unique fixed point $x^* \in X$ (i.e., $Tx^* = x^*$). Moreover, for any $x_0 \in X$, the sequence defined by $x_{n+1} = Tx_n$ converges to x^*, and the following error estimate holds:

$$d(x_n, x^*) \le \frac{k^n}{1-k}\, d(x_1, x_0).$$

3. Existence Result

We start our result with the nonlinear case of the problem (1.1)-(1.2). In fact at first we consider the conformable fractional boundary value problem

$$\begin{aligned} &\mathcal{T}_\upsilon \zeta(\tau) = \hbar(\tau, \zeta(\tau)) \quad 1 < \upsilon < 2, 0 \le \tau \le \wp, \\ &\zeta(0) = \alpha, \quad \zeta(\wp) = \beta. \end{aligned} \tag{3.1}$$

We state the following Lemma which converts the nonlinear conformable fractional boundary value problem to an equivalent integral equation.

Lemma 3.1. *If $\zeta(\tau)$ is a solution of conformable fractional boundary value problem (3.1), then it is a solution of integral equation*

$$\zeta(\tau) = \int_0^\tau (\tau - \varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma, \zeta(\varsigma))d\varsigma - \frac{1}{\wp}\int_0^\wp \tau(\wp - \varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma, \zeta(\varsigma))d\varsigma + \frac{(\beta - \alpha)}{\wp}\tau + \alpha, \tag{3.2}$$

too.

Proof. Using Lemma 2.3 and Definition 2.6 and integrating of order υ from the differential equation (1.1) we get

$$\zeta(\tau) = \int_0^\tau (\tau - \varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma, \zeta(\varsigma))d\varsigma + c_1 + c_2\tau. \tag{3.3}$$

From the first boundary condition we have

$$\zeta(0) = c_1 = \alpha,$$

and from the second boundary condition we get

$$\int_0^\wp (\wp - \varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma, \zeta(\varsigma))d\varsigma + \alpha + c_2\wp = \beta, \tag{3.4}$$

so

$$c_2 = \frac{\beta - \alpha}{\wp} - \frac{1}{\wp}\int_0^\wp (\wp - \varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma, \zeta(\varsigma))d\varsigma. \tag{3.5}$$

Now by replacing the amount of c_2 in (3.4) we will have

$$\zeta(\tau) = \int_0^\tau (\tau - \varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma, \zeta(\varsigma))d\varsigma - \frac{1}{\wp}\int_0^\wp \tau(\wp - \varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma, \zeta(\varsigma))d\varsigma + \frac{(\beta - \alpha)}{\wp}\tau + \alpha, \tag{3.6}$$

and this complete the proof. □

Now we are ready to prove the existence and uniqueness of the solution for the conformable fractional boundary value problem (3.1). We shall use the Banach fixed point theorem (Theorem 2.9) over the fractional integral equation (3.2), In fact we will show that the operator which defined based on the tranfared integral equation is a contraction mapping which is one of the main conditin of the Banach fixed point theorem. After this we can apply numerical methods to find the approximate solution. More details are in the following theorem.

Theorem 3.2. *Let*

(H1): $\hbar \in \mathcal{C}([0, \wp])$ *and* $\tau^{\upsilon-2}\hbar \in \mathcal{L}^1[0, \wp])$,

(H2): *For any ζ and ξ in $\mathcal{C}([0,\wp])$ there exists a constant Λ such that for each $\tau \in [0,\wp]$ we have $|\hbar(\tau,\zeta(\tau)) - \hbar(\tau,\xi(\tau))| \le \Lambda|\zeta(\tau)-\xi(\tau)|$*

If $\Lambda \le \frac{\upsilon(\upsilon-1)}{2\wp^{\upsilon}}$ then conformable fractional boundary value problem (3.1) has a unique solution $\zeta \in \mathcal{C}([0,\wp])$.

