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Ritz-optimization approach for solving various fractional problems

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Abstract

This work presents a composite numerical approach for solving three classes of problems by integrating several computational techniques. The proposed method combines an optimization framework with the Ritz method, the Legendre–Gauss quadrature rule, and fractional Mott functions (FMFs) as basis functions. In addition, derivative matrices of the FMFs are incorporated into the numerical algorithm.
The core idea of the proposed approach begins with the application of the Ritz method while enforcing the given conditions. Subsequently, the Legendre–Gauss quadrature rule, together with the optimization technique, is employed to approximate all functions involved in the problem. In particular, the hidden layer of the optimization framework is constructed using FMFs and their associated derivative matrices. Finally, by applying a classical optimization procedure, the problem is reduced to a system of algebraic equations.
Furthermore, an error analysis of the integrals arising in the formulation is carried out. To demonstrate the effectiveness and applicability of the proposed scheme, several representative problems are solved.

Keywords. Ritz method, Optimization method, Fractional Mott functions, Variable-order fractional derivative, Legendre-Gauss quadrature rule.
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1. Introduction

In recent years, fractional calculus has brought about a profound transformation in the modeling of many phenomena in science and engineering, including dynamic viscoelasticity, fluid mechanics, signal processing, finance, hydrology, diffusion processes, and continuum mechanics, among others [6, 11, 25, 27, 31].

Meanwhile, fractional derivatives have appeared in differential equations that are difficult, or even impossible, to solve using analytical methods. Consequently, numerous numerical techniques have been developed for approximating solutions to fractional partial differential equations and fractional partial integro-differential equations. Existing numerical approaches include nonstandard finite difference and Chebyshev collocation methods [3], parametric cubic spline techniques [35], shifted Legendre polynomials with operational matrices [37], the reproducing kernel Hilbert space method [18], the Genocchi hybrid collocation method [12], and the combination of Euler scaling functions with optimization techniques [16].

Numerical methods are attractive and efficient tools for solving complex equations. Motivated by this, we propose a novel numerical scheme based on the integration of optimization techniques and the Ritz method, in which we use fractional Mott functions. The construction of the operational matrix, together with the Ritz framework, plays a crucial role in obtaining highly accurate approximate solutions. These polynomials have relatively few terms compared to other polynomials, which reduces the CPU time required in the approximation algorithm. Several studies addressing these methodologies can be found in [29, 30, 33, 38].

The primary objective of the proposed method is to develop a highly accurate numerical approach for solving a wide range of problems. To this end, we introduce a new scheme that combines the high-precision operational matrix of fractional Mott functions, the Ritz method, and the Legendre–Gauss quadrature rule. Furthermore, to demonstrate the

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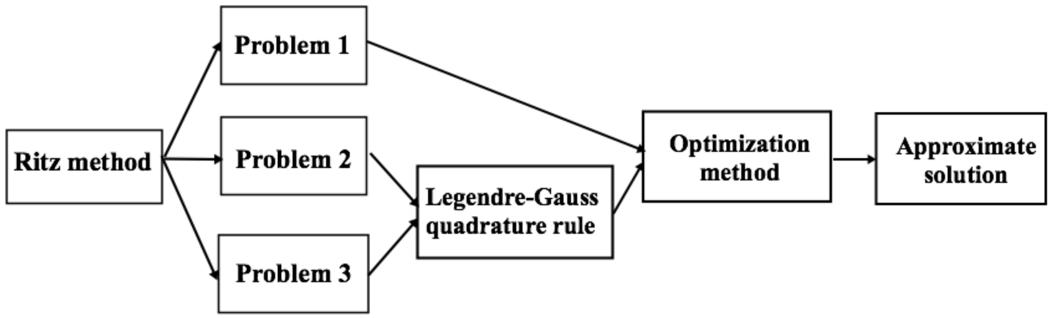

FIGURE 1. Diagram of the approach for finding the approximate solution for three types of problems.

effectiveness and versatility of the proposed approach, we solve variable-order (VO) time-fractional partial differential equations, time-fractional Fredholm partial integro-differential equations, and VO time-fractional Volterra partial integro-differential equations. It is worth noting that variable-order fractional derivatives have significantly enhanced the modeling of many natural phenomena, as their memory effects vary with time and spatial location [21]. Several applications of variable-order fractional derivatives can be found in [7, 10, 20, 26, 34].

Numerical methods, as powerful tools for solving complex problems involving variable-order fractional derivatives in engineering and physics, have attracted considerable attention from researchers in recent years. Therefore, in recent years various approximating methods have been introduced, which we stated some of them:

- Heydari and Razzaghi in [22] applied third-kind Chebyshev cardinal functions for the VO-time fractional RLW-Burgers equation.
- Abdelkawy et al. in [1] provided the shifted fractional Jacobi collocation method for VO-fractional functional differential equations.
- Authors in [2] introduced Vieta-Fibonacci operational matrices for solving VO-fractional integro-differential equations.
- El-Gindy et al. in [17] suggested the combination of the spectral method and shifted fractional Gegenbauer polynomials for VO-fractional integro-differential equations.
- Marasi and Derakhshan in [28] applied Haar wavelet collocation method for variable order fractional integro-differential equations.
- Authors in [19] used the meshless method for solving the three-dimensional VO-time fractional diffusion equation.

Many other related studies can be found in [23, 32] and the references therein. Therefore, the diversity of numerical methods in this field has motivated us to develop an accurate and efficient numerical scheme for solving the proposed problems. The remainder of the manuscript is organized as follows:

- We begin by introducing the fractional Mott functions and constructing the matrices used in the methodology.
- Section 3 presents the numerical scheme for solving the problems mentioned in Eqs. (5.1), (5.2), and (5.3).
- The error analysis of integral that emerged in Problems 2 and 3 is discussed in section 4.
- In Section 5, some numerical examples and comparisons are presented to indicate the efficiency of the method.
- End the paper with the conclusions and remarks.

Also, to demonstrate the whole process of the methodology, we draw a diagram of the method in Figure 1.

1.1. **Problems statement.** The aim of this work is to investigate the approximate solution of three classes of problems:

- VO-time fractional partial differential equations:

$$D_t^{\gamma_1(x,t)}u(x,t) = \Psi\left(x,t,u,u_x,u_{xx}\right), \quad 0 \le x,t \le 1, \quad 0 < \gamma_1(x,t) \le 1. \tag{1.1}$$

- Time fractional Fredholm partial integro-differential equations [4]:

$$D_t^{\gamma_2}u(x,t) + \mu_1 u_x(x,t) + \mu_2 u_{xx}(x,t) + \int_0^1 \mathcal{K}(x,t,\eta)u_{xx}(x,\eta)d\eta = \mathcal{F}\left(x,t,u\right), \tag{1.2}$$

$$0 \le x,t \le 1, \quad 0 < \gamma_2 \le 1. \tag{1.3}$$

- VO-time fractional Volterra partial integro-differential equations:

$$D_t^{\gamma_3(x,t)}u(x,t) + \Theta(x,t)u_x(x,t) + \int_0^t \mathcal{K}(x,t,\eta)\psi(x,\eta,u(x,\eta))d\eta = f\left(x,t\right), \tag{1.4}$$

$$0 \le x,t \le 1, \quad 0 < \gamma_3(x,t) \le 1,$$

subject to the initial and boundary conditions

$$u(x,0) = \mathfrak{F}(x), \quad u(0,t) = \mathfrak{G}_0(t), \quad u(1,t) = \mathfrak{G}_1(t).$$

Here, the unknown function is $u(x,t)$, which is obtained with the help of the numerical process. Also, μ_1 and μ_2 are constant and, $\mathcal{K}(.,.,.)$, $\Theta(.,.)$, $f\left(.,.\right)$, $\mathcal{F}\left(.,.,.\right)$, $\mathfrak{F}(.)$, $\mathfrak{G}_0(.)$, and $\mathfrak{G}_1(.)$ are known functions. In addition, $D_t^{\gamma(x,t)}$ demonstrate the VO-Caputo fractional derivative of order $\gamma(x,t)$ with the respect to variable t, which is defined as follows [12]:

$$D_t^{\gamma(x,t)}f(x,t) = \begin{cases} \frac{1}{\Gamma(n-\gamma(x,t))}\int_0^t (t-\eta)^{n-\gamma(x,t)-1}\frac{\partial^n f(x,\eta)}{\partial \eta^n}d\eta, & n-1 < \gamma(x,t) < n, \\ \frac{\partial^n f(x,\eta)}{\partial \eta^n}, & \gamma(x,t) = n \in \mathbb{N}. \end{cases} \tag{1.5}$$

2. Preliminaries and notation

This section introduces the essential concepts required for the proposed method.

