Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-12
DOI:10.22034/cmde.2025.62567.2768

Stability analysis, bifurcation analysis and exact solution of van der Waals form for the fluid granular matter

Temesgen Desta Leta[1,2], Nafiseh Esfandian[3], Jamilu Sabi'u[4], and Hadi Rezazadeh[5,*]

[1]Reading College, Nanjing University of Information Science and Technology, 219 Ningliu, Nanjing 210044, P R China.
[2]Department of Mathematics, Dilla University, Dilla 419, Ethiopia.
[3]Department of Electrical Engineering, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran.
[4]Department of Mathematics, Northwest University, Kano, Nigeria.
[5]Faculty of Engineering Technology, Amol University of Special Modern Technologies, Amol, Iran.

Abstract

This paper analyses one of the most popular models in nature and industry, the conventional type of van der Waals for granular fluids. The recognition of dynamic and static properties of such models is crucial in various areas of business, from engineering to pharmaceuticals and other physical phenomena, such as those explored in geophysics. The phase separation processis described in this model.This paper analyses one of the most popular models in nature and industry, the conventional type of van der Waals for granular fluids. The dynamic bifurcation methodis used to detect the existence of smooth and non-smooth dynamic bahaviours, and periodic, solitary, singular soliton, anti-kink, and kink wave profiles have been produced. As a result, the rich dynamic properties of the model help to utilize the extended direct algebra method to obtain exact wave solutions. Theachieved solutionswere compared with different approaches. As per the authors' knowledge, no such work has yet been published in the literature.

Keywords. Van der Waals for fluid granular matter, New extended direct algebraic method, Exact solutions, Bifurcation.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Granulation is the process that primary powder particles engage with a broader particle form. The effects of the procedure are referred to as granules. Grain substances are sand, cereals, sugar, pills,asteroids, broken coal, and cosmetics. In various sectors of the production process, from medicine to civil engineering, as well as in some important physical phenomena, including those investigated in the geosciences, it is crucial to understand the static and dynamic properties of this type of matter, see [5, 25]. The above places considerable emphasis on researching the physical characteristics of this model and, for this reason, some of the new methods are used to obtain exact and solitary wave solutions.Various successful and efficient methods exist to obtain exact and solitary wave solutions for nonlinear partial differential equations (NLPDEs); the generalized exponential rational function method [6], Lie symmetry analysis [8], the generalized Kudryashov method [9], the (G''/G)- expansion method [1], the discrete tanh method [11], the extended Tanh–Coth method and the modified simple equation method [2], the method of functional variable [12], the sine and cosine method [23], the $\exp_a$-function and unified methods [10], the generalized logistic equation method [19], the Hirota bilinear method [17], the improved tanh $(\Phi(\xi)/2)$-expansion method [18], the perturbation-iteration algorithm [13], the residual power series method [24], the sub-equation method [4], the rational sine-Gordon expansion method [14], the bifurcation methods [15], the inverse engineering scheme [28], and the modified simple equation method [21].

Received: 19 July 2024; Accepted: 15 January 2026.
* Corresponding author. Email: h.rezazadeh@ausmt.ac.ir.

The standard form of the van der Waals for fluidized granular matter, considering the macroscopic equation for mass, momentum, and energy is given as

$$\begin{aligned} &D_t n + n\nabla.u = 0, \\ &D_t u + (mn)^{-1}\nabla.p = 0, \\ &nD_t T + \nabla.J + p : \nabla u + \omega = 0, \end{aligned} \tag{1.1}$$

where the material derivative D_t is described by $D_t = \frac{\partial}{\partial t} + u.\nabla$,the number of density is $n = \partial^{-1} dvf(v)$, the flow velocity is $u = \frac{1}{n}\partial^{-1} dvvf(v)$, the granular temperature is defined as $T = \frac{m}{dn}\partial^{-1} dvC^2 f(v)$ and the heat flux $J = -k(n,t)\nabla T$. Furthermore, the peculiar velocity $C = v - u$and $k$is the thermal conductivity.

However, from the Equation (1.1), density and horizontal momentum act by the conservation equations, and the velocity field is low to ignore the nonlinear terms in v, therefore the horizontal momentum and density can be vertically measured with$j(x,t)$and $\rho(x,t)$ that satisfies the equations of continuity

$$\begin{aligned} &\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}, \\ &\frac{\partial j(x,t)}{\partial t} = \frac{\partial \varphi}{\partial x}. \end{aligned} \tag{1.2}$$

By virtue of Equations (1.1) and (1.2), the van der Waals form is quantified to the dominant order [16]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2} - \eta\frac{\partial u}{\partial t} - u^3 - \varepsilon u\right) = 0, \tag{1.3}$$

where u is the field that defines the vertical average density correction, η is the effective viscosity, x is the coordinate that shows the horizontal direction of the granular structure, and ε is the bifurcation parameter. So, we want to scrutinize the wave solution of the form

$$u(x,t) = u(\xi), \tag{1.4}$$

where $\xi = kx + \varpi t$ and $u : \mathrm{R} \to \mathrm{R}$ is a smooth function. Eq. (1.3) has been discussed in [3] using the Painlevé analysis and the Tanh expansion method, in [16] using the six separate methods, in [27] using the modified Kudryashov method and the hyperbolic function method,also the exponential rational function method and Generalized Kudryashov method where adopted in [7] to derive the exact solutions of the model.

Substituting Eq. (1.4) into Eq. (1.3) we have an ordinary differential equation (ODE)$u = u(\xi)$ as follows

$$k^4\ddot{u} + \varpi\eta k^2\dot{u} - (k\varepsilon - \varpi^2)u + k^2u^3 = 0. \tag{1.5}$$

Let $\dot{u} = y$ to obtain the system

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{\varpi\eta}{k^2}y + \frac{(k\varepsilon - \varpi^2)}{k^4}u - \frac{1}{k^2}u^3, \tag{1.6}$$

with the renowned first integral, we're going to get

$$H(u,y) = y^2 + \frac{2\varpi\eta}{k^2}uy - \frac{(k\varepsilon - \varpi^2)}{k^4}u^2 + \frac{1}{2k^2}u^4 + \frac{2C}{k^4} \equiv h, \tag{1.7}$$

where h is an energy level. We know that the study of exact travel wave solutions of Eq. (1.1) in different parametric spaces is still not considered before. This paper aims to implement the extended direct algebra method first proposed by Rezazadeh [22] in an attempt to discover exact wave solutions and the dynamical system method for studying the dynamic properties of the system (1.6) under the three-dimensional space parameter for the van der Waals fluidized granular matter.