Proof. To apply the Banach fixed point theorem over the fractional integral equation (3.2), we define the operator $\Xi : \mathcal{C}([0,\wp]) \to \mathcal{C}([0,\wp])$ by

$$\Xi(\zeta)(\tau) = \int_0^{\tau} (\tau-\varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma,\zeta(\varsigma))d\varsigma - \frac{1}{\wp}\int_0^{\wp} \tau(\wp-\varsigma)\varsigma^{\upsilon-2}\hbar(\varsigma,\zeta(\varsigma))d\varsigma + \frac{(\beta-\alpha)}{\wp}\tau + \alpha. \tag{3.7}$$

Using Assuption (H1) we get

$$\begin{aligned}|\Xi(\zeta)(\tau) - \Xi(\xi)(\tau)| &\le \left|\int_0^{\tau} (\tau-\varsigma)\varsigma^{\upsilon-2}[\hbar(\varsigma,\zeta(\varsigma)) - \hbar(\varsigma,\xi(\varsigma))]d\varsigma - \frac{1}{\wp}\int_0^{\wp} \tau(\wp-\varsigma)\varsigma^{\upsilon-2}[\hbar(\varsigma,\zeta(\varsigma)) - \hbar(\varsigma,\xi(\varsigma))]d\varsigma\right| \\ &\le \Lambda\int_0^{\tau} (\tau-\varsigma)\varsigma^{\upsilon-2}|\zeta(\varsigma)-\xi(\varsigma)|d\varsigma + \frac{\Lambda}{\wp}\int_0^{\wp} \tau(\wp-\varsigma)\varsigma^{\upsilon-2}|\zeta(\varsigma)-\xi(\varsigma)|d\varsigma \\ &\le \Lambda\frac{2\wp^{\upsilon}}{\upsilon(\upsilon-1)}\|\zeta-\xi\|_{\infty}.\end{aligned}$$

Hence

$$\|\Xi(\zeta)(\tau) - \Xi(\xi)(\tau)\| \le \Lambda\frac{2\wp^{\upsilon}}{\upsilon(\upsilon-1)}\|\zeta-\xi\|_{\infty} \le \|\zeta-\xi\|_{\infty}.$$

Consequently, by the assumption of the theorem, the operator is a contraction mapping , and using the Banach fixed point theorem, the operator has a unique fixed point which is the unique solution of the conformable fractional boundary value problem (3.1). ☐

Corollary 3.3. *Under conditions*

$(H'1)$: *$\theta \in \mathcal{C}([0,\wp])$ and $\tau^{\upsilon-2}\theta \in \mathcal{L}^1[0,\wp])$,*

$(H'2)$: *$\|\theta\| \le \frac{\upsilon(\upsilon-1)}{2\wp^{\upsilon}}$,*

Conformable fractional boundary value problem (1.1)-(1.2) has a unique solution $\zeta \in \mathcal{C}([0,\wp])$.

Proof. If we set $\hbar(\tau,\zeta(\tau)) = \rho(\tau) - \theta(\tau)\,\zeta(\tau)$ we get the linear conformable fractional boundary value problem. It is enough to show that all conditions of Theorem 3.2 are holding. Obviously condition $(H'1)$ is direct result of condition $(H1)$. On the other hand for all $\tau \in [0,\wp]$ and $\zeta,\xi \in \mathcal{C}([0,\wp])$ we have

$$\hbar(\tau,\zeta(\tau)) - \hbar(\tau,\xi(\tau))| \le |\theta||\zeta(\tau)-\xi(\tau)|, \tag{3.8}$$

hence if $\|\theta\| \le \frac{\upsilon(\upsilon-1)}{2\wp^{\upsilon}}$, then all conditions of Theorem 3.2 are holding and Conformable fractional boundary value problem (1.1)-(1.2) will have a unique solution $\zeta \in \mathcal{C}([0,\wp])$. ☐

Now considering the above corollary, we can have the desired integral equation for the conformable fractional boundary value problem as below.

