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Analyzing approximate solutions of the Cauchy problem for Helmholtz equation

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Abstract

In this article, we address the challenge of recovering solutions to the Helmholtz equation within a confined three-dimensional space, utilizing information gathered from a section of the boundary. This scenario pertains to the Cauchy problem. By employing the Carleman function, we derive an explicit approximate solution. To tackle this problem, we leverage the Carleman function, a powerful tool in the study of partial differential equations. The Carleman estimate allows us to transition from boundary data to an approximate solution in the interior of the domain.

Keywords. Cauchy problem, Carleman function, Regularized solution, Ill-posed problems, Conditionally posed problems.

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1. Introduction

GThe study of ill-posed problems occupies a central position in modern applied mathematics, particularly due to their prevalence in physics, engineering, geophysics, medical imaging, and a wide range of inverse modeling tasks. Among these problems, the Cauchy problem for the Helmholtz equation represents one of the most challenging and widely investigated cases. This problem arises naturally in models describing wave propagation, acoustics, electromagnetism, elasticity, and quantum mechanics, where it is often necessary to reconstruct information inside a domain using only partial measurements from its boundary. The difficulty stems from the fact that the Cauchy problem does not satisfy stability with respect to the given data; even small perturbations in boundary information may lead to disproportionately large deviations in the reconstructed solution. This sensitivity necessitates the development of robust regularization techniques capable of transforming the originally unstable problem into one that admits meaningful approximate solutions. A powerful and widely used approach to address such instability is based on Carleman-type methods. Carleman functions, initially introduced in the context of analytic continuation and complex analysis, have become essential tools in the analysis of partial differential equations, particularly elliptic equations such as the

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Helmholtz equation. Their ability to impose exponential weighting makes them suitable for suppressing the amplification of errors, thereby enabling stable reconstruction of interior solutions from incomplete or imprecise boundary data. Over several decades, significant progress has been made in extending Carleman-based methods to multidimensional domains, various boundary configurations, and systems of equations. This progress has deepened our understanding of unique continuation, stability estimates, and the construction of regularized approximation formulas. In this article, we analyze approximate solutions of the Cauchy problem for the Helmholtz equation using an explicit Carleman function–based approach. Our goal is to investigate the behavior of these approximations, evaluate their stability, and establish quantitative estimates that demonstrate how the proposed method compensates for the intrinsic ill-posedness of the problem. The results presented here contribute to the broader theory of inverse and ill-posed problems and offer practical insights for computational applications in scientific and engineering disciplines where the Helmholtz equation plays a fundamental role.

The analysis of approximate solutions to the Cauchy problem for the Helmholtz equation is deeply connected to the classical theory of ill-posed problems, whose foundations were established by the pioneering works of Tikhonov and his collaborators [53, 54]. Their regularization philosophy influenced subsequent developments in elliptic and inverse problems, including the study of Cauchy-type formulations for the Laplace and Helmholtz equations. Early contributions by Lavrent'ev [42, 43] and Ivanov [21] highlighted the characteristic instability inherent to such problems, motivating the formulation of continuation methods and stability estimates. The Carleman function approach, introduced in the context of quasi-analytic classes by Carleman [13], and later adapted for elliptic equations by Goluzin and Krylov [20], has since become a cornerstone technique. Substantial progress was made through the works of Yarmukhamedov [56, 57], who derived explicit Carleman-type formulas for the Laplace equation and established their applicability to the Cauchy problem. These developments laid the groundwork for contemporary extensions to the Helmholtz operator.

In recent years, numerous studies have advanced the Carleman-based regularization of the Helmholtz equation, especially through matrix factorization techniques and multidimensional generalizations. Notable contributions include the works of Juraev and coauthors [22–41], who formulated stable continuation algorithms, derived integral representations, and explored the behavior of approximate solutions in bounded and unbounded domains. Their results demonstrate how exponential weights can suppress high-frequency instabilities and yield convergence under minimal smoothness assumptions. Parallel investigations, such as those by Efendiev and colleagues [6, 7, 14–16], have contributed to the spectral analysis of related operator families, enriching the theoretical framework surrounding wave propagation, almost periodic structures, and differential operator pencils.

Beyond elliptic systems, research into elasticity [45–47], biharmonic-type equations [48], and other inverse or nonlocal problems [1, 3, 4, 18, 19] shows that similar analytical challenges and stabilization mechanisms arise across many branches of mathematical physics. Methods inspired by Carleman estimates have also found applications in numerical modeling, fractional kinetic equations [55], and even imaging sciences [44]. Collectively, these works underscore the broad applicability and ongoing relevance of Carleman-type techniques in addressing the instability of Cauchy problems. The present study builds directly on this rich body of literature, integrating classical regularization theory with modern analytical tools to construct, justify, and evaluate approximate solutions for the Helmholtz equation within three-dimensional bounded domains.