This article is structured as follows: In section 2, we analyze the stability of equilibrium points and investigate the bifurcation of the phase portraits system (1.6). In section 3, first, we describe the extended direct algebra method for solving NLPDEs, then we apply this method to seek exact wave solutions for Eq. (1.3). In section 4, the physical explanations of our exact wave solutions of Eq. (1.3) are presented. In the last section 5, the conclusion has been drawn.

2. Stability analysis of equilibrium points and phase portraits

Now we consider the corresponding regular system of Equation (1.6) for zero viscosity of the VdW inviscid model as follows

$$\frac{du}{d\zeta} = k^4 y, \quad \frac{dy}{d\zeta} = (k\varepsilon - \varpi^2)u - k^2u^3, \tag{2.1}$$

where $d\xi = ud\zeta$ with the first integral

$$H(u,y) = y^2 - \frac{(k\varepsilon - \varpi^2)}{k^4}u^2 + \frac{1}{2k^2}u^4 + \frac{2\mathrm{C}}{k^4} \equiv h. \tag{2.2}$$

System (2.1) is a planer dynamical system defined in three-parameter space (k, ε, ϖ). Although all moving wave solutions are determined by these phase orbits characterized by the system's vector fields (2.1), as the parameters change, in the phase plane we will analyze the bifurcations of the phase portraits of this system. In the next subsections, we discuss the stability of the equilibrium points and the bifurcation of the phase portraits, depending on the change in the parameter.

2.1. Stability analysis of equilibrium point.

The equilibrium points of system (2.1) satisfy $Q_3(u) \equiv (k\varepsilon - \varpi^2)u - k^2u^3$. It is clear that $Q_3'(u) = 0$ when $u = u_{1,2} = \mp\frac{\sqrt{3(\varpi^2 - k\varepsilon)}}{k}$. The cubic equation has three real roots when $\frac{2}{3\sqrt{3}}(k\varepsilon - \varpi^2) < 0$, otherwise, there's a real zero solution. We have two hyperbolic fixed points in the first case and a center fixed point in the second case.

Let one of the singular points of Eq. (1.5), then the linearized system Eq. (1.5) has $(u^*, 0)$, the characteristic values at the singular point $(u^*, 0)$ namely $\lambda_{1,2} = \pm k^2\sqrt{Q_3'(u^*)}$. Consequently, we know the following according to the qualitative theory of dynamic systems. If $Q_3'(u^*) > 0$ or (< 0); then $(u^*, 0)$ is a saddle point (or a center point). If $Q_3'(u^*) = 0$, $(u^*, 0)$ is a degenerate saddle point.

2.2. Bifurcation of phase portraits.

Due to the symmetry of $u \to -u$ on Eq. (2.2), we will assume without loss of generality $\varpi < \sqrt{k\varepsilon}$. The following phase portraits of the system are obtained from the above analysis.

From the above phase portraits,there exist a family of periodic orbits around the center $E_0(\varphi_0, 0)$ (see Figure 1 (a)-(c), (v) and (vi)) and the two equilibrium points $E_{1,2}(u_{1,2}, 0)$ (see Figure 1(d)). Besides, there exists a homoclinic orbit (see Figure 1(c)-1(e)) and two heteroclinic orbits connecting two saddle points at$E_{1,2}(u_{1,2}, 0)$(see Figure 1(a) and (f)). These periodic orbits, homoclinic,heteroclinic orbits,and open orbits give as respectively, periodic, solitary, kink and anti-kink and compacton wave profiles to the system (1.6).

Correlating to the energy level $H(u, y) = h$ we have, $h_0(0, 0) = 0$,

$$h_1(u_1, 0) = \frac{1}{\kappa^6}\left(-3\varpi^4 + 6\varpi^2\varepsilon\kappa - 3\varepsilon^2\kappa^2 + 4C\sqrt{3(\varpi^2 - \kappa\varepsilon)}\kappa\right),$$

$$h_2(u_2, 0) = -\frac{1}{\kappa^6}\left(3\varpi^4 - 6\varpi^2\varepsilon\kappa + 3\varepsilon^2\kappa^2 + 4C\sqrt{3(\varpi^2 - \kappa\varepsilon)}\kappa\right).$$

In the next section, we apply the method of the extended direct algebra method to determine the traveling wave solutions of Eq. (1.3).

3. Method of solution

For any given NLPDEs

$$R(x, t, B_t, B_x, B_{tt}, B_{xx}, ...) = 0, \tag{3.1}$$

We utilize the transformation,

$$B(x, t) = u(\xi), \quad \xi = kx - \varpi t, \tag{3.2}$$

where k and ϖ are constants. So, Eq. (2.1) transform into the next nonlinear ODE

$$G(u, u', u'', u''', ...) = 0, \tag{3.3}$$

where$G$ is the polynomial function of its arguments, and

$$u' = \frac{du}{d\xi}.$$

The next target is to investigate the solutions of Eq. (3.3) as

$$u(\zeta) = \sum_{i=0}^{N} b_i \Phi^i(\xi), \tag{3.4}$$

where N is the positive integer obtained by balancing the highest derivative term with the highest nonlinear term of the nonlinear ODE, and the constantsare also to be determined. Eq. (3.4) is considered to be ageneralization of the ansatz method used in [26]. The $\Phi(\xi)$satisfies

$$\Phi'(\xi) = Ln(A)\left(q_1 + q_2\Phi(\xi) + q_3\Phi^2(\xi)\right), \tag{3.5}$$

where$q_1, q_2, q_3$are to be determined. Lastly, the solutions obtained using the relations between q_1, q_2 and q_3 as demonstrated in [26].