2.1. **Fractional Mott functions.** The fractional Mott functions are defined using the analytical formula of the Mott polynomials [24] together with the simple change of variable $x = \xi^\alpha$ as follows:

$$S_m^\alpha(\xi) = (-1)^m\left(\frac{\xi^\alpha}{2}\right)^m (m-1)!\sum_{k=0}^{h(\frac{m}{2})}\binom{m}{k}(-1)^k\frac{\xi^{-2k\alpha}}{(m-2k-1)!}, \quad m = 0,1,2,\dots, \tag{2.1}$$

where

$$h(\frac{m}{2}) = \begin{cases} \frac{m}{2}, & \text{if M is even,} \\ \frac{m}{2}-\frac{1}{2}, & \text{if M is odd.} \end{cases}$$

Any functions f defined over $[0,1]$ can be expanded by fractional Mott functions as:

$$f(x,t) \simeq \sum_{m=0}^{M_1}\sum_{n=0}^{M_2} f_{mn}S_m^\alpha(x)S_n^\beta(t) = S^{\alpha T}(x)FS^\beta(t). \tag{2.2}$$

Here, $S^\alpha(x)$ and $S^\beta(t)$ are M_1+1 and M_2+1 column vectors, respectively. And also, F is $(M_1+1)\times(M_2+1)$ coefficient matrix, in which the proposed matrix and vector are introduced as follows:

$$S^\alpha(x) = \left[\begin{array}{cccc} S_0^\alpha(x) & S_1^\alpha(x) & \dots & S_{M_1}^\alpha(x) \end{array}\right]^T, \quad S^\beta(t) = \left[\begin{array}{cccc} S_0^\beta(t) & S_1^\beta(t) & \dots & S_{M_2}^\beta(t) \end{array}\right]^T,$$

and

$$F = \left[f_{mn}\right], \quad m = 0,1,\dots,M_1, \quad n = 0,1,\dots,M_2.$$

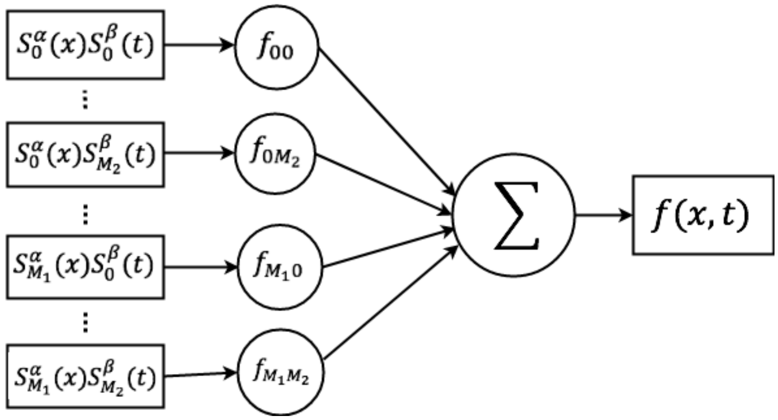

FIGURE 2. Diagram for finding the approximation of functions.

Each component of the coefficient matrix is determined by utilizing regular grid points. Moreover, the diagram of the one layer of the neural network method is demonstrated in Figure 2.

2.2. **Derivative matrix.** In the algorithm of the numerical solution, we need to approximate $(x[x-1]S^\alpha(x))'$ with respect to FMFs. Hence, we consider

$$(x[x-1]S^\alpha(x))' \simeq \mathcal{P}S^\alpha(x), \tag{2.3}$$

where the component of $\mathcal{P}$ is obtained as:

$$\mathcal{P} = [\mathcal{P}_{ij}]_{(M_1+1)\times(M_1+1)}, \quad \mathcal{P}_{ij} = \begin{cases} (i-1)\alpha+2, & j=i+1, \\ -((i-1)\alpha+1), & j=i, \\ 0, & Otherwise, \end{cases} \quad i,j=1,2,\dots,M_1+1.$$

Also can be written the following formula:

$$(x[x-1]S^\alpha(x))'' \simeq \mathcal{Q}S^\alpha(x), \tag{2.4}$$

where

$$\mathcal{Q} = [\mathcal{Q}_{ij}]_{(M_1+1)\times(M_1+1)}, \quad \mathcal{Q}_{ij} = \begin{cases} 2, & j=i=1, \\ ((i-1)\alpha+1)((i-1)\alpha+2), & j=i, \\ -((i-1)\alpha+1)((i-1)\alpha), & j=i-1, \\ 0, & Otherwise, \end{cases} \quad i,j=2,\dots,M_1+1.$$

2.3. **VO-fractional derivative matrix.** In the process of numerical algorithms, we need to approximate the VO-fractional derivative of an unknown function $D_t^{\gamma(x,t)}u(x,t)$. So, we consider

$$D_t^{\gamma(x,t)}\left(tS^\beta(t)\right) \simeq t^{1-\gamma(x,t)}\mathcal{R}_{\gamma(x,t)}S^\beta(t), \quad 0<\gamma(x,t)\le 1, \tag{2.5}$$

where $\mathcal{R}_{\gamma(x,t)}$ indicates the VO-fractional derivative matrix. Hence, to compute the components of this matrix, we use the analytical form of FMFs as follows:

$$\begin{aligned} D_t^{\gamma(x,t)} S_m^\beta(t) &= \left(\frac{-1}{2}\right)^m (m-1)! \sum_{k=0}^{h(\frac{m}{2})} \binom{m}{k} (-1)^k \frac{D_t^{\gamma(x,t)} \left(t^{(m-2k)\beta+1}\right)}{(m-2k-1)!} \\ &= \left(\frac{-1}{2}\right)^m (m-1)! \sum_{k=0}^{h(\frac{m}{2})} \binom{m}{k} (-1)^k \\ &\times \frac{\Gamma((m-2k)\beta+2)}{\Gamma((m-2k)\beta+2-\gamma(x,t))(m-2k-1)!} t^{(m-2k)\beta+1-\gamma(x,t)} \\ &= t^{1-\gamma(x,t)} \sum_{k=0}^{h(\frac{m}{2})} \omega_{m,k}^\beta(x,t) t^{(m-2k)\beta}. \end{aligned} \tag{2.6}$$

In this part of approximating process, we consider

$$t^{(m-2k)\beta} \simeq \sum_{i=0}^{M_2} a_i S_i^\beta(t).$$

Then, the combination of Eq. (2.6) with the above relation leads to the following relation:

$$\begin{aligned} D_t^{\gamma(x,t)} S_m^\beta(t) &\simeq t^{1-\gamma(x,t)} \sum_{i=0}^{M_2} \sum_{k=0}^{h(\frac{m}{2})} \omega_{m,k}^\beta(x,t) a_i S_i^\beta(t) \\ &= t^{1-\gamma(x,t)} \sum_{i=0}^{M_2} r_{m,k,i}^\beta(x,t) S_i^\beta(t), \quad r_{m,k,i}^\beta(x,t) = \sum_{k=0}^{h(\frac{m}{2})} \omega_{m,k}^\beta(x,t) a_i. \end{aligned} \tag{2.7}$$

Therefore, with regard to the above relation, we obtain the following formula:

$$D_t^{\gamma(x,t)} S_m^\beta(t) \simeq t^{1-\gamma(x,t)} \left[r_{m,k,0}^\beta(x,t), r_{m,k,1}^\beta(x,t), \ldots, r_{m,k,M_2}^\beta(x,t) \right] S^\beta(t).$$

It is worth mentioning that, by changing the value of $m = 0, 1, \ldots, M_2$, the rows of the proposed matrix are achieved. By applying the aforementioned process for $\gamma(x,t) = 0.5$ with $M_2 = 2$, we get

$$D_t^{0.5} \left(t S^1(t)\right) \simeq t^{0.5} \mathcal{R}_{0.5} S^1(t),$$

accompany with

$$\mathcal{R}_{0.5} = \begin{bmatrix} 1.128379167095513 & 0 & 0 \\ 0 & 1.504505556127350 & 0 \\ 0 & 0 & 1.805406667352820 \end{bmatrix}.$$

3. Procedure of the method

This section presents the combination of the optimization and the Ritz methods for solving the proposed problems. Hence, to reach the goal, we consider the following formula by using the Ritz method and the problem's conditions:

$$u(x,t) \simeq tx(x-1) S^{\alpha T}(x) A S^\beta(t) + \mathfrak{F}(x) + (1-x)\left[\mathfrak{G}_0(t) - \mathfrak{G}_0(0)\right] + x\left[\mathfrak{G}_1(t) - \mathfrak{G}_1(0)\right]. \tag{3.1}$$

Next, to approximate the other functions that appeared in the problems, we use Eq. (3.1) and the derivative matrix as follows:

$$u_x(x,t) \simeq t S^{\alpha T}(x) \mathcal{P}^T A S^\beta(t) + \mathfrak{F}'(x) - \left[\mathfrak{G}_0(t) - \mathfrak{G}_0(0)\right] + \left[\mathfrak{G}_1(t) - \mathfrak{G}_1(0)\right], \tag{3.2}$$

and

$$u_{xx}(x,t) \simeq t S^{\alpha T}(x) \mathcal{Q}^T A S^\beta(t) + \mathfrak{F}''(x). \tag{3.3}$$

Moreover, from Eq. (3.1) and the fractional derivative matrix, we get

$$D_t^{\gamma(x,t)}u(x,t) \simeq x(x-1)t^{1-\gamma(x,t)}S^{\alpha T}(x)A\mathcal{R}_{\gamma(x,t)}S^{\beta}(t) + (1-x)D_t^{\gamma(x,t)}\mathfrak{G}_0(t) + xD_t^{\gamma(x,t)}\mathfrak{G}_1(t). \tag{3.4}$$

In the next step, we substitute the above approximate functions in the problems.