The present study builds upon a broad spectrum of mathematical and physical research related to analytic continuation, inverse problems, and the structural properties of differential equations. Foundational interpolation and approximation techniques, such as those proposed in [50], provide essential tools for constructing stable numerical schemes used in solving ill-posed boundary value problems. The classical Carleman formula for the Helmholtz equation [8] and the generalized continuation methods pioneered by Goluzin and Krylov [20] offer a theoretical backbone for analyzing instability phenomena arising in Cauchy-type problems. These works are complemented by the comprehensive treatment of Carleman formulas in complex analysis found in [5], which deepens the functional-analytic framework underpinning modern regularization approaches. Research in related areas of quantum theory and heat propagation further enriches the mathematical landscape. For instance, the investigation of unusual quantum entanglement within Schrödinger-type systems [9] and the integro-differential formulation of the heat equation [10] highlight the pervasive role of continuation and factorization techniques across physical models. Additional insights are provided by studies on stationary Schrödinger equations [11] and the behavior of magnon dynamics governed by the Klein–Gordon equation [12], both of which demonstrate the utility of decoupling and transformation methods analogous to those used for the Helmholtz equation. In the realm of applied operator theory, the approximate factorization of matrix polynomials [2] and spectral analyses of PT-symmetric Schrödinger operators on graphs [17] provide further evidence of the interconnectedness between spectral stability, analytic continuation, and inverse boundary problems. Moreover, numerical approaches developed for solving Volterra-type integral equations [49] contribute valuable techniques for addressing the computational challenges inherent in ill-posed problems. Collectively, these studies form a coherent theoretical and methodological foundation upon which the current analysis of approximate solutions to the Cauchy problem for the Helmholtz equation is constructed.

The scientific novelty of this study lies in the development and rigorous justification of a new analytical framework for constructing approximate solutions to the ill-posed Cauchy problem for the Helmholtz equation within a three-dimensional bounded domain. While Carleman-type methods have long been recognized as powerful tools for stabilizing inverse and continuation problems, the present work extends their applicability by introducing a refined Carleman function specifically adapted to the geometry of a composite boundary consisting of both flat and curved surfaces. This tailored construction enables the derivation of explicit integral representations that are not only theoretically grounded but also computationally viable for practical implementations involving incomplete or noisy boundary data. A key element of novelty is the establishment of precise quantitative estimates that describe the convergence and stability behavior of the proposed approximate solutions. These estimates demonstrate how the newly constructed Carleman function effectively suppresses the instability typically inherent to Cauchy problems, providing a sharper and more robust stability mechanism compared to classical formulations used for the Helmholtz operator. Furthermore, the paper introduces an improved regularization strategy that ensures uniform convergence away from boundary layers, allowing for a more accurate reconstruction of interior solutions even in the presence of small perturbations to the boundary data. Another original contribution is the detailed analysis of the sensitivity of the approximate solution to variations in the regularization parameter. By deriving explicit asymptotic relationships, the study clarifies the optimal selection of this parameter, thereby enhancing the practical relevance of the method. Collectively, these results advance the theoretical understanding of Carleman-based regularization techniques and broaden their applicability to more complex geometric configurations and multidimensional settings, marking a significant step forward in the field of ill-posed problems and inverse boundary value analysis.

2. Fundamental aspects and the formulation of the Cauchy problem

This part focuses on developing a set of essential solutions for the Helmholtz equation, which is defined by an entire function possessing specific characteristics.

Let $\mathbb{R}^3$ represent a three-dimensional real Euclidean space,

$$\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \quad \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3,$$
$$\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2, \quad \eta' = (\eta_1, \eta_2) \in \mathbb{R}^2,$$
$$r = |y - x|, \quad \alpha = |y' - x'|, \quad w = i\sqrt{u^2 + \alpha^2} + \eta_3, \quad u \geq 0.$$

Let G be a bounded, simply-connected region in $\mathbb{R}^3$, characterized by a boundary that is piecewise smooth. This boundary includes the plane T defined by $\eta_3 = 0$, along with a smooth surface S located in the half-space where $\eta_3 > 0$. Therefore, we express the boundary of G as $\partial G{=}S\bigcup T$.

In this context, we investigate the Helmholtz equation within the specified domain G.

$$(\triangle + \lambda^2)W(\eta) = 0, \tag{2.1}$$

where $\lambda > 0$, $\triangle - N$ represents the Laplace operator.

We define $\mathrm{N}(w)$ as a entire function that yields real values when w is real (where $w{=}u + iv$; with u and v being real numbers) and adheres to the subsequent criteria:

$$\mathrm{N}(u) \neq 0, \ \sup_{v\geq 1}\left|v^p \mathrm{N}^{(p)}(w)\right| = \mathrm{M}(u,p) < \infty, \quad -\infty < u < \infty, \ \ p = 0,1,2,3. \tag{2.2}$$

We establish a function $\Psi(\eta,\lambda;\xi)$ under the condition that $\eta \neq \xi$ through the subsequent equation:

$$\Psi(\eta,\lambda;\xi) = -\frac{1}{2\pi^2\ K(\xi_3)} \int_0^{\infty} \mathrm{Im}\frac{\mathrm{N}(w)}{w-\xi_3}\frac{\cos(\lambda u)}{\sqrt{u^2+\alpha^2}}du, \quad w{=}i\sqrt{u^2+\alpha^2}+\ \eta_3, \tag{2.3}$$

The function $\Psi(\eta,\lambda;\xi)$ can be represented as

$$\Psi(\eta,\lambda;\xi) = -\frac{e^{i\lambda r}}{4\pi r} + \varphi(\eta,\lambda;\xi). \tag{2.4}$$

The expression $-\frac{e^{i\lambda r}}{4\pi r}$ serves as the primary solution to the Helmholtz equation in three-dimensional space $\mathbb{R}^3$. This solution is characterized by the Hankel function of the first kind. Additionally, $\varphi(\eta,\lambda;\xi$ represents the regular solution of the Helmholtz equation concerning the variable ?, with the specific inclusion of the point $\eta = \xi$.

Cauchy problem 1. Consider that $W(\eta) \in C^2(G) \bigcap C^1(G)$. We have:

$$W(\eta)|_S = f(\eta), \quad \partial_n W(\eta)|_S = g(\eta), \quad \eta \in S. \tag{2.5}$$

In this context, $f(\eta)$ and $g(\eta)$ represent specified continuous vector functions defined on S. The operator ∂_n denotes differentiation with respect to the outward normal direction at the boundary ∂G.

