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An approach for solving the generalized fractional Burgers-Fisher equation

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Abstract

This study introduces an approach for finding an approximate solution to the time fractional generalized Burgers-Fisher equation. The core idea of the method is to transform the nonlinear partial differential equation into a linear one through two dimensional Haar wavelet with iteration technique. Subsequently, the Haar wavelet collocation method is employed to address the linear equation derived in the prior step. Numerical simulations are conducted to rigorously evaluate the performance of the proposed algorithm. The results demonstrate that the scheme is not only computationally efficient but also highly accurate across various parameter configurations, including different fractional orders (α), nonlinearity strengths (η), and coefficients (ξ, β). Consequently, this work establishes the presented Haar wavelet iterative method as a powerful and reliable tool for solving this important class of nonlinear fractional differential equations.

Keywords. Fractional PDEs, Fractional derivatives and integrals, Haar wavelet, Operational matrix, Collocation method.
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1. Introduction

Many physical phenomena in engineering and applied sciences are modeled using nonlinear equations. These equations play a crucial role in describing complex systems such as fluid dynamics, heat transfer, wave propagation, and other dynamic processes. Fractional partial differential equations (FPDEs) have gained increasing attention in the modeling of such phenomena, as they provide more accurate descriptions of systems with memory and hereditary properties. However, solving these equations often presents significant challenges due to their inherent nonlinearity and fractional order, which makes traditional methods less effective.

This work focuses on finding numerical solutions for the generalized one-dimensional time-fractional Burgers-Fisher equation, given by:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \xi u^\eta \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} + \beta u(1 - u^\eta), \tag{1.1}$$

where ξ, μ, and β are constant parameters, $0 < \alpha \leq 1$ denotes the fractional order, and the time derivative is defined in the Caputo sense. The exact solution for $\alpha = 1$ is ([10, 18])

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2} \tanh \left[\frac{-\xi\eta}{2(\eta+1)} \left(x - \left(\frac{\xi}{\eta+1} + \frac{\beta(\eta+1)}{\xi} \right) t \right) \right] \right)^{1/\eta}.$$

This equation models a combination of reaction, convection, and diffusion processes and finds applications in diverse areas such as fluid dynamics, heat conduction, and the study of capillary-gravity waves.

Notably, the Burgers-Fisher equation generalizes several important models. For instance, when $\xi = 0$ and $\eta = 1$, it reduces to the time-fractional Fisher's equation, a fundamental model in mathematical biology:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \beta u(1 - u). \tag{1.2}$$

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More generally, the time-fractional Fisher's equation is expressed as:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + F(u), \quad u(x,0) = \varphi(x), \tag{1.3}$$

where $u(x,t)$ represents the population density at position x and time $t > 0$. The nonlinear function $F(u)$ is typically continuous and satisfies the conditions $F(0) = F(1) = 0$ and $F'(0) > 0 > F'(1)$, which are characteristic of logistic growth models in population dynamics.

The generalized Burgers-Fisher equation has been the focus of extensive research, leading to the development of numerous analytical and numerical techniques for its solution. Among these, wavelet methods have gained significant prominence. The properties of orthogonal waveletsparticularly their ability to represent functions efficiently using compactly supported basesmake them exceptionally well-suited for solving both ordinary and partial differential equations. This advantage is further amplified in the context of fractional partial differential equations (FPDEs), along with other fractional equations, where wavelet-based techniques have consistently proven to deliver highly accurate and computationally efficient solutions. [1, 4, 8, 12, 13, 16]

A diverse array of wavelet approaches has been applied to this problem. For instance, Kumar et al. [11] employed a discontinuous Legendre wavelet Galerkin method, Singh et al. [22] utilized a B-spline collocation method, and Saeed et al. [20] implemented a CAS wavelet quasilinearization technique. Other notable methods include the Adomian decomposition method [10], the homotopy perturbation method (HPM) [18], the Haar wavelet Picard method (HWPM) [19], a Legendre wavelet technique [9], a Taylor wavelets method [5], a Chebyshev wavelet approach [2], a multiwavelets method [21], a wavelet-based lifting scheme [6], a spline solution [15], a differential quadrature method [14], and a combined Haar wavelet and optimal homotopy asymptotic method [7].

