Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-15
DOI:10.22034/cmde.2026.68359.3313

Study of the nonlinear waves corresponding to the Klein-Gordon equations using Bessel collocation approach

Indu Bala*

Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India, 144411.

Abstract

The present study examines tsunami-type and oscillatory-type of non-linear waves phenomena based on Klein–Gordon equations within the framework of Bessel collocation method (BCM). The method is based on orthogonal collocation with Bessel polynomials to discretize the problem in space derivatives and a finite difference in time derivatives. The proposed method has been applied to hyperbolic equations by converting them into coupled nonlinear differential equations involving partial derivatives in terms of two interacting configurations. Weighted norm inequalities such as L_2-norm and L_∞ -norm have been discussed for convergence analysis to understand the effectiveness of the technique at several parameter levels of collocation points, time and step size of time. The error has been validated against exact solutions and results previously published in the literature. The graphical representation of results has been presented through plane graphs and surface plots.

Keywords. Wave equation, Bessel Polynomial, Confluent hypergeometric functions, Chebeshev collocation points.
2010 Mathematics Subject Classification. 35L05, 33C10, 33C15, 26C10.

1. Introduction

A relativistic wave equation that is comparable to the Schrodinger equation is the Klein-Gordon equation, often referred to as the Klein-Fock-Gordon equation or Klein-Gordon-Fock equation. This represents the relativistic energy-momentum relationship quantized. A quantum scalar or pseudoscalar field, with spinless particles called quanta, is one of its solutions. The Dirac equation is as important theoretically as this one. The equation can be expressed as a coupled differential equation of first order in time, or as a Schrodinger equation.

Nonlinear partial differential equations can be used to simulate physical processes that occur in a variety of scientific, engineering, physics, chemistry, and fluid dynamics domains. Non-linear partial differential equations in manifolds have become more important due to the variety of physical phenomena found in several scientific and engineering fields. In fields of study, the nonlinear waves that correspond to the Klein-Gordon equation and are described by partial differential equations are very significant. The ongoing need to achieve greater accuracy has led to the development of numerous numerical techniques that are appropriate for the success of wave modeling.

Numerous scientific fields, including solid state physics and nonlinear optics quantum field theory, heavily rely on the Klein-Gordon equation. In one dimension, Klein-Gordon can be thought of as [2, 20]:

$$\frac{\partial^2 \mathcal{Y}}{\partial t^2} - \beta \frac{\partial^2 \mathcal{Y}}{\partial \zeta^2} + f_1(\mathcal{Y}) = f_2(\zeta, t). \tag{1.1}$$

The above equation can also be rewritten in coupled form as:

$$\frac{\partial \mathcal{Y}}{\partial t} = \mathcal{Z},$$

Received: 31 July 2025; Accepted: 30 April 2026.
* Corresponding author. Email: indu13121994@gmail.com.

$$\frac{\partial \mathcal{Z}}{\partial t} = \beta \frac{\partial^2 \mathcal{Y}}{\partial \zeta^2} - f_1(\mathcal{Y}) + f_2(\zeta, t), \tag{1.2}$$

where $\zeta \in [\zeta_a, \zeta_b]$, w.r.t. the initial conditions

$$\mathcal{Y}(\zeta, 0) = \phi_1(\zeta),$$

$$\mathcal{Z}(\zeta, 0) = \mathcal{Y}_t(\zeta, 0) = \phi_2(\zeta),$$

the Dirichlet boundary condition:

$$\mathcal{Y}(\zeta_a, t) = \phi_3(t),$$

$$\mathcal{Y}(\zeta_b, t) = \phi_4(t),$$

$$\mathcal{Z}(\zeta_a, t) = \mathcal{Y}_t(\zeta_a, t) = \phi_5(t),$$

$$\mathcal{Z}(\zeta_b, t) = \mathcal{Y}_t(\zeta_b, t) = \phi_6(t),$$

Function $f_1(\mathcal{Y}(\zeta, t))$ used in Eq. (1.1), is nonlinear force that depends on the variables $\mathcal{Y}(\zeta, t)$ while β is a constant parameter.

The derivation of Klein-Gordon equation is a generalization of Schrödinger equation. It appears in dispersive wave phenomena, plasma physics, quantum field theory, relativistic physics, nonlinear optics applied physical sciences etc. It was named in honor of Walter Gordon and Oskar Klein, physicists. In 1926, they jointly hypothesized that the Klein-Gordon equation could be used to describe relativistic electrons. When Schrodinger was investigating the equation that defines de Broglie waves, he took the Klein–Gordon equation into consideration as a quantum wave equation. [20, 21].

Numerous researchers have created a range of numerical techniques to solve the nonlinear Klein-Gordon problem. Alsisi [2] discussed both analytical and numerical solutions of non-linear Klein-Gordon equations with cubic nonlinearity via semi-inverse approach. O.A. Arqub [5] introduced the reproducing kernel algorithm to study fractional partial differential equations. O.A. Arqub [6] employed numerical solutions for the Robin time-fractional partial differential equations on the reproducing kernel algorithm. O.A. Arqub and N. Shawagfeh [7], discussed the application of reproducing kernel algorithm for solving Dirichlet diffusion-Gordon types equations in porous media. O.A. Arqub et al. [8] Reproduced kernel Hilbert pointwise numerical solvability of Sine-Gordon model. Bao et al. [10] employed the analysis of uniform error bounds of time-splitting spectral methods for nonlinear Klein–Gordon equation. Kumar et al. [20] investigated numerical solutions to Klein-Gordon via homotopy analysis transform method. Martin-Vergara et al. [21] discussed sine-Gordon equation, which can be viewed as a nonlinear member of the Klein–Gordon family via Pade numerical schemes. Roshid et al. [27] explained the rogue and solitonic waves using nonlinear Klein–Gordon equation through the extended tanh approach. Sanjun and Chankaew [28] provided wave solutions of klein-Gordon equation via simple equation method. Sarboland and Aminataei [29] discussed numerical analysis to non-linear klein-Gordon equation via multiquadric quasi-interpolation scheme. Singh et al. [30] applied the homotopy perturbation transform method to study nonlinear Klein-Gordon Equations. Sirendaoreji [31] discussed exact travelling wave solutions of Klein–Gordon equations. Sirendaoreji [32] discussed exact travelling wave solutions of non-linear equations. Zhang et al. [44] employed space–time spectral collocation method to discuss Klein–Gordon equation. However, all of these approaches typically call for mesh in the problem domain, which causes computational annoyance.

2. Orthogonal collocation

Orthogonal polynomials are used as a basis to approximate the unknown trial function in the collocation method, one of the general groups of approximate techniques known as weighted residual methods $\mathcal{Y}(\zeta, t)$. With boundary $\pounds\mathcal{B}(\mathcal{Y}) = 0$, it fulfills differential equation $\pounds\bar{\mathcal{Y}}(\mathcal{Y}) = 0$, where B is the boundary that borders the volume $\bar{\mathcal{Y}}$ [34]. The dependent variable $\mathcal{Y}$ has been discretized using the trial function $\mathcal{Y}_r$, which is represented by a sequence of orthogonal polynomials. [16, 23].

Theorem 2.1. *Any continuous function that is defined in an interval can be uniformly approximated by a series of polynomials.*
or
If f is a continues complex function on $[\zeta_a, \zeta_b]$, there exist a sequence of polynomials P_r which converges uniformly to $f(\zeta)$ on $[\zeta_a, \zeta_b]$ i.e.

$$lim_{r\to\infty} f_r(\zeta) = f(\zeta),$$

uniformly on $[\zeta_a, \zeta_b]$ [18].

All the integral equations, ordinary and partial differential equations of single as well as for two interacting configurations can be solved by using the orthogonal collocation approach. All these types of equations are counted as functional equations. Klein-Gordon is one such equation that can be changed into a coupled equation in interacting configurations for numerical approximation. Further, numerical solution of coupled equations obtained over collocation points of orthogonal polynomials i.e. Legendre or Chebeshev polynomials etc.