It is necessary to reconstruct the vector function $W(\eta)$ within the area G, utilizing its values $f(\eta)$ and $g(\eta)$ on the boundary S.

In Equation (2.3), selecting

$$\mathrm{N}(w) = \exp(\sigma w), \quad \mathrm{N}(\xi_3) = \exp(\sigma x_3), \quad \sigma > 0, \tag{2.6}$$

we get

$$\Psi_\sigma(\eta,\lambda;\xi) = -\frac{e^{-\sigma\xi_3}}{2\pi^2} \int_0^{\infty} \mathrm{Im}\frac{\exp(\sigma \mathrm{w})}{w-\xi_3}\frac{\cos(\lambda \mathrm{u})}{\sqrt{u^2+\alpha^2}}du, \tag{2.7}$$

For a function $W(\eta) \in C^2(G) \bigcap C^1(G)$, and for any $\xi \in G$, the Green's integral formula is applicable:

$$W(\xi) = \int_{\partial G} \Big[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta, \qquad \xi \in G, \tag{2.8}$$

In this section and the following parts, we refer to functions that are confined to compact regions within the domain G, denoted as $C(\xi)$.

Theorem 2.1. *Let $W(\eta) \in C^2(G) \bigcap C^1(G)$ it satisfy the inequality*

$$|W(\eta)| + |\partial_n W(\eta)| \leq M, \quad \eta \in T. \tag{2.9}$$

If

$$W_\sigma(\xi) = \int_S \Big[g(\eta)\Psi_\sigma(\eta,\lambda;\xi) - f(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta, \qquad \xi \in G, \tag{2.10}$$

subsequently, the estimates presented are accurate

$$|W(\xi) - W_\sigma(\xi)| \leq C(\xi)\sigma M e^{-\sigma\xi_3}, \quad \sigma > 1, \quad \xi \in G, \tag{2.11}$$

$$\big|\partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_\sigma(\xi)\big| \leq C(\xi)\sigma M e^{-\sigma\xi_3}, \quad \sigma > 1, \quad \xi \in G, \quad j = \overline{1,3}. \tag{2.12}$$

Proof. To begin, we will assess the inequality presented in (2.11). By applying the integral expression from (2.8) in conjunction with the relation established in (2.10), we derive the following

$$\begin{aligned} W(\xi) &= \int_{\partial G} \Big[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta \\ &= \int_{S} \Big[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta \end{aligned}$$

$$+\int_T \Big[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta$$
$$= W_\sigma(\xi) + \int_T \Big[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta, \qquad \xi \in G. \tag{2.13}$$

Considering the inequality (2.9), we evaluate the subsequent

$$|W(\xi) - W_\sigma(\xi)| \le \left| \int_T \Big[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta \right|$$
$$\le M \int_T \Big[\Big|\Psi_\sigma(\eta,\lambda;\xi)| + |\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big|\Big] ds_\eta, \qquad \xi \in G. \tag{2.14}$$

We estimate the integrals $\int_T |\Psi_\sigma(\eta,\lambda;\xi)|ds_\eta$ and $\int_T |\partial_n\Psi_\sigma(\eta,\lambda;\xi)|ds_\eta, \quad j=\overline{1,3}$ on the part T of the plane $\eta_3 = 0$. By isolating the imaginary component of Equation (2.7), we derive:

$$\Psi_\sigma(\eta,\lambda;\xi) = \frac{e^{\sigma(\eta_3-\xi_3)}}{2\pi^2}\Bigg[-\int_0^\infty \frac{\cos\sigma\sqrt{u^2+\alpha^2}}{u^2+r^2}\cos(\lambda u)du$$
$$+\int_0^\infty \frac{(\eta_3-\xi_3)\sin\sigma\sqrt{u^2+\alpha^2}}{u^2+r^2}\frac{\cos(\lambda u)}{\sqrt{u^2+\alpha^2}}du\Bigg], \quad \eta \ne \xi,\ \xi_3 > 0. \tag{2.15}$$

Considering Equation (2.15), we obtain

$$\int_T \Big|\Psi_\sigma(\eta,\lambda;\xi)\Big| ds_y \le C(\xi)\sigma e^{-\sigma\xi_3}, \qquad \sigma > 1, \quad \xi \in G. \tag{2.16}$$

In order to evaluate the second integral, we apply the equality

$$\partial_n\Psi_\sigma(\eta,\lambda;\xi) = \partial_{\eta_1}\Psi_\sigma(\eta,\lambda;\xi)\cos\alpha_1 + \partial_{\eta_2}\Psi_\sigma(\eta,\lambda;\xi)\cos\beta_1 + \partial_{\eta_3}\Psi_\sigma(\eta,\lambda;\xi)\cos\gamma_1 \tag{2.17}$$

Here $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ are the coordinates of the unit external normal n at the point y of the boundary ∂G.
Starting from the equality in (2.15) and applying partial derivatives concerning $\eta_j, \quad j=\overline{1,3}$, we derive the following approximations:

$$\int_T \Big|\partial_{\eta_1}\Psi_\sigma(\eta,\lambda;\xi)\Big| ds_\eta \le C(\xi)\sigma e^{-\sigma\xi_3}, \qquad \sigma > 1, \quad \xi \in G, \tag{2.18}$$
$$\int_T \Big|\partial_{\eta_2}\Psi_\sigma(\eta,\lambda;\xi)\Big| ds_\eta \le C(\xi)\sigma e^{-\sigma\xi_3}, \qquad \sigma > 1, \quad \xi \in G, \tag{2.19}$$
$$\int_T \Big|\partial_{\eta_3}\Psi_\sigma(\eta,\lambda;\xi)\Big| ds_\eta \le C(\xi)\sigma e^{-\sigma\xi_3}, \qquad \sigma > 1, \quad \xi \in G. \tag{2.20}$$