Corollary 3.4. *Let $\theta \in \mathcal{C}([0,\wp])$ and $1 < \upsilon < 2$, then the conformable fractional boundary value problem (1.1)-(1.2) is equivalent to the integral equation*

$$\zeta(\tau) + \int_0^{\tau} \Gamma(\tau,\varsigma)\zeta(\varsigma)d\varsigma + \int_0^{\wp} \Pi(\tau,\varsigma)\zeta(\varsigma)d\varsigma = \psi(\tau), \tag{3.9}$$

where

$$\psi(\tau) = \alpha + \frac{(\beta-\alpha)}{\wp}\tau + \int_0^{\tau} (\tau-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma - \frac{1}{\wp}\int_0^{\wp} \tau(\wp-\varsigma)\varsigma^{\upsilon-2}\varrho(\varsigma)d\varsigma, \tag{3.10}$$

$$\Gamma(\tau,\varsigma) = (\tau-\varsigma)\varsigma^{\upsilon-2}\theta(\varsigma), \quad \Pi(\tau,\varsigma) = -\frac{\tau}{\wp}(\wp-\varsigma)\varsigma^{\upsilon-2}\theta(\varsigma). \tag{3.11}$$

Proof. The proof is straightforward; it is enough to replace relations (3.10) and (3.11) into relation (3.9). ☐

4. Numerical Study

In this section, we'll discuss a method for finding numerical solutions to linear Volterra-Fredholm integral equations of the second kind. This method is based on the trapezoidal rule.

We'll divide the interval $[0, \wp]$ into K equal subintervals. The width of each subinterval, h, is calculated as $(b-a)/K$. The endpoints of these subintervals are denoted as τ_j, where $\tau_j = a + j \cdot h$.

We'll use the trapezoidal rule to approximate definite integrals. This rule estimates the integral of a function $\varrho(\tau)$ over the interval $[a, b]$ by approximating the area under the curve of the function with trapezoids

$$\int_a^b \varrho(\tau) d\tau \approx \frac{h}{2}\left[\varrho(a) + 2\sum_{i=1}^{K} \varrho(\tau_i) + \varrho(b)\right]. \tag{4.1}$$

Applying this rule to Equation (3.9) gives us the following equation for $\psi(\tau_j)$:

$$\begin{aligned}\psi(\tau_j) = \zeta(\tau_j) &+ \frac{h}{2}\left[\Gamma(\tau_j,\tau_0)\zeta(\tau_0) + 2\sum_{i=1}^{j-1}\Gamma(\tau_j,\tau_i)\zeta(\tau_i) + \Gamma(\tau_j,\tau_i)\zeta(\tau_i)\right] \\ &+ \frac{h}{2}\left[\Pi(\tau_j,\tau_0)\zeta(\tau_0) + 2\sum_{i=1}^{K-1}\Pi(\tau_j,\tau_i)\zeta(\tau_i) + \Pi(\tau_j,\tau_K)\zeta(\tau_K)\right],\end{aligned} \tag{4.2}$$

$$\begin{aligned}\Rightarrow \forall j = 0,\dots,K, \quad \psi_j = \zeta_j &+ \frac{h}{2}\left[\Gamma_{j0}\zeta_0 + 2\sum_{i=1}^{j-1}\Gamma_{ji}\zeta_i + \Gamma_{jj}\zeta_j\right] \\ &+ \frac{h}{2}\left[\Pi_{j0}\zeta_0 + 2\sum_{i=1}^{K-1}\Pi_{ji}\zeta_i + \Pi_{jK}\zeta_K\right].\end{aligned} \tag{4.3}$$

This equation can be rewritten in a more concise form:

$$\frac{h}{2}(\Gamma_{j0} + \Pi_{j0})\zeta_0 + h\sum_{i=1}^{j-1}(\Gamma_{ji} + \Pi_{ji})\zeta_i + \frac{h}{2}(\frac{2}{h} + \Gamma_{jj} + 2\Pi_{jj})\zeta_j + h\sum_{i=j+1}^{K-1}\Pi_{ji}\zeta_i + \frac{h}{2}\Pi_{jK}\zeta_K = \psi_j. \tag{4.4}$$