The primary aim of the present work is to implement a method that combines a Haar wavelet collocation technique with the an iteration scheme. This approach is designed to demonstrate the capability of these methods in handling nonlinear fractional equations of arbitrary order and can be adapted to various types of nonlinearities. Our numerical results confirm that the proposed method is highly efficient and provides accurate solutions for the generalized time-fractional Burgers-Fisher equation.

The structure of this paper is as follows. In section 2, we review the fundamental definitions of fractional derivatives and integrals, which are essential for understanding the generalized Burgers-Fisher equation with a fractional time derivative. Section 3 provides a brief introduction to wavelet theory. This is followed by a detailed explanation of the Haar wavelet and its properties in section 4. Section 5 is dedicated to presenting the proposed numerical method, which combines the Haar wavelet collocation technique with the iteration method for solving the fractional generalized Burgers-Fisher equation. Finally, in section 6, we present two numerical examples to demonstrate the effectiveness and accuracy of our method. The results show that the proposed approach is both reliable and efficient for solving this class of fractional partial differential equations.

2. Fractional derivative and integral

The Caputo fractional derivative of order $\alpha > 0$ is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}} d\tau, \quad \text{for } x > 0,$$

where $n = \lceil \alpha \rceil$ denotes the ceiling of the derivative order, and the function f is n-times differentiable.

The fractional integral (of Riemann–Liouville type) of order $\alpha > 0$ is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2.1}$$

For $n-1 < \alpha \leq n$, where n is a natural number, the following properties hold for the Riemann–Liouville integral and the Caputo derivative:

$$D^\alpha (I^\alpha f(x)) = f(x), \tag{2.2}$$

$$I^{\alpha}(D^{\alpha}f(x)) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0^+)}{k!}x^k, \tag{2.3}$$

$$I^{\alpha}(I^{\beta}f(x)) = I^{\alpha+\beta}f(x), \quad \alpha,\beta > 0, \tag{2.4}$$

$$I^{\beta}(I^{\alpha}f(x)) = I^{\alpha+\beta}f(x), \tag{2.5}$$

$$I^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}. \tag{2.6}$$

For more details on fractional derivatives and integrals, see [17].

3. Multiresolution Analysis in $L^2(\mathbb{R})$

Wavelets are families of functions generated through the translation and dilation of a single, fixed function $\psi(t)$ known as wavelet. The wavelets is defined by:

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right), \quad a,b\in\mathbb{R}, \quad a\neq 0, \tag{3.1}$$

where the parameter a is a dilation or scaling parameter that controls the wavelet's width, and b is a translation parameter that determines its position along the time axis.

For practical computation, a discrete version of this transform is essential. This is achieved by discretizing the parameters a and b. A standard approach is to let the scale vary dyadically by setting $a = a_0^m$ for some fixed $a_0 > 1$ (typically $a_0 = 2$) and $m \in \mathbb{Z}$. To ensure the discretized wavelets provide a complete cover of the real line at each scale, the translation parameter is proportionally discretized as $b = nb_0a_0^m$, where $b_0 > 0$ is fixed and $n \in \mathbb{Z}$. Substituting these discrete parameters into the wavelet definition (3.1) yields the discrete wavelet family:

$$\psi_{m,n}(t) = a_0^{-m/2}, \psi\left(a_0^{-m}t - nb_0\right), \quad m,n\in\mathbb{Z}. \tag{3.2}$$

For any function $f \in L^2(\mathbb{R})$, its discrete wavelet coefficients are then given by the inner products $\langle f, \psi_{m,n}\rangle$.

