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Inverse Sturm-Liouville problem on a time scale with turning point

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Abstract

In this paper, an inverse Sturm-Liouville problem(ISLP) defined on a time scale with a single turning point is investigated. We first consider the Weyl function, and the solutions of the problems on the right side of the turning point. Subsequently, we introduce new solutions and a novel Weyl function to the left of the turning point. By incorporating the properties of time scales, this approach ensures that the inverse problem admits a uniquely determined solution.

Keywords. Sturm-Liouville operator, Inverse problem, Time scales, Weyl function.
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1. INTRODUCTION

Linear differential equations classified under the Sturm-Liouville framework provide a versatile tool for mathematical and physical studies, with numerous practical applications documented in the literature [1, 9, 16, 29]. In many applied areas, including fluid dynamics, quantum mechanics, and elasticity theory, inverse problems naturally arise. An example in quantum physics is that Schrödinger's formula can be reformulated in the Sturm-Liouville form, where the admissible energies of the system manifest as the operator's eigenvalues. Consequently, obtaining the solution to the inverse problem means determining the potential energy function using the specified energy eigenvalues [27]. Over time, the study of the ISLP has progressed through numerous contributions. Initial work established the fundamental concept and demonstrated solvability under certain conditions [5]. Later developments introduced methods using two distinct spectra, as well as techniques based on spectral functions [8, 14, 20, 21, 26]. While these results shaped the field considerably, additional aspects were explored in subsequent comprehensive analyses [24]. More recent research has addressed boundary conditions(BC) and uniqueness, showing that specifying eigenvalues along with corresponding weight numbers can uniquely determine the inverse problem [7].

Differential equations featuring turning points play a crucial role in physics and other applied sciences because of their broad practical applications [28]. The initial framework for this topic was laid out by Eberhard, Freiling, and Schneider, who has provided methods for linking the solutions of differential equations possessing complex parameters, which are of second order and have multiple turning points [10]. Subsequent studies expanded on these foundations. For example, ISLP with an indefinite weight function and a single turning point were analyzed, yielding uniqueness results [3, 4]. Later work examined equations on finite intervals with multiple turning points, emphasizing their importance within inverse spectral theory [25]. This body of research was further consolidated in a comprehensive monograph by Wasow, which systematically addresses turning points and their applications in mathematical analysis, engineering, and physics [31].

Differential operators on time scales have found extensive applications across various fields of science and engineering. They serve as a powerful framework for describing phenomena such as population dynamics in discrete settings and control processes in continuous domains [6]. The theory of time scales, first proposed in 1988, provides a unified mathematical foundation that bridges discrete and continuous dynamical systems [17]. Following this development,

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Sturm-Liouville eigenvalue problems in the setting of time scales were primarily analyzed in their direct form, leaving the inverse case mostly unaddressed [2]. Subsequent work established the completeness and orthogonality of the related eigenfunctions in the relevant function space [15]. Previous results were extended by analyzing the variation of eigenvalues under normalized separated BC, showing that they continuously depend on the boundary parameters [19]. In the advancement of the classical theory, differential forms with coefficients of definite measure gained attention, thereby expanding the discussion of Sturm-Liouville operators within the scope of time scales [11].

Recent developments in Sturm-Liouville problems on time scales have led to notable progress in inverse spectral theory. In particular, Bohner and Koyunbakan investigated Borg-type results for a time scale consisting of a subset of integers, proposing methods involving Δ-derivatives on discrete intervals [8]. Subsequently, Ozkan and Adalar examined the half-ISLP on a time scale formed by merging a standard interval with an additional discrete set, establishing a Hochstadt–Lieberman-type theorem under Robin BC and analyzing several specific cases [30]. This type of half-inverse problem represents an important extension of the classical theory first proposed by Hochstadt and Lieberman [18]. More recently, Kuznetsova treated time scales as finite collections of intervals and isolated points, proving uniqueness results via spectral mappings combined with recovery algorithms that employ matrices [22, 23].

In this study, our work builds upon the foundational research conducted by Yurko and Freiling [12, 13]. Their research primarily concentrated on solutions featuring turning points, though it did not consider time scales. Our approach represents the solutions to the right of the turning point in accordance with their methods. The real challenge emerges when transitioning from the I_+ to I_-. To bridge this gap, we employ the jump condition, a crucial concept derived from the definition of Δ-derivatives within the framework of time scales. As a result, we are able to obtain the relevant solutions within the interval I_-.

A central aspect of our study is the derivation of a novel Weyl function specifically designed for the I_-. By employing the transfer matrix together with the newly defined Weyl function, the problem is shown to admit a unique reconstruction, ensuring its well-posedness. It is important to note that our modifications do not alter the characteristic function or the associated eigenvalues. As identified in [12], these quantities were already obtained based on the I_+. Therefore, we adjust the solutions on the I_- such that they do not impact the eigenvalues and the characteristic function.

Before presenting the main results, let's first consider the concepts of time scales. A closed subset of $\mathbb{R}$ under the conventional Euclidean topology is called a *time scale*. We assume T is a time scale and define the jump functions $\sigma : T \to T$ and $\sigma_- : T \to T$ by

$$\sigma(t) = \begin{cases} \inf\{a \in T : a > t\}, & \text{if } t \neq \max T, \\ t, & \text{if } t = \max T, \end{cases}$$

$$\sigma_-(t) = \begin{cases} \sup\{a \in T : a < t\}, & \text{if } t \neq \min T, \\ t, & \text{if } t = \min T. \end{cases}$$

Let $t \in T$ be a point in a time scale T. The point t is classified as follows

- Left-dense: if $\sigma_-(t) = t$,
- Left-isolated: if $\sigma_-(t) < t$,
- Isolated: if $\sigma_-(t) < t < \sigma(t)$,
- Right-dense: if $\sigma(t) = t$,
- Right-isolated: if $\sigma(t) > t$,
- Dense: if $\sigma(t) = t = \sigma_-(t)$.