Problem 3.1. The error relation for this problem is obtained as follows:

$$\begin{aligned} w^{\gamma_1(x,t)}(x,t) &= x(x-1)t^{1-\gamma_1(x,t)}S^{\alpha T}(x)A\mathcal{R}_{\gamma_1(x,t)}S^{\beta}(t) \\ &+ (1-x)D_t^{\gamma_1(x,t)}\mathfrak{G}_0(t) + xD_t^{\gamma_1(x,t)}\mathfrak{G}_1(t) \\ &- \Psi\begin{bmatrix} x,t, \\ tx(x-1)S^{\alpha T}(x)AS^{\beta}(t) + \mathfrak{F}(x) + (1-x)\left[\mathfrak{G}_0(t)-\mathfrak{G}_0(0)\right] + x\left[\mathfrak{G}_1(t)-\mathfrak{G}_1(0)\right], \\ tS^{\alpha T}(x)\mathcal{P}^T AS^{\beta}(t) + \mathfrak{F}'(x) - \left[\mathfrak{G}_0(t)-\mathfrak{G}_0(0)\right] + \left[\mathfrak{G}_1(t)-\mathfrak{G}_1(0)\right], \\ tS^{\alpha T}(x)\mathcal{Q}^T AS^{\beta}(t) + \mathfrak{F}''(x) \end{bmatrix}. \end{aligned} \tag{3.5}$$

Therefore, the error vector at points

$$x_m = \frac{2m-1}{2(M_1+1)}, \quad t_n = \frac{2n-1}{2(M_2+1)}, \quad m = 1,2,\ldots,M_1+1, \quad n = 1,2,\ldots,M_2+1,$$

is achieved as follows:

$$\mathbf{W}^{\gamma_1} = \left[w^{\gamma_1(x_m,t_n)}(x_m,t_n)\right]_{[(M_1+1)(M_2+1)]\times 1}, \quad m = 1,2,\ldots,M_1+1, \quad n = 1,2,\ldots,M_2+1.$$

Then, we define the error function for Problem 1 as follows:

$$E^{\gamma_1} = \frac{1}{2}\mathbf{W}^{\gamma_1}(\mathbf{W}^{\gamma_1})^T. \tag{3.6}$$

As a result, to obtain the components of unknown matrix A, we consider the following optimization problem:

$$\mathbf{\Phi}^{\gamma_1} = \min_A E^{\gamma_1}. \tag{3.7}$$

Then, for finding the minimum value A, we utilized the following necessary conditions:

$$\frac{\partial}{\partial a_{mn}}\mathbf{\Phi}^{\gamma_1} = 0, \quad m = 1,2,\ldots,M_1+1, \quad n = 1,2,\ldots,M_2+1. \tag{3.8}$$

Finally, by replacing the obtained matrix in Eq. (3.1), the approximate solution is achieved.

Problem 3.2. Due to Problem 3.2, Eq. (3.3) and Legendre–Gauss quadrature rule (Canuto et al. 2007), we introduced the following relation:

$$\int_0^1 \mathcal{K}(x,t,\eta)u_{xx}(x,\eta)d\eta \simeq \sum_{s=1}^{\mathfrak{S}} w_s\mathcal{K}(x,t,\sigma_s)\left[\sigma_s S^{\alpha T}(x)\mathcal{Q}^T AS^{\beta}(\sigma_s) + \mathfrak{F}''(x)\right] = \Upsilon^{\alpha,\beta}(x,t,\sigma_s).$$

So, the error relation of problem 2 is calculated as follows:

$$\begin{aligned} w^{\gamma_2}(x,t) &= x(x-1)t^{1-\gamma_2}S^{\alpha T}(x)A\mathcal{R}_{\gamma_2}S^{\beta}(t) + (1-x)D_t^{\gamma_2}\mathfrak{G}_0(t) + xD_t^{\gamma_2}\mathfrak{G}_1(t) \\ &+ \mu_1\left[tS^{\alpha T}(x)\mathcal{P}^T AS^{\beta}(t) + \mathfrak{F}'(x) - \left[\mathfrak{G}_0(t)-\mathfrak{G}_0(0)\right] + \left[\mathfrak{G}_1(t)-\mathfrak{G}_1(0)\right]\right] \\ &+ \mu_2\left[tS^{\alpha T}(x)\mathcal{Q}^T AS^{\beta}(t) + \mathfrak{F}''(x)\right] \\ &+ \sum_{s=1}^{\mathfrak{S}} w_s\mathcal{K}(x,t,\sigma_s)\left[\sigma_s S^{\alpha T}(x)\mathcal{Q}^T AS^{\beta}(\sigma_s) + \mathfrak{F}''(x)\right] \\ &- \mathcal{F}\left(x,t,tx(x-1)S^{\alpha T}(x)AS^{\beta}(t) + \mathfrak{F}(x)\right. \\ &\left.+ (1-x)\left[\mathfrak{G}_0(t)-\mathfrak{G}_0(0)\right] + x\left[\mathfrak{G}_1(t)-\mathfrak{G}_1(0)\right]\right). \end{aligned} \tag{3.9}$$

From Eq. (3.9) and points (x_m, t_n), we get the error vector

$$\mathbf{W}^{\gamma_2} = [w^{\gamma_2}]_{[(M_1+1)(M_2+1)]\times 1}, \quad m = 1, 2, \ldots, M_1 + 1, \quad n = 1, 2, \ldots, M_2 + 1,$$

And also, the following error function is deduced:

$$E^{\gamma_2} = \frac{1}{2}\mathbf{W}^{\gamma_2}(\mathbf{W}^{\gamma_2})^T, \tag{3.10}$$

In view of the above formula, we consider the following optimization problem:

$$\mathbf{\Phi}^{\gamma_2} = \min_A E^{\gamma_2}. \tag{3.11}$$

Hence, the by considering the following conditions the minimum value of A is funded:

$$\frac{\partial}{\partial a_{mn}}\mathbf{\Phi}^{\gamma_2} = 0, \quad m = 1, 2, \ldots, M_1 + 1, \quad n = 1, 2, \ldots, M_2 + 1. \tag{3.12}$$

Therefore, an approximate solution is obtained.