The most common and structured method for constructing useful wavelet families is through a Multiresolution Analysis (abbreviated by MRA). An MRA provides a rigorous framework for building orthonormal wavelet bases with desired properties, such as having compact support, vanishing moments, and regularity, which are crucial for applications in signal processing and numerical analysis.

Definition 3.1 (Multiresolution Analysis)**.** An MRA of $L^2(\mathbb{R})$ is a sequence $\{V_m : m \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ which satisfy:

- **(i):** (Nested) $V_m \subset V_{m+1}$, for all $m \in \mathbb{Z}$;
- **(ii):** (Density) $\bigcup_{m\in\mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R})$;
- **(iii):** (Separation) $\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$;
- **(iv):** (Scaling) $f(t) \in V_m$ if and only if $f(2t) \in V_{m+1}$ for all $m \in \mathbb{Z}$;
- **(v):** (Orthonormal basis) there is a function ϕ in V_0 such that the system $\{\phi(t-n) : n \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

The function ϕ is called a *scaling function* of the given MRA.

4. The Haar wavelet

$h_i(x)$ which is called the i-th uniform Haar wavelet, defined for $x \in [0, 1)$, is given by:

$$h_i(x) = \begin{cases} 1, & \text{for } a(i) \le x < b(i), \\ -1, & \text{for } b(i) \le x < c(i), \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $a(i) = \frac{k-1}{m}$, $b(i) = \frac{k-0.5}{m}$, $c(i) = \frac{k}{m}$, and $i = 2^j + k + 1$. Here, $j = 0, 1, 2, \ldots, J$ is the dilation parameter, $m = 2^{j+1}$ is the number of basis functions for a given j, and $k = 0, 1, 2, \ldots, 2^j - 1$ is the translation parameter. The maximum level of resolution is J. In particular, the Haar scaling function is $h_1(x) = \chi_{[0,1)}(x)$, where $\chi_{[0,1)}(x)$ is the characteristic function on the interval $[0, 1)$. We define the collocation points as $x_j = \frac{j-0.5}{m}$ for $j = 1, 2, 3, \dots, m$.

The next step involves the construction of an operational matrix for fractional integration. This matrix is derived by computing the Riemann-Liouville fractional integral of the Haar wavelet basis functions defined in Eq. (4.1). The Riemann-Liouville fractional integral of order $\alpha > 0$ for the Haar wavelets is defined as follows:

$$P_{\alpha,1}(x) = I^{\alpha}_{a(1)} h_1(x) = \frac{1}{\Gamma(\alpha)} \int_{a(1)}^{x} (x-s)^{\alpha-1} h_1(s) ds, \tag{4.2}$$

$$\begin{aligned} P_{\alpha,i}(x) &= I^{\alpha}_{a(i)} h_i(x) = \frac{1}{\Gamma(\alpha)} \int_{a(i)}^{x} (x-s)^{\alpha-1} h_i(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \begin{cases} \int_{a(i)}^{x} (x-s)^{\alpha-1} ds, & \text{for } a(i) \le x < b(i), \\ \int_{a(i)}^{b(i)} (x-s)^{\alpha-1} ds - \int_{b(i)}^{x} (x-s)^{\alpha-1} ds, & \text{for } b(i) \le x < c(i), \\ \int_{a(i)}^{b(i)} (x-s)^{\alpha-1} ds - \int_{b(i)}^{c(i)} (x-s)^{\alpha-1} ds, & \text{for } x \ge c(i). \end{cases} \end{aligned} \tag{4.3}$$

Simplifying these expressions yields:

$$P_{\alpha,1}(x) = \frac{(x - a(1))^{\alpha}}{\Gamma(\alpha+1)}, \tag{4.4}$$

and

$$P_{\alpha,i}(x) = \frac{1}{\Gamma(\alpha+1)} \begin{cases} (x-a(i))^{\alpha}, & \text{for } a(i) \le x < b(i), \\ (x-a(i))^{\alpha} - 2(x-b(i))^{\alpha}, & \text{for } b(i) \le x < c(i), \\ (x-a(i))^{\alpha} - 2(x-b(i))^{\alpha} + (x-c(i))^{\alpha}, & \text{for } x \ge c(i). \end{cases} \tag{4.5}$$