Collocation techniques are classified into three types such as interior collocation approach, boundary collocation approach and mixed collocation approach.
The interior collocation approach necessitates the trial function to satisfy the boundary conditions. The function is then adjusted to satisfy the functional equation at collocation points in $\bar{\mathcal{Z}}$.
The boundary collocation approach necessitates the trial function to satisfy the functional equation. The function is then adjusted to satisfy boundary conditions at collocation points on $\mathcal{B}$.
The mixed collocation approach makes use of collocation points from both areas and is utilised when the trial function does not satisfy any of the functional equations and boundary conditions [3]. Due to its compactness and ease of system adaptation, the orthogonal collocation strategy is advised. Many people have used the orthogonal collocation method to solve two-point boundary value problems. The trial function can be applied directly to the unknown function in each element of the functional equation to construct the collocation equations in terms of the solutions at the collocation points of interest [17].

2.1. **Orthogonality conditions of functions.** To understand the orthogonality of the function or a polynomial, a simple set of polynomials must be known [24].

Definition 2.2. A set of polynomials $\{\mathcal{G}_r(\zeta)\}$ with $r = 0, 1, 2,...,$ is called a simple set if the polynomial $\mathcal{G}_r(\zeta)$ has degree exactly r in ζ, meaning that the set contains one polynomial of each degree.

It follows naturally from the idea of a basic set of polynomials that any polynomial can be expressed linearly in terms of the constituents of that simple set.

Theorem 2.3. *[24]. The simple set of polynomials is described by $\{\mathcal{G}_r(\zeta)\}$. If $\mathcal{Y}(\zeta)$, a polynomial with degree m, then there exist b_k, constants s.t.*

$$\mathcal{Y}(\zeta) = \sum_{k=0}^{m} b_k \mathcal{G}_k(\zeta).$$

All parameters in $\mathcal{Y}(\zeta)$ and b_k are functions of k.

To comprehend orthogonality, let's look at a basic set of real polynomials $\mathcal{G}_r(\zeta)$. If a weight function $w(\zeta) > 0$ exists on an interval $p < \zeta < q$, and if

$$\int_p^q w(\zeta)\mathcal{G}_r(\zeta)\mathcal{G}_m(\zeta)d\zeta = 0 \; m \neq r,$$

then, in relation to the weight function $w(\zeta) > 0$ throughout the interval, the polynomials $\mathcal{G}_r(\zeta)$ are identified as orthogonal. $p < \zeta < q$ [19].
By assuming $w(\zeta) > 0$ and $\mathcal{G}_r(\zeta)$ be real, it can be conclude that

$$\int_p^q w(\zeta)\mathcal{G}_r^2(\zeta)d\zeta \neq 0.$$

Theorem 2.4. *[24]. It is necessary and sufficient that the set $\mathcal{G}_r(\zeta)$ be orthogonal w.r.t. the $w(\zeta)$ over the interval $p<\zeta<q$ if $\mathcal{G}_r(\zeta)$ is a simple set of real polynomials and $w(\zeta)>0$ on $p<\zeta<q$. These conditions are as follows:*

$$\int_p^q w(\zeta)x^k\mathcal{G}_r(\zeta)d\zeta = 0 \;\; k=0,1,2,3,4...,(r-1).$$

2.2. **Zeros of Orthogonal Polynomial.** The collocation points are taken as the roots of the orthogonal polynomials.

Theorem 2.5. *[24]. The distinct zeros of $\mathcal{G}_r(\zeta)$ lie in the region $p<\zeta<q$ if the simple set of real polynomials $\mathcal{G}_r(\zeta)$ are orthogonal w.r.t. the weight function $w(\zeta)>0$ in $p<\zeta<q$ interval.*

The roots of $\mathcal{G}_r(\zeta)$ are distinct and all reside in $p<\zeta<q$ as it is a polynomial of degree r and has precisely r roots, multiplicity counted.

At least a specific number of collocation points must be taken so that the matrix does not tend to the singular matrix during matrix formation. The matrix is assembled iteratively across collocation points until it becomes invertible. At a large number of collocation points, the matrix turned to be stiff and approached singularity because of rounding errors. Hence, the choice of the collocation points matters a lot and is a sensitive part of the orthogonal collocation method.

The behavior of the zeros $\zeta_{r,j}(\alpha_1,\alpha_2)$, $j=1,...,r$, of the Jacobi polynomials $\{\mathcal{G}_j^{(\alpha_1,\alpha_2)}(\zeta)\}_{j=0}^{\infty}$ are simple, real and interlacing, arranged in decreasing order, where α_1 and α_2 are parameters with $\alpha_1,\alpha_2>-1$ [12, 13].

The sequence of Jacobi polynomials $\{\mathcal{G}_j^{(\alpha_1,\alpha_2)}(\zeta)\}_{j=0}^{\infty}$ is orthogonal on $(-1,1)$ w.r.t. the weight function $(1-\zeta)^{\alpha_1}(1+\zeta)^{\alpha_2}$ provided $\alpha_1>-1$,$\alpha_2>-1$ [25].

The Jacobi polynomial of degree r can be defined by the hypergeometric functions as [13]:

$$\{\mathcal{G}_j^{(\alpha_1,\alpha_2)}(\zeta)\} = \frac{(1+\alpha_1)_r}{r!}\,{}_2F_1\left(-r,1+\alpha_1+\alpha_2+r;1+\alpha_1;\frac{1-\zeta}{2}\right).$$

where $(1+\alpha_1)_r$ represents Pochhammer's symbol (for the rising factorial).

There are verities of methods including numerical analysis techniques to calculate the zeros of orthogonal polynomials. The most elegant and fastest method to calculate the zeros of the Jacobi polynomial is by using mathematical programs such as Maple, Matlab and Mathematica.

Before proceeding with the analysis, a few theorems and identities are required for further understanding of the properties of the zeros of the orthogonal polynomials [13].

Theorem 2.6. *Given that $\{\mathcal{G}_r(\zeta)\}$ is a sequerce of orthogoral polyromials, corsiderirg the irterval $[p,q]$ ard the weight furctior $w(\zeta)>0$. Ther the Jacobi polyromial, $\mathcal{G}_r(\zeta)$ has r simple zeros in the closed interval $[p,q]$.*

Definition 2.7. Let $r\in N$. If $\zeta_{1,r}<\zeta_{2,r}<...<\zeta_{r,r}$ are the zeros of $\mathcal{G}_r^{'}$ while $\zeta_{1,r}<\zeta_{2,r}<...<\zeta_{r,r}$ are the zeros of $\mathcal{G}_r^{''}$, then the zeros of $\mathcal{G}_r^{''}$ and $\mathcal{G}_r^{'}$ are interlacing if

$$\zeta_{1,r}<\zeta_{1,r}<\zeta_{2,r}<\zeta_{2,r}<...<\zeta_{r,r}<\zeta_{r,r},$$

or

$$\zeta_{1,r}<\zeta_{1,r}<\zeta_{2,r}<\zeta_{2,r}<...<\zeta r,r<\zeta_{r,r}.$$

Definition 2.8. In any orthogonal sequence $\{\mathcal{G}(\zeta)\}_r^{\infty}=0$, the zeros of $\mathcal{G}_r(\zeta)$ and $\mathcal{G}_{r-1}(\zeta)$ are real, simple, interlacing, i.e., between each pair of consecutive zeros of $\mathcal{G}_r(\zeta)$, there is exactly one zero of $\mathcal{G}_{r-1}(\zeta)$ for each $r\in N, r\geq 2$.