Define

$$T^0 = \begin{cases} T - \{\sup T\}, & \text{provided that } \sup T \text{ is isolated from the left,} \\ T, & \text{Otherwise.} \end{cases}$$

Let $t \in T^0$, Δ-differentiable at the point t is said of a function g defined on T if, for all $\epsilon > 0$, one can find $\delta > 0$ so that

$$\left|(g(\sigma(t)) - g(a)) - (\sigma(t) - a)g^{\Delta}(t)\right| \leq \epsilon \left|\sigma(t) - a\right|,$$

for every $a \in T \cap (t-\delta, t+\delta)$. The quantity $g^{\Delta}(t)$, which is the value at the point t of the function g, is referred to as the Δ-derivative.
We define

$$C^n_{rd}(T) := \{g : g \text{ is rd-continuous n-order } \Delta\text{-differentiable}\}.$$

For higher-order derivatives ($n \geq 2$), we suppose $g^{\Delta^{n-1}}$ is defined on $T^{0^{n-1}}$, where $b^{n-1} = \underbrace{bb\dots b}_{n-1}$. If $g^{\Delta^{n-1}}$ is again Δ-differentiable on $T^{0^n} := (T^{0^{n-1}})^0$, the Δ-derivative of order n is defined as

$$g^{\Delta^n} := \left(g^{\Delta^{n-1}}\right)^{\Delta}.$$

Let $n \geq 1$. We denote the set of functions whose Δ-derivatives of order n exist as

$$C^n(T) := \left\{g : g^{\Delta^n} \text{ exists and } g^{\Delta^n} \in C(T^{0^n})\right\}.$$

We rely on Propositions 1.1 through 1.3, rigorously proved in [6], which form the theoretical foundation of our study. These propositions provide fundamental results within the framework of equations defined on time scales, allowing a unified analysis of differential and difference equations.

Proposition 1.1. *Consider $g : T \to \mathbb{R}$ is a function and if $t \in T^0$, accordingly, the following can be stated:*

(i) Whenever $g(t)$ is Δ-differentiable at the point t, the function is continuous there.

(ii) A function g is Δ-differentiable at a right-isolated point $t \in T$ if and only if it is continuous at t. Hence, we obtain

$$g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t}.$$

(iii) If $t \in T$ belong to the set of right-dense points, in this case g is Δ-differentiable at t, if and only if the following limit exists

$$\lim_{a \to t, a \in T} \frac{g(t) - g(a)}{t - a} = g^{\Delta}(t).$$

Notably, for some $\epsilon > 0$, let $(t - \epsilon, t + \epsilon) \subset \mathbb{T}$, then g is Δ-differentiable at this point if and only if it is, at t, differentiable, and we write

$$g^{\Delta}(t) = g'(t).$$

Prior to presenting the next proposition, we introduce the concept of the Δ-integral. An *antiderivative* of a function g on T is a function $G : T \to \mathbb{R}$ such that $G^{\Delta}(t) = g(t)$ holds for all $t \in T^0$. Regarding this function, the Δ-integral of g is given by

$$\int_a^b g(t)\Delta t = G(b) - G(a), \quad \text{for all } a, b \in T.$$

Proposition 1.2. *If $g \in C_{rd}$ and $t \in T^0$, then*

$$\int_t^{\sigma(t)} g(x)\Delta x = \mu(t)g(t).$$

According to the definition given in [6], For $a, b \in T$, the interval is expressed as

$$[a, b] := \{t \in T : a \leq t \leq b\}.$$

Proposition 1.3. *Assume $a, b \in T$ and g belongs to C_{rd}:*

(i) In the case where $T = \mathbb{R}$, we have

$$\int_a^b g(t)\Delta t = \int_a^b g(t)\, dt,$$

(ii) If only isolated points are present in $[a,b]$, *then*

$$\int_a^b g(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} g(t)\mu(t), & \text{if } a<b,\\ 0, & \text{if } a=b,\\ -\sum_{t\in[b,a)} g(t)\mu(t), & \text{if } a>b. \end{cases} \tag{1.1}$$

Here, the right-hand side involves the ordinary Riemann integral from elementary calculus.

We take into account the boundary value problem(BVP) L given by

$$\begin{aligned} &\ell y := -y^{\Delta\Delta}(x) + q(x)y\big(\sigma(x)\big) = \lambda R^2(x)y\big(\sigma(x)\big), \quad x\in T^{0^2}, \\ &T = [0,b_1]\cup\{x_1\}\cup[a_1,1], \quad 0<b_1<x_1<a_1<1, \end{aligned} \tag{1.2}$$

$$U(y) = y^{\Delta}(0) - hy(0) = 0, \tag{1.3}$$

$$V(y) = y^{\Delta}(1) + h_1 y(1) = 0. \tag{1.4}$$

The spectral parameter, in this case, is expressed as $\lambda=\rho^2$. Both q and R^2 are real-valued functions. Here, h and h_1 are assumed to be real constants. Let us assume that

$$R^2(x) = (x-x_1)^{l_1} R_0(x),$$

where $l_1\in\mathbb{N}$, and $R_0>0$ for x in T, and R_0 is twice continuously differentiable on $[0,1]$. The zero x_1 of R^2 is called turning point.

In this paper, at the point b_1, according to the definition of the Δ-derivative, the jump conditions listed below are derived:

$$y(b_1) = y(a_1) - y^{\Delta}(a_1)\,\mu(b_1) + \big(\,q(b_1) - \lambda R^2(b_1)\big)\,y(a_1)\,\mu^2(b_1), \tag{1.5}$$

$$y^{\Delta}(b_1) = y^{\Delta}(a_1) + \big(\lambda R^2(b_1) - q(b_1)\big)\,y(a_1)\,\mu(b_1). \tag{1.6}$$

Considering the preceding discussions, the research should focus on three main directions. These are presented as follows:

(i) Adjustment of solutions in I_-.
(ii) Introduction of the Weyl function in I_-.
(iii) Uniqueness theorems for the inverse problem on time scale.