It is possible to expand a square-integrable function $y(x)$ defined over $[0, 1]$, by wavelets as

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \psi_{n,m}(x), \tag{4.6}$$

where

$$c_{n,m} = \langle y(x), \psi_{n,m}(x) \rangle.$$

If we truncate the series (4.6), we obtain:

$$y(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = \mathbf{C}^T \Psi(x), \tag{4.7}$$

here, the coefficient vector $\mathbf{C} = [c_{ij}]^T$, $i = 1, 2, \dots, 2^{k-1}$, $j = 0, 1, \dots, M-1$ and the wavelet function vector $\mathbf{\Psi}(x) = [\psi_{ij}(x)]^T$, $i = 1, 2, \dots, 2^{k-1}$, $j = 0, 1, \dots, M-1$ are $m' = 2^{k-1} \times M$ column vectors.

For clarity, Eq. (4.7) may be rewrite as

$$y(x) \approx \sum_{i=1}^{m'} c_i \psi_i, \text{ where } c_i = c_{n,m}, \psi_i(x) = \psi_{n,m}(x). \tag{4.8}$$

Furthermore, a function $u(x,t) \in L_2([0,1] \times [0,1])$ may be also approximated as

$$u(x,t) = \mathbf{\Psi}^T(x) U \mathbf{\Psi}(t), \tag{4.9}$$

where $U = [u_{ij}]_{m' \times m'} = [\langle \psi_i(x), \langle u(x,t), \psi_j(t) \rangle \rangle]_{m' \times m'}$. To evaluate the coefficients $u_{i,j}$, we use the wavelet collocation method.

Using the collocation points $x(j) = (j - 0.5)/m$ for $j = 1, 2, \dots, m$, we define the Haar wavelet matrix $H_{m\times m}$ as:

$$H_{m\times m} = \begin{pmatrix} h_1(x(1)) & h_1(x(2)) & \cdots & h_1(x(m)) \\ h_2(x(1)) & h_2(x(2)) & \cdots & h_2(x(m)) \\ \vdots & \vdots & \ddots & \vdots \\ h_m(x(1)) & h_m(x(2)) & \cdots & h_m(x(m)) \end{pmatrix}.$$

For Haar wavelet the Equation (4.8) can be written in vector form as $\mathbf{y} = \mathbf{c}H$, where $\mathbf{c} = [c_1, c_2, \dots, c_m]$ and

$$\mathbf{y} = [y(x(1)), y(x(2)), \dots, y(x(m))].$$

The coefficient vector can be computed by $\mathbf{c} = \mathbf{y}H^{-1}$, where H^{-1} is the inverse of H. Similarly, the fractional integration matrix P^α is constructed by evaluating Eqs. (4.4) and (4.5) at the collocation points:

$$P^\alpha_{m\times m} = \begin{pmatrix} P_{\alpha,1}(x(1)) & P_{\alpha,1}(x(2)) & \cdots & P_{\alpha,1}(x(m)) \\ P_{\alpha,2}(x(1)) & P_{\alpha,2}(x(2)) & \cdots & P_{\alpha,2}(x(m)) \\ \vdots & \vdots & \ddots & \vdots \\ P_{\alpha,m}(x(1)) & P_{\alpha,m}(x(2)) & \cdots & P_{\alpha,m}(x(m)) \end{pmatrix}.$$