Problem 3.3. Similar to the previous part, we use the Legendre-Gauss quadrature rule for approximating an integral part. Hence, by taking $\eta = t\xi$, we have

$$\begin{aligned}\int_0^t \mathcal{K}(x,t,\eta)\psi(x,\eta,u(x,\eta))d\eta &= \int_0^1 t\mathcal{K}(x,t,t\xi)\psi(x,t\xi,u(x,t\xi))d\xi \\ &\simeq \sum_{r=1}^{\mathfrak{R}} w_r t\mathcal{K}(x,t,t\sigma_r)\psi(x,t\sigma_r,u(x,t\sigma_r)) \\ &\simeq \sum_{r=1}^{\mathfrak{R}} w_r t\mathcal{K}(x,t,t\sigma_r)\times\psi\begin{pmatrix} x, t\sigma_r, \\ t\sigma_r x(x-1)S^{\alpha T}(x)AS^{\beta}(t\sigma_r) + \mathfrak{F}(x) \\ +(1-x)\left[\mathfrak{G}_0(t\sigma_r) - \mathfrak{G}_0(0)\right] + x\left[\mathfrak{G}_1(t\sigma_r) - \mathfrak{G}_1(0)\right]\end{pmatrix} \\ &= \hat{\Upsilon}^{\alpha,\beta}(x,t,\sigma_r).\end{aligned} \tag{3.13}$$

Uncorrected Proof

By substituting the above approximation relations in Problem 3.3, the error relation is concluded:

$$\begin{aligned} w^{\gamma_3(x,t)}(x,t) &= x(x-1)t^{1-\gamma_3(x,t)}S^{\alpha T}(x)A\mathcal{R}_{\gamma_3(x,t)}S^{\beta}(t) \\ &+ (1-x)D_t^{\gamma_3(x,t)}\mathfrak{G}_0(t) + xD_t^{\gamma_3(x,t)}\mathfrak{G}_1(t) \\ &+ \Theta(x,t)\left[tS^{\alpha T}(x)\mathcal{P}^T AS^{\beta}(t) + \mathfrak{F}'(x) - [\mathfrak{G}_0(t) - \mathfrak{G}_0(0)] + [\mathfrak{G}_1(t) - \mathfrak{G}_1(0)]\right] \\ &+ \sum_{r=1}^{\mathfrak{R}} w_r t\mathcal{K}(x,t,t\sigma_r)\times\psi\begin{pmatrix} x, t\sigma_r, \\ t\sigma_r x(x-1)S^{\alpha T}(x)AS^{\beta}(t\sigma_r) + \mathfrak{F}(x) \\ +(1-x)\left[\mathfrak{G}_0(t\sigma_r) - \mathfrak{G}_0(0)\right] + x\left[\mathfrak{G}_1(t\sigma_r) - \mathfrak{G}_1(0)\right]\end{pmatrix} \\ &- f(x,t). \end{aligned} \tag{3.14}$$

Therefore, from Eq. (3.14) and points (x_m, t_n), can be deduced the following vector:

$$\mathbf{W}^{\gamma_3} = \left[w^{\gamma_3(x_m,t_n)}\right]_{[(M_1+1)(M_2+1)]\times 1}, \quad m = 1, 2, \ldots, M_1 + 1, \quad n = 1, 2, \ldots, M_2 + 1.$$

In view of the vector mentioned above, we consider the following error function:

$$E^{\gamma_3} = \frac{1}{2}\mathbf{W}^{\gamma_3}(\mathbf{W}^{\gamma_3})^T, \tag{3.15}$$

where the optimization problem corresponding to Eq. (3.15) is defined as follows:

$$\mathbf{\Phi}^{\gamma_3} = \min_A E^{\gamma_3}. \tag{3.16}$$

Now, to get the elements of the unknown vector A, we consider the following conditions:

$$\frac{\partial}{\partial a_{mn}}\mathbf{\Phi}^{\gamma_3} = 0, \quad m = 1, 2, \ldots, M_1 + 1, \quad n = 1, 2, \ldots, M_2 + 1. \tag{3.17}$$

By solving the system resulting from the above conditions, matrix A is determined. As a result, the approximate solution is obtained according to this matrix and Eq. (3.1).

4. Error analysis of integration

This section provides the error of integration that appeared in Problems 4.1 and 4.2. To achieve this goal, we use the error of the Legendre-Gauss quadrature rule and the error of the approximate solution in Sobolev space.

Problem 4.1. From the Legendre-Gauss quadrature rule error [8], we have

$$\left| \int_0^1 \mathcal{K}(x,t,\eta) u_{xx}(x,\eta) d\eta - \Psi(x,t,\sigma_s) \right| \leq \frac{(\mathfrak{S}+1)\mathfrak{S}^3((\mathfrak{S}-1)!)^4}{(2\mathfrak{S}+1)((2\mathfrak{S})!)^3} Z^{\mathfrak{S}}(x,t), \tag{4.1}$$

where

$$\Psi(x,t,\sigma_s) = \sum_{s=1}^{\mathfrak{S}} w_s \mathcal{K}(x,t,\sigma_s) u_{xx}(x,\sigma_s), \quad Z^{\mathfrak{S}}(x,t) = \sup_{y \in (0,1)} \left| \frac{\partial^{2\mathfrak{S}} [\mathcal{K}(x,t,y) u_{xx}(x,y)]}{\partial y^{2\mathfrak{S}}} \right|$$

Also, according to Eq. (4.1) can be written:

$$\begin{aligned} \left\| \int_0^1 \mathcal{K}(x,t,\eta) u_{xx}(x,\eta) d\eta - \Upsilon^{\alpha,\beta}(x,t,\sigma_s) \right\|_{L^2(\Delta)} &\leq \left\| \int_0^1 \mathcal{K}(x,t,\eta) u_{xx}(x,\eta) d\eta - \Psi(x,t,\sigma_s) \right\|_{L^2(\Delta)} \\ &+ \left\| \Psi(x,t,\sigma_s) - \Upsilon^{\alpha,\beta}(x,t,\sigma_s) \right\|_{L^2(\Delta)}. \end{aligned} \tag{4.2}$$

By assuming

$$\max_{x \in \Delta} |\mathcal{K}(x,t,\sigma_s)| = L,$$

and using the error function in Sobolev space $H^{\mu}([0,1])$ for $1 \leq l \leq \mu$ [? ?]

$$\|f - f_{M_1}\|_{L^2([0,1])} \leq C \alpha^{2l-\frac{1}{2}-\mu} M_1^{2l-\frac{1}{2}-\mu} |f|_{H^{\mu;M_1\alpha}([0,1])},$$

we obtain the upper bound of the second norm that emerged in Eq. (4.2) as follows:

$$\begin{aligned} \|\Psi(x,t,\sigma_s) - \Upsilon^{\alpha,\beta}(x,t,\sigma_s)\|_{L^2(\Delta)} &= \left\| \sum_{s=1}^{\mathfrak{S}} w_s \mathcal{K}(x,t,\sigma_s) u_{xx}(x,\sigma_s) - \sum_{s=1}^{\mathfrak{S}} w_s \mathcal{K}(x,t,\sigma_s) \left[\sigma_s S^{\alpha T}(x) \mathcal{Q}^T A S^{\beta}(\sigma_s) + \mathfrak{F}''(x) \right] \right\|_{L^2(\Delta)} \\ &\leq \sum_{s=1}^{\mathfrak{S}} w_s L \left\| u_{xx}(x,\sigma_s) - \left[\sigma_s S^{\alpha T}(x) \mathcal{Q}^T A S^{\beta}(\sigma_s) + \mathfrak{F}''(x) \right] \right\|_{L^2([0,1])} \\ &\leq \sum_{s=1}^{\mathfrak{S}} w_s L C \alpha^{2l-\frac{1}{2}-\mu} M_1^{2l-\frac{1}{2}-\mu} |u_{xx}(.,\sigma_s)|_{H^{\mu;M_1}([0,1])}. \end{aligned} \tag{4.3}$$

Therefore, from Eqs. (4.1) and (4.3), we obtain

$$\begin{aligned} \left\| \int_0^1 \mathcal{K}(x,t,\eta) u_{xx}(x,\eta) d\eta - \Upsilon^{\alpha,\beta}(x,t,\sigma_s) \right\|_{L^2(\Delta)} &\leq \frac{(\mathfrak{S}+1)\mathfrak{S}^3((\mathfrak{S}-1)!)^4}{(2\mathfrak{S}+1)((2\mathfrak{S})!)^3} D^{\mathfrak{S}} \\ &+ \sum_{s=1}^{\mathfrak{S}} w_s L C \alpha^{2l-\frac{1}{2}-\mu} M_1^{2l-\frac{1}{2}-\mu} |u_{xx}(.,\sigma_s)|_{H^{\mu;M_1}([0,1])}, \end{aligned} \tag{4.4}$$

where

$$D^{\mathfrak{S}} = \left(\int_0^1 \int_0^1 [Z^{\mathfrak{S}}(x,t)]^2 dx dt \right)^{\frac{1}{2}}.$$

Problem 4.2. To compute the error of approximation in Eq. (3.13), we consider

$$\left\|\int_0^1 t\mathcal{K}(x,t,t\xi)\psi(x,t\xi,u(x,t\xi))d\xi - \hat{\Upsilon}^{\alpha,\beta}(x,t,\sigma_r)\right\|_{L^2(\Delta)} \leq \left\|\int_0^1 t\mathcal{K}(x,t,t\xi)\psi(x,t\xi,u(x,t\xi))d\xi - \hat{\Psi}(x,t,\sigma_r)\right\|_{L^2(\Delta)} + \left\|\hat{\Psi}(x,t,\sigma_r) - \hat{\Upsilon}^{\alpha,\beta}(x,t,\sigma_r)\right\|_{L^2(\Delta)}, \tag{4.5}$$

where

$$\hat{\Psi}(x,t,\sigma_r) = \sum_{r=1}^{\mathfrak{R}} w_r t\mathcal{K}(x,t,t\sigma_r)\psi(x,t\sigma_r,u(x,t\sigma_r)).$$