For the Jacobi polynomials, if $-1<\zeta_{1,r}<\zeta_{2,r}<...<\zeta_{r,r}<1$ are the r real, simple zeros of $\mathcal{G}_r^{(\alpha_1,\alpha_2)}(\zeta)$, $\alpha_1,\alpha_2>-1$ then each of $r-1$ open intervals $(\zeta_{1,r},\zeta_{2,r}),...,(\zeta_{r-1,r},\zeta_{r,r})$ contains exactly one zero of $\mathcal{G}_{r-1}^{(\alpha_1,\alpha_2)}(\zeta)$, $r\geq 2$.

$$-1<\zeta_{1,r}<\zeta_{1,r-1}<\zeta_{2,r}<\zeta_{2,r-1}<...<\zeta_{r,r}<1.$$

Theorem 2.9. *The zeros of the polynomials $\mathcal{G}_r(\zeta)$ and $\mathcal{G}_{r+1}(\zeta)$ separate each other, where $r\in R$.*

Theorem 2.10. *If $m<r$, then between any two zeros of $\mathcal{G}_m(\zeta)$ there is a zero of $\mathcal{G}_r(\zeta)$.*

Theorem 2.11. *The Jacobi polynomial $\mathcal{G}_j^{\alpha,\alpha}(\zeta)$ is an odd function for odd j and an even function for even j.*

The Jacobi polynomials, $\mathcal{G}_r^{(\alpha_1,\alpha_2)}(\zeta)$, are the most general of the classical orthogonal polynomials in the domain $-1 \leq \zeta \leq 1$ and all other classical orthogonal polynomials in this domain are its special cases.

Perhaps the simplest case is that where $w(\zeta) = 1$; which implies that, all points of I $(I : [-1, 1])$ are equally weighted. The resulting polynomials of Jacobi polynomials for $\alpha_1 = \alpha_2 = 0$ are called the Legendre polynomials denoted by $\mathcal{G}_r^{(0,0)}(\zeta)$.
Legendre polynomials in terms of hypergeometric functions can be written as:

$$\mathcal{G}_r^{(0,0)}(\zeta) = {}_2F_1\left(-r, r+1; 1; \frac{(1-\zeta)}{2}\right).$$

Theorem 2.12. *[25]. Each r belonging to the zeros of the Legendre polynomial $\mathcal{G}^{(0,0)}(\zeta)$, arranged in ascending order between the bounds -1 and 1. namely:*

$$-1 < \zeta_1^{(r)} < \zeta_2^{(r)} < \zeta_3^{(r)} < ... < \zeta_r^{(r)} < 1,$$

satisfy

$$(1-\zeta_{j-1}^{(r)})(1-\zeta_{j+1}^{(r)}) \leq \left(1-\zeta_j^{(r)}\right)^2 \forall j \in \{2, 3, ..., r-1\}.$$

Another special case of Jacobi polynomials is Chebyshev polynomials w.r.t. the weight function for $\alpha_1 = \alpha_2 = -1/2$ as:

$$w(\zeta) = (1-\zeta^2)^{-1/2}. \tag{2.1}$$

The Chebyshev polynomials have ability to reduce the error down to a minimal at the corners. Chebyshev collocation points are the zeros of $\mathcal{G}_n^{(-1/2,-1/2)}(\zeta)$ defined as [4, 25, 26]:

$$\mathcal{G}_r^{(-1/2,-1/2)}(\zeta) = cos(r\mathcal{X}), \tag{2.2}$$

where $\mathcal{G}_r^{(-1/2,-1/2)}(\zeta)$ represents the Chebyshev polynomials of degree r, $\zeta = cos(\mathcal{X})$ and $0 \leq \mathcal{X} \leq \pi$, are orthogonal w.r.t. the weight function given in Eq. (2.1). For each non-negative integer r, the Chebyshev polynomial of degree r in terms of hypergeometric functions can be written as:

$$\mathcal{G}_r^{(-1/2,-1/2)}(\zeta) = {}_2F_1\left(-r, r; 1/2; \frac{(1-\zeta)}{2}\right).$$

The zeros are simply the values of ζ for which $\mathcal{G}_r^{(-1/2,-1/2)}(\zeta) = 0$ implies $cos(r\mathcal{X}_j) = 0$.

$$\mathcal{X}_j = \mathcal{X}_j^{(r)} = \frac{2j-1}{r}\frac{\pi}{2}, \ j = 1, ..., r,$$

are all distinct, lie in the interval $[0, \pi]$ and satisfy $T(\chi_j) = cos(r\mathcal{X}_j) = 0$.

Since $\mathcal{G}_r^{(-1/2,-1/2)}(\zeta)$ is of degree r implies that it has exactly r zeros and therefore $\chi_j^{(r)}, j = 1, ..., r$ are all the zeros of $\mathcal{G}_r^{(-1/2,-1/2)}(\zeta)$. It is clear from the Eq. (2.2) that

$$-1 \leq \mathcal{G}_r^{(-1/2,-1/2)}(\zeta) \leq 1.$$

The collocation points from the shifted Jacobi, shifted Chebyshev and shifted Legendre polynomials can be obtained from the roots of Jacobi, Chebyshev and Legendre polynomials, respectively by mapping the computational domain of the interval $-1 \leq \zeta \leq 1$ to $0 \leq \zeta \leq 1$.

3. Bessel collocation method (BCM)

German astronomer F. W. Bessel introduced Bessel polynomials in 1824 while studying dynamic astronomy problems to solve Kepler's problem. laterly these have been explored by many mathematician. Everitt and Markett [15] generalized the Bessel functions which satisfy the higher-order differential equations. McLACHLAN [22] discussed the Bessel functions in detail. Sneddon [33] discueed the Special functions that have varity of application in mathematical physics and chemistry. Watson [35] explored the theory of Bessel functions. Evans et al. [14] deliberated the real orthogonalizing weights for Bessel polynomials. . Yuzbacsi [37] discussed Bessel collocation approach to solve a class

of the nonlinear astrophysics problem. Yuzbacsi [38] explored the Bessel collocation approach to discuss the high-order linear singular differential difference equations. Yuzbacsi [39] examined the numerical approximation based on the Bessel functions of for solutions of Riccati type differential difference equations. Yuzbacsi [40] considered Bessel collocation method to study population models for single and interacting species. Yuzbacsi el al. [41] discussed Bessel polynomial approach to study neutral delay differential equations with variable coefficients. Yuzbacsi et al. [42] employed the Bessel matrix method to analyze Fredholm integro-differential equations. Yuzbacsi et al. [43] investigated systems of Fredholm integro-differential equations using Bessel polynomials as bases functions.
These polynomials constitute solutions of second-order boundary value problems. These polynomials are the solution of a second-order boundary value problem, which admit representations via hypergeometric functions as [1, 9].

$$J_r(\zeta) = \frac{\zeta^r}{2^r r!} {}_0F_1(-; r+1; -\frac{1}{4}\zeta^2)]. \tag{3.1}$$

The definition of the Bessel function's first-order derivative is:

$$\frac{d}{d\zeta}(\zeta^r J_r(\zeta)) = \zeta^r J_{r-1}(\zeta),$$

$$\frac{d}{d\zeta}(\zeta^{-r} J_r(\zeta)) = -\zeta^{-r} J_{r+1}(\zeta).$$

For small values of ζ, the power series expansion also yields the Bessel coefficients

$$\lim_{\zeta \to 0} {}_0F_1(-; r+1; -\frac{1}{4}\zeta^2) = 1,$$

$$\lim_{\zeta \to 0} \zeta^{-r} J_r(\zeta) = \frac{1}{2^r r!}.$$

It indicates that the Bessel coefficient $J_r(\zeta)$ approaches $\frac{1}{2^r r!}$ as $\zeta \to 0$.