By merging the evaluations from Equations (2.16) through (2.20) and incorporating (2.14), we derive an estimation represented by (2.11).
Next, we will demonstrate the validity of inequality (2.12). To accomplish this, we differentiate the Equations (2.8) and (2.10) concerning $\xi_j, \quad (j=\overline{1,3})$, leading us to the following results:

$$\partial_{\xi_j}W(\xi) = \int_{\partial G}\Big[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta$$
$$= \int_S \Big[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta$$
$$+ \int_T \Big[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta \tag{2.21}$$
$$\partial_{\xi_j}W_\sigma(\xi) = \int_S \Big[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi)\Big] ds_\eta, \qquad \xi \in G, \quad j=\overline{1,3}. \tag{2.22}$$

Considering (2.21) and inequality (2.9), we derive the subsequent estimation

$$\begin{aligned}\left|\partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_\xi(\xi)\right| &\le \left|\int_T \left[\partial_n W(\xi)\partial_{\xi_j}\Psi_\xi(\eta,\lambda;\xi) - W(\xi)\partial_{\xi_j}(\partial_n \Psi_\xi(\eta,\lambda;\xi))\right] ds_\eta\right| \\ &\le M\int_T \left[\left|\partial_{\xi_j}\Psi_\xi(\eta,\lambda;\xi)\right| + \left|\partial_{\xi_j}(\partial_n \Psi_\xi(\eta,\lambda;\xi))\right|\right] ds_\eta, \quad \xi\in G, \quad j{=}\overline{1,3}. \end{aligned} \tag{2.23}$$

In this process, we evaluate the integrals $\int_T \left|\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\right| ds_\eta$, $\int_T \left|\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right| ds_\eta$, $j{=}\overline{1,3}$ along the section T of the plane defined by $\eta_3{=}\ 0$.

To estimate the first integral, From equality (2.15), taking partial derivatives with respect to ξ_j, $\quad j{=}\overline{1,3}$, we obtain the following estimates:

$$\int_T \left|\partial_{\xi_1}\Psi_\sigma(\eta,\lambda;\xi)\right| ds_\eta \le C\left(\xi\right)\sigma e^{-\sigma\xi_3}, \quad \sigma{>}\ 1, \quad \xi\in G. \tag{2.24}$$

$$\int_T \left|\partial_{\xi_2}\Psi_\sigma(\eta,\lambda;\xi)\right| ds_\eta \le C\left(\xi\right)\sigma e^{-\sigma\xi_3}, \quad \sigma{>}\ 1, \quad \xi\in G. \tag{2.25}$$

$$\int_T \left|\partial_{\xi_3}\Psi_\sigma(\eta,\lambda;\xi)\right| ds_\eta \le C\left(\xi\right)\sigma e^{-\sigma\xi_3}, \quad \sigma{>}\ 1, \quad \xi\in G. \tag{2.26}$$

In order to evaluate the secondary integrals, we employ the equivalence

$$\begin{aligned}\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi)) &= \partial_{\xi_j}(\partial_{\eta_1}\Psi_\sigma\left(\eta,\lambda;\xi\right))\cos\alpha_1 + \partial_{\xi_j}(\partial_{\eta_2}\Psi_\sigma\left(\eta,\lambda;\xi\right))\cos\beta_1 \\ &\quad + \partial_{\xi_j}(\partial_{\eta_3}\Psi_\sigma(\eta,\lambda;\xi))\cos\gamma_1, \quad s{=}\alpha^2, \quad j{=}\overline{1,3}.\end{aligned} \tag{2.27}$$

Consequently, considering Equations (2.17) and (2.27), we derive the ensuing approximations:

$$\int_T \left|\partial_{\xi_1}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right| ds_\eta \le C\left(\xi\right)\sigma e^{-\sigma\xi_3}, \quad \sigma{>}\ 1, \quad \xi\in G. \tag{2.28}$$

$$\int_T \left|\partial_{\xi_2}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right| ds_\eta \le C\left(\xi\right)\sigma e^{-\sigma\xi_3}, \quad \sigma{>}\ 1, \quad \xi\in G. \tag{2.29}$$

$$\int_T \left|\partial_{\xi_3}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right| ds_\eta \le C\left(\xi\right)\sigma e^{-\sigma\xi_3}, \quad \sigma{>}\ 1, \quad \xi\in G. \tag{2.30}$$

At this point, we integrate the evaluations from (2.24)–(2.26), alongside those from (2.27)–(2.30), while considering (2.23), which leads us to derive the estimate (2.12). Thus, Theorem 2.1 is proved. □

Corollary 2.2. *For every $\xi\in G$ the following equalities hold:*

$$\lim_{\sigma\to\infty} W_\sigma(\xi) = W(\xi), \quad \lim_{\sigma\to\infty}\partial_{\xi_j} W_\sigma(\xi) = \partial_{\xi_j} W(\xi), \quad \xi\in G, \quad j{=}\overline{1,3}.$$

We define the set $\overline{G}_\varepsilon$ as follows:

$$\overline{G}_\varepsilon{=}\{(\xi_1,\xi_2,\xi_3)\in G, \quad a{>}\xi_3\ge\varepsilon, \quad a{=}\max_T \psi(\xi'), \quad 0<\varepsilon{<}a\}.$$

It is apparent that the set $\overline{G}_?$ is a compact subset of G.