From Eqs. (3.1) and (3.13), we obtain

$$\left\|\hat{\Psi}(x,t,\sigma_r) - \hat{\Upsilon}^{\alpha,\beta}(x,t,\sigma_r)\right\|_{L^2(\Delta)} = \left\|\sum_{r=1}^{\mathfrak{R}} w_r t\mathcal{K}(x,t,t\sigma_r)\psi(x,t\sigma_r,u(x,t\sigma_r)) - \sum_{r=1}^{\mathfrak{R}} w_r t\mathcal{K}(x,t,t\sigma_r)\psi \begin{pmatrix} x, t\sigma_r, \\ t\sigma_r x(x-1)S^{\alpha T}(x)AS^{\beta}(t\sigma_r) + \mathfrak{F}(x) \\ +(1-x)\left[\mathfrak{G}_0(t\sigma_r) - \mathfrak{G}_0(0)\right] + x\left[\mathfrak{G}_1(t\sigma_r) - \mathfrak{G}_1(0)\right] \end{pmatrix}\right\|_{L^2(\Delta)}. \tag{4.6}$$

Considering the following assumption

$$\max_{x\in\Delta} |\mathcal{K}(x,t,t\sigma_r)| = K,$$

and the error function norm defined in Sobolev space [8, 13]

$$\|g - g_{M_1,M_2}\|_{L^2(\Delta)} \leq C\alpha^{2l-\frac{1}{2}-\mu} M^{2l-\frac{1}{2}-\mu} |g|_{H^{\mu;M\alpha}(\Delta)}.$$

And also, by establishing the Lipschitz condition for ψ with Lipschitz constant δ, we conclude

$$\begin{aligned} &\left\|\hat{\Psi}(x,t,\sigma_r) - \hat{\Upsilon}^{\alpha,\beta}(x,t,\sigma_r)\right\|_{L^2(\Delta)} \\ &\leq K\delta \sum_{r=1}^{\mathfrak{R}} w_r \left\| u(x,t\sigma_r)) - \begin{pmatrix} t\sigma_r x(x-1)S^{\alpha T}(x)AS^{\beta}(t\sigma_r) + \mathfrak{F}(x) \\ +(1-x)\left[\mathfrak{G}_0(t\sigma_r) - \mathfrak{G}_0(0)\right] + x\left[\mathfrak{G}_1(t\sigma_r) - \mathfrak{G}_1(0)\right] \end{pmatrix}\right\|_{L^2(\Delta)} \\ &\leq KC\delta \sum_{r=1}^{\mathfrak{R}} w_r \alpha^{2l-\frac{1}{2}-\mu} M^{2l-\frac{1}{2}-\mu} |g|_{H^{\mu;M\alpha}(\Delta)}. \end{aligned} \tag{4.7}$$

Also, regarding the Legendre-Gauss quadrature rule error [8], we have

$$\left|\int_0^1 t\mathcal{K}(x,t,t\xi)\psi(x,t\xi,u(x,t\xi))d\xi - \hat{\Psi}(x,t,\sigma_r)\right| \leq \frac{(\mathfrak{R}+1)\mathfrak{R}^3((\mathfrak{R}-1)!)^4}{(2\mathfrak{R}+1)((2\mathfrak{R})!)^3} W^{\mathfrak{R}}(x,t), \tag{4.8}$$

where

$$W^{\mathfrak{R}}(x,t) = \sup_{y\in(0,1)} |[t\mathcal{K}(x,t,ty)\psi(x,ty,u(x,ty))]^{(2\mathfrak{R})}|.$$

Consequently, by replacing Eqs. (4.7) and (4.8) in Eq. (4.5), we have

$$\left\|\int_0^1 t\mathcal{K}(x,t,t\xi)\psi(x,t\xi,u(x,t\xi))d\xi - \hat{\Upsilon}^{\alpha,\beta}(x,t,\sigma_r)\right\|_{L^2(\Delta)} \leq \frac{(\mathfrak{R}+1)\mathfrak{R}^3((\mathfrak{R}-1)!)^4}{(2\mathfrak{R}+1)((2\mathfrak{R})!)^3} B^{\mathfrak{R}} + KC\delta \sum_{r=1}^{\mathfrak{R}} w_r \alpha^{2l-\frac{1}{2}-\mu} M^{2l-\frac{1}{2}-\mu} |g|_{H^{\mu;M\alpha}(\Delta)}, \tag{4.9}$$

where

$$B^{\Re} = \left(\int_0^1 \int_0^1 [W^{\Re}(x,t)]^2 dx dt \right)^{\frac{1}{2}}.$$

5. Numerical results

In this section, we apply the proposed numerical algorithm presented in the previous section to provide the approximate solution of six numerical examples. To indicate the efficiency of the numerical scheme, we calculate the maximum absolute error L_∞ and root-mean-square error L_2 in some examples in which $u(x,t)$ is the exact solution and $u_{M_1 M_2}(x,t)$ is the approximate solution:

$$\begin{aligned} L_\infty &= \max_{1 \le i \le M_1, 1 \le j \le M_2} |u(x_i,t_j) - u_{M_1M_2}(x_i,t_j)|, \\ L_2 &= \left(\sum_{i=1}^{M_1} \sum_{j=1}^{M_2} |u(x_i,t_j) - u_{M_1M_2}(x_i,t_j)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Example 5.1. Consider the VO-time fractional convection-diffusion equation of order $0 < \gamma(x,t) \le 1$:

$$D_t^{\gamma(x,t)} u(x,t) + x u_x(x,t) + u_{xx}(x,t) = 2t^{\gamma(x,t)} + 2x^2 + 2,$$

subject to the initial condition $u(x,0) = x^2$ and boundary conditions

$$u(0,t) = \frac{2\Gamma(\gamma(x,t)+1)}{\Gamma(2\gamma(x,t)+1)} t^{2\gamma(x,t)}, \quad u(1,t) = 1 + \frac{2\Gamma(\gamma(x,t)+1)}{\Gamma(2\gamma(x,t)+1)} t^{2\gamma(x,t)}.$$

The exact solution of this problem is $u(x,t) = x^2 + \frac{2\Gamma(\gamma(x,t)+1)}{\Gamma(2\gamma(x,t)+1)} t^{2\gamma(x,t)}$ [39]. From the method procedure and problem conditions, we obtain

$$u(x,t) \simeq tx(x-1) S^{\alpha T}(x) A S^{\beta}(t) + x^2 + \left[\frac{2\Gamma(\gamma(x,t)+1)}{\Gamma(2\gamma(x,t)+1)} t^{2\gamma(x,t)} \right].$$

Then, with the help of operational matrices of derivative, we deduce

$$\begin{aligned} &u_x(x,t) \simeq t S^{\alpha T}(x) \mathcal{P}^T A S^{\beta}(t) + 2x, \\ &u_{xx}(x,t) \simeq t S^{\alpha T}(x) \mathcal{Q}^T A S^{\beta}(t), \end{aligned}$$

and

$$D_t^{\gamma(x,t)} u(x,t) \simeq x(x-1) t^{1-\gamma(x,t)} S^{\alpha T}(x) A \mathcal{R}_{\gamma(x,t)} S^{\beta}(t) + 2t^{\gamma(x,t)}.$$

Hence, by substituting the above approximate relations in the problem, we have

$$\begin{aligned} e_{mn} &= x_m(x_m-1) t_n^{1-\gamma(x_m,t_n)} S^{\alpha T}(x_m) A \mathcal{R}_{\gamma(x_m,t_n)} S^{\beta}(t_n) + 2t_n^{\gamma(x_m,t)} \\ &\quad - x_m \left[t_n S^{\alpha T}(x_m) \mathcal{P}^T A S^{\beta}(t_n) + 2x_m \right] + t S^{\alpha T}(x_m) \mathcal{Q}^T A S^{\beta}(t_n) - 2t_n^{\gamma(x_m,t_n)} - 2x_m^2 - 2. \end{aligned}$$

Consequently, according to the optimization problem created from this problem for various values of $\gamma(x,t) = \beta$, we get the exact solution. Also, different numerical approaches have also examined this example [9, 36, 39]. Hence, we compared the results of these methods with the proposed method in Table 1. Actually, this table demonstrates that by using a few numbers of basis functions ($M_1 = M_2 = 2$), we get the approximate solution with high precision. Furthermore, a comparison between the exact solution and approximate solutions for $M_1 = M_2 = 2$ and $\alpha = 1, \beta = 0.5$ with $\gamma(x,t) = \frac{8-x+t^3}{80}$ are plotted in Figure 3.