2. REPRESENTATION OF SOLUTIONS

In this paper, by selecting the time scale T, we define I_+ as the set of all x for which $R^2(x)>0$, and I_- as the set of x satisfying $R^2(x)<0$ and we write:

$$I = I_-\cup I_+.$$

There are four distinct types of turning points. However, we consider only one type, namely when l_1 is odd and $R^2(x)(x-x_1)^{-l_1}>0$ in I. We set

$$\nu_1 = \frac{1}{2+l_1},$$

$$\theta_1 = \begin{cases} 1, & \text{if } \nu_1>\frac14,\\ 1-\epsilon_0 \quad (\text{with } \epsilon_0>0 \ \text{ as small as desired}), & \text{if } \nu_1=\frac14,\\ 4\nu_1, & \text{if } \nu_1<\frac14, \end{cases}$$

$$\xi(x) = \begin{cases} 0, & x\in I_+,\\ 1, & x\in I_-, \end{cases}$$

$$\gamma_1 = 2\sin\left(\frac{\pi}{2}\nu_1\right),$$

$$K_-(x)K_-^*(x)=\exp\left(-\frac{\pi}{2}i\,\xi(x)\right)=\begin{cases}1, & x\in I_+,\\ -i, & x\in I_-,\end{cases}$$

$$R_+^2(x)=\max(R^2(x),0),\quad R_-^2(x)=\max(-R^2(x),0).$$

We consider the sectors S_k corresponding to $k=0,1$:

$$S_k=\left\{\rho|\arg\rho\in\left[\frac{k\pi}{4},\frac{(k+1)\pi}{4}\right]\right\}. \tag{2.1}$$

Since the eigenvalues of problem L are real, then k can be chosen as 0 or 1. In I_+, we have

$$\begin{aligned}\int_0^x |R(t)|\,\Delta t&=\int_0^{b_1}|R(t)|\,\Delta t+\int_{b_1}^{\alpha_1}|R(t)|\,\Delta t+\int_{\alpha_1}^{x}|R(t)|\,\Delta t\\ &=\int_0^{b_1}|R(t)|\,dt+\mu(b_1)\,|R(b_1)|+\mu(a_1)\,|R(a_1)|+\int_{\alpha_1}^{x}|R(t)|\,dt.\end{aligned}$$

To simplify the form of the solutions, we consider the following symbols:

$$\tau = \mu(b_1)\,|R(b_1)|+\mu(a_1)\,|R(a_1)|, \tag{2.2}$$

$$\gamma_x = -i\left(\int_0^{b_1}|R(t)|dt+\tau+\int_{a_1}^{x}|R(t)|dt\right), \tag{2.3}$$

$$\eta_x = \int_0^x |R(t)|dt. \tag{2.4}$$

According to the fundamental solutions (FSS) in [12], in I_+, for j=0,1, we write

$$Z_1^{(j)}(x,\rho)=(-i\rho)^j|R(x)|^{j-\frac{1}{2}}\exp\left[-i\rho\left(\int_0^{b_1}|R(t)|dt+\tau+\int_{a_1}^{x}|R(t)|dt\right)\right]K_-(x)k(x,\rho), \tag{2.5}$$

$$Z_2^{(j)}(x,\rho)=(i\rho)^j|R(x)|^{j-\frac{1}{2}}\exp\left[i\rho\left(\int_0^{b_1}|R(t)|dt+\tau+\int_{a_1}^{x}|R(t)|dt\right)\right]K_-^*(x)k(x,\rho), \tag{2.6}$$

and in I_-, we write

$$Z_1^{(j)}(x,\rho)=(-i\rho)^j|R(x)|^{j-\frac{1}{2}}\left(\exp(\frac{i\pi}{2})\right)^j\exp\left(\rho\int_0^x|R(x)|\,dt\right)K_-(x)k(x,\rho), \tag{2.7}$$

$$Z_2^{(j)}(x,\rho)=(i\rho)^j|R(x)|^{j-\frac{1}{2}}\left(\exp(\frac{i\pi}{2})\right)^j\exp\left(\rho\int_0^x|R(x)|\,dt\right)K_-^*(x)k(x,\rho). \tag{2.8}$$

Following the notation of [12], we denote by $\varphi(x,\lambda)$ the solution of (1.2) on the interval [0,1] satisfying the initial conditions

$$\varphi(0,\lambda)=1,\quad \varphi'(0,\lambda)=h.$$

The aim is to obtain the explicit form of the function $\varphi(x,\lambda)$ on the interval I_- by applying (1.5)–(1.6). However, first, according to the definition of $\varphi(x,\lambda)$ on I_+, we have:

$$\varphi(a_1,\lambda)=\frac{1}{2}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho), \tag{2.9}$$

$$\varphi^{\Delta}(a_1,\lambda)=-\frac{1}{2}(i\rho)|R(0)|^{\frac{1}{2}}|R(a_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho). \tag{2.10}$$

Lemma 2.1. *Based on* (1.5)–(1.6)*, the solutions* $\varphi(b_1,\lambda)$ *and* $\varphi^{\Delta}(b_1,\lambda)$ *are obtained as:*

$$\varphi(b_1,\lambda)=\frac{1}{2}\rho^2|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right), \tag{2.11}$$

$$\varphi^{\Delta}(b_1,\lambda)=-\frac{1}{2}\rho^2|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{3}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right). \tag{2.12}$$

Proof. By considering (1.5) and (2.9)–(2.10), we get

$$\begin{aligned}\varphi(b_1,\lambda)&=\Big(1+\big(q(b_1)-\lambda R^2(b_1)\big)\mu^2(b_1)\Big)\frac{1}{2}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\\&\quad+\mu(b_1)(i\rho)\frac{1}{2}|R(0)|^{\frac{1}{2}}|R(a_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\\&=\frac{1}{2}\rho^2|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\\&\quad\times\frac{\Big(1+\big(q(b_1)-\lambda R^2(b_1)\big)\mu^2(b_1)\Big)+\mu(b_1)(i\rho)|R(a_1)|}{\rho^2}.\end{aligned}$$

If $\rho\to\infty$, (2.11) is proven. By considering (1.6), we have:

$$\begin{aligned}\varphi^{\Delta}(b_1,\lambda)&=-\frac{(i\rho)}{2}|R(0)|^{\frac{1}{2}}|R(a_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\\&\quad+\frac{1}{2}\Big(\big(\lambda R^2(b_1)-q(b_1)\big)\mu(b_1)\Big)|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\\&=-\frac{1}{2}\rho^2|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\\&\quad\times\frac{\Big[\rho-\Big(\big(\lambda R^2(b_1)-q(b_1)\big)\mu(b_1)\Big)(i)^{-1}|R(a_1)|^{-1}\Big]}{\rho^2}.\end{aligned}$$