So

$$h\sum_{i=1}^{j-1}(\Gamma_{ji} + \Pi_{ji})\zeta_i + \frac{h}{2}(\frac{2}{h} + \Gamma_{jj} + 2\Pi_{jj})\zeta_j + h\sum_{i=j+1}^{K-1}\Pi_{ji}\zeta_i = \psi_j - \frac{h}{2}(\Gamma_{j0} + \Pi_{j0})\zeta_0 - \frac{h}{2}\Pi_{jK}\zeta_K. \tag{4.5}$$

Finally, we get a system of $K-1$ equations, which is:

$$\mathbf{BZ} = \mathbf{C},$$

where $\mathbf{Z} = (\zeta_1, \zeta_2, \dots, \zeta_{K-1})$, $\mathbf{B} = (b_{ji}), j, i = 1, 2, \dots, K-1$ and $\mathbf{C} = (c_j), j = 1, 2, \dots, K-1$

$$b_{ji} = \begin{cases} \frac{h}{2}(\Gamma_{jj} + 2\Pi_{jj})/2 + 1, & \text{if } i = j, \\ h(\Gamma_{ji} + \Pi_{ji}), & \text{if } i = 1 : j-1, \\ h\Pi_{ji}, & \text{if } i = j+1 : K-1, \end{cases} \tag{4.6}$$

and

$$c_j = \psi_j - \frac{h}{2}(\Gamma_{j0} + \Pi_{j0})\alpha - \frac{h}{2}\Pi_{jK}\beta \quad \text{if } j = 1, ..., K-1. \tag{4.7}$$

5. Algorithm of the method

The numerical solution procedure is summarized in the form of pseudocode and presented in the following algorithm:

Input:

- Number of subintervals K,
- Kernel functions $k_1(\tau, \varsigma)$, $k_2(\tau, \varsigma)$,

- Source function $f(\tau)$,
- Step size $h = 1/K$.

Output:

- Approximate solution values $\zeta_i \approx \zeta(\tau_i), \quad i = 0, 1, \ldots, K$.

Step 1: Grid generation

- Set $h \leftarrow 1/K$.
- For $i = 0, 1, \ldots, K$, define grid points $\tau_i \leftarrow ih$.

Step 2: Initialization

(3) Prescribe the initial value ζ_0.

Step 3: Assembly of the algebraic system

(4) For $i = 1$ to M do

(5) Approximate the Volterra integral using the composite trapezoidal rule:

$$V_i \leftarrow \frac{h}{2}\left[k_1(\tau_i, 0)\zeta_0 + 2\sum_{j=1}^{i-1} k_1(\tau_i, \tau_j)\zeta_j + k_1(\tau_i, \tau_i)\zeta_i\right].$$

(6) Approximate the Fredholm integral using the composite trapezoidal rule:

$$F_i \leftarrow \frac{h}{2}\left[k_2(\tau_i, 0)\zeta_0 + 2\sum_{j=1}^{K-1} k_2(\tau_i, \tau_j)\zeta_j + k_2(\tau_i, \tau_M)\zeta_M\right].$$

(7) Form the discrete equation:

$$\zeta_i = f(\tau_i) - V_i + F_i.$$

Step 4: Solution of the linear system

(8) Collect all equations for $i = 1, 2, \ldots, K$.

(9) Solve the resulting linear system for $\{\zeta_1, u_2, \ldots, \zeta_K\}$.