Now, using wave transformation $U(\xi) = u(x,t)$ where$\xi = kx + \varpi t$ in Eq. (1.3) and integrating twice and setting the constant to be zero to have

$$k^4 u'' + \varpi\eta k^2 u' - (k\varepsilon - \varpi^2)u + k^2 u^3 = 0. \tag{3.6}$$

Balancing the highest derivative with the highest nonlinear terms of Eq. (3.6), we obtain$N = 1$ andthen proceed as;

$$U(\xi) = b_0 + b_1\Phi(\xi). \tag{3.7}$$

We substitute Eq. (3.7) into Eq. (1.6) and equate the coefficients of $\Phi^i(\xi)$for $(i = 0, 1, 2, \cdots, N)$to zero to obtain the system:

$$\begin{aligned}
&\Phi^0(\xi) : \varpi^2 b_0 + k^2 b_0^3 - k\varepsilon b_0 + k^2\eta\varpi b_1 Ln(A) q_1 + k^4 b_1 Ln(A)^2 q_1 q_2 = 0,\\
&\Phi^1(\xi) : b_0 + \alpha b_1 Ln(A)^2 q_1 q_2 + \beta b_1 Ln(A) q_1 + \sigma b_0^3 = 0,\\
&\Phi^2(\xi) : -k\varepsilon b_1 + \varpi^2 b_1 + 2k^4 b_1 Ln(A)^2 (q_1 q_3 + q_2^2) + k^2\eta\varpi b_1 Ln(A) q_2 + 3k^2 b_0^2 b_1 = 0,\\
&\Phi^3(\xi) : 2k^4 b_1 Ln(A)^2 q_3^2 + k^2 b_1^3 = 0.
\end{aligned}$$

After solving the system, we have

Case-1: When

$$b_0 = \pm\frac{1}{2}\frac{\left(q_2 + \sqrt{q_2^2 - 4q_3q_1}\right)b_1}{2q_3}, \quad \varpi = \mp\frac{3b_1^2\sqrt{q_2^2 - 4q_3q_1}}{2q_3^2\eta Ln(A)}, \quad k = \pm\frac{\sqrt{-2}b_1}{2q_3 Ln(A)},$$

$$\varepsilon = \mp\frac{\sqrt{-2}b_1^3\left((9 - 2\eta^2)(q_2^2 - 4q_1q_3)\right)}{4q_3^3\eta^2 Ln(A)}, \qquad b_1 = b_1.$$

$$u(x,t) = \pm\frac{1}{2}\frac{\left(q_2 + \sqrt{q_2^2 - 4q_3q_1}\right)b_1}{2q_3} + b_1\Phi(\xi), \tag{3.8}$$

$$\xi = \frac{\pm b_1}{2q_3 Ln(a)}\left(\sqrt{-2}x - \frac{3}{2}\frac{b_1\sqrt{q_2^2 - 4q_3q_1}}{q_3\eta}t\right).$$

From Eqs. (3.2), (3.4), and (3.8) along withthe solutions of Eq. (3.5), we get the new type of exact solution of the van der Waals Equation (1.3) as follows.

When $\Lambda = q_2^2 - 4q_1q_3 < 0$ and $q_3 \neq 0$ then

$$u_1^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} + i\tan_A\left(\frac{\sqrt{-\Lambda}}{2}\xi\right)\right),$$

$$u_2^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} - i\cot_A\left(\frac{\sqrt{-\Lambda}}{2}\xi\right)\right),$$

$$u_3^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} + i\left(\tan_A\left(\sqrt{-\Lambda}\xi\right) \pm \sqrt{pq}\sec_A\left(\sqrt{-\Lambda}\xi\right)\right)\right),$$

$$u_4^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} + i\left(-\cot_A\left(\sqrt{-\Lambda}\xi\right) \pm \sqrt{pq}\csc_A\left(\sqrt{-\Lambda}\xi\right)\right)\right),$$

$$u_5^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{4q_3}\left(1 + i\left(\tan_A\left(\frac{\sqrt{-\Lambda}}{4}\xi\right) - \cot_A\left(\frac{\sqrt{-\Lambda}}{4}\xi\right)\right)\right),$$

where $\xi = \frac{\pm b_1}{2q_3 Ln(A)}\left(\sqrt{-2}x - \frac{3}{2}\frac{b_1\sqrt{\Lambda}}{q_3\eta}t\right)$..
When $\Lambda = q_2^2 - 4q_1q_3 > 0 \text{and} q_3 \neq 0 \text{then}$

$$u_6^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} - \tanh_A\left(\frac{\sqrt{\Lambda}}{2}\xi\right)\right),$$

$$u_7^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} - \coth_A\left(\frac{\sqrt{\Lambda}}{2}\xi\right)\right),$$

$$u_8^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} + \left(-\tanh_A\left(\sqrt{\Lambda}\xi\right) \pm i\sqrt{pq}\sec h_A\left(\sqrt{\Lambda}\xi\right)\right)\right),$$

$$u_9^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{2q_3}\left(\pm\frac{1}{2} + \left(-\coth_A\left(\sqrt{\Lambda}\xi\right) \pm \sqrt{pq}\csc h_A\left(\sqrt{\Lambda}\xi\right)\right)\right),$$

$$u_{10}^{\pm}(x,t) = \frac{\sqrt{\Lambda}b_1}{4q_3}\left(\pm 1 - \left(\tanh_A\left(\frac{\sqrt{\Lambda}}{4}\xi\right) + \coth_A\left(\frac{\sqrt{\Lambda}}{4}\xi\right)\right)\right),$$

where$\xi = \frac{\pm b_1}{2q_3 Ln(A)}\left(\sqrt{-2}x - \frac{3}{2}\frac{b_1\sqrt{\Lambda}}{q_3\eta}t\right)$.
When $q_1q_3 > 0$ and $q_2 = 0$ then

$$u_{11}^{\pm}(x,t) = b_1\sqrt{\frac{q_1}{q_3}}\left(\pm\frac{i}{2} + \tan_A\left(\frac{\pm b_1}{2Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right)\right),$$

$$u_{12}^{\pm}(x,t) = b_1\sqrt{\frac{q_1}{q_3}}\left(\pm\frac{i}{2} - \cot_A\left(\frac{\pm b_1}{2Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right)\right),$$