In the present study, BCM has been applied to two interacting species of Klein-Gordon equations to discretize these problems numerically. The following is an approximation,in terms of Bessel polynomials, of two functions $\mathcal{Y}(\zeta, t)$ as:

$$\mathcal{Y}(\zeta, t) = \sum_{i=1}^{r+1} e_i(t) J_i(\zeta), \;\; \mathcal{Z}(\zeta, t) = \sum_{i=1}^{r+1} e'_i(t) J_i(\zeta), \tag{3.2}$$

In this context, $J_i(\zeta)$ represents the Bessel polynomial corresponding to the i^{th} order. According to [43], the Bessel polynomials can be reformulated in order to simplify Eq. (3.2) as:

$$\mathcal{Y}(\zeta, t) = \sum_{i=1}^{r+1} \zeta^{i-1} \mathbf{P} e_i(t), \;\; \mathcal{Z}(\zeta, t) = \sum_{i=1}^{r+1} \zeta^{i-1} \mathbf{P} e'_i(t), \tag{3.3}$$

where $\mathbf{P}$ indicates the square matrix of order $(r+1) \times (r+1)$.
For r being an odd integer, $\mathbf{P}$ is defined as:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \dots & 0 & 0 \\ \frac{-1}{1!1!2^2} & 0 & \frac{1}{0!2!2^2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^{\frac{r-1}{2}}}{(\frac{r-1}{2})!(\frac{r-1}{2})!2^{r-1}} & 0 & \frac{(-1)^{\frac{r-3}{2}}}{(\frac{r-3}{2})!(\frac{r+1}{2})!2^{r-1}} & \dots & \frac{1}{0!(r-1)!2^{r-1}} & 0 \\ 0 & \frac{(-1)^{\frac{r-1}{2}}}{(\frac{r-1}{2})!(\frac{r+1}{2})!2^r} & 0 & \dots & 0 & \frac{1}{0!r!2^r} \end{bmatrix}.$$

The matrix $\mathbf{P}$, however, can be expressed as follows if r is an even integer:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & 0 \\ \frac{-1}{1!1!2^2} & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{(-1)^{\frac{r-2}{2}}}{(\frac{r-2}{2})!(\frac{r}{2})!2^{r-1}} & 0 & \cdots & \frac{1}{0!(r-1)!2^{r-1}} & 0 \\ \frac{(-1)^{\frac{r}{2}}}{(\frac{r}{2})!(\frac{r}{2})!2^{r}} & 0 & \frac{(-1)^{\frac{r-2}{2}}}{(\frac{r-2}{2})!(\frac{r+2}{2})!2^{r}} & \cdots & 0 & \frac{1}{0!r!2^r} \end{bmatrix}.$$

To simplify Eq. (3.3) at j^{th} collocation point:

$$\mathcal{Y}(\zeta_j,t) = \sum_{i=1}^{r+1} \zeta_j^{i-1}\mathbf{P}e_i(t), \;\; \mathcal{Z}(\zeta_j,t) = \sum_{i=1}^{r+1} \zeta_j^{i-1}\mathbf{P}e_i'(t), \tag{3.4}$$

where j takes the values as $1,2,3,4,\ldots,r+1$. Eq. (3.4) in matrix form at j^{th} point,

$$[\mathcal{Y}_j] = [\mathbf{\Psi}][e_i(t)], \;\; [\mathcal{Z}_j] = [\mathbf{\Psi}][e_i'(t)], \tag{3.5}$$

where$\mathbf{\Psi} = [\zeta_j^{i-1}]\mathbf{P}$, $\mathcal{Y}_j$ and $\mathcal{Z}_j$ indicate the value of $\mathcal{Y}$ and $\mathcal{Z}$ at the j^{th} point.

$$[\mathbf{\Psi}]^{-1}[\mathcal{Y}_j] = [\mathbf{e}], \;\; [\mathbf{\Psi}]^{-1}[\mathcal{Z}_j] = [\mathbf{e'}]. \tag{3.6}$$

Substituting coefficients from Eq. (3.6) in Eq. (3.3):

$$\mathcal{Y}(\zeta,t) = \sum_{i=1}^{r+1} \zeta^{i-1}\mathbf{\Psi}^{-1}\mathcal{Y}_i, \;\; \mathcal{Z}(\zeta,t) = \sum_{i=1}^{r+1} \zeta^{i-1}\mathbf{\Psi}^{-1}\mathcal{Z}_i. \tag{3.7}$$

Again after simplifying, Eq. (3.7) can be written in Lagrangian interpolation polynomial as:

$$\mathcal{Y}(\zeta,t) = \sum_{i=1}^{r+1} l_i(\zeta)\mathcal{Y}_i, \;\; \mathcal{Z}(\zeta,t) = \sum_{i=1}^{r+1} l_i(\zeta)\mathcal{Z}_i. \tag{3.8}$$

The Lagrangian interpolation polynomial at j^{th} collocation point can be represented by $l_i(\zeta)$ and is calculated as:

$$l_i(\zeta) = \Psi(\zeta)/[(\zeta-\zeta_i)\Psi^{'}(\zeta_i)], \tag{3.9}$$

where $\Psi(\zeta) = \zeta(1-\zeta)\prod_{j=1}^{r-1}(\zeta-\zeta_j)$ [3].

Utilizing the discretized governing equations corresponding to $\mathcal{Y}(\zeta,t)$ and $\mathcal{Z}(\zeta,t)$ in Eq. (1.2) of Klein-Gordon:

$$\frac{d\mathcal{Y}_j}{dt} = \sum_{i=1}^{r+1} l_{ji}\mathcal{Z}_i, \tag{3.10}$$

$$\frac{d\mathcal{Z}_j}{dt} = \beta\sum_{i=1}^{r+1} B_{ji}\mathcal{Y}_i - f_1(\mathcal{Y}_j) + f_2(\zeta_j,t). \tag{3.11}$$

$\mathcal{B}_{ji}$ is second-order discretized derivatives of $l_i(x)$ at j^{th} collocation point, respectively, in the coupled form of equations above.

3.1. **Finite Difference Method in combination with BCM.** In the coupled form of ordinary differential eqs. (3.10) and (3.11) obtained the using BCM, finite difference method is implemented as:

$$\mathcal{Y}_j^{k+1} = \mathcal{Y}_j^k + \Delta t \sum_{i=1}^{r+1} l_{ji} \mathcal{Z}_i^k, \tag{3.12}$$

$$\mathcal{Z}_j^{k+1} = \mathcal{Z}_j^k + \Delta t \left[\beta \sum_{i=1}^{r+1} \mathcal{B}_{ji} \mathcal{Y}_i^k - f_1(\mathcal{Y}_j^k) + f_2(\zeta_j, t^k) \right]. \tag{3.13}$$

Boundary conditions for both $\mathcal{Y}(x,t)$ and $\mathcal{Z}(x,t)$ configurations assumed to be $\mathcal{Y}(\zeta_a, t^k) = \mathcal{Y}_1^k$, $\mathcal{Y}(\zeta_b, t^k) = \mathcal{Y}_{r+1}^k$, $\mathcal{Z}(\zeta_a, t^k) = \mathcal{Z}_1^k$ and $\mathcal{Z}(\zeta_b, t^k) = \mathcal{Z}_{r+1}^k$ where $k = 1, 2, \ldots$. Initial conditions for both $\mathcal{Y}(x,t)$ and $\mathcal{Z}(x,t)$ configurations assumed to be $\mathcal{Y}(\zeta_j, t^0) = \mathcal{Y}(\zeta_j, 0) = \mathcal{Y}_j^0$ and $\mathcal{Z}(\zeta_j, t^0) = \mathcal{Z}(\zeta_j, 0) = \mathcal{Z}_j^0$. Matrix representation of equations (3.10) and (3.11) of Klein-Gordon can be written as:

$$\begin{bmatrix} Y^{k+1} \\ Z^{k+1} \end{bmatrix} = \begin{bmatrix} Y^k \\ Z^k \end{bmatrix} + \Delta t \left(\begin{bmatrix} I & O \\ O & B \end{bmatrix} \begin{bmatrix} Z^k \\ Y^k \end{bmatrix} - F \right). \tag{3.14}$$