Step 5: Error evaluation (optional)

(10) For $i = 0, 1, \ldots, K$, compute the absolute error

$$E_i = \left|\zeta_{\text{exact}}(\tau_i) - \zeta_i\right|.$$

End Algorithm

5.1. **Absolute Error.** The accuracy of the proposed numerical method is assessed by computing the absolute error between the analytical solution $\zeta(\tau_i)$ and the numerical approximation $\zeta_h(\tau_i)$. The point-wise absolute error is defined as

$$E_{abs} = |\zeta(\tau_i) - \zeta_h(\tau_i)|,$$

where τ_i denote the computational grid points with step size $h = \frac{1}{K}$. This definition is used to evaluate the numerical error at each node of the discretized domain.

6. Graphical Simulations

In this part, we give three examples to illustrate the method given in section 4 to solve Eqs. (1.1)-(1.2). The exact solutions of all three examples are known and used to show that the numerical solution obtained with the proposed method is correct. The proposed numerical solution is coded in Mathematica 13.2 to simulate the numerical results.

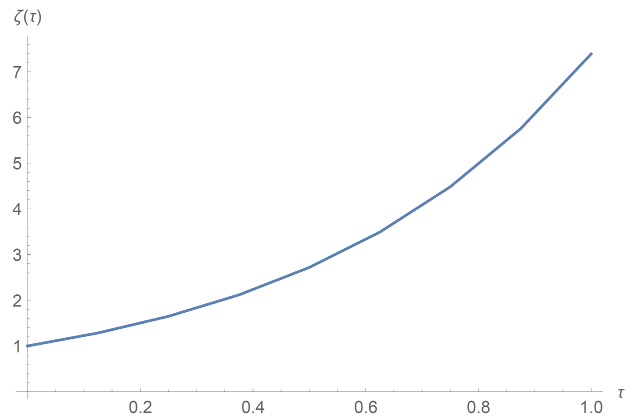

FIGURE 1. Numerical plots of $\zeta(\tau)$ at $\upsilon = \frac{3}{2}$ for $K = 8$ by using the proposed method.

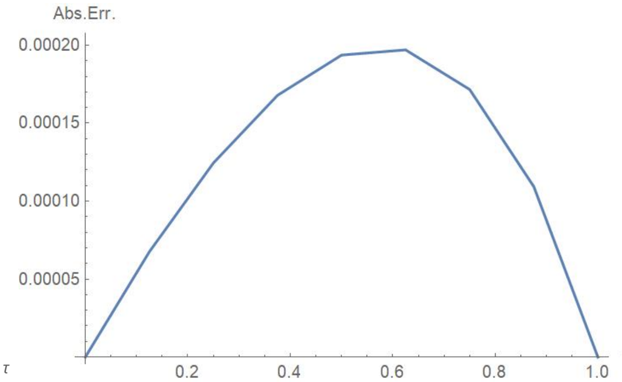

FIGURE 2. Plot of the absolute error for Example 6.1.

Example 6.1. As a first example, consider the following fractional boundary value problem:

$$\begin{aligned} &\mathcal{T}_{\frac{3}{2}}\zeta(\tau) + \frac{1}{10}\tau^{\frac{1}{2}}\zeta(\tau) = \frac{41}{10}\tau^{\frac{1}{2}}e^{2\tau}, \qquad 0 < \tau < 1, \\ &\zeta(0) = 1, \quad \zeta(1) = e^2. \end{aligned} \tag{6.1}$$

Here $\upsilon = \frac{3}{2}$, $\theta(\tau) = \frac{1}{10}\tau^{\frac{1}{2}}, \varrho(\tau) = \frac{41}{10}\tau^{\frac{1}{2}}e^{2\tau}, \wp = 1, \alpha = 1$ and $\beta = e^2$. The exact solution is $\zeta(\tau) = e^{2\tau}$.

The approximate solution and absolute error are given in Figures 1 and 2 for $K = 8$, respectively. Also the CPU time for computations of Figures 1 and 2 are respectively 0.140625 and 0.293576 in seconds.