For example, with $m = 4$ and $\alpha = 0.9$, the Haar wavelet operational matrix of fractional integration is:

$$P^{0.9}_{4\times 4} = \begin{pmatrix} 0.2305 & 0.4941 & 0.7421 & 0.9811 \\ 0.2305 & 0.4941 & 0.2812 & -0.0071 \\ 0.2305 & 0.0331 & -0.0156 & -0.0091 \\ 0 & 0 & 0.2305 & 0.0331 \end{pmatrix}.$$

Theorem 4.1 ([3]). *Suppose the function $u_m(x,t)$, derived via the Haar wavelet method, approximates the exact solution $u(x,t)$. Then, the error of this approximation satisfies the following bound:*

$$\|u(x,t) - u_m(x,t)\|_E \leq \frac{K}{\sqrt{3m}},$$

where $\|u(x,t)\|_E = \left(\int_0^1 \int_0^1 u^2(x,t)dxdt \right)^{1/2}$.

5. The Proposed Method

We apply the Picard iteration method to Eq. (1.1), which yields the iterative scheme:

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - \mu \frac{\partial^2 u_{r+1}}{\partial x^2} = -\xi u_r^\eta \frac{\partial u_r}{\partial x} + \beta u_r(1 - u_r^\eta),$$

where ξ, μ, and β are parameters, $0 < \alpha \leq 1$, and r denotes the iteration index. This equation is solved subject to the initial and boundary conditions:

$$u_{r+1}(x,0) = g(x), \quad u_{r+1}(0,t) = f_0(t), \quad u_{r+1}(1,t) = f_1(t),$$

for $t \geq 0$ and $0 < x < 1$.

Using the Haar wavelet method, we assume that the second-order spatial derivative can be expanded as:

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j}^{r+1} h_i(x) h_j(t) = \mathbf{H}^T(x)\, \mathbf{C}^{r+1}\, \mathbf{H}(t), \tag{5.1}$$

where $m = 2M$ is the number of basis functions, $\mathbf{H}(x)$ and $\mathbf{H}(t)$ are Haar function vectors, and $\mathbf{C}^{r+1}$ is the $m \times m$ coefficient matrix to be determined.

Applying the double integral operator I_x^2 (with respect to x) to both sides of Eq. (5.1) gives:

$$u_{r+1}(x,t) = (\mathbf{P}_x^2)^T\, \mathbf{C}^{r+1}\, \mathbf{H}(t) + p(t)x + q(t), \tag{5.2}$$

where $\mathbf{P}_x^2$ is the operational matrix of double integration.

Using the boundary conditions, $u_{r+1}(0,t) = f_0(t)$ and $u_{r+1}(1,t) = f_1(t)$, we solve for $q(t)$ and $p(t)$:

$$\begin{aligned} &\text{At } x = 0: \quad u_{r+1}(0,t) = q(t) = f_0(t), \\ &\text{At } x = 1: \quad u_{r+1}(1,t) = (\mathbf{P}_x^2(1))^T\, \mathbf{C}^{r+1}\, \mathbf{H}(t) + p(t) + f_0(t) = f_1(t). \end{aligned}$$

Solving for $p(t)$ yields $p(t) = f_1(t) - f_0(t) - (\mathbf{P}_x^2(1))^T\, \mathbf{C}^{r+1}\, \mathbf{H}(t)$. Substituting $p(t)$ and $q(t)$ back into Eq. (5.2) gives the expression for u_{r+1}:

$$u_{r+1}(x,t) = (\mathbf{P}_x^2)^T\, \mathbf{C}^{r+1}\, \mathbf{H}(t) + x\left(f_1(t) - f_0(t) - (\mathbf{P}_x^2(1))^T\, \mathbf{C}^{r+1}\, \mathbf{H}(t)\right) + f_0(t). \tag{5.3}$$

Next, we approximate the nonlinear part on the RHS of the original equation using a 2D Haar wavelet expansion:

$$\begin{aligned} S(x,t) &= -\xi u_r^\eta \frac{\partial u_r}{\partial x} + \beta u_r(1 - u_r^\eta) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} m_{i,j} h_i(x) h_j(t) = \mathbf{H}^T(x)\, \mathbf{M}\, \mathbf{H}(t), \end{aligned} \tag{5.4}$$

where the coefficients are given by $m_{i,j} = \langle h_i(x), \langle S(x,t), h_j(t)\rangle\rangle$.