$$\begin{aligned} u_{13}^{\pm}(x,t) = b_1\sqrt{\frac{q_1}{q_3}}\Bigg(\pm\frac{i}{2} + \Bigg(&\tan_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right) \\ &\pm\sqrt{pq}\sec_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right)\Bigg)\Bigg),\end{aligned}$$

$$\begin{aligned} u_{14}^{\pm}(x,t) = b_1\sqrt{\frac{q_1}{q_3}}\Bigg(\pm\frac{i}{2} + \Bigg(&-\cot_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right) \\ &\pm\sqrt{pq}\csc_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right)\Bigg)\Bigg),\end{aligned}$$

$$\begin{aligned} u_{15}^{\pm}(x,t) = \frac{b_1}{2}\sqrt{\frac{q_1}{q_3}}\Bigg(\pm i + \Bigg(&\tan_A\left(\frac{\pm b_1}{4Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right) \\ &-\cot_A\left(\frac{\pm b_1}{4Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}t}\right)\right)\Bigg)\Bigg).\end{aligned}$$

When $q_1q_3 < 0$ and $q_2 = 0$ then

$$u_{16}^{\pm}(x,t) = b_1\sqrt{-\frac{q_1}{q_3}}\left(\pm\frac{1}{2} - \tanh_A\left(\frac{\pm b_1}{2Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right),$$

$$u_{17}^{\pm}(x,t) = b_1\sqrt{-\frac{q_1}{q_3}}\left(\pm\frac{1}{2} - \coth_A\left(\frac{\pm b_1}{2Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right),$$

$$u_{18}^{\pm}(x,t) = b_1\sqrt{-\frac{q_1}{q_3}}\left(\pm\frac{1}{2} + \left(-\tanh_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right.\right.$$
$$\left.\left.\pm i\sqrt{pq}\sec h_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right)\right),$$

$$u_{19}^{\pm}(x,t) = b_1\sqrt{-\frac{q_1}{q_3}}\left(\pm\frac{1}{2} + \left(-\coth_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}\csc h_A\left(\frac{\pm b_1}{Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right)\right),$$

$$u_{20}^{\pm}(x,t) = \frac{b_1}{2}\sqrt{-\frac{q_1}{q_3}}\left(\pm 1 + \left(\tanh_A\left(\frac{\pm b_1}{2Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right.\right.$$
$$\left.\left.+\coth_A\left(\frac{\pm b_1}{2Ln(A)}\sqrt{\frac{q_1}{q_3}}\left(\sqrt{-2}x - \frac{3b_1}{\eta}\sqrt{-\frac{q_1}{q_3}}t\right)\right)\right)\right).$$

5) When $q_2 = 0$ and $q_3 = q_1$, then

$$u_{21}^{\pm}(x,t) = b_1\left(\pm\frac{i}{2} + \tan_A\left(\frac{\pm b_1}{2Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right)\right),$$

$$u_{22}^{\pm}(x,t) = b_1\left(\pm\frac{i}{2} - \cot_A\left(\frac{\pm b_1}{2Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right)\right),$$

$$u_{23}^{\pm}(x,t) = b_1\left(\pm\frac{i}{2} + \left(\tan_A\left(\frac{\pm b_1}{Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right) \pm\sqrt{pq}\sec_A\left(\frac{\pm b_1}{Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right)\right)\right),$$

$$u_{24}^{\pm}(x,t) = b_1\left(\pm\frac{i}{2} + \left(-\cot_A\left(\frac{\pm b_1}{Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right) \pm\sqrt{pq}\csc_A\left(\frac{\pm b_1}{Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right)\right)\right),$$

$$u_{25}^{\pm}(x,t) = \frac{b_1}{2}\left(\pm i + \left(\tan_A\left(\frac{\pm b_1}{4Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right) - \cot_A\left(\frac{\pm b_1}{4Ln(A)}\left(\sqrt{-2}x - \frac{3ib_1}{\eta}t\right)\right)\right)\right).$$

6) When$q_2 = 0$and$q_3 = -q_1$, then

$$u_{16}^{\pm}(x,t) = b_1\left(\pm\frac{1}{2} - \tanh_A\left(\frac{\pm b_1}{2Ln(A)}\left(\sqrt{2}x - \frac{3ib_1}{\eta}t\right)\right)\right),$$

$$u_{17}^{\pm}(x,t) = b_1\left(\pm\frac{1}{2} - \coth_A\left(\frac{\pm b_1}{2Ln(A)}\left(\sqrt{2}x - \frac{3ib_1}{\eta}t\right)\right)\right),$$

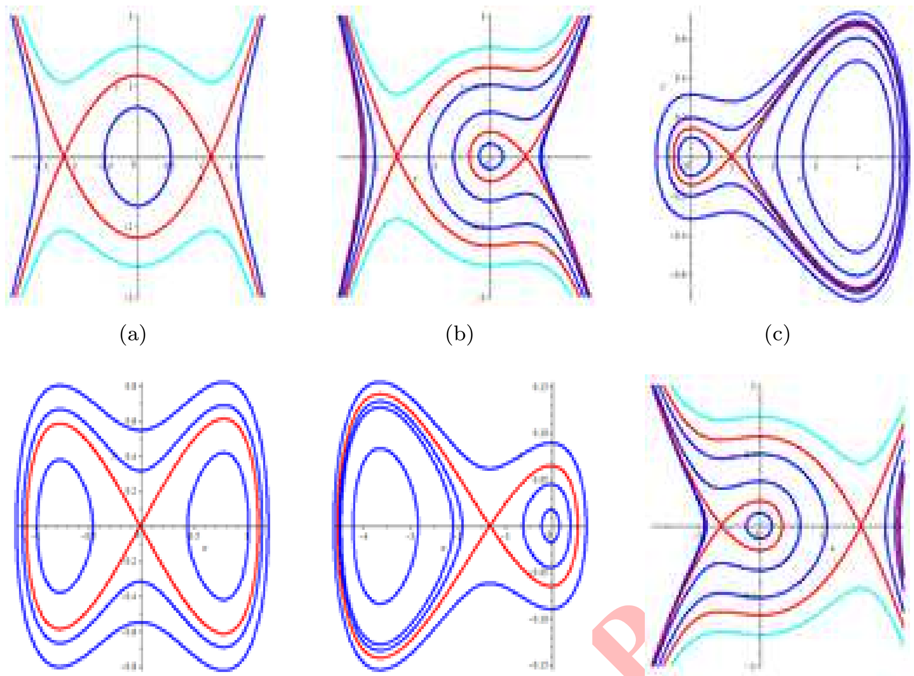
$$u_{18}^{\pm}(x,t) = b_1\left(\pm\tfrac{1}{2} + \left(-\tanh_A\left(\tfrac{\pm b_1}{Ln(A)}\left(\sqrt{2}x - \tfrac{3ib_1}{\eta}t\right)\right) \pm i\sqrt{pq}\sec h_A\left(\tfrac{\pm b_1}{Ln(A)}\left(\sqrt{2}x - \tfrac{3ib_1}{\eta}t\right)\right)\right)\right),$$