Example 6.2. As a second example, consider the following fractional boundary value problem:

$$\begin{aligned} &\mathcal{T}_{\frac{4}{3}}\zeta(\tau) + \tau^{\frac{2}{3}}\zeta(\tau) = \tau^{\frac{2}{3}}(1-\pi^2)\cos\pi\tau, \quad 0 < \tau < 1, \\ &\zeta(0) = 1, \quad \zeta(\wp) = -1 \end{aligned}$$

Here $\upsilon = \frac{4}{3}$, $\theta(\tau) = \tau^{\frac{2}{3}}, \varrho(\tau) = \tau^{\frac{2}{3}}(1-\pi^2)\cos\pi\tau$, $\wp = 1, \alpha = 1$ and $\beta = -1$. The exact solution is $\zeta(\tau) = \cos\pi\tau$. The approximate solution and absolute error are given in Figures 3 and 4 for $K = 16$, respectively. The CPU times for computations of Figures 3 and 4 are 0.344057 and 0.163978.

Example 6.3. Finaly, consider the following fractional boundary value problem:

$$\begin{aligned} &\mathcal{T}_{\frac{5}{3}}\zeta(\tau) - 6\tau^{\frac{4}{3}}\zeta(\tau) = 9\tau^{\frac{13}{3}}e^{t^3}, \qquad 0 < \tau < \wp, \\ &\zeta(0) = 1, \quad \zeta(\wp) = e, \end{aligned}$$

Here $\upsilon = \frac{5}{3}$, $\theta(\tau) = -6\tau^{\frac{4}{3}}, \varrho(\tau) = 9\tau^{\frac{13}{3}}e^{t^3}, \wp = 1, \alpha = 1$ and $\beta = e$. The exact solution is $\zeta(\tau) = e^{t^3}$.

The approximate solution and absolute error are given in Figures 5 and 6 for $K = 32$, respectively. The CPU times for computations of Figures 5 and 6 are 0.0268252 and 0.101379.

7. Conclusion

In this paper, we investigated a class of linear conformable fractional differential equations subject to Dirichlet boundary conditions, establishing a rigorous theoretical and computational framework for their analysis. By reformulating the boundary value problem as an equivalent Volterra-type integral equation, we used the Banach contraction

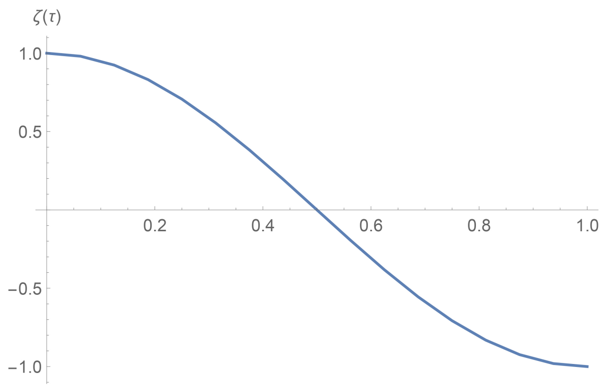

FIGURE 3. Numerical plots of $\zeta(\tau)$ at $\upsilon = \frac{4}{3}$ for $K = 16$ by using the proposed method.

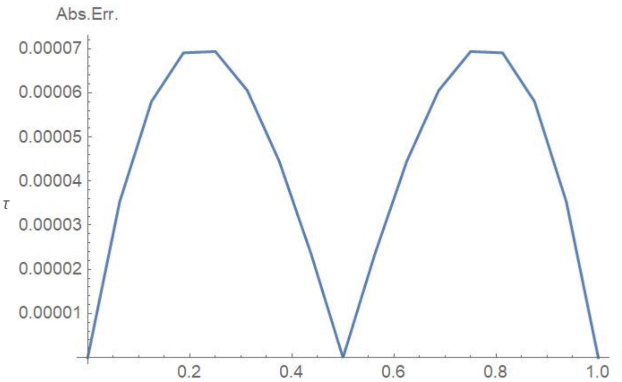

FIGURE 4. Plot of the absolute error for Example 6.2.