Corollary 2.3. *When ξ is an element of $\xi\in\overline{G}_\varepsilon$, the sets of functions $\{W_\sigma(\xi)\}$ and $\{\partial_{\xi_j} W_\sigma(\xi)\}$ exhibit uniform convergence as $\sigma\to\infty$. This means that:*

$$W_\sigma(\xi) \Longrightarrow W(\xi), \quad \partial_{\xi_j} W_\sigma(\xi) \Longrightarrow \partial_{\xi_j} W(\xi), \quad j{=}\overline{1,3}.$$

It is important to recognize that the set $E_\varepsilon{=}G\backslash\overline{G}_\varepsilon$ acts as a boundary layer in this context, similar to the principles of singular perturbation theory, in which uniform convergence is absent.

3. Evaluation of the Resilience of the Solution to the Cauchy Problem

Let's consider the surface S defined by the equation

$$\eta_3 = \psi(\eta'), \quad \eta' \in \mathbb{R}^2,$$

where $\psi(\eta')$ represents a single-valued function that meets the criteria set by the Lyapunov conditions.

Let's introduce the following

$$a = \max_T \psi(\eta'), \quad b = \max_T \sqrt{1 + \psi'^2(\eta')}.$$

Theorem 3.1. *Let $W(\eta) \in C^2(G) \bigcap C^1(G)$ and adhere to condition (2.9), and also on a smooth surface S, the inequality holds true*

$$|W(\eta)| + |\partial_n W(\eta)| \le \delta, \quad \eta \in S. \tag{3.1}$$

Consequently, the subsequent evaluation holds true

$$|W(\xi)| \le C(\xi)\, \sigma M^{1-\frac{\xi_3}{a}} \delta^{\frac{\xi_3}{a}}, \quad \sigma > 1, \quad \xi \in G, \tag{3.2}$$

$$\left|\partial_{\xi_j} W(\xi)\right| \le C(\xi)\, \sigma M^{1-\frac{\xi_3}{a}} \delta^{\frac{\xi_3}{a}}, \quad \sigma > 1, \quad \xi \in G, \quad j = \overline{1,3}. \tag{3.3}$$

Proof. To begin, let's evaluate inequality (3.2). By applying the integral expression (2.8), we obtain

$$\begin{aligned} W(\eta) = &\int_S \left[\partial_n W(\eta) \Psi_\sigma(\eta, \lambda; \xi) - W(\eta) \partial_n \Psi_\sigma(\eta, \lambda; \xi)\right] ds_\eta \\ &+ \int_T \left[\partial_n W(\eta) \Psi_\sigma(\eta, \lambda; \xi) - W(\eta) \partial_n \Psi_\sigma(\eta, \lambda; \xi)\right] ds_\eta, \quad \xi \in G. \end{aligned} \tag{3.4}$$

We estimate the following

$$\begin{aligned} |W(\eta)| \le &\left|\int_S \left[\partial_n W(\eta) \Psi_\sigma(\eta, \lambda; \xi) - W(\eta) \partial_n \Psi_\sigma(\eta, \lambda; \xi)\right] ds_\eta\right| \\ &+ \left|\int_T \left[\partial_n W(\eta) \Psi_\sigma(\eta, \lambda; \xi) - W(\eta) \partial_n \Psi_\sigma(\eta, \lambda; \xi)\right] ds_\eta\right|, \quad \xi \in G. \end{aligned} \tag{3.5}$$

Taking into account inequality (3.1), we evaluate the initial integral of inequality (3.2).

$$\left|\int_S \left[\partial_n W(\eta) \Psi_\sigma(\eta, \lambda; \xi) - W(\eta) \partial_n \Psi_\sigma(\eta, \lambda; \xi)\right] ds_\eta\right| \le \delta \int_S \left[|\Psi_\sigma(\eta, \lambda; \xi)| + |\partial_n \Psi_\sigma(\eta, \lambda; \xi)|\right] ds_\eta, \quad \xi \in G. \tag{3.6}$$

To accomplish this, we evaluate the integrals $\int_S |\Psi_\sigma(\eta, \lambda; \xi)|\, ds_\eta$ and $\int_S |\partial_n \Psi_\sigma(\eta, \lambda; \xi)|\, ds_\eta$ on a differentiable surface S.

Given the equality (2.15), we have

$$\int_S |\Psi_\sigma(\eta, \lambda; \xi)|\, ds_y \le C(\xi) \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G. \tag{3.7}$$

In order to evaluate the second integral, as indicated by Equation (2.17), we derive

$$\int_S |\partial_{\eta_1} \Psi_\sigma(\eta, \lambda; \xi)|\, ds_\eta \le C(\xi)\, \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G, \tag{3.8}$$

$$\int_S |\partial_{\eta_2} \Psi_\sigma(\eta, \lambda; \xi)|\, ds_\eta \le C(\xi)\, \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G, \tag{3.9}$$

$$\int_S |\partial_{\eta_3} \Psi_\sigma(\eta, \lambda; \xi)|\, ds_\eta \le C(\xi)\, \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G, \tag{3.10}$$

From (3.7)–(3.10), bearing in mind (3.6), we obtain

$$\left|\int_S [\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)]\, ds_\eta\right| \le \delta\int_S [|\Psi_\sigma(\eta,\lambda;\xi)| + |\partial_n\Psi_\sigma(\eta,\lambda;\xi)|]\, ds_\eta$$
$$\le C(\xi)\sigma\delta e^{\sigma(a-\xi_3)}, \quad \sigma>1, \quad \xi\in G. \tag{3.11}$$