$$u_{19}^{\pm}(x,t) = b_1\left(\pm\tfrac{1}{2} + \left(-\coth_A\left(\tfrac{\pm b_1}{Ln(A)}\left(\sqrt{2}x - \tfrac{3ib_1}{\eta}t\right)\right) \pm\sqrt{pq}\csc h_A\left(\tfrac{\pm b_1}{Ln(A)}\left(\sqrt{2}x - \tfrac{3ib_1}{\eta}t\right)\right)\right)\right),$$

$$u_{20}^{\pm}(x,t) = \tfrac{b_1}{2}\left(\pm 1 + \left(\tanh_A\left(\tfrac{\pm b_1}{2Ln(A)}\left(\sqrt{2}x - \tfrac{3ib_1}{\eta}\sqrt{-\tfrac{q_1}{q_3}}t\right)\right) + \coth_A\left(\tfrac{\pm b_1}{2Ln(A)}\left(\sqrt{2}x - \tfrac{3ib_1}{\eta}t\right)\right)\right)\right).$$

7) When $q_1 = 0$and $q_2 \neq 0$then

$$u_{35}^{\pm}(x,t) = \frac{q_2b_1}{q_3}\left(\pm\frac{1}{2} - \frac{p}{\cosh_A(q_2\xi) - \sinh_A(q_2\xi) + p}\right),$$



(a) (b) (c)

(d) (e) (f)

Figure 1. (a)$\kappa < 0, \varpi < 0, \varepsilon = 0; u_1 = -u_2$. (b)$\kappa < 0, \varepsilon < 0; u_1 > 0 > \sqrt{3(k\varepsilon - \omega^2)}$. (c)$\kappa > 0, \omega > 0, \varepsilon < 0; u_1 > u^* = 1 > 0$. (d)$\kappa > 0, \varpi < 0, \varepsilon = 0; u_1 = -u_2$. (e)$\kappa > 0, \varepsilon > 0; -\sqrt{3(k\varepsilon - \omega^2)} < u^* < 0$. (f)$\kappa > 0, \varepsilon > 0; u_1 > 0 > \sqrt{3(k\varepsilon - \omega^2)}$.

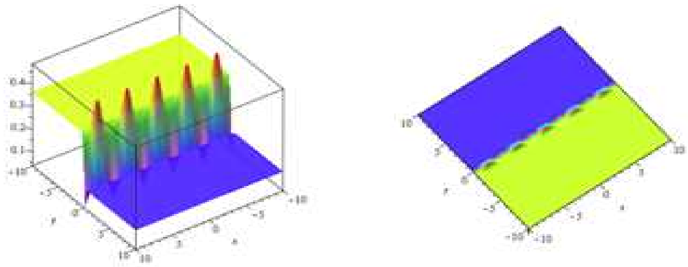
$$u_{36}^{\pm}(x,t) = \frac{q_2 b_1}{q_3}\left(\pm\frac{1}{2} - \frac{\sinh_A(q_2\xi) + \cosh_A(q_2\xi)}{\sinh_A(q_2\xi) + \cosh_A(q_2\xi) + q}\right),$$

where$\xi = \frac{\pm b_1}{2q_3 Ln(A)}\left(\sqrt{-2}x - \frac{3}{2}\frac{b_1 q_2}{q_3 \eta}t\right)$.
8)When$q_2 = \lambda$,$q_3 = m\lambda(m \neq 0)$and$q_1 = 0$then

$$u_{37}^{\pm}(x,t) = b_1\left(\pm\frac{1}{2m} + \frac{pA^{\frac{\pm b_1}{2mLn(A)}\left(\sqrt{-2}x - \frac{3}{2}\frac{b_1}{m\eta}t\right)}}{p - mqA^{\frac{\pm b_1}{2mLn(A)}\left(\sqrt{-2}x - \frac{3}{2}\frac{b_1}{m\eta}t\right)}}\right).$$

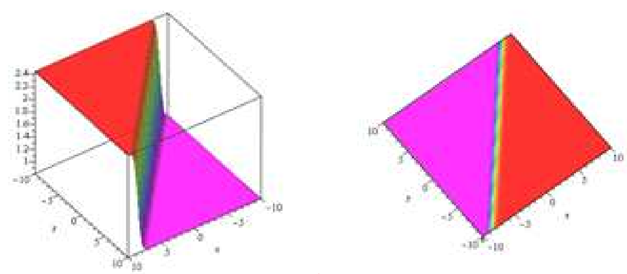
4. Physical explanation

This work's solutions take the form of elliptic functions. Furthermore, elliptic functions are an extension of solutions to trigonometric and hyperbolic functions. As a result, the solutions provided here are more robust than those of the hyperbolic function technique, the modified Kudryashov method [3], the exponential rational function method (ERFM), and the generalized Kudryashov method [16].The physical explanations can be illustrated by plotting the contour and 3D graphs of some of the derived solutions using appropriate constant values. The plots of $\left|u_6^{+}(x,t)\right|$, $\left|u_{13}^{-}(x,t)\right|$ and $\left|u_{16}^{+}(x,t)\right|$ are shown below (fig(1-4)):



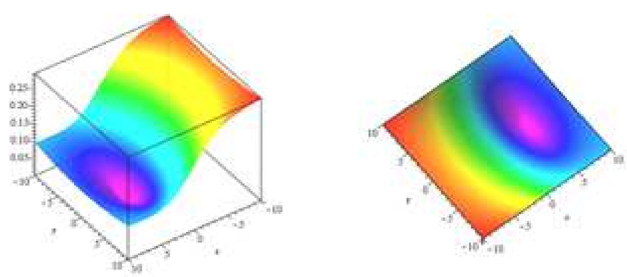
(a) (b)

FIGURE 2. 3D and contour plot exact solution of the $\left|u_6^+(x,t)\right|$ with $q_1 = 2, q_2 = 1, q_3 = -2.5, A = 2.7, p = 0.95, q = 0.98$ $\eta = 1$ and $b_1 = 1.2$ when $-10 \leq x, t \leq 10$.