If $\rho\to\infty$, (2.12) is proven. ☐

Based on Lemma 2.1, $\varphi(x,\lambda)$ was determined at b_1. Now, given this solution and its Δ-derivative. Utilizing the proposition below, one can obtain both the solution and its Δ-derivative within the interval I_-

Proposition 2.2. *Let* $\rho\in S_k$ *of* (2.1) *and* $|\rho|\to\infty$*, with* $x\in I_-$*, then*

$$\begin{aligned}\varphi(x,\lambda)&=\frac{-\rho^2}{4i}\,|R(0)|^{\frac{1}{2}}\,|R(a_1)|^{-\frac{1}{2}}\,|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})\exp\left(\rho\int_x^{b_1}|R(t)|\,dt\right)K_-(a_1)K_-^*(b_1)k(a_1,\rho)\\&\quad\times k^{-1}(b_1,\rho)|R(x)|^{-\frac{1}{2}}K_-(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right),\end{aligned} \tag{2.13}$$

$$\begin{aligned}\varphi^{\Delta}(x,\lambda)&=\frac{\rho^3}{4i}\,|R(0)|^{\frac{1}{2}}\,|R(a_1)|^{-\frac{1}{2}}\,|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})\exp\left(\rho\int_x^{b_1}|R(t)|\,dt\right)K_-(a_1)K_-(b_1)k(a_1,\rho)\\&\quad\times k^{-1}(b_1,\rho)\,|R(x)|^{\frac{1}{2}}\,K_-^*(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right).\end{aligned} \tag{2.14}$$

Proof. From the $Z_1(x,\rho)$ and $Z_2(x,\rho)$ on I_- and (2.7)–(2.8), the solution $\varphi(x,\lambda)$ is to be linear combination of these $Z_1(x,\rho)$ and $Z_2(x,\rho)$. Thus, for $x\in I_-$, there exist constants G and H so that

$$\varphi(x,\lambda)=GZ_1(x,\rho)+HZ_2(x,\rho). \tag{2.15}$$

Therefore, we write:

$$\begin{cases}\varphi(b_1,\lambda)=GZ_1(b_1,\rho)+HZ_2(b_1,\rho),\\ \varphi^{\Delta}(b_1,\lambda)=GZ_1^{\Delta}(b_1,\rho)+HZ_2^{\Delta}(b_1,\rho).\end{cases} \tag{2.16}$$

Using Cramer's rule, (2.11) and (2.12), the coefficients can be obtained as follows:

$$\begin{aligned}G=&-\frac{\rho^2}{4i}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}|R(b_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})\exp(-\rho\eta_{b_1})K_-(a_1)K_-^*(b_1)k(a_1,\rho)k^{-1}(b_1,\rho)\\ &\times\left[|R(b_1)|\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right)-\frac{-R^2(b_1)\mu(b_1)|R(a_1)|^{-1}+O(\frac{1}{\rho})}{\rho}\right],\end{aligned}$$

and

$$\begin{aligned}H=&-\frac{\rho^2}{4i}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}|R(b_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})\exp(\rho\eta_{b_1})K_-(a_1)K_-(b_1)k(a_1,\rho)k^{-1}(b_1,\rho)\\ &\times\left[|R(b_1)|\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right)+\frac{-R^2(b_1)\mu(b_1)|R(a_1)|^{-1}+O(\frac{1}{\rho})}{\rho}\right].\end{aligned}$$

By substituting G and H in (2.15), we can get

$$\begin{aligned}\varphi(x,\lambda)=&-\frac{\rho^2}{4i}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})\exp\left(\rho\int_x^{b_1}|R(t)|dt\right)K_-(a_1)K_-^*(b_1)\\ &\times k(a_1,\rho)k^{-1}(b_1,\rho)|R(x)|^{-\frac{1}{2}}K_-(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right)\\ &\times\left(\exp\left(-2\rho\int_x^{b_1}|R(t)|dt\right)+\frac{K_-(b_1)K_-^*(x)}{K_-^*(b_1)K_-(x)}\right).\end{aligned}$$

If $\rho\to\infty$, then the first term within the parentheses vanishes, and (2.13) is proven. On the other hand, by taking the Δ-derivative of (2.15), we can write:

$$\varphi^{\Delta}(x,\lambda)=GZ_1^{\Delta}(x,\rho)+HZ_2^{\Delta}(x,\rho). \tag{2.17}$$

By substituting G and H in (2.17), we can obtain the Δ-derivative of $\varphi(x,\lambda)$ for $x\in I_-$, as presented here:

$$\begin{aligned}\varphi^{\Delta}(x,\lambda)=&\left[\frac{-\rho^2}{4i}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})(-\rho\eta_{b_1})K_-(a_1)K_-^*(b_1)k(a_1,\rho)k^{-1}(b_1,\rho)(\rho)\right.\\ &\left.\times|R(x)|^{\frac{1}{2}}K_-(x)k(x,\rho)\exp(\rho\eta_x)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right)\right]\\ &+\left[\frac{-\rho^2}{4i}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})(\rho\eta_{b_1})K_-(a_1)K_-(b_1)k(a_1,\rho)k^{-1}(b_1,\rho)(-\rho)\right.\\ &\left.\times|R(x)|^{\frac{1}{2}}K_-^*(x)k(x,\rho)\exp(-\rho\eta_x)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right)\right]\\ =&\frac{\rho^3}{4i}|R(0)|^{\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_a)\exp\left(\rho\int_x^{b_1}|R(t)|\,dt\right)K_-(a_1)K_-(b_1)k(a_1,\rho)\\ &\times k^{-1}(b_1,\rho)|R(x)|^{\frac{1}{2}}K_-^*(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right)\\ &\times\left(-\frac{K_-^*(b_1)K_-(x)}{K_-(b_1)K_-^*(x)}\exp\left(-2\rho\int_x^{b_1}|R(t)|\,dt\right)+1\right).\end{aligned}$$

If $\rho\to\infty$, then the first term within the parentheses vanishes, and (2.14) is proven. □