The following is known

$$|W(\xi) - W_\sigma(\xi)| \le \left|\int_T [\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n\Psi_\sigma(\eta,\lambda;\xi)]\, ds_\eta\right|$$
$$\le M\int_T [|\Psi_\sigma(\eta,\lambda;\xi)| + |\partial_n\Psi_\sigma(\eta,\lambda;\xi)|]\, ds_\eta$$
$$\le C(\xi)\,\sigma M e^{-\sigma\xi_3}, \quad \sigma>1, \quad \xi\in G. \tag{3.12}$$

Taking into account (3.11)–(3.12) and reflecting on (3.5), we find that

$$|W(\xi)| \le \frac{C(\xi)\,\sigma}{2}\left(\delta e^{\sigma a} + M\right)e^{-\sigma\xi_3}, \quad \sigma>1, \quad \xi\in G, \tag{3.13}$$

Choosing σ from the equality

$$\sigma = \frac{1}{a}\ln\frac{M}{\sigma}, \tag{3.14}$$

we derive an approximation (3.2).

Let us now demonstrate the validity of inequality (3.3). To achieve this, we will compute the partial derivative of the integral expression (2.8) concerning the variable ξ_j, $j=\overline{1,3}$:

$$\partial_{\xi_j}W(\xi) = \int_{\partial G}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta$$
$$= \int_S\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta$$
$$+ \int_T\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta$$
$$= \partial_{\xi_j}W_\sigma(\xi) + \int_T\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta, \quad \xi\in G, \quad j=\overline{1,3}. \tag{3.15}$$

Here

$$\partial_{\xi_j}W_\sigma(\xi) = \int_S\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta,\ \xi\in G, \quad j=\overline{1,3}. \tag{3.16}$$

Taking into account equation (3.16), we evaluate the subsequent analysis

$$\left|\partial_{\xi_j}W(\xi)\right| \le \left|\int_{\partial G}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta\right|$$
$$\le \left|\int_S\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta\right|$$
$$+ \left|\int_T\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta\right|$$
$$\le \left|\partial_{\xi_j}W_\sigma(\xi)\right| + \left|\int_T\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta\right|, \ \xi\in G, \quad j=\overline{1,3}. \tag{3.17}$$

Considering the inequality (3.1), we can evaluate the initial integral of inequality (3.17).

$$\left|\int_S\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\sigma(\eta,\lambda;\xi)\right)\right] ds_\eta\right|$$

$$\leq \delta \int_S \left[\left| \partial_{\xi_j} \Psi_\sigma(\eta, \lambda; \xi) \right| + \left| \partial_{\xi_j} \left(\partial_n \Psi_\sigma(\eta, \lambda; \xi) \right) \right| \right] ds_\eta, \quad \xi \in G, \quad j = \overline{1,3}. \tag{3.18}$$

To achieve this, we evaluate the integrals $\int_S \left| \partial_{\xi_j} \Psi_\sigma(\eta, \lambda; \xi) \right| ds_\eta$ and $\int_S \left| \partial_{\xi_j} \left(\partial_n \Psi_\sigma(\eta, \lambda; \xi) \right) \right| ds_\eta$, $j=\overline{1,3}$ on a smooth surface S.

Taking into account equation (2.15), we derive

$$\int_S \left| \partial_{\xi_1} \Psi_\sigma(\eta, \lambda; \xi) \right| ds_\eta \leq C\left(\xi\right) \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G. \tag{3.19}$$

$$\int_S \left| \partial_{\xi_2} \Psi_\sigma(\eta, \lambda; \xi) \right| ds_\eta \leq C\left(\xi\right) \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G. \tag{3.20}$$

$$\int_S \left| \partial_{\xi_3} \Psi_\sigma(\eta, \lambda; \xi) \right| ds_\eta \leq C\left(\xi\right) \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G. \tag{3.21}$$

To estimate the second integral, based on equality (2.27), we have

$$\int_S \left| \partial_{\xi_1} \left(\partial_n \Psi_\sigma(\eta, \lambda; \xi) \right) \right| ds_\eta \leq C\left(\xi\right) \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G. \tag{3.22}$$

$$\int_S \left| \partial_{\xi_2} \left(\partial_n \Psi_\sigma(\eta, \lambda; \xi) \right) \right| ds_\eta \leq C\left(\xi\right) \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G. \tag{3.23}$$

$$\int_S \left| \partial_{\xi_3} \left(\partial_n \Psi_\sigma(\eta, \lambda; \xi) \right) \right| ds_\eta \leq C\left(\xi\right) \sigma e^{\sigma(a-\xi_3)}, \quad \sigma > 1, \quad \xi \in G. \tag{3.24}$$

From (3.19)–(3.24), bearing in mind (3.18), we obtain

$$\left| \int_S \left[\partial_n W(\eta) \partial_{\xi_j} \Psi_\sigma(\eta, \lambda; \xi) - W(\eta) \partial_{\xi_j} (\partial_n \Psi_\sigma(\eta, \lambda; \xi)) \right] ds_\eta \right| \leq C(\xi) \sigma \delta e^{\sigma(a-\xi_3)}, \sigma > 1, \quad \xi \in G, \quad j = \overline{1,3}. \tag{3.25}$$

The information available is as follows:

$$\left| \int_T \left[\partial_n W(\eta) \partial_{\xi_j} \Psi_\sigma(\eta, \lambda; \xi) - W(\eta) \partial_{\xi_j} (\partial_n \Psi_\sigma(\eta, \lambda; \xi)) \right] ds_\eta \right| \leq C(\xi) \sigma M e^{-\sigma \xi_3}, \;\; \xi \in G. \tag{3.26}$$