(a) (b)

FIGURE 3. 3D and contour plot exact solution of the $\left|u_{13}^-(x,t)\right|$ with $q_1 = 1, q_2 = 1.5, A = 2.8, p = 1.3, q = 1.2, \eta = 4$ and $b_1 = 2$ when $-10 \leq x, t \leq 10$.



(a) (b)

FIGURE 4. 3D and contour plot exact solution of the $\left|u_{16}^+(x,t)\right|$ with $q_1 = 0.5, q_3 = -1.5, A = e, p = 1, q = 1$ $\eta = 0.9$ and $b_1 = 0.2$ when $-1 \leq x, t \leq 1$.

FIGURE LEGENDS

5. CONCLUSION

Nonlinear wave equations with constant coefficients have different applications and can be solved by many researchers. Nowadays numerous methods are used to derive thetraveling wave solutions of these equations. In this paper, we applied a method of bifurcation and extended direct algebraic methods to obtain different dynamical behaviours of stability properties and exact traveling wave solutions of the usual type of van der Waals equation with the constant coefficient for the fluid granular matter respectively. By slightly changing the parameters of the model, we simulate different phase trajectories including smooth and nonsmooth phase portraits. Also, we obtained equilibrium points and their stability by using the theories of the singularity of the first type. The smooth periodic orbits and homoclinic orbits of the model provide periodic and solitary wave solutions while compacton wave profiles and heteroclinic wave profiles give rise to anti-kink, kink, and breaking wave solutions of thismodel.Byutilizing the extended direct algebra method, we demonstrate the existence of more than forty-eight traveling wave solutions for the normal van der Waals model for the fluidized granular matter.The solutions achieved are newer and also more general than in the literature. Results obtained show that these methods are verysimple, reliable, and efficient to solve NLPDEs.

Acknowledgment

The authors would like to express their sincere thanks to the editor and anonymous reviewers for constructive comments and suggestions to improve the quality of this paper.

Funding

Not applicable.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

The authors declare that they have no competing interests.

6. Appendix A:

6.1. **Existence of bounded traveling wave solutions of an inviscid VdW model.** We next prove the existence of periodic and solitary wave solution of Eq. (2.1) depending on the Jacobian elliptic functions. Thus, by multiplying Eq. (2.1) byu' and integrating once, we get that $u$must satisfy

$$(u')^2 = \frac{1}{2k^2}\left[-u^4 + \frac{2\left(k\varepsilon - \varpi^2\right)}{k^2}u^2 - 4k^2C\right], \tag{6.1}$$

whereC_u is the required non-zero integration. For simplicity, we take$P\left(t\right) = t^4 - \frac{2(k\varepsilon-\varpi^2)}{k^2}t^2 - 4k^2C_t$. Although the solutions of Eq. (6.1) is heavily dependent on the roots of $P(t)$. For $0 > C_t > \frac{(k\varepsilon-\varpi^2)^2}{4k^6}$, we have

$$\begin{aligned} P\left(t\right) &= -\left(t^2 - \tfrac{k\varepsilon-\varpi^2}{k^2}\right)^2 - \tfrac{\left(k\varepsilon-\varpi^2\right)^2}{k^4} - 4k^2C \\ &= -\left(t^2 - \tfrac{k\varepsilon-\varpi^2}{k^2} - \sqrt{\tfrac{(k\varepsilon-\varpi^2)^2-4k^6C}{k^4}}\right)\left(t^2 - \tfrac{k\varepsilon-\varpi^2}{k^2} + \sqrt{\tfrac{(k\varepsilon-\varpi^2)^2-4k^6C}{k^4}}\right), \end{aligned}$$

and

$$\frac{k\varepsilon-\varpi^2}{k^2} - \sqrt{\frac{\left(k\varepsilon-\varpi^2\right)^2 - 4k^6C}{k^4}} > 0, \qquad \frac{k\varepsilon-\varpi^2}{k^2} + \sqrt{\frac{\left(k\varepsilon-\varpi^2\right)^2 - 4k^6C}{k^4}} > 0.$$

Hence, $P(t)$ has real and symmetric roots $\pm u_{\mathrm{m}}$ and $\pm u_{\mathrm{M}}$.We assume that $u_m < 0 < u_{\mathrm{M}}$. (see Figure 1(d)). Hence, we can write

$$(u')^2 = \frac{1}{2k^2}\left[\left(u^2 - u_{\mathrm{m}}{}^2\right)\left(u_{\mathrm{M}}{}^2 - u^2\right)\right] \tag{6.2}$$

Since the right-hand side of (6.2) is nonnegative, then we obtain that$u_{\mathrm{m}} < u < u_{\mathrm{M}}$ and u_i^2's (i = m, M) satisfy

$$\begin{cases} u_{\mathrm{m}}{}^2 + u_{\mathrm{M}}{}^2 = \frac{2\left(k\varepsilon - \varpi^2\right)}{k^2} > 0, \\ -u_{\mathrm{m}}{}^2 u_{\mathrm{M}}{}^2 = 4k^2 C < 0. \end{cases} \tag{6.3}$$

Define $\rho(\xi) = \frac{u(\xi)}{u_{\mathrm{M}}}$ and $k_1^2 = \frac{u_{\mathrm{M}}{}^2 - u_{\mathrm{m}}{}^2}{u_{\mathrm{M}}{}^2}$, then (6.2) becomes

$$(\rho')^2 = \frac{u_{\mathrm{M}}{}^2}{2k^2}\left[\left(\rho^2 - \frac{u_{\mathrm{m}}{}^2}{u_{\mathrm{M}}{}^2}\right)\left(1 - \rho^2\right)\right], \tag{6.4}$$