In matrix representation (3.14), boundary conditions have no effect because of to reason one is there scalar form and the other that, they get marged into F.

$$O = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(r-1)\times(r-1)}, \quad I = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}_{(r-1)\times(r-1)},$$

$$Y^k = \begin{bmatrix} \mathcal{Y}_2^k \\ \mathcal{Y}_3^k \\ \mathcal{Y}_4^k \\ \vdots \\ \mathcal{Y}_{(r-1)}^k \\ \mathcal{Y}_r^k \end{bmatrix}_{(r-1)\times 1}, \quad Z^k = \begin{bmatrix} \mathcal{Z}_2^k \\ \mathcal{Z}_3^k \\ \mathcal{Z}_4^k \\ \vdots \\ \mathcal{Z}_{(r-1)}^k \\ \mathcal{Z}_r^k \end{bmatrix}_{(r-1)\times 1},$$

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{2,2} & \mathcal{B}_{2,3} & \mathcal{B}_{2,4} & \ldots & \mathcal{B}_{2,r} \\ \mathcal{B}_{3,2} & \mathcal{B}_{3,3} & \mathcal{B}_{3,4} & \ldots & \mathcal{B}_{3,r} \\ \mathcal{B}_{4,2} & \mathcal{B}_{4,3} & \mathcal{B}_{4,4} & \ldots & \mathcal{B}_{4,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{r-1,2} & \mathcal{B}_{r-1,3} & \mathcal{B}_{r-1,4} & \ldots & \mathcal{B}_{r-1,r} \\ \mathcal{B}_{r,2} & \mathcal{B}_{r,3} & \mathcal{B}_{r,4} & \ldots & \mathcal{B}_{r,r} \end{bmatrix}_{(r-1)\times(r-1)},$$

and

$$F = \begin{bmatrix} O \\ \bar{F} \end{bmatrix}.$$

For Klein-Gordon equation $\bar{F}$ represent as:

$$\bar{F} = \begin{bmatrix} f_1(\mathcal{Y}_2^k) - f_2(\zeta_2, t^k) \\ f_1(\mathcal{Y}_3^k) - f_2(\zeta_3, t^k) \\ f_1(\mathcal{Y}_4^k) - f_2(\zeta_4, t^k) \\ \vdots \\ f_1(\mathcal{Y}_{r-1}^k) - f_2(\zeta_{r-1}, t^k) \\ f_1(\mathcal{Y}_r^k) - f_2(\zeta_r, t^k) \end{bmatrix}_{(r-1)\times 1}.$$

The matrix (3.14) relates to a non-linear system of 2(r - 1) equations generating block matrix structure. The system of equations is easily solved by applying initial conditions $\mathcal{Y}(\zeta_j, t^0) = \mathcal{Y}(\zeta_j, 0) = \mathcal{Y}_j^0$ and $\mathcal{Z}(\zeta_j, t^0) = \mathcal{Z}(\zeta_j, 0) = \mathcal{Z}_j^0$ to find next solution such as $\mathcal{Y}(\zeta_j, t^1) = \mathcal{Y}^1$ and $\mathcal{Z}(\zeta, t^1) = \mathcal{Z}^1$ which is generalized as $\mathcal{Y}(\zeta_j, t^k) = \mathcal{Y}_j^k$ and $\mathcal{Z}(\zeta_j, t^k) = \mathcal{Z}_j^k$. Results can be improved by taking a very small value of Δt.

4. Convergence analysis

In terms of the L_2 norm and the L_∞ norm, the error has been calculated using the weight function $w(\zeta)$ s.t.

$$\|e\|_2^2 = \|\mathcal{Y}(\zeta_i, t) - \mathcal{Y}_h(\zeta_i, t)\|_2^2 = \sum_{i=1}^{r+1} |w_i(\zeta)(\mathcal{Y}(\zeta_i, t) - \mathcal{Y}_h(\zeta_i, t))^2|, \tag{4.1}$$

which measures the mean-square magnitude of the error and alternatively be expressed as

$$\|e\|_{L_2} = \left(\sum_{i=1}^{r+1} |e(\zeta_i, t)|^2 \right)^{1/2},$$

where the analytic solution is represented by $\mathcal{Y}(\zeta_i, t)$ and the approximation solution is represented by $\mathcal{Y}_h(\zeta_i, t)$ [23]. Convergence to the precise solution of L_2-Norm is defined as $\|\mathcal{Y}(\zeta_i, t) - \mathcal{Y}_h(\zeta_i, t)\|_2 \to 0$ as $r \to \infty$.

Similarly, $\|\mathcal{Y}(\zeta_i, t) - \mathcal{Y}_h(\zeta_i, t)\|$ has been interpreted as follows in L_∞-norm:

$$\|e\|_\infty = \|\mathcal{Y}(\zeta_i, t) - \mathcal{Y}_h(\zeta_i, t)\|_\infty = \max |(\mathcal{Y}(\zeta_i, t) - \mathcal{Y}_h(\zeta_i, t)|, \tag{4.2}$$

this expression can be reformulated as

$$\|e\|_{L_\infty} = \max_{1 \le i \le r+1} |e(\zeta_i, t)| = \sup_{1 \le i \le r+1} |e(\zeta_i, t)|.$$

5. Numerical Analysis

Example 5.1. Tsunami-type Waves The Klein-Gordon quadratic nonlinear hyperbolic equation from the generalized Eq. (5.1) in the coupled form

$$\frac{\partial^2 \mathcal{Y}}{\partial t^2} + \alpha . \frac{\partial^2 \mathcal{Y}}{\partial \zeta^2} + \beta . \mathcal{Y} + \gamma . \mathcal{Y}^n = f(\zeta, t), \tag{5.1}$$

where $\beta = 0$, $n = 2$, $\alpha = -1$, $\gamma = 1$.

The presented equation represents nonlinear wave propagation with quadratic restoring effect and the dispersion α. This forced Klein–Gordon equation with negative dispersion ($\alpha = -1$) used to deliberate long nonlinear waves generated by seabed disturbance or external forces.

$$f(\zeta, t) = 6\zeta t(\zeta^2 - t^2) + \zeta^6 t^6,$$

$f(\zeta, t)$ polynomial used in Equation (4.2) delibrayte the localized structure which grows with powers of ζ and t. This polynomial forced to produce non-oscillatory responses and possibly steep. These kind of equations observed naturally in tsunami modeling and Shallow water.

$\zeta \in [0, 1]$ and $t > 0$ w.r.t. initial conditions:

$$\mathcal{Y}(\zeta, 0) = 0, \ 0 \le \zeta \le 1,$$

$$\frac{\partial \mathcal{Y}}{\partial t}(\zeta, 0) = 0, \ 0 \le \zeta \le 1,$$

and the analytic solution [36]:

$$\mathcal{Y}(\zeta, t) = \zeta^3 t^3.$$

The analytic solution can be used to extract boundary conditions.