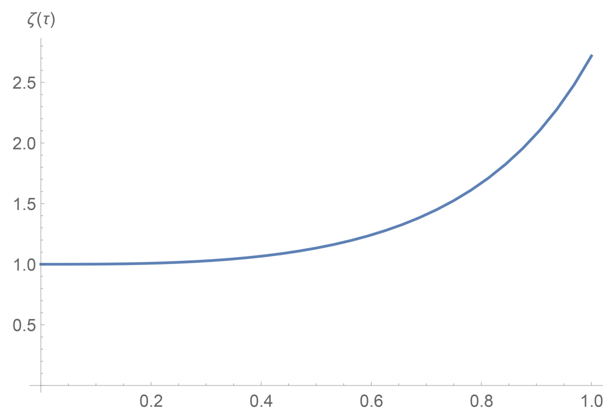

FIGURE 5. Numerical plots of $\zeta(\tau)$ at $\upsilon = \frac{5}{3}$ for $K = 32$ by using the proposed method.

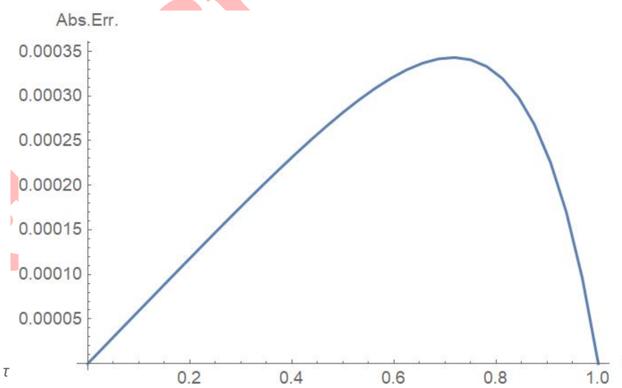

FIGURE 6. Plot of the absolute error for Example 6.3.

principle to prove the existence and uniqueness of solutions for the general nonlinear case under appropriate Lipschitz-type conditions on the nonlinear term. This theoretical foundation naturally encompassed the linear case as a direct corollary, thereby unifying the treatment of both linear and nonlinear problems within a single analytical framework.

From a computational perspective, we developed a numerical scheme based on discretizing the equivalent integral formulation rather than directly approximating the differential operator—a strategy that circumvents certain stability issues associated with fractional derivative approximations near boundary points. The proposed method demonstrated robust convergence behavior, as evidenced by three carefully selected numerical examples spanning homogeneous and nonhomogeneous problems with varying fractional orders. The computed solutions exhibited acceptable absolute errors across different mesh refinements, confirming both the accuracy and practical applicability of our approach. This work contributes to the emerging literature on conformable fractional calculus in several meaningful ways. First, while analytical studies of conformable boundary value problems have proliferated in recent years, numerical treatments—particularly those grounded in integral reformulations—remain relatively scarce. Our methodology bridges this gap by providing a stable, implementable algorithm with provable convergence properties. Second, the integral-equation approach offers inherent advantages in handling boundary conditions, as they are naturally incorporated into the kernel structure rather than imposed as external constraints. Third, the framework is readily extensible to other boundary conditions (e.g., Neumann or Robin types) and higher-order conformable equations. Nevertheless, certain limitations warrant acknowledgment. The current analysis assumes sufficient smoothness of the forcing term and relies on the contraction property, which may restrict applicability to problems with strong nonlinearities. Additionally, the conformable derivative itself—while computationally convenient—exhibits limitations in modeling certain memory-dependent phenomena compared to classical Caputo or Riemann–Liouville derivatives. Future research directions include: (i) extending the methodology to systems of conformable fractional equations; (ii) developing adaptive step-size strategies to improve computational efficiency; (iii) analyzing problems with nonsmooth data or singular kernels; and (iv) comparing the conformable framework against other fractional derivatives in boundary value contexts to better delineate its domain of applicability.

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