Then introduce a new term χ using $\rho^2 = 1 - k_1^2 \sin^2 \chi$, from (6.3) and (6.4), we get

$$(\chi')^2 = \frac{u_{\mathrm{M}}{}^2}{2k^2}\left[1 - k_1^2 \sin^2 \chi\right]. \tag{6.5}$$

We get this basedon the description of the snoidal Jacobi elliptical function

$$\int_0^{\chi(\xi)} \frac{dt}{\sqrt{1 - k_1^2 \sin^2 \xi}} = \frac{1}{\sqrt{2}} \frac{u_{\mathrm{M}}}{\mathrm{k}} \xi, \tag{6.6}$$

has a solution

$$\sin\left(\chi\left(\xi\right)\right) = sn\left(\frac{1}{\sqrt{2}} \frac{u_{\mathrm{M}}}{\mathrm{k}} \xi, k_1\right).$$

hence, using the relation $k_1^2 sn^2 + dn^2 = 1$, we obtain

$$\rho\left(\xi\right) = \sqrt{1 - k_1^2 \sin^2 \xi} = \sqrt{1 - k_1^2 sn^2\left(\frac{1}{\sqrt{2}} \frac{u_{\mathrm{M}}}{\mathrm{k}} \xi, k_1\right)} = dn\left(\frac{1}{\sqrt{2}} \frac{u_{\mathrm{M}}}{\mathrm{k}} \xi, k_1\right)$$

And$z\left(0\right) = 1$.
Substituting the form of z(ξ) to the definition $\rho(\xi) = \frac{u(\xi)}{u_{\mathrm{M}}}$, we get the dnoidal wave solution

$$u_{A1}\left(\xi\right) = u_{\mathrm{M}} dn\left(\frac{1}{\sqrt{2}} \frac{u_{\mathrm{M}}}{\mathrm{k}} \xi, k_1\right). \tag{6.7}$$

Alreadydn has a period of $2K$,that is $dn\left(u, K\right) = dn\left(u + 2K, k_1\right)$, for which the elliptic integral of the first kind is defined by $K = (k_1)$, hence we obtained thefundamental period

$$T_u = \frac{2\sqrt{2}}{k u_1} K\left(k_1\right), \tag{6.8}$$

Then, from (6.3), we get $0 < u_{\mathrm{m}} < \frac{\sqrt{k\varepsilon - \varpi^2}}{k} < u_{\mathrm{M}} < \frac{\sqrt{2(k\varepsilon - \varpi^2)}}{\mathrm{k}}$, and fundamental period T_u can be seen as a function of a variable u_2 only, that is

$$T_u\left(\mu_2\right) = \frac{2\sqrt{2}k}{\sqrt{2\left(k\varepsilon - \varpi^2\right) - k^2 u_{\mathrm{m}}{}^2}} K\left(k_1\left(u_2\right)\right), \tag{6.9}$$

with$k_1^2\left(u_{\mathrm{m}}\right) = \frac{2\left(k\varepsilon - \varpi^2\right) - 2k^2 u_{\mathrm{m}}{}^2}{2(k\varepsilon - \varpi^2) - k^2 u_{\mathrm{m}}{}^2}$.
Next, we will show that $T_u > \frac{\pi\sqrt{2}}{\sqrt{k\varepsilon - \varpi^2}}$. Note that if $u_2 \to 0$, we have that $k_1\left(u_{\mathrm{m}}\right) \to 1^-$, and then $K\left(k_1\left(u_{\mathrm{m}}\right)\right) \to +\infty$.. Therefore, $T_u \to +\infty$ as $u_{\mathrm{m}} \to 0$. On the other hand, if $u_{\mathrm{m}} \to \frac{\sqrt{k\varepsilon - \varpi^2}}{k}$, we have that $k_1\left(u_{\mathrm{m}}\right) \to 0^+$, which imply that

$K(k_1(u_m)) \to \frac{\pi}{2}$. Therefore, $T_u \to \frac{\pi\sqrt{2}}{\sqrt{k\varepsilon-\varpi^2}}$ as $u_m \to \frac{\sqrt{k\varepsilon-\varpi^2}}{k}$. Moreover, since the function $u_m \in \left(0, \frac{\sqrt{k\varepsilon-\varpi^2}}{k}\right) \to T_u(u_m)$ decreases strictly, it follows that $T_u > \frac{\pi\sqrt{2}}{\sqrt{k\varepsilon-\varpi^2}}$.

Remark 6.1. If $u_m \to 0^+$(see Figure 1 (c) and (d)), we obtain that $u_m \to \frac{\sqrt{2(k\varepsilon-\varpi^2)}}{k}, k_1(u_m) \to 1^-$. Then, based on$dn(x,1) = \sec(x)$, the relation (6.8) forfeit its periodicity and we get a single hump waveform with "infinity periodic"

$$u_{A2}\left(\xi; \frac{\sqrt{2(k\varepsilon-\varpi^2)}}{k}, 0\right) \to \frac{\sqrt{2(k\varepsilon-\varpi^2)}}{\mathrm{k}}\operatorname{sech}\left(\sqrt{k\varepsilon-\varpi^2}\xi\right),$$

and

$$u_{A3}\left(\xi; \frac{\sqrt{2(k\varepsilon-\varpi^2)}}{\mathrm{k}}, 0\right) \to \frac{k\varepsilon-\varpi^2}{2} - \frac{\sqrt{2(k\varepsilon-\varpi^2)}}{k}\operatorname{sech}^2\left(\sqrt{k\varepsilon-\varpi^2}\xi\right),$$

that are wave solutions of Eq. (1.3). Similarly, we can obtain bounded traveling wave solutions of Eq. (1.3) such as (anti-)kink solution, explicit solutions, and rational solutions as shown in the main result of section 3.

References

[1] R. Abazari and S. J. Jamshidzadeh, *Exact solitary wave solutions of the complex Klein–Gordon equation*, Optik – International Journal for Light and Electron Optics, *126* (2015), 1970–1975.