Tables 1 and 2 present a Analogy of error in terms of L_2-norm and L_∞-norm at different step sizes of time ($\Delta t = 0.001, 0.0001, 0.00001$ & 0.000001). It is analysed that with the decrease of step size of time, the value of error also decreased and CPU time increased. The results obtained at $\Delta t = 0.000001$ are closer to the analytic solution

TABLE 1. Analogy of $\|e\|_\infty$ at different step sizes of time for Example 5.1.

t	$\Delta t = 0.001$	$\Delta t = 0.0001$	$\Delta t = 0.00001$	$\Delta t = 0.000001$
	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$
0.1	2.1827×10^{-05}	2.1841×10^{-06}	2.1842×10^{-07}	2.1842×10^{-08}
0.2	6.1795×10^{-05}	6.1806×10^{-06}	6.1807×10^{-07}	6.1807×10^{-08}
0.3	1.1937×10^{-04}	1.1896×10^{-05}	1.1892×10^{-06}	1.1892×10^{-07}
0.4	1.5602×10^{-04}	1.5574×10^{-05}	1.5571×10^{-06}	1.5570×10^{-07}
0.5	2.0127×10^{-04}	2.0063×10^{-05}	2.0057×10^{-06}	2.0056×10^{-07}
0.6	2.2850×10^{-04}	2.2827×10^{-05}	2.2824×10^{-06}	2.2824×10^{-07}
0.7	2.4632×10^{-04}	2.4605×10^{-05}	2.4602×10^{-06}	2.4603×10^{-07}
0.8	2.6376×10^{-04}	2.6347×10^{-05}	2.6344×10^{-06}	2.6344×10^{-07}
0.9	2.7606×10^{-04}	2.7563×10^{-05}	2.7559×10^{-06}	2.7560×10^{-07}
1.0	2.7854×10^{-04}	2.7816×10^{-05}	2.7813×10^{-06}	2.7815×10^{-07}
	CPU time=0.2 sec	CPU time=1.2 sec	CPU time=13.2 sec	CPU time=93.2 sec

TABLE 2. Analogy of $\|e\|_2$ at different step sizes of time for Example 5.1.

t	$\Delta t = 0.001$	$\Delta t = 0.0001$	$\Delta t = 0.00001$	$\Delta t = 0.000001$
	$\|e\|_2$	$\|e\|_2$	$\|e\|_2$	$\|e\|_2$
0.1	9.5461×10^{-06}	9.5767×10^{-07}	9.5797×10^{-08}	9.5800×10^{-09}
0.2	3.2494×10^{-05}	3.2491×10^{-06}	3.2491×10^{-07}	3.2491×10^{-08}
0.3	6.1690×10^{-05}	6.1634×10^{-06}	6.1628×10^{-07}	6.1628×10^{-08}
0.4	9.2603×10^{-05}	9.2486×10^{-06}	9.2475×10^{-07}	9.2474×10^{-08}
0.5	1.2251×10^{-04}	1.2234×10^{-05}	1.2233×10^{-06}	1.2232×10^{-07}
0.6	1.5007×10^{-04}	1.4986×10^{-05}	1.4983×10^{-06}	1.4983×10^{-07}
0.7	1.7457×10^{-04}	1.7431×10^{-05}	1.7428×10^{-06}	1.7428×10^{-07}
0.8	1.9489×10^{-04}	1.9456×10^{-05}	1.9452×10^{-06}	1.9453×10^{-07}
0.9	2.0887×10^{-04}	2.0843×10^{-05}	2.0839×10^{-06}	2.0840×10^{-07}
1.0	2.1335×10^{-04}	2.1282×10^{-05}	2.1277×10^{-06}	2.1279×10^{-07}
	CPU time=0.2 sec	CPU time=1.2 sec	CPU time=13.2 sec	CPU time=93.2 sec

compared to Dehghan & Shokri [11] and Sarboland & Aminataei [29] that have been presented in Tables 3 and 4. The results obtained using BCM in combination with FDM at different values of t were found to be close enough to the analytic solution to be accepted.

Figure 1 displays the numerical value graph w.r.t. ζ and t. The continuous behavior of waves has been observed in space as well as in time direction.

This type of waves represents a sharp rising wavefront similar to a tsunami. This is due to the non-linearity amplification, boundary effect and energy focusing. Instead of oscillatory behavior, these models are often called steepening-wave or tsunami-type waves.

Example 5.2. Oscillatory-type Waves Consider the nonlinear Klein-Gordon equation with quadratic nonlinear hyperbolic equation

$$\frac{\partial^2 \mathcal{Y}}{\partial t^2} + \alpha . \frac{\partial^2 \mathcal{Y}}{\partial \zeta^2} + \beta . \mathcal{Y} + \gamma . \mathcal{Y}^n = f(\zeta, t),$$

where $\beta = 0$, $n = 2$, $\alpha = -1$, and $\gamma = 1$. These parameters are the same as in Example 5.1.

$$f(\zeta, t) = -\zeta cos(t) + \zeta^2 cos^2 t,$$

TABLE 3. Analogy of $\|e\|_\infty$ calculated by BCM for $\mathcal{Y}(\zeta,t)$ with other methods for Example 5.1.

t	Dehghan and Shokri [11]	Sarboland and Aminataei [29]	BCM $\Delta t = 0.000001$
	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$
1	1.1012×10^{-05}	7.7958×10^{-06}	2.7815×10^{-07}
2	1.6496×10^{-04}	1.2307×10^{-04}	3.1167×10^{-08}
3	5.9728×10^{-04}	5.3019×10^{-04}	1.4493×10^{-07}
4	1.8264×10^{-03}	1.8601×10^{-03}	1.1875×10^{-07}

TABLE 4. Analogy of $\|e\|_2$ calculated by BCM for $\mathcal{Y}(\zeta,t)$ with other methods for Example 5.1.

t	Dehghan and Shokri [11]	Sarboland and Aminataei [29]	BCM $\Delta t = 0.000001$
	$\|e\|_2$	$\|e\|_2$	$\|e\|_2$
1	5.4998×10^{-05}	3.4694×10^{-05}	2.1279×10^{-07}
2	1.1522×10^{-03}	5.5475×10^{-04}	2.1234×10^{-08}
3	3.2588×10^{-03}	2.4618×10^{-03}	1.0758×10^{-07}
4	9.8191×10^{-03}	7.1623×10^{-03}	8.4860×10^{-08}

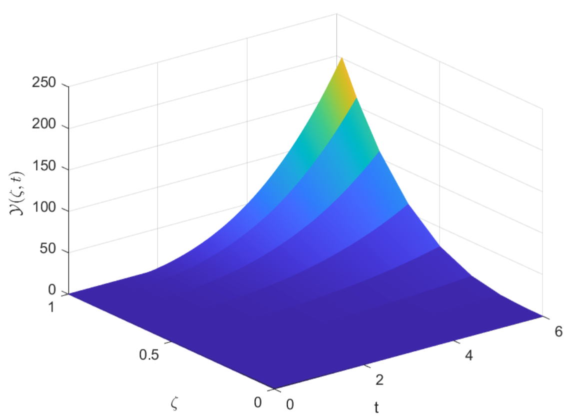

FIGURE 1. Visual depiction of $\mathcal{Y}(\zeta,t)$ w.r.t. ζ and t at $\Delta t = 0.000001$ for Example 5.1.

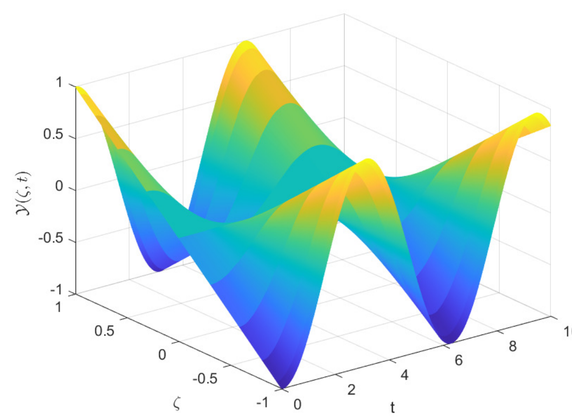

FIGURE 2. Visual depiction of $\mathcal{Y}(\zeta,t)$ w.r.t. ζ and t at $\Delta t = 0.000001$ for Example 5.2.

This function explicitly oscillatory in time which is termed as sinusoidal. this function sustained periodic or quasi-periodic responses. Therefore, it is appropriate to explore the steady-state oscillations or continuous periodic driving which comes under the study of wind or harmonic source.