Taking into account (3.25)–(3.26) and keeping (3.17) in mind, we arrive at the following conclusion

$$\left| \partial_{\xi_j} W(\xi) \right| \leq \frac{C\left(\xi\right) \sigma}{2} \left(\sigma e^{\sigma a} + M \right) e^{-\sigma \xi_3}, \quad \sigma > 1, \quad \xi \in G, \quad j = \overline{1,3}. \tag{3.27}$$

By selecting σ based on the Equation (3.14), we derive an estimate given by (3.3). Therefore, Theorem 3.1 is proved. □

Let $W(\eta) \in C^2(G) \bigcap C^1(G)$. Instead of using the functions $f\left(\eta\right)$ and $g(\eta)$ defined on S, let us consider their respective approximations, denoted as $f_\delta(\eta)$ and $g_\delta(\eta)$, which incur a small error represented $0 < \delta < 1$,

$$\max_S \left| f(\eta) - f_\delta(\eta) \right| \leq \delta, \quad \max_S \left| g(\eta) - g_\delta(\eta) \right| \leq \delta \tag{3.28}$$

We put

$$W_{\sigma(\delta)}(\xi) = \int_S \left[g_\delta(\eta) \Psi_\sigma(\eta, \lambda; \xi) - f_\delta(\eta) \partial_n \Psi_\sigma(\eta, \lambda; \xi) \right] ds_\eta, \;\; \xi \in G. \tag{3.29}$$

Theorem 3.2. *Consider the $W(\eta) \in C^2(G) \cap C^1(G)$ situated on the plane defined by ?$_3$= 0, which meets the criteria outlined in condition (2.9). The subsequent evaluations hold true*

$$\left|W(\xi) - W_{\sigma(\delta)}(\xi)\right| \le C(\xi)\sigma M^{1-\frac{\xi_3}{a}}\delta^{\frac{\xi_3}{a}}, \ \sigma > 1, \ \xi \in G, \tag{3.30}$$

$$\left|\partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_{\sigma(\delta)}(\xi)\right| \le C(\xi)\sigma M^{1-\frac{\xi_3}{a}}\delta^{\frac{\xi_3}{a}}, \ \sigma > 1, \ \xi \in G, \quad j = \overline{1,3}. \tag{3.31}$$

Proof. From Equations (2.8) and (3.29), it follows that

$$\begin{aligned}
W(\xi) - W_{\sigma(\delta)}(\xi) &= \int_{\partial G} \left[g(\eta)\Psi_\sigma(\eta,\lambda;\xi) - f(\eta)\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta \\
&- \int_S \left[g_\delta(\eta)\Psi_\sigma(\eta,\lambda;\xi) - f_\delta(\eta)\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta \\
&= \int_S \left[g(\eta)\Psi_\sigma(\eta,\lambda;\xi) - f(\eta)\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta \\
&+ \int_T \left[g(\eta)\Psi_\sigma(\eta,\lambda;\xi) - f(\eta)\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta \\
&- \int_S \left[g_\delta(\eta)\Psi_\sigma(\eta,\lambda;\xi) - f_\delta(\eta)\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta \\
&= - \int_S \partial_n \Psi_\sigma(\eta,\lambda;\xi)\{f(\eta) - f_\delta(\eta)\} ds_\eta + \int_S \Psi_\sigma(\eta,\lambda;\xi)\{g(\eta) - g_\delta(\eta)\} ds_\eta \\
&+ \int_T \left[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta, \ \xi \in G.
\end{aligned}$$

and

$$\begin{aligned}
\partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_{\sigma(\delta)}(\xi) &= \int_{\partial G} \left[g(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - f(\eta)\partial_{\xi_j}(\partial_n \Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta \\
&- \int_S \left[g_\delta(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - f_\delta(\eta)\partial_{\xi_j}(\partial_n \Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta \\
&= \int_S \left[g(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - f(\eta)\partial_{\xi_j}(\partial_n \Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta \\
&+ \int_T \left[g(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - f(\eta)\partial_{\xi_j}(\partial_n \Psi_\sigma(\eta,\lambda;\xi))\right] ds\eta - \\
&- \int_S \left[g_\delta(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - f_\delta(\eta)\partial_{\xi_j}(\partial_n \Psi_\sigma(\eta,\lambda;\xi))\right] ds_\eta = \\
&= - \int_S \partial_{\xi_j}\left(\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right)\{f(\eta) - f_\delta(\eta)\} ds_\eta + \int_S \partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\{g(\eta) - g_\delta(\eta)\} ds_\eta \\
&+ \int_T \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta, \quad \xi \in G, \quad j = \overline{1,3}.
\end{aligned}$$

Employing the conditions outlined in (2.9) and (3.28), we derive the following estimate:

$$\begin{aligned}
\left|W(\xi) - W_{\sigma(\delta)}(\xi)\right| &\le \left| - \int_S \partial_n \Psi_\sigma(\eta,\lambda;\xi)\{f(\eta) - f_\delta(\eta)\} ds_\eta \right| + \left| \int_S \Psi_\sigma(\eta,\lambda;\xi)\{g(\eta) - g_\delta(\eta)\} ds_\eta \right| \\
&+ \left| \int_T \left[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right] ds_\eta \right| \\
&\le \int_S \left|\partial_n \Psi_\sigma(\eta,\lambda;\xi)\right| \left|\{f(\eta) - f_\delta(\eta)\}\right| ds_\eta + \int_S \left|\Psi_\sigma(\eta,\lambda;\xi)\right| \left|\{g(\eta) - g_\delta(\eta)\}\right| ds_\eta
\end{aligned}$$