Substituting the approximations from Eqs. (5.1) and (5.4) into the iterative differential equation yields:

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} = \mu\, \mathbf{H}^T(x)\, \mathbf{C}^{r+1}\, \mathbf{H}(t) + \mathbf{H}^T(x)\, \mathbf{M}\, \mathbf{H}(t). \tag{5.5}$$

Applying the operator I_t^α to Eq. (5.5) and using $u_{r+1}(x,0) = g(x)$ gives:

$$u_{r+1}(x,t) = \mu\, \mathbf{H}^T(x)\, \mathbf{C}^{r+1}\, \mathbf{P}_t^\alpha + \mathbf{H}^T(x)\, \mathbf{M}\, \mathbf{P}_t^\alpha + g(x), \tag{5.6}$$

where $\mathbf{P}_t^\alpha = I_t^\alpha \mathbf{H}(t)$ is the operational matrix for fractional integration of order α with respect to t.

We now have two expressions for $u_{r+1}(x,t)$: one from the boundary conditions (Eq. (5.3)) and one from the initial condition and integration of the differential Equation (5.6). Equating them yields:

$$(\mathbf{P}_x^2)^T \mathbf{C}^{r+1}\mathbf{H}(t) - x(\mathbf{P}_x^2(1))^T \mathbf{C}^{r+1}\mathbf{H}(t) - \mu \mathbf{H}^T(x)\mathbf{C}^{r+1}\mathbf{P}_t^\alpha + \mathbf{K}(x,t) - \mathbf{H}^T(x)\mathbf{M}\mathbf{P}_t^\alpha = 0, \tag{5.7}$$

where $\mathbf{K}(x,t) = x(f_1(t) - f_0(t)) + f_0(t) - g(x)$.

To solve numerically, we discretize the problem at the collocation points x_i, t_j for $i,j = 1,2,\ldots,m$. Let H_x and H_t be the $m \times m$ Haar matrices at these points, and let P_x^2, P_t^α be the corresponding operational matrices. Let X be the $m \times m$ matrix where each row is the vector of collocation points x_i. The discrete form of Eq. (5.7) is:

$$\left((\mathsf{P}_x^2)^T - \mathsf{X} \circ (\mathsf{P}_x^2(1))^T\right) \mathbf{C}^{r+1}\mathsf{H}_t - \mu \mathsf{H}_x^T \mathbf{C}^{r+1}\mathsf{P}_t^\alpha = \mathsf{H}_x^T \mathbf{M}\mathsf{P}_t^\alpha - \mathsf{K}, \tag{5.8}$$

where K is the matrix $\mathbf{K}(x_i, t_j)$ and $\circ$ denotes the Hadamard (element-wise) product.

Multiplying both sides of Eq. (5.8) by $(\mathsf{H}_x^T)^{-1}$ from the left and by $(\mathsf{H}_t)^{-1}$ from the right yields a Sylvester equation for the coefficient matrix $\mathbf{C}^{r+1}$:

$$\underbrace{(\mathsf{H}_x^T)^{-1}\left((\mathsf{P}_x^2)^T - \mathsf{X} \circ (\mathsf{P}_x^2(1))^T\right)}_{\mathbf{A}} \mathbf{C}^{r+1} - \mu\, \mathbf{C}^{r+1} \underbrace{\mathsf{P}_t^\alpha (\mathsf{H}_t)^{-1}}_{\mathbf{B}} = \underbrace{(\mathsf{H}_x^T)^{-1}\left(\mathsf{H}_x^T \mathbf{M}\mathsf{P}_t^\alpha - \mathsf{K}\right)(\mathsf{H}_t)^{-1}}_{\mathbf{D}}. \tag{5.9}$$

This equation is of the form $\mathbf{AX} + \mathbf{XB} = \mathbf{D}$, which can be solved for the unknown matrix $\mathbf{X} = \mathbf{C}^{r+1}$.