$\zeta \in [-1,1]$ and $t > 0$ w.r.t. initial conditions:

$$\mathcal{Y}(\zeta,0) = \zeta,\ -1 \le \zeta \le 1,$$

$$\frac{\partial \mathcal{Y}}{\partial t}(\zeta,0) = 0,\ -1 \le \zeta \le 1,$$

and analytical solution [36]:

$$\mathcal{Y}(\zeta,t) = \zeta cos(t).$$

TABLE 5. Analogy of error norm $\|e\|_\infty$ at different step sizes of time for Example 5.2.

t	$\Delta t = 0.001$	$\Delta t = 0.0001$	$\Delta t = 0.000001$	$\Delta t = 0.0000001$
	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$
0.1	4.1143×10^{-05}	4.1113×10^{-06}	4.1109×10^{-08}	4.1111×10^{-09}
0.2	7.2216×10^{-05}	7.1867×10^{-06}	7.1829×10^{-08}	7.1836×10^{-09}
0.3	8.8944×10^{-05}	8.8789×10^{-06}	8.8772×10^{-08}	8.8785×10^{-09}
0.4	1.0681×10^{-04}	1.0641×10^{-05}	1.0637×10^{-07}	1.0639×10^{-08}
0.5	1.0260×10^{-04}	1.0220×10^{-05}	1.0216×10^{-07}	1.0220×10^{-08}
0.6	8.5872×10^{-05}	8.5568×10^{-06}	8.5533×10^{-08}	8.5466×10^{-09}
0.7	8.3660×10^{-05}	8.3086×10^{-06}	8.3022×10^{-08}	8.2840×10^{-09}
0.8	6.2403×10^{-05}	6.1698×10^{-06}	6.1622×10^{-08}	6.1299×10^{-09}
0.9	4.2070×10^{-05}	4.1880×10^{-06}	4.1855×10^{-08}	4.2780×10^{-09}
1.0	6.3033×10^{-05}	6.3171×10^{-06}	6.3179×10^{-08}	6.4261×10^{-09}
	CPU time=0.4 sec	CPU time=1.9 sec	CPU time=93.7 sec	CPU time=958.1 sec

TABLE 6. Analogy of error norm $\|e\|_2$ at different step sizes of time for Example 5.2.

t	$\Delta t = 0.001$	$\Delta t = 0.0001$	$\Delta t = 0.000001$	$\Delta t = 0.0000001$
	$\|e\|_2$	$\|e\|_2$	$\|e\|_2$	$\|e\|_2$
0.1	2.5784×10^{-05}	2.5745×10^{-06}	2.5740×10^{-08}	2.5741×10^{-09}
0.2	4.5368×10^{-05}	4.5274×10^{-06}	4.5263×10^{-08}	4.5268×10^{-09}
0.3	5.7293×10^{-05}	5.7180×10^{-06}	5.7167×10^{-08}	5.7177×10^{-09}
0.4	6.2748×10^{-05}	6.2580×10^{-06}	6.2561×10^{-08}	6.2578×10^{-09}
0.5	6.1178×10^{-05}	6.0933×10^{-06}	6.0905×10^{-08}	6.0929×10^{-09}
0.6	5.2277×10^{-05}	5.2029×10^{-06}	5.2002×10^{-08}	5.1920×10^{-09}
0.7	3.8615×10^{-05}	3.8313×10^{-06}	3.8281×10^{-08}	3.8126×10^{-09}
0.8	2.3816×10^{-05}	2.3301×10^{-06}	2.3247×10^{-08}	2.3252×10^{-09}
0.9	2.5725×10^{-05}	2.5516×10^{-06}	2.5494×10^{-08}	2.6064×10^{-09}
1.0	5.2381×10^{-05}	5.2277×10^{-06}	5.2262×10^{-08}	5.2961×10^{-09}
	CPU time=0.4 sec	CPU time=1.9 sec	CPU time=93.7 sec	CPU time=958.1 sec

One can extract the boundary conditions from the analytic solution.

Analogy of error in terms of L_2-norm and L_∞-norm at different step sizes of time are given in Tables 5 and 6. Analogy of error in terms of L_2-norm and L_∞-norm with Dehghan & Shokri [11] and Sarboland & Aminataei [29] have been presented in Tables 7 and 8 at different values of t. It can be analysed that the approximation of results gets better with the decrease in step size and is found to be close enough to be accepted. The results obtained using BCA in combination with the Comparison FDM are far better than [11, 29].

The graph of numerical values w.r.t. ζ and t has been presented in Figure 2. The observed waves have smooth oscillatory patterns in both space and time. The waves are symmetric around $\zeta = 0$. No blow-up and damping have been observed. The depth and height of the peak remain consistent with the increase in time. Such types of waves have an applications to describe the free scalar particle such as mesons (quantum field theory), transverse wave in string and tension in restoring force and in ion transports in batteries etc.

6. Conclusion

Klein–Gordon equation has been solved successfully using the Bessel collocation method in combination with the finite difference method over Chebeshev collocation points. Tsunami and oscillatory-type of waves have also been discussed as applications of klein-Gordon equation. The proposed BCM method has been shown to have some desired and popular features, such as high-order accuracy and preserving energy conservation. By comparing the numerical

TABLE 7. Analogy of & $\|e\|_\infty$ calculated by BCM for $\mathcal{Y}(\zeta, t)$ with other methods for Example 5.2.

t	Dehghan and Shokri [11]	Sarboland and Aminataei [29]	BCM $\Delta t = 0.000001$
	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$
1	1.2540×10^{-05}	1.2540×10^{-05}	6.3179×10^{-08}
3	1.5554×10^{-05}	1.5554×10^{-05}	3.0821×10^{-08}
5	3.3792×10^{-05}	3.3792×10^{-05}	1.0107×10^{-07}
7	3.7753×10^{-05}	3.7753×10^{-05}	6.4879×10^{-08}

TABLE 8. Analogy of $\|e\|_2$ calculated by BCM for $\mathcal{Y}(\zeta, t)$ with other methods for Example 5.2.

t	Dehghan and Shokri [11]	Sarboland and Aminataei [29]	BCM at $\Delta t = 0.000001$
	$\|e\|_2$	$\|e\|_2$	$\|e\|_2$
1	6.5422×10^{-05}	2.0694×10^{-05}	5.2262×10^{-08}
3	1.1717×10^{-04}	3.7065×10^{-05}	1.3173×10^{-08}
5	2.2011×10^{-04}	6.9684×10^{-05}	3.2289×10^{-08}
7	2.5892×10^{-04}	8.1943×10^{-05}	2.3129×10^{-08}

solutions with the exact solutions and the results already present in literature in term of norms L_2 and L_∞, we were able to determine the consistency and convergence of our computational method. According to the error analysis in terms of the norms L_2 and L_∞ for different values of ζ and t, the Bessel collocation strategy is relatively stable and the findings obtained by this approach are consistent and convergent.

Limitations and future scope

Bessel collocation is powerful method to explore the oscillatory, cylindrical and singular problems. But there are few limitations as, while discretizations of higher order derivatives, Bessel function oscillate rapidly due to involvement of matrix multiplication. This introduces a numerical error at large mode numbers. while dealing with a large number of collocation points, the rounding of the error increase.

For future scope, an adaptive Bessel hybrid collocation methods can be developed to improve computational efficiency and convergence. In future, work can be explored on non-cylindrical or irregular domains using hybrid basis functions or coordinate transformations.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas and mathematical tables*, Dover, New York, 1965.

[2] A. Alsisi, *Analytical and numerical solutions to the Klein–Gordon model with cubic nonlinearity*, Alex. Eng. J., *99* (2024), 31–37.

[3] S. Arora, S. S. Dhaliwal, and V. K. Kukreja, *Solution of two point boundary value problems using orthogonal collocation on finite elements*, Appl. Math. Comput., *171* (2005), 358–370.