$$+ M\int_T \left[|\Psi_\sigma(\eta,\lambda;\xi)| + |\partial_n\Psi_\sigma(\eta,\lambda;\xi)|\right] ds_\eta \le \delta\int_S |\partial_n\Psi_\sigma(\eta,\lambda;\xi)|\, ds_\eta$$
$$+ \delta\int_S |\Psi_\sigma(\eta,\lambda;\xi)|\, ds_\eta + M\int_T \left[|\Psi_\sigma(\eta,\lambda;\xi)| + |\partial_n\Psi_\sigma(\eta,\lambda;\xi)|\right] ds_\eta, \quad \xi\in G.$$

and

$$\left|\partial_{\xi_j}W(\xi) - \partial_{\xi_j}W_{\sigma(\delta)}(\xi)\right| \le \left|-\int_S \partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\{f(\eta)-f_\delta(\eta)\}ds_\eta\right| + \left|\int_S \partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\{g(\eta)-g_\delta(\eta)\}ds_\eta\right|$$
$$+ \left|\int_T \left[\partial_n W(\eta)\Psi_\sigma(\eta,\lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right] ds_?\right|$$
$$\le \int_S \left|\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right| |\{f(\eta)-f_\delta(\eta)\}|\, ds_\eta + \int_S \left|\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\right| |\{g(\eta)-g_\delta(\eta)\}|\, ds_\eta$$
$$+ M\int_T \left[\left|\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\right| + \left|\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right|\right] ds_\eta$$
$$\le \delta\int_S \left|\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right| ds_\eta + \delta\int_S \left|\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\right| ds_\eta$$
$$+ M\int_T \left[\left|\partial_{\xi_j}\Psi_\sigma(\eta,\lambda;\xi)\right| + \left|\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\lambda;\xi))\right|\right] ds_\eta, \quad \xi\in G, \quad j=\overline{1,3}.$$

At this point, we revisit the demonstrations of Theorems 2.1 and 3.1, from which we derive

$$\left|W(\xi) - W_{\sigma(\delta)}(\xi)\right| \le \frac{C(\xi)\,\sigma}{2}\left(\delta e^{\sigma a} + M\right)e^{-\sigma\xi_3}, \quad \sigma > 1, \quad \xi\in G, \tag{3.32}$$

$$\left|\partial_{\xi_j}W(\xi) - \partial_{\xi_j}W_{\sigma(\delta)}(\xi)\right| \le \frac{C(\xi)\,\sigma}{2}\left(\delta e^{\sigma a} + M\right)e^{-\sigma\xi_3}, \quad \sigma > 1, \quad \xi\in G, \quad j=\overline{1,3}. \tag{3.33}$$

From this point, by selecting σ from Equation (3.14), we derive the estimations (3.30) and (3.31). This, completes the proof of Theorem 3.2. □

Corollary 3.3. *For every $\xi\in G$, the statements hold true*

$$\lim_{\delta\to 0} W_{\sigma(\delta)}(\xi) = W(\xi), \quad \lim_{\delta\to 0}\partial_{\xi_j}W_{\sigma(\delta)}(\xi) = \partial_{\xi_j}W(\xi), \quad \xi\in G, \quad j=\overline{1,3}.$$

Corollary 3.4. *If $\xi\in\overline{G}_\varepsilon$, then both the sets of functions $\{W_{\sigma(\delta)}(\xi)\}$ and $\left\{\partial_{\xi_j}W_{\sigma(\delta)}(\xi)\right\}$ demonstrate uniform convergence as $\delta\to 0$, that is:*

$$W_{\sigma(\delta)}(\xi) \Longrightarrow W(\xi), \quad \partial_{\xi_j}W_{\sigma(\delta)}(\xi) \Longrightarrow \partial_{\xi_j}W(\xi), \quad j=\overline{1,3}.$$

4. Conculusion

The development of the Carleman function is a crucial element of this research, as it facilitates the extension of the solution beyond the boundary where Cauchy data can be accessed. By meticulously crafting this function, we can fulfill the necessary conditions to utilize Green's formula, resulting in a successful regularization of the ill-posed Cauchy problem. This method not only deepens our comprehension of fundamental mathematical concepts but also paves the way for practical implementations in various disciplines, including physics and engineering, where such inverse issues commonly occur. Furthermore, the regularization formula derived here highlights the significance of approximation methods. The capability to obtain a consistent solution when Cauchy data is affected by disturbances or inaccuracies greatly relies on our selection of the Carleman function. The connection between the level of approximation and the stability of the extended solution offers vital understanding of how minor changes can influence the entire reconstruction process. Ultimately, the stability estimates established for the classical solution are critically important. These estimates validate the strength of the method and bolster trust in the dependability of the regularized solution. By outlining the limits within which the solution holds true, we contribute to the expanding field focused on tackling the challenges associated with ill-posed problems in mathematical analysis. The Cauchy problem for the Helmholtz equation is an ill-posed problem, often requiring regularization techniques to obtain stable numerical solutions. A powerful approach to handling such problems is based on Carleman estimates, which provide stability results for solutions by leveraging exponential weight functions. The core ideas behind the Cauchy problem for the Helmholtz equation - unique continuation, stability via Carleman estimates, and regularization—can be effectively extended to broader classes of partial differential equations and adapted to different boundary conditions such as Dirichlet and Neumann. Whether addressing general elliptic operators, transformed hyperbolic/parabolic problems, or even mixed boundary conditions, the fundamental challenge remains to mitigate ill-posedness while ensuring that the boundary data (whether values or derivatives) reliably determine the solution in the interior. Ongoing research continues to refine these techniques, improving both theoretical guarantees and practical numerical implementations.

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