The iterative process begins with an initial guess $u_0(x,t)$. For $r = 0$, the right-hand side $\mathbf{M}$ is computed based on u_0, and the Sylvester equation is solved to find $\mathbf{C}^1$, which is used to compute $u_1(x,t)$ from Eq. (5.6) or (5.3). This process is repeated for $r = 1, 2, \ldots$ until convergence is achieved.

5.1. **Numerical Examples.** To show the efficacy of the presented method, two numerical examples of the generalized Burgers-Fisher equation is solved for various parameter values of ξ, β, η, and α, with μ fixed at 1. The results are compared against those from the Taylor wavelet collocation method [5] to benchmark performance and demonstrate the capability of the present approach.

Example 5.1. We consider Eq. (1.1) with the following initial/boundary conditions:

$$u(x,0)=\left(\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\xi\eta}{2(1+\eta)}x\right)\right)^{\frac{1}{\eta}},$$
$$u(0,t)=\left(\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\xi\eta}{2(1+\eta)}\left[-\left(\frac{\xi^2+\beta(1+\eta)^2)}{\xi(1+\eta)}\right)t\right]\right)\right)^{\frac{1}{\eta}},$$
$$u(1,t)=\left(\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\xi\eta}{2(1+\eta)}\left[1-\left(\frac{\xi^2+\beta(1+\eta)^2)}{\xi(1+\eta)}\right)t\right]\right)\right)^{\frac{1}{\eta}},$$

which the exact solution for $\alpha=1$ is

$$u(x,t)=\left(\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\xi\eta}{2(1+\eta)}\left[x-\left(\frac{\xi^2+\beta(1+\eta)^2}{\xi(1+\eta)}\right)t\right]\right)\right)^{\frac{1}{\eta}}.$$

An initial approximation, denoted as $u_0(x,t)$, is chosen to initiate the iterative process. The numerical method was subsequently applied for various parameter values of ξ and β, with the fractional order and nonlinearity parameter fixed at $\alpha=1$ and $\eta=1$, respectively. Here we let $\mu=1$.

The absolute errors of the approximate solutions corresponding to these parameter variations are presented in Table 1. To benchmark the performance of the present method, its results were compared against those obtained using the Taylor wavelet collocation technique [5].

Table 1. Comparison of the approximate solutions obtained using the proposed method with Taylor wavelet collocation method [5] for $M=8, K=2$, and $J=4$ in Example 5.1.

	$\xi=\beta=0.5$		$\xi=\beta=0.01$	
(x,t)	The present method	TWCM [5]	The present method	TWCM [5]
$(\frac{1}{32},\frac{1}{32})$	1.3671e-12	2.1744e-10	1.0113e-16	1.2973e-16
$(\frac{7}{32},\frac{7}{32})$	2.1893e-11	2.5308e-09	6.4302e-16	1.9987e-15
$(\frac{13}{32},\frac{13}{32})$	1.2781e-09	1.4702e-08	7.4590e-16	3.2177e-15
$(\frac{19}{32},\frac{19}{32})$	6.3492e-09	8.0278e-08	4.5673e-16	3.7963e-15
$(\frac{25}{32},\frac{25}{32})$	1.2101e-09	1.9275e-08	8.8403e-16	5.7517e-15
$(\frac{31}{32},\frac{31}{32})$	6.5063e-11	8.8047e-09	8.8736e-17	9.1127e-16

Example 5.2. For $\beta=0$, Eq. (1.1) is reduced to the generalized Burger equation. We have taken different values of ξ,η. Here $\xi=1$ and $\beta=0$. The results of the present method were compared with the Taylor wavelet collocation method [5] in