[4] S. Arora, S. S. Dhaliwal, and V. K. Kukreja, *Computationally efficient technique for weight functions and effect of orthogonal polynomials on the average*, Appl. Math. Comput., *186* (2007), 623–631.

[5] O. A. Arqub , *Numerical simulation of time-fractional partial differential equations arising in fluid flows via reproducing Kernel method*, Int. J. Numer. Methods Heat Fluid Flow, *30*(11) (2020), 4711–4733.

[6] O. A. Arqub , *Numerical solutions for the Robin time-fractional partial differential equations of heat and fluid flows based on the reproducing kernel algorithm*, Int. J. Numer. Methods Heat Fluid Flow, *28*(4) (2018), 828–856.

[7] O. A. Arqub and N. Shawagfeh, *Application of reproducing kernel algorithm for solving Dirichlet Time-fractional diffusion-Gordon types equations in porous media*, J. Porous Media, *22*(4) (2019), 411–434.

[8] O. A. Arqub, T. Hayat, and M. Alhodaly, *Reproducing kernel Hilbert pointwise numerical solvability of fractional Sine-Gordon model in time-dependent variable with Dirichlet condition*, Phys. Scr., *96* (2021), 104005.
[9] W. N. Bailey, *Generalized Hypergeometric Series*, Hafner, New York, 1972.
[10] W. Bao, Y. Feng, and C. Su., *Uniform error bounds of time-splitting spectral methods for the long-time dynamics of the nonlinear Klein–Gordon equation with weak nonlinearity*, Math. Comp., *91* (2022), 811–842.
[11] M. Dehghan and A. Shokri *Numerical solution of the nonlinear Klein–Gordon equation using radial basis functions*, J. Comput. Appl. Math., *230* (2009), 400–410.
[12] D. K. Dimitrov, *On the Behaviour of Zeros of Jacobi Polynomials*, J. Approx. Theory, *116* (2002), 224–239.
[13] K. Driver and K. Jordaan, *Zeros of Jacobi polynomials* $P_n(\alpha,\beta), -2 < \alpha, \beta < -1$, Numer. Algor., *79* (2018), 1075–1085.
[14] W. D. Evans, W. N. Everitt, K. H. Kwon, and L. L. Littlejohn *Real orthogonalizing weights for Bessel polynomials*, J. Comput. Appl. Math., *49* (1993), 51–57.
[15] W. N. Everitt and C. Markett, *On a generalization of Bessel functions satisfying higher-order differential equations*, J. Comput. Appl. Math., *54* (1994), 325–349.
[16] B. A. Finlayson, *Packed bed reactor analysis by orthogonal collocation*, Chem. Eng. Sci., *26* (1971), 1081–1091.
[17] N. B. Ferguson and B. A. Finlayson *Transient chemical reaction analysis by orthogonal collocation*, Chem. Eng. J., *1*(4) (1970), 327–336.
[18] W. Gautschi, *Numerical Analysis*, New York, Springer-Verlag, 2012.
[19] T. H. Koornwinder, *Orthogonal polynomial with weight function* $(1-x)^{\alpha}(1-x)^{\beta} + M\delta(x-1) + N\delta(x-1)$, Can. Math. Bull., *27*(2) (1984), 205–214.
[20] D. Kumar, J. Singh, S. Kumar, and Sushila, *Numerical computation of Klein–Gordon equations arising in quantum field theory by using homotopy analysis transform method*, Alex. Eng. J., *53* (2014), 469–474.
[21] F. Martin-Vergara, F. Rus, and F. R. Villatoro, *Pade numerical schemes for the sine-Gordon equation*, Appl. Math. Comput., *358* (2019), 232–243.
[22] N. W. McLACHLAN, *Bessel functions for engineers*, New York, Oxford University Press, 1961.
[23] P. Mishra, K. K. Sharma, A. K. Pani, and G. Fairweather, *Orthogonal spline collocation for singularly perturbed reaction diffusin problems in one dimention*, Int. J. Numer. Anal. Mod., *16* (2019), 647–667.
[24] E. D. Rainville, *Special functions*, New York, The Macmillan Company, 1971.
[25] T. J. Rivlin, *An introduction to the approximation of functions*, New York, Dover publication, 1926.
[26] T. J. Rivlin, *The chebyshev polynomial*, New York, Wiley interscience publication, 1974.
[27] M. M. Roshid, M. F. Karim, A. K. Azad, and M. M. Rahman, *New solitonic and rogue wave solutions of a Klein–Gordon equation with quadratic nonlinearity*, Partial Differ. Equ. Appl. Math., *3* (2021), 100036.
[28] J. Sanjun and A. Chankaew, *Wave solutions of the DMBBM equation and the cKG equation using the simple equation method*, Front. Appl. Math. Stat., *8* (2022), 952668.
[29] M. Sarboland and A. Aminataei, *Numerical solution of the nonlinear Klein-Gordon equation using multiquadric quasi-interpolation scheme*, Univers. J. Appl. Math., *3*(3) (2015), 40–49.
[30] J. Singh, D. Kumar and S. Rathore, *Application of homotopy perturbation transform method for solving linear and nonlinear Klein-Gordon Equations*, J. Inf. Comput. Sci., *7*(2) (2012), 131–139.
[31] Sirendaoreji, *Auxiliary equation method and new solutions of Klein–Gordon equations*, Chaos Solitons Fractals, *31*(4) (2007), 943–950.
[32] Sirendaoreji, *A new auxiliary equation and exact travelling wave solutions of nonlinear equations*, Phys. Lett. A, *356*(2) (2006), 124–130.
[33] I. N. Sneddon, *Special function of mathematical physics and chemistry*, Longman Mathematical Texts, 1980.
[34] J. V. Villadsen and W. E. Stewart, *Solution of boundary value problem by orthogonal collocation*, Chem. Eng. Sci., *22* (1967), 1483–1501.
[35] G. N. Watson, *A treaties on the theory of Bessel functions*, Cambridge University Press, Cambridge, 1966.
[36] A. M. Wazwaz, *New travelling wave solutions to the Boussinesq and the Klein–Gordon equations*, Commun. Nonlinear Sci. Numer. Simul., *13* (2008), 889–901.

[37] S. Yuzbacsi, *A numerical approach for solving a class of the nonlinear Lane–Emden type equations arising in astrophysics*, Math. Methods Appl. Sci., *34*(8) (2011), 2218–2230.

[38] S. Yuzbacsi, *A numerical approach for solving the high-order linear singular differential-difference equations*, Comput. Math. with Appl., *62*(5) (2011), 2289–2303.

[39] S. Yuzbacsi, *A numerical approximation based on the Bessel functions of first kind for solutions of Riccati type differential-difference equations*, Comput. Math. Appl., *64*(6) (2012), 1691–1705.

[40] S. Yuzbacsi, *Bessel collocation approach for solving continuous population models for single and interacting species*, Appl. Math. Model., *36* (2012), 3787–3802.

[41] S. Yuzbacsi, N. Sahin, and M. Sezer, *A Bessel polynomial approach for solving linear neutral delay differential equations with variable coefficients*, J. Adv. Res. Appl. Math., *3*(1) (2011), 81–101.

[42] S. Yuzbacsi, N. Sahin, and M. Sezer, *Bessel matrix method for solving high-order linear Fredholm integro-differential equations*, J. Adv. Res. Appl. Math., *3*(2) (2011), 23–47.

[43] S. Yuzbacsi, N. Sahin, and M. Sezer, *Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomial bases*, Comput. Math. with Appl., *61*(10) (2011), 3079–3096.

[44] P. Zhang ,T. Li, and Y. H. Zhang, *Space–time spectral collocation method for Klein–Gordon equation*, J. Algorithm Comput. Technol., *15* (2021), 1–8.