Example 5.2. Consider the VO-time fractional nonlinear parabolic problem of order $0 < \gamma(x,t) \le 1$:

$$D_t^{\gamma(x,t)} u(x,t) - u_{xx}(x,t) - u^2(x,t) = \exp(x)\sin(\pi x) - \exp(2x)\sin(\pi x)^2 + \pi^2 \exp(x)\sin(\pi x),$$

subject to the initial condition $u(x,0) = \sin(\pi x)$ and boundary conditions

$$u(0,t) = u(1,t) = 0.$$

TABLE 1. The absolute error of the present method and methods in [9, 36, 39] for different values of x with $t = 0.5$, $\alpha = 1$ and $\gamma(x,t) = \beta = 0.5$ for Example 5.1.

x	Present method $M_1 = M_2 = 2$	Haar wavelet [9] $m = 64$	Sinc-Legendre [36] $m = 25$	Chebyshev wavelet [39] $m = 6$
0.1	0	1.210×10^{-3}	6.462×10^{-6}	1.110×10^{-16}
0.2	0	1.259×10^{-3}	1.578×10^{-5}	1.110×10^{-16}
0.3	0	1.865×10^{-3}	2.272×10^{-5}	2.220×10^{-16}
0.4	0	7.412×10^{-3}	2.674×10^{-5}	2.220×10^{-16}
0.5	0	1.000×10^{-3}	2.759×10^{-5}	0
0.6	0	7.460×10^{-3}	2.534×10^{-5}	0
0.7	0	1.724×10^{-3}	2.035×10^{-5}	2.220×10^{-16}
0.8	3.6734×10^{-40}	4.990×10^{-3}	1.320×10^{-5}	0
0.9	0	4.990×10^{-3}	1.320×10^{-5}	0

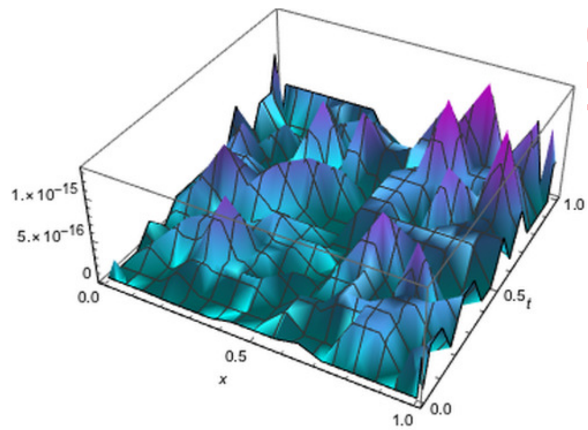

FIGURE 3. Comparison between the exact solution and approximate solutions with $M_1 = M_2 = 2$, $\alpha = 1, \beta = 0.5$ and $\gamma(x,t) = \frac{8-x+t^3}{80}$ for Example 5.1.

The exact solution to this problem when $\gamma(x,t) = 1$ is $u(x,t) = \sin(\pi x)\exp(x)$ [14]. Due to the method procedure and problem conditions, we get

$$u(x,t) \simeq tx(x-1)S^{\alpha T}(x)AS^{\beta}(t) + \sin(\pi x).$$

In Table 2, we report the absolute error of the present method and Euler collocation method [14] at some selected nodes. Also, to show the effect of values α and β in the calculation process, we present the maximum absolute error L_∞ and root-mean-square error L_2 in Table 3. Moreover, the curves of the approximate solution for $\gamma(x,t) = 0.7, 0.8, 0.9, 1$ are plotted in Figure 4. As we expected, as $\gamma(x,t)$ approaches 1, the approximate solutions get close to the exact solution corresponding to $\gamma(x,t) = 1$.

Example 5.3. Consider the 2D-time fractional Fredholm integro-differential equation [4]:

$$D_t^{\gamma}u(x,t) - u_x(x,t) - u_{xx}(x,t) + \int_0^1 \cos(x+t)u_{xx}(x,\eta)d\eta = f(x,t),$$

subject to the initial condition $u(x,0) = 0$ and Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0.$$

TABLE 2. The absolute error of the present method and Euler collocation method [14] for different values of t with $M_1 = M_2 = 2$ and $\gamma(x,t) = 1$ for Example 5.2.

	Present method			Ref. [14]		
x	$t = 0.2$	$t = 0.6$	$t = 0.8$	$t = 0.2$	$t = 0.6$	$t = 0.8$
0	0	0	0	0	0	0
0.1	2.7866×10^{-4}	1.1635×10^{-3}	1.4663×10^{-3}	2.3621×10^{-3}	1.5577×10^{-3}	2.3148×10^{-3}
0.2	1.6862×10^{-4}	4.4262×10^{-4}	1.2113×10^{-3}	5.7168×10^{-4}	2.9672×10^{-3}	3.2345×10^{-3}
0.3	5.0543×10^{-4}	1.6785×10^{-4}	3.2961×10^{-3}	7.8378×10^{-4}	6.4874×10^{-3}	7.7339×10^{-3}
0.4	6.1352×10^{-4}	2.0808×10^{-3}	4.0572×10^{-3}	1.0709×10^{-3}	7.8842×10^{-3}	9.6333×10^{-3}
0.5	6.2350×10^{-4}	2.1198×10^{-3}	4.1717×10^{-3}	1.0216×10^{-3}	8.1451×10^{-3}	1.0028×10^{-2}
0.6	6.1352×10^{-4}	2.0808×10^{-3}	4.0572×10^{-3}	1.0709×10^{-3}	7.8842×10^{-3}	9.6333×10^{-3}
0.7	5.0543×10^{-4}	1.6785×10^{-3}	3.2961×10^{-3}	7.8378×10^{-4}	6.4874×10^{-3}	7.7339×10^{-3}
0.8	1.6862×10^{-4}	4.4262×10^{-4}	1.2113×10^{-3}	5.7168×10^{-4}	2.9672×10^{-3}	3.2345×10^{-3}
0.9	2.7866×10^{-4}	1.1635×10^{-3}	1.4663×10^{-3}	2.3621×10^{-4}	1.5577×10^{-3}	2.3148×10^{-3}
1.0	3.2101×10^{-40}	3.2101×10^{-40}	3.2101×10^{-40}	1.4987×10^{-39}	1.7338×10^{-39}	1.9395×10^{-39}

TABLE 3. Errors for diverse values of α and β with $M_1 = M_2 = 3$ and $\gamma(x,t) = 1$ for Example 5.2.

	$\alpha = \beta = 0.5$	$\alpha = 0.5, \beta = 1$	$\alpha = 1, \beta = 0.5$
L_2-error	6.2622×10^{-2}	6.2215×10^{-2}	2.9022×10^{-3}
L_∞-error	3.0723×10^{-2}	3.0511×10^{-2}	1.9701×10^{-3}

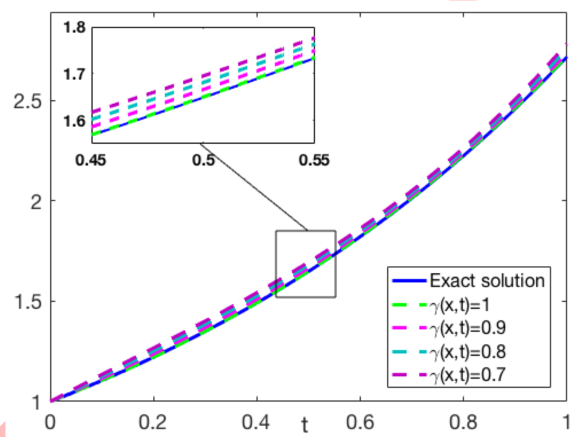

FIGURE 4. Comparison between the exact solution ($\gamma(x,t) = 1$) and approximate solutions with $M_1 = M_2 = 2$ and $x = 0.5$ for Example 5.2.

$f(x,t)$ is chosen, such that the exact solution is $u(x,t) = (1-x)^2 \ln(x+1) t^{3\gamma}$. In view of the presented method, we obtain the approximate solution as follows:

$$u(x,t) \simeq tx(x-1) S^{\alpha T}(x) A S^{\beta}(t).$$

Now, we solve this problem for different values of α, β and M_1. In Table 4, the error is listed for diverse values of components that appeared in the method. In Table 4 it can be observed that the numerical solutions converge to the exact solution as the value of M_1 increases.