The following new conditions for the solution $\varphi(x,\lambda)$ under jump conditions are obtained:

$$\varphi(0,\lambda)=\alpha_0,\ \varphi^{\Delta}(0,\lambda)=\beta_0, \tag{2.18}$$

where $\beta_0=h\alpha_0$, and both are nonzero. Consequently, $\varphi(x,\lambda)$ constitutes a solution to problem L.
For $x\in I_-$, based on (2.13) and (2.14), the fact that these two relations are bounded can be expressed as follows:

$$|\varphi(x,\lambda)|\le C\,\rho^2|R(x)|^{-\frac{1}{2}}\exp(\tau\operatorname{Im}\rho)\exp\left(|\operatorname{Re}\rho|\int_x^{b_1}|R(t)|\,dt\right), \tag{2.19}$$

$$|\varphi^{\Delta}(x,\lambda)|\le C\,|\rho^3||R(x)|^{\frac{1}{2}}\exp(\tau\operatorname{Im}\rho)\exp\left(|\operatorname{Re}\rho|\int_x^{b_1}|R(t)|\,dt\right). \tag{2.20}$$

Here and elsewhere, the constant C is taken to mean a positive constants for bounding and estimating relationships.

3. WEYL FUNCTION

In this section, we introduce the Weyl function. To this end, we require two solutions $S(x,\lambda)$ and $\Phi(x,\lambda)$ of (1.2). Following the notation of [12], we denote by $S(x,\lambda)$ the solution of (1.2) on the interval [0,1] satisfying the initial conditions

$$S(0,\lambda)=0,\quad S'(0,\lambda)=1.$$

Now we want to obtain the explicit expression of the solution $S(x,\lambda)$ by considering (1.5)–(1.6) for $x\in I_-$, similarly, we have the following as in Lemma 2.1:

$$S(b_1,\lambda)=\rho|R(0)|^{-\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right), \tag{3.1}$$

$$S^{\Delta}(b_1,\lambda)=-\frac{1}{2}\rho|R(0)|^{-\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(\rho\gamma_{a_1})K_-(a_1)k(a_1,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right). \tag{3.2}$$

Proposition 3.1. *Let* $\rho\in S_k$ *of* (2.1) *and* $|\rho|\to\infty$, *with* $x\in I_-$, *then*

$$\begin{aligned}S(x,\lambda)=\frac{-\rho}{4i}\,|R(0)|^{-\frac{1}{2}}\,|R(a_1)|^{-\frac{1}{2}}\,|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})\exp\left(\rho\int_x^{b_1}|R(t)|\,dt\right)K_-(a_1)K_-(b_1)\\ \times k(a_1,\rho)k^{-1}(b_1,\rho)|R(x)|^{-\frac{1}{2}}K_-^*(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right),\end{aligned} \tag{3.3}$$

$$\begin{aligned}S^{\Delta}(x,\lambda)=\frac{\rho^2}{4i}\,|R(0)|^{-\frac{1}{2}}\,|R(a_1)|^{-\frac{1}{2}}\,|R(b_1)|^{\frac{1}{2}}\exp(\rho\gamma_{a_1})\exp\left(\rho\int_x^{b_1}|R(t)|\,dt\right)K_-(a_1)K_-(b_1)\\ \times k(a_1,\rho)k^{-1}(b_1,\rho)\,|R(x)|^{\frac{1}{2}}\,K_-^*(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right).\end{aligned} \tag{3.4}$$

Proof. Analogous to the proof for Proposition 2.2, and by using the (3.1)–(3.2), this proposition is also proven. □

For $x\in I_-$, using (3.3) and (3.4), the boundedness of these two relations is deduced from the following relation:

$$|S(x,\lambda)|\le C\,|\rho||R(x)|^{-\frac{1}{2}}\exp(\tau\operatorname{Im}\rho)\exp\left(|\operatorname{Re}\rho|\int_x^{b_1}|R(t)|\,dt\right), \tag{3.5}$$

$$|S^{\Delta}(x,\lambda)|\le C\,\rho^2|R(x)|^{\frac{1}{2}}\exp(\tau\operatorname{Im}\rho)\exp\left(|\operatorname{Re}\rho|\int_x^{b_1}|R(t)|\,dt\right). \tag{3.6}$$

The following new conditions for the solution $S(x,\lambda)$ under jump conditions are obtained:

$$S(0,\lambda)=\alpha_0',\ \ S^{\Delta}(0,\lambda)=\beta_0'. \tag{3.7}$$

Let $\Phi(x,\lambda)$ be the solution of (1.2) satisfying the boundary condition $V(\Phi)=0,\ U(\Phi)=1$ for $x\in[0,1]$, as defined in [12]. In this paper, using (1.5)–(1.6), we obtain expressions for $\Phi(x,\lambda)$ and $\Phi^{\Delta}(x,\lambda)$ on $x\in I_-$. In a manner similar to Lemma 2.1, this leads to the following:

$$\Phi(b_1,\lambda)=\frac{\rho}{i}|R(0)|^{-\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(-\rho\gamma_{a_1})K_-^*(a_1)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right), \tag{3.8}$$

$$\Phi^{\Delta}(b_1,\lambda)=-\frac{\rho}{i}|R(0)|^{-\frac{1}{2}}|R(a_1)|^{-\frac{1}{2}}\exp(-\rho\gamma_{a_1})K_-^*(a_1)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right). \tag{3.9}$$

Proposition 3.2. *Let $\rho\in S_k$ of (2.1) and $|\rho|\to\infty$, with $x\in I_-$, then*

$$\begin{aligned}\Phi(x,\lambda)=&\frac{\rho}{2}\left|R(0)\right|^{-\frac{1}{2}}\left|R(a_1)\right|^{-\frac{1}{2}}\left|R(b_1)\right|^{\frac{1}{2}}\exp(-\rho\gamma_{a_1})\exp\left(\rho\int_x^{b_1}\left|R(t)\right|dt\right)K_-^*(a_1)K_-(b_1)\\ &\times k^{-1}(b_1,\rho)|R(x)|^{-\frac{1}{2}}K_-^*(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\Phi^{\Delta}(x,\lambda)=&-\frac{\rho^2}{2}\left|R(0)\right|^{-\frac{1}{2}}\left|R(a_1)\right|^{-\frac{1}{2}}\left|R(b_1)\right|^{\frac{1}{2}}\exp(-\rho\gamma_{a_1})\exp\left(\rho\int_x^{b_1}\left|R(t)\right|dt\right)K_-^*(a_1)K_-(b_1)\\ &\times k^{-1}(b_1,\rho)|R(x)|^{\frac{1}{2}}K_-^*(x)k(x,\rho)\left(-R^2(b_1)\mu^2(b_1)+O(\frac{1}{\rho})\right).\end{aligned} \tag{3.11}$$