[2] R. Abazari, *Application of extended tanh function method on KdV-Burgers equation with forcing term*, Romanian Journal of Physics, *59* (2014), 3–11.

[3] A. Abourabia and A. Morad, *Exact traveling wave solutions of the van der Waals normal form for fluidized granular matter*, Physica A: Statistical Mechanics and its Applications, *437* (2015), 333–350.

[4] L. Akinyemi, M. Şenol, and O. S. Iyiola, *Exact solutions of the generalized multidimensional mathematical physics models via sub-equation method*, Mathematics and Computers in Simulation, *182* (2021), 211–233.

[5] M. Argentina, M. Clerc, and R. Soto, *Van der Waals–like transition in fluidized granular matter*, Physical Review Letters, *89* (2002), 044301.

[6] E. Atilgan, M. Senol, A. Kurt, and O. Tasbozan, *New wave solutions of time-fractional coupled Boussinesq–Whitham–Broer–Kaup equation as a model of water waves*, China Ocean Engineering, *33* (2019), 477–483.

[7] S. Bibi, N. Ahmed, U. Khan, and S. T. Mohyud-Din, *Some new exact solitary wave solutions of the van der Waals model arising in nature*, Results in Physics, *9* (2018), 648–655.

[8] R. Cimpoiasu, *Multiple invariant solutions of the 3D potential Yu–Toda–Sasa–Fukuyama equation via symmetry technique*, International Journal of Modern Physics B, *34* (2020), 2050188.

[9] R. Cimpoiasu and A. S. Pauna, *Complementary wave solutions for the long-short wave resonance model via the extended trial equation method and the generalized Kudryashov method*, Open Physics, *16* (2018), 419–426.

[10] K. Hosseini, M. Osman, M. Mirzazadeh, and F. Rabiei, *Investigation of different wave structures to the generalized third-order nonlinear Schrödinger equation*, Optik – International Journal for Light and Electron Optics, *206* (2020), 164259.

[11] A. Houwe, M. Inc, S. Doka, B. Acay, and L. Hoan, *The discrete tanh method for solving the nonlinear differential-difference equations*, International Journal of Modern Physics B, *34* (2020), 2050177.

[12] M. Inc, H. Rezazadeh, J. Vahidi, M. Eslami, M. A. Akinlar, M. N. Ali, and Y.-M. Chu, *New solitary wave solutions for the conformable Klein–Gordon equation with quantic nonlinearity*, AIMS Mathematics, *5* (2020), 6972–6984.

[13] A. Kurt, M. Şenol, O. Tasbozan, and M. Chand, *Two reliable methods for the solution of fractional coupled Burgers' equation arising as a model of polydispersive sedimentation*, Applied Mathematics and Nonlinear Sciences, *4* (2019), 523–534.

[14] T. D. Leta, A. El Achab, W. Liu, and J. Ding, *Application of bifurcation method and rational sine-Gordon expansion method for solving 2D complex Ginzburg–Landau equation*, International Journal of Modern Physics B, *34* (2020), 2050079.

[15] T. D. Leta and J. Li, *Dynamical behavior of traveling wave solutions of a long waves–short waves resonance model*, Qualitative Theory of Dynamical Systems, *18* (2019), 741–760.

[16] D. Lu, A. R. Seadawy, and M. M. Khater, *Bifurcations of new multi-soliton solutions of the van der Waals normal form for fluidized granular matter via six different methods*, Results in Physics, *7* (2017), 2028–2035.

[17] J. Manafian, *Novel solitary wave solutions for the (3+1)-dimensional extended Jimbo–Miwa equations*, Computers and Mathematics with Applications, *76* (2018), 1246–1260.

[18] J. Manafian and S. Heidari, *Periodic and singular kink solutions of the Hamiltonian amplitude equation*, Advances in Mathematical Models and Applications, *4* (2019), 134–149.

[19] Z. Pinar, H. Rezazadeh, and M. Eslami, *Generalized logistic equation method for Kerr law and dual power law Schrödinger equations*, Optical and Quantum Electronics, *52* (2020), 1–16.

[20] G. Rahman, K. S. Nisar, B. Ghanbari, and T. Abdeljawad, *On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals*, Advances in Difference Equations, *2020* (2020), 1–19.

[21] N. Raza and A. Javid, *Dynamics of optical solitons with Radhakrishnan–Kundu–Lakshmanan model via two reliable integration schemes*, Optik – International Journal for Light and Electron Optics, *178* (2019), 557–566.

[22] H. Rezazadeh, *New solitons solutions of the complex Ginzburg–Landau equation with Kerr law nonlinearity*, Optik – International Journal for Light and Electron Optics, *167* (2018), 218–227.

[23] J. Sabi'u, A. Jibril, and A. M. Gadu, *New exact solution for the (3+1) conformable space–time fractional modified Korteweg–de Vries equations via Sine-Cosine Method*, Journal of Taibah University for Science, *13* (2019), 91–95.

[24] M. Senol, *Analytical and approximate solutions of (2+1)-dimensional time-fractional Burgers–Kadomtsev–Petviashvili equation*, Communications in Theoretical Physics, *72* (2020), 055003.

[25] R. Soto, M. Argentina, and M. G. Clerc, *Van der Waals-like transition in fluidized granular matter*, in Granular Gas Dynamics, Springer, 2003, 317–333.

[26] M. Usman, A. Hussain, H. Ali, F. Zaman, and N. Abbas, *Dispersive modified Benjamin–Bona–Mahony and Kudryashov–Sinelshchikov equations: non-topological, topological, and rogue wave solitons*, International Journal of Mathematics and Computer in Engineering, *3*(1) (2025), 21–34.

[27] A. Zafar, B. Khalid, A. Fahand, H. Rezazadeh, and A. Bekir, *Analytical behaviour of travelling wave solutions to the van der Waals model*, International Journal of Applied and Computational Mathematics, *6* (2020), 1–16.

[28] A. Zubair and N. Raza, *Optical bright, dark and dipole solitons with derivative nonlinearity in the presence of parity-time-symmetric lattices*, Modern Physics Letters B, *34* (2020), 2050174.