Example 5.4. Consider the 2D-time fractional Fredholm integro-differential equation [4]:

$$D_t^{\gamma} u(x,t) + \cos(u(x,t)) - 2u_x(x,t) - u_{xx}(x,t) - \int_0^1 \eta^{\gamma} \sinh(x+t) u_{xx}(x,\eta) d\eta = f(x,t),$$

TABLE 4. Errors for diverse values of M_1 and $\beta = \gamma$ with $M_2 = 3$, $\mathfrak{S} = 7$ and $\alpha = 1$ for Example 5.3.

γ	β	M_1	L_2-error	L_∞-error
0.25	0.25	3	3.1349×10^{-2}	1.4544×10^{-2}
		5	6.6768×10^{-3}	3.4184×10^{-3}
		7	2.2273×10^{-4}	1.2881×10^{-4}
0.5	0.25	3	2.3298×10^{-2}	1.2551×10^{-2}
		5	5.5593×10^{-3}	3.2297×10^{-3}
		7	1.5456×10^{-4}	7.4093×10^{-5}
0.5	0.5	3	2.3424×10^{-2}	1.2594×10^{-2}
		5	5.6261×10^{-3}	3.2770×10^{-3}
		7	1.3866×10^{-4}	7.9209×10^{-5}
1	1	3	1.4227×10^{-2}	8.8895×10^{-3}
		5	3.7474×10^{-3}	2.5650×10^{-3}
		7	6.7011×10^{-5}	3.5732×10^{-5}
		10	1.2842×10^{-6}	8.8331×10^{-7}

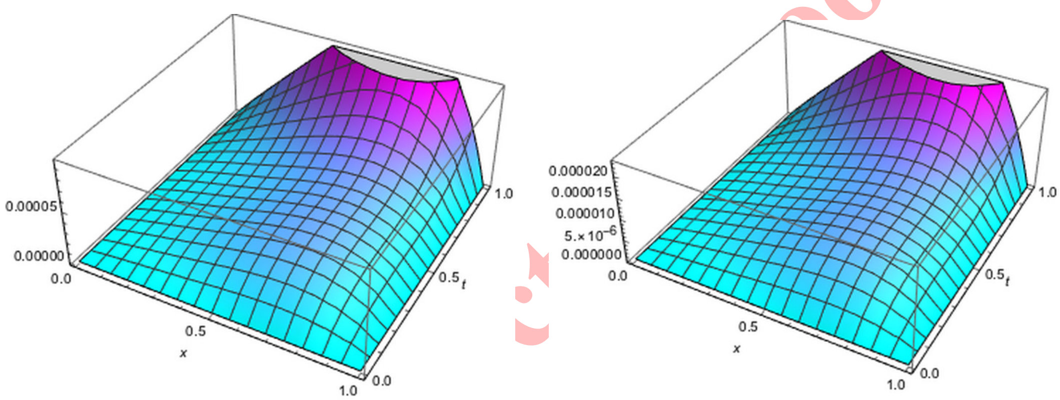

FIGURE 5. The absolute error for $\mathfrak{S} = 5$ (left) $\mathfrak{S} = 7$ (right) with $M_1 = M_2 = 2$, $\alpha = \beta = 1$ and $\gamma = 0.9$ for Example 5.4.

subject to the initial condition $u(x, 0) = x \exp(x - 1)$, and Dirichlet boundary conditions

$$u(0, t) = t^{\gamma+2}, \quad u(1, t) = 1 + t^{\gamma+2}.$$

$f(x, t)$ is chosen, such that the exact solution is $u(x, t) = x \exp(x-1) + t^{\gamma+2}$. By implementing the proposed approach for this problem, the approximate solution is obtained as follows:

$$u(x, t) \simeq tx(x - 1)S^{\alpha T}(x)AS^{\beta}(t) + x \exp(x - 1) + t^{\gamma+2}.$$

The results of this problem are displayed in Tables 5 and 6 and Figure 5. Also, the comparison of the absolute error obtained by the proposed method and the method in [4] are displayed in Table 5. It should be noted that with a small number of basic functions ($M_1 = M_2 = 2$), we reached a high accuracy of the exact solution. Now, to demonstrate the behavior of the approximate solution for different choices of α, β and γ, we exhibit the absolute errors for $M_1 = M_2 = 2$ and $\mathfrak{S} = 7$ in Table 6. In addition, plots of the absolute error for $\mathfrak{S} = 5$ (left) and $\mathfrak{S} = 7$ (right) are represented in Figure 5. In this figure can be seen that the value of $\mathfrak{S}$ affects the accuracy of the approximate solutions.

Example 5.5. Consider the 2D-VO-time fractional Volterra integro-differential equation:

$$D_t^{\gamma(x,t)}u(x, t) - u_{xx}(x, t) - \int_0^t \cos(x(t - \eta))u(x, \eta)d\eta = f(x, t),$$

TABLE 5. The absolute error of the present method and method in [4] with $M_1 = M_2 = 2$, $\alpha = \beta = \gamma = 1$ and $\mathfrak{S} = 7$ for Example 5.4.

x	t	Present method	Ref. [4]
0.2	0.25	1.2882×10^{-41}	$1.1098062724 \times 10^{-8}$
	0.5	1.2882×10^{-41}	$1.3475516667 \times 10^{-8}$
	0.75	1.2882×10^{-41}	$2.3987084829 \times 10^{-8}$
	1	1.2882×10^{-41}	$4.6951033335 \times 10^{-8}$
0.4	0.25	3.1468×10^{-41}	$1.5559383669 \times 10^{-8}$
	0.5	3.1468×10^{-41}	$3.2416693935 \times 10^{-8}$
	0.75	3.1468×10^{-41}	$1.4942838745 \times 10^{-7}$
	1	3.1468×10^{-41}	$6.2012285851 \times 10^{-8}$
0.6	0.25	1.5819×10^{-41}	$2.2169823188 \times 10^{-8}$
	0.5	1.5819×10^{-41}	$1.3613113948 \times 10^{-7}$
	0.75	1.5819×10^{-41}	$2.1304279661 \times 10^{-7}$
	1	1.5819×10^{-41}	$7.6931983681 \times 10^{-8}$
0.8	0.25	9.3890×10^{-41}	$2.6399003035 \times 10^{-8}$
	0.5	9.3890×10^{-41}	$1.4999640176 \times 10^{-8}$
	0.75	9.3890×10^{-41}	$2.8080854547 \times 10^{-7}$
	1	9.3890×10^{-41}	$7.7035008652 \times 10^{-8}$

TABLE 6. The absolute error of the present method for $M_1 = M_2 = 2$, $\gamma = 0.5$ and $\mathfrak{S} = 7$ of Example 5.4.

$x = t$	$\beta = 1$	$\beta = 0.5$	$\beta = 0.25$
0	0	0	0
0.1	2.0391×10^{-5}	1.1705×10^{-5}	2.2477×10^{-5}
0.2	5.8426×10^{-5}	5.7417×10^{-5}	5.6516×10^{-5}
0.3	1.0024×10^{-4}	9.7383×10^{-5}	9.5553×10^{-5}
0.4	1.4274×10^{-4}	1.4214×10^{-4}	1.4133×10^{-4}
0.5	1.8484×10^{-4}	1.9189×10^{-4}	1.9397×10^{-4}
0.6	2.6557×10^{-4}	2.4382×10^{-4}	2.4921×10^{-4}
0.7	2.6731×10^{-4}	2.8876×10^{-4}	2.9512×10^{-4}
0.8	2.9468×10^{-4}	3.0391×10^{-4}	3.0632×10^{-4}
0.9	2.5545×10^{-4}	2.3975×10^{-4}	2.3523×10^{-4}
1.0	8.0938×10^{-41}	8.0938×10^{-41}	8.0938×10^{-41}

subject to the initial condition $u(x, 0) = (1 - x^4)\exp(x)$ and boundary conditions

$$u(0, t) = \exp(t), \quad u(1, t) = 0.$$

$f(x, t)$ is chosen, such that the exact solution is $u(x, t) = (1 - x^4)\exp(xt)$ for a particular value of $\gamma(x, t) = 1$ [5]. Due to the proposed method and the problem conditions, we get

$$u(x, t) \simeq tx(x - 1)S^{\alpha T}(x)AS^{\beta}(t) + (1 - x^4)\exp(x) + (x - 1)(\exp(t) - 1).$$

By considering the proposed approach, we present the results in Tables 7 and 8. In these tables, the error of the approximate solutions for various choices of elements that appeared in the method is represented. It can be seen in Table 7 that the error decreases as the number of basis functions increases (M_1, M_2).