Proof. Analogous to the proof for Proposition 2.2, and by (3.8)–(3.9), this proposition is also proven. ☐

It is important to note that our modifications do not affect the characteristic function or the eigenvalues. We have the solutions on the right side of the turning point, and according to the definition in [12], the characteristic function is defined in terms of these solutions as:

$$\Delta(\lambda)=\varphi^{\Delta}(1,\lambda)+h_1\varphi^{\Delta}(1,\lambda). \tag{3.12}$$

Therefore, we adjust the solutions on the left side of the turning point in such a way that they do not affect the eigenvalues and the characteristic function of problem L. According to (3.10) and based on the jump condition, we introduce a new Weyl function for I_- as $M(\lambda)=\Phi(0,\lambda)$. In [12], the Weyl function for the problem on I_+ was denoted by $m(\lambda)$. Now we want to obtain the relationship between $M(\lambda)$ and $m(\lambda)$.

Proposition 3.3. *Let $\rho\in S_k$ of (2.1) and $|\rho|\to\infty$, then*

$$M(\lambda)=\frac{\left[\left(1+\left(q(b_1)-\lambda R^2(b_1)\right)\mu^2(b_1)\right)\Phi(a_1)-\mu(b_1)\Phi^{\Delta}(a_1)\right]\left(\alpha_0'\beta_0-\alpha_0\beta_0'\right)+D(b_1)}{\left(\beta_0-h\alpha_0\right)S(b_1)+\left(h\alpha_0'-\beta_0'\right)\varphi(b_1)}, \tag{3.13}$$

where

$$\begin{aligned}&\Phi(a_1)=\Phi(a_1,\lambda),\ \ \Phi^{\Delta}(a_1)=\Phi^{\Delta}(a_1,\lambda),\ \ S(b_1)=S(b_1,\lambda),\\ &\varphi(b_1)=\varphi(b_1,\lambda),\ \ D(b_1)=\alpha_0S(b_1)-\alpha_0'\varphi(b_1).\end{aligned}$$

along with

$$m(\lambda)=\frac{\Phi(x,\lambda)-S(x,\lambda)}{\varphi(x,\lambda)},\quad x\in I_+.$$

Proof. In I_+, since the effect of the Weyl function is as follow:

$$\begin{aligned}\Phi(a_1,\lambda) &= S(a_1,\lambda)+m(\lambda)\,\varphi(a_1,\lambda),\\ \Phi^{\Delta}(a_1,\lambda) &= S^{\Delta}(a_1,\lambda)+m(\lambda)\,\varphi^{\Delta}(a_1,\lambda).\end{aligned}$$

In light of these two derived relations and (1.5), we have

$$\Phi(b_1,\lambda)=\Big(1+\big(q(b_1)-\lambda R^2(b_1)\big)\mu^2(b_1)\Big)\big(S(a_1)+m\varphi(a_1)\big)-\mu(b_1)\big(S^{\Delta}(a_1)+m\varphi^{\Delta}(a_1)\big), \tag{3.14}$$

where

$$m=m(\lambda),\;\; S(a_1)=S(a_1,\lambda),\;\; \varphi(a_1)=\varphi(a_1,\lambda),\;\; S^{\Delta}(a_1)=S^{\Delta}(a_1,\lambda),\;\; \varphi^{\Delta}(a_1)=\varphi^{\Delta}(a_1,\lambda).$$

We have:

$$U(\Phi)=\Phi^{\Delta}(0,\lambda)-h\Phi(0,\lambda)=1 \implies \Phi^{\Delta}(0,\lambda)=1+M(\lambda)h.$$

Because in the interval I_-, the solution $\Phi(x,\lambda)$ is a linear combination of the solution $\varphi(x,\lambda)$ and $S(x,\lambda)$, we have

$$\Phi(x,\lambda)=A_1S(x,\lambda)+A_2\varphi(x,\lambda). \tag{3.15}$$

At $x=0$, this implies

$$\begin{aligned}M(\lambda) &= A_1\alpha_0'+A_2\alpha_0,\\ 1+M(\lambda)h &= A_1\beta_0'+A_2\beta_0.\end{aligned}$$

Using Cramer's rule yields, we can get

$$A_1=\frac{M(\lambda)\beta_0-\alpha_0\big(1+M(\lambda)h\big)}{\alpha_0'\beta_0-\alpha_0\beta_0'}, \tag{3.16}$$

$$A_2=\frac{\alpha_0'\big(1+M(\lambda)h\big)-M(\lambda)\beta_0'}{\alpha_0'\beta_0-\alpha_0\beta_0'}. \tag{3.17}$$

By these values A_1 and A_2 in (3.16)–(3.17) and by placing it in (3.15), we have

$$\Phi(x,\lambda)=\frac{M(\lambda)\beta_0-\alpha_0\big(1+M(\lambda)h\big)}{\alpha_0'\beta_0-\alpha_0\beta_0'}S(x,\lambda)+\frac{\alpha_0'\big(1+M(\lambda)h\big)-M(\lambda)\beta_0'}{\alpha_0'\beta_0-\alpha_0\beta_0'}\varphi(x,\lambda), \tag{3.18}$$

$$\Phi(b_1,\lambda)=\frac{M(\lambda)\beta_0-\alpha_0\big(1+M(\lambda)h\big)}{\alpha_0'\beta_0-\alpha_0\beta_0'}S(b_1,\lambda)+\frac{\alpha_0'\big(1+M(\lambda)h\big)-M(\lambda)\beta_0'}{\alpha_0'\beta_0-\alpha_0\beta_0'}\varphi(b_1,\lambda). \tag{3.19}$$

Thus, considering (3.14) and (3.19), the following result is obtained:

$$\begin{aligned}&\Big(1+\big(q(b_1)-\lambda R^2(b_1)\big)\mu^2(b_1)\Big)\big(S(a_1)+m\varphi(a_1)\big)-\mu(b_1)\big(S^{\Delta}(a_1)+m\varphi^{\Delta}(a_1)\big)\\ &=\frac{\big(M(\lambda)\beta_0-\alpha_0-M(\lambda)h\alpha_0\big)S(b_1)+\big(\alpha_0'+M(\lambda)h\alpha_0'-M(\lambda)\beta_0'\big)\varphi(b_1)}{\alpha_0'\beta_0-\alpha_0\beta_0'}\\ &=\frac{M(\lambda)\big((\beta_0-h\alpha_0)S(b_1)+(h\alpha_0'-\beta_0')\varphi(b_1)\big)-\alpha_0S(b_1)+\alpha_0'\varphi(b_1)}{\alpha_0'\beta_0-\alpha_0\beta_0'}.\end{aligned}$$

Finally, $M(\lambda)$ is obtained from the above relation and it is proved. □

4. INVERSE PROBLEM

The concept of the inverse problem is introduced in this section. It is demonstrated that if the Weyl functions associated with two BVP coincide, then their corresponding inverse problem solutions are identical. Let $M(\lambda)$ denote a given Weyl function, and suppose the objective is to reconstruct the potential function q; this constitutes the *inverse problem.* In order to establish a uniqueness result for this problem, we introduce a another BVP $\tilde{L}$, which has the same structure as L, but with potentially distinct coefficients $\tilde{q}, \tilde{h}, \tilde{h}_1$. In what follows, we denote by $\tilde{\alpha}$ the quantity corresponding to α in problem $\tilde{L}$. We define a matrix $P(x,\lambda) = [P_{ij}(x,\lambda)]_{i,j=1}^{2}$, for $x \in I_-$ called the *transfer matrix*, such that

$$P(x,\lambda)\begin{bmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}^{\Delta}(x,\lambda) & \tilde{\Phi}^{\Delta}(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi^{\Delta}(x,\lambda) & \Phi^{\Delta}(x,\lambda) \end{bmatrix}. \tag{4.1}$$

Considering the linearly independent solutions, the following result is obtained:

$$\det\begin{bmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}^{\Delta}(x,\lambda) & \tilde{\Phi}^{\Delta}(x,\lambda) \end{bmatrix} = \tilde{h}_2 \neq 0. \tag{4.2}$$

Clearly, $\tilde{h}_2 \neq 0$ because the Wronskian of two linearly independent solutions is a nonzero constant. Using (4.1) we calculate

$$P_{11}(x,\lambda) = \frac{1}{\tilde{h}_2}\left(\varphi(x,\lambda)\tilde{\Phi}^{\Delta}(x,\lambda) - \Phi(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\right), \tag{4.3}$$

$$P_{12}(x,\lambda) = \frac{1}{\tilde{h}_2}\left(-\varphi(x,\lambda)\tilde{\Phi}(x,\lambda) + \Phi(x,\lambda)\tilde{\varphi}(x,\lambda)\right), \tag{4.4}$$

$$P_{21}(x,\lambda) = \frac{1}{\tilde{h}_2}\left(\varphi^{\Delta}(x,\lambda)\tilde{\Phi}^{\Delta}(x,\lambda) - \Phi^{\Delta}(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\right), \tag{4.5}$$

$$P_{22}(x,\lambda) = \frac{1}{\tilde{h}_2}\left(-\varphi^{\Delta}(x,\lambda)\tilde{\Phi}(x,\lambda) + \Phi^{\Delta}(x,\lambda)\tilde{\varphi}(x,\lambda)\right). \tag{4.6}$$

In $|\rho| \to \infty$, by substituting (2.13)–(2.14) and (3.10)–(3.11) into (4.3)–(4.6), it follows that:

$$|P_{11}(x,\lambda) - 1| \leq O(\frac{1}{\rho}), \quad |P_{12}(x,\lambda)| \leq O(\frac{1}{\rho}),$$
$$|P_{21}(x,\lambda)| \leq O(\frac{1}{\rho}), \quad |P_{22}(x,\lambda) - 1| \leq O(\frac{1}{\rho}).$$

Theorem 4.1. *If $M \equiv \tilde{M}$, then $h = \tilde{h}$ and $h_1 = \tilde{h}_1$ and $S(x,\lambda) \equiv \tilde{S}(x,\lambda)$ for $x \in I_-$.*

Proof. Since the initial conditions for the model system $\tilde{L}$ are chosen to coincide with those of L, we have:

$$\alpha_0 = \tilde{\alpha}_0, \quad \beta_0 = \tilde{\beta}_0, \quad \alpha_0' = \tilde{\alpha}_0', \quad \beta_0' = \tilde{\beta}_0'. \tag{4.7}$$

Given the $\beta_0 = h\alpha_0$ and $\tilde{\beta}_0 = \tilde{h}\tilde{\alpha}_0$, it directly follows that:

$$h = \tilde{h}.$$

From [12], it follows that:

$$\varphi(1,\lambda) \equiv \tilde{\varphi}(1,\lambda), \quad \varphi^{\Delta}(1,\lambda) \equiv \tilde{\varphi}^{\Delta}(1,\lambda).$$

And in view of (3.12), we obtain:

$$h_1 = \tilde{h}_1.$$

From (3.16) and according to $\beta_0 = h\alpha_0$, we have:

$$A_1 = \frac{-\alpha_0}{\alpha_0'\beta_0 - \alpha_0\beta_0'}. \tag{4.8}$$

From (4.7) and (4.8), we get $A_1 = \tilde{A}_1$. Similarly, from $M \equiv \tilde{M}$, (3.17), and (4.7), we obtain $A_2 = \tilde{A}_2$.