Example 5.6. Consider the 2D-VO-time fractional Volterra integro-differential equation:

$$D_t^{\gamma(x,t)} u(x, t) = f(x, t) + (\cos^2(x) + sin(x))u_{xx}(x, t) + \int_0^t \exp(x + t + \eta)u^2(x, \eta)d\eta,$$

TABLE 7. The absolute error of the present method for different values of β and $M_1 = M_2$ with $\alpha = 1$ and $\mathfrak{S} = 7$ for Example 5.5.

$x = t$	$\beta = 0.25$		$\beta = 0.5$		$\beta = 1$	
	$M_1 = M_2 = 2$	$M_1 = M_2 = 4$	$M_1 = M_2 = 2$	$M_1 = M_2 = 4$	$M_1 = M_2 = 2$	$M_1 = M_2 = 4$
0	0	0	0	0	0	0
0.1	4.2429×10^{-3}	2.4037×10^{-3}	1.5688×10^{-3}	1.3461×10^{-4}	5.1594×10^{-4}	1.3961×10^{-6}
0.2	2.6791×10^{-3}	1.6264×10^{-3}	6.8838×10^{-5}	5.7951×10^{-5}	1.0398×10^{-3}	2.0053×10^{-5}
0.3	3.3775×10^{-3}	6.6466×10^{-4}	4.0077×10^{-3}	1.7821×10^{-6}	3.9814×10^{-3}	1.5348×10^{-5}
0.4	6.4689×10^{-3}	2.1478×10^{-4}	5.3672×10^{-3}	1.6646×10^{-5}	4.2387×10^{-3}	6.8795×10^{-6}
0.5	1.1651×10^{-3}	6.7274×10^{-6}	4.7280×10^{-5}	1.0784×10^{-5}	1.1046×10^{-3}	3.0738×10^{-6}
0.6	1.0818×10^{-2}	1.3559×10^{-4}	1.0569×10^{-2}	1.7142×10^{-5}	1.0206×10^{-2}	3.8904×10^{-6}
0.7	1.9129×10^{-2}	9.3155×10^{-5}	1.7317×10^{-2}	4.2114×10^{-5}	1.5382×10^{-2}	5.9112×10^{-5}
0.8	1.0402×10^{-2}	1.1399×10^{-4}	8.8783×10^{-3}	1.5516×10^{-4}	7.1956×10^{-3}	1.3669×10^{-4}
0.9	1.3859×10^{-2}	2.9631×10^{-5}	1.3028×10^{-2}	1.3050×10^{-5}	1.2159×10^{-2}	2.5764×10^{-5}
1.0	0	0	0	0	0	0

TABLE 8. Errors for diverse values of $\alpha = \beta = 1$ and $\mathfrak{S} = 9$ with $M_1 = M_2 = 4$ for Example 5.5.

	$\gamma(x,t) = 0.5$	$\gamma(x,t) = 0.9$	$\gamma(x,t) = 1 - 0.5\exp(-xt)$
L_2-error	1.5511×10^{-4}	1.5391×10^{-4}	1.3769×10^{-4}
L_∞-error	1.3785×10^{-4}	1.3703×10^{-4}	1.3758×10^{-4}

TABLE 9. The absolute error of the present method and fractional Genocchi collocation method [15] for different values of $\mathfrak{S}$ with $\alpha = \beta = \gamma(x,t) = 1$ and $M_1 = M_2 = 2$ for Example 5.6.

$x = t$	Present method			Fractional Genocchi collocation method [15]	
	$\mathfrak{S} = 3$	$\mathfrak{S} = 5$	$\mathfrak{S} = 7$	$M_1 = 2, M_2 = 6$	$M_1 = 2, M_2 = 9$
0	0	0	0	0	0
0.1	1.3907×10^{-6}	6.0090×10^{-11}	9.3881×10^{-19}	3.14×10^{-8}	1.78×10^{-11}
0.2	1.4728×10^{-6}	6.4160×10^{-11}	6.0149×10^{-19}	7.23×10^{-8}	4.38×10^{-11}
0.3	2.2295×10^{-6}	9.9780×10^{-11}	2.8431×10^{-18}	1.16×10^{-7}	8.05×10^{-11}
0.4	6.9997×10^{-6}	3.2770×10^{-10}	7.0009×10^{-18}	1.88×10^{-7}	8.05×10^{-11}
0.5	4.6633×10^{-6}	2.8551×10^{-10}	4.9476×10^{-18}	2.59×10^{-7}	1.64×10^{-10}
0.6	1.6285×10^{-5}	5.0820×10^{-10}	1.3016×10^{-17}	3.02×10^{-7}	2.02×10^{-10}
0.7	6.3557×10^{-5}	2.3923×10^{-9}	5.3355×10^{-17}	3.40×10^{-7}	2.27×10^{-10}
0.8	1.2633×10^{-4}	4.9567×10^{-9}	1.0682×10^{-16}	3.75×10^{-7}	2.23×10^{-10}
0.9	1.5033×10^{-4}	6.0083×10^{-9}	1.2734×10^{-16}	2.42×10^{-7}	1.42×10^{-10}
1.0	0	0	0	6.61×10^{-43}	6.65×10^{-45}

subject to the initial condition $u(x, 0) = x$ and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = \cos(t).$$

$f(x, t)$ is chosen, such that the exact solution is $u(x, t) = x\cos(t)$ for a particular value of $\gamma(x, t) = 1$ [15]. By considering the proposed approach and problem conditions, we conclude

$$u(x, t) \simeq tx(x-1)S^{\alpha T}(x)AS^{\beta}(t) + x\cos(t).$$

In Table 9, we exhibit the absolute error for $M_1 = M_2 = 2$ and then the outcomes compared with the results of fractional-order Genocchi method [15] for $M_1 = 2, M_2 = 6, 9$. From Table 9, it can be concluded that the numerical solutions are in high agreement with the analytical solution as $\mathfrak{S}$ increases. Furthermore, the approximate solution for different values of $\gamma(x, t)$ is listed in Table 10. It is observed in Table 10 that as $\gamma(x, t)$ approaches 1, the approximate solutions converge to the exact solution for $\gamma(x, t) = 1$.

TABLE 10. The approximate solution for different values of $\gamma(x,t)$ with $M_1 = M_2 = 2$, $\alpha = \beta = 1$ and $\mathfrak{S} = 7$ for Example 5.6.

$x = t$	$\gamma(x,t) = 0.3$	$\gamma(x,t) = 0.5$	$\gamma(x,t) = 0.7$	$\gamma(x,t) = 0.9$	Exact solution
0	0	0	0	0	0
0.1	9.8504×10^{-2}	9.8703×10^{-2}	9.8964×10^{-2}	9.9298×10^{-2}	9.9500×10^{-2}
0.2	1.9241×10^{-1}	1.9315×10^{-1}	1.9411×10^{-1}	1.9530×10^{-1}	1.9601×10^{-1}
0.3	2.7943×10^{-1}	2.8095×10^{-1}	2.8285×10^{-1}	2.8521×10^{-1}	2.8660×10^{-1}
0.4	3.5741×10^{-1}	3.5983×10^{-1}	3.6277×10^{-1}	3.6634×10^{-1}	3.6842×10^{-1}
0.5	4.2439×10^{-1}	4.2769×10^{-1}	4.3157×10^{-1}	4.3617×10^{-1}	4.3879×10^{-1}
0.6	4.7863×10^{-1}	4.8260×10^{-1}	4.8713×10^{-1}	4.9231×10^{-1}	4.9520×10^{-1}
0.7	5.1856×10^{-1}	5.2279×10^{-1}	5.2747×10^{-1}	5.3261×10^{-1}	5.3538×10^{-1}
0.8	5.4285×10^{-1}	5.4667×10^{-1}	5.5077×10^{-1}	5.5511×10^{-1}	5.5736×10^{-1}
0.9	5.5038×10^{-1}	5.5287×10^{-1}	5.5547×10^{-1}	5.5811×10^{-1}	5.5944×10^{-1}
1.0	5.4030×10^{-1}	5.4030×10^{-1}	5.4030×10^{-1}	5.4030×10^{-1}	5.4030×10^{-1}

6. CONCLUSION

In this paper, we focus on combining the optimization method, fractional Mott functions, and the Ritz method to solve three classes of problems. These problems involve Fredholm and Volterra partial integro-differential equations, for which the Legendre–Gauss quadrature rule is employed to obtain numerical approximations. In addition, derivative matrices are utilized to efficiently handle the proposed formulations.

By applying the proposed technique, the original problems are reduced to systems of algebraic equations, leading to highly accurate approximate solutions. Furthermore, an error analysis of the integrals appearing in the problems is presented. The implementation of the proposed approach, together with the derivative matrices, results in a new numerical method that can be applied to a wide range of problems. Finally, numerical examples are provided to demonstrate the efficiency and accuracy of the proposed method.

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