From (3.15), we have $\Phi(x,\lambda) = A_1 S(x,\lambda) + A_2\varphi(x,\lambda)$ and $\tilde{\Phi}^{\Delta}(x,\lambda) = \tilde{A}_1\tilde{S}^{\Delta}(x,\lambda) + \tilde{A}_2\tilde{\varphi}^{\Delta}(x,\lambda)$. Since $A_1 = \tilde{A}_1$ and $A_2 = \tilde{A}_2$ and according to (4.3), we have:

$$\begin{aligned}\varphi(x,\lambda)\tilde{\Phi}^{\Delta}(x,\lambda) - \Phi(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda) &= \varphi(x,\lambda)\Big(\tilde{A}_1\tilde{S}^{\Delta}(x,\lambda) + \tilde{A}_2\tilde{\varphi}^{\Delta}(x,\lambda)\Big) - \Big(A_1 S(x,\lambda) + A_2\varphi(x,\lambda)\Big)\tilde{\varphi}^{\Delta}(x,\lambda)\\ &= A_1\Big(\varphi(x,\lambda)\tilde{S}^{\Delta}(x,\lambda) - S(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\Big),\end{aligned}$$

and we obtain:

$$P_{11}(x,\lambda) = \frac{A_1}{\tilde{h}_2}\Big(\varphi(x,\lambda)\tilde{S}^{\Delta}(x,\lambda) - S(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\Big).$$

On the other hand, according to (4.2), we have:

$$\begin{aligned}\tilde{h}_2 &= \tilde{\varphi}(x,\lambda)\tilde{\Phi}^{\Delta}(x,\lambda) - \tilde{\Phi}(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\\ &= \tilde{A}_1\Big(\tilde{\varphi}(x,\lambda)\tilde{S}^{\Delta}(x,\lambda) - \tilde{S}(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\Big).\end{aligned}$$

Thus, we finally obtain

$$P_{11}(x,\lambda) = \frac{A_1}{\tilde{A}_1}\cdot\frac{\Big(\varphi(x,\lambda)\tilde{S}^{\Delta}(x,\lambda) - S(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\Big)}{\Big(\tilde{\varphi}(x,\lambda)\tilde{S}^{\Delta}(x,\lambda) - \tilde{S}(x,\lambda)\tilde{\varphi}^{\Delta}(x,\lambda)\Big)}.$$

Since $A_1 = \tilde{A}_1$ and the numerator and denominator of the second fraction are the Wronskians between the solutions of L and $\tilde{L}$, we can evaluate them at $x = 0$. Since this Wronskian is constant for any λ and independent of x, the numerator and denominator are equal. Finally, we obtain:

$$P_{11}(x,\lambda) \equiv 1.$$

Analogously, for (4.4)–(4.6), it is shown that:

$$P_{12}(x,\lambda) \equiv 0, \qquad P_{21}(x,\lambda) \equiv 0, \qquad P_{22}(x,\lambda) \equiv 1.$$

And these obtained relations show that the transfer matrix is the identity matrix; hence we have:

$$\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda),\ \Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda),\ \varphi^{\Delta}(x,\lambda) \equiv \tilde{\varphi}^{\Delta}(x,\lambda),\ \Phi^{\Delta}(x,\lambda) \equiv \tilde{\Phi}^{\Delta}(x,\lambda).$$

Since $h = \tilde{h}$ and $M \equiv \tilde{M}$, and $A_1 = \tilde{A}_1$, $A_2 = \tilde{A}_2$ and $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$, $\Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda)$, finally from (3.18) we get $S(x,\lambda) \equiv \tilde{S}(x,\lambda)$. ☐

Theorem 4.2. *For $x \in I_-$, assume $M \equiv \tilde{M}$, then $q = \tilde{q}$.*

Proof. To prove the equality $q = \tilde{q}$ on I_-, we consider:

$$I_{\varphi} = \left\{x \in I_- \mid \varphi\big(\sigma(x),\lambda\big) \neq 0\right\}.$$

Because $\varphi(x,\lambda)$ satisfies (1.2), it follows that:

$$\begin{aligned}q(x)\varphi\big(\sigma(x),\lambda\big) &= \lambda R^2(x)\varphi\big(\sigma(x),\lambda\big) + \varphi^{\Delta\Delta}(x,\lambda),\\ \tilde{q}(x)\tilde{\varphi}\big(\sigma(x),\lambda\big) &= \lambda R^2(x)\tilde{\varphi}\big(\sigma(x),\lambda\big) + \tilde{\varphi}^{\Delta\Delta}(x,\lambda).\end{aligned}$$

Thus, we have:

$$q(x)\varphi\big(\sigma(x),\lambda\big) = \tilde{q}(x)\tilde{\varphi}\big(\sigma(x),\lambda\big).$$

Since $\varphi\big(\sigma(x),\lambda\big) \equiv \tilde{\varphi}\big(\sigma(x),\lambda\big)$ and $\varphi\big(\sigma(x),\lambda\big) \neq 0$, it follows that

$$q(x) = \tilde{q}(x).$$

If $x \notin I_\varphi$, since $\varphi(x,\lambda)$ and $\Phi(x,\lambda)$ are linearly independent, and in view of (4.1), we have $\Phi(\sigma(x),\lambda) \neq 0$. Consequently:

$$q(x)\Phi\big(\sigma(x),\lambda\big) = \lambda R^2(x)\Phi\big(\sigma(x),\lambda\big) + \Phi^{\Delta\Delta}(x,\lambda),$$
$$\tilde{q}(x)\tilde{\Phi}\big(\sigma(x),\lambda\big) = \lambda R^2(x)\tilde{\Phi}\big(\sigma(x),\lambda\big) + \tilde{\Phi}^{\Delta\Delta}(x,\lambda).$$

Therefore, we write:

$$q(x)\Phi\big(\sigma(x),\lambda\big) = \tilde{q}(x)\tilde{\Phi}\big(\sigma(x),\lambda\big).$$

Since $\Phi\big(\sigma(x),\lambda\big) \equiv \tilde{\Phi}\big(\sigma(x),\lambda\big)$ and $\Phi\big(\sigma(x),\lambda\big) \neq 0$, it follows that

$$q(x) = \tilde{q}(x).$$

□

5. CONCLUSION

Initially, we considered the Weyl function in the interval I_+. Leveraging this analysis, we subsequently obtained the corresponding Weyl function in the interval I_-. Utilizing the Weyl function defined in I_-, we rigorously established the uniqueness of the related ISLP.

For prospective research directions, it is of significant interest to delve into more intricate settings, such as those involving multiple turning points or diverse BC frameworks. Moreover, an alternative approach could involve assuming that the solution is known prior to the turning point; then, by employing the appropriate jump conditions at that point, one can systematically derive the solution past the turning point. Following this methodological extension, a comprehensive study of the characteristic function and the eigenvalue distribution in this generalized context would likely yield profound theoretical insights and enrich the spectral theory associated with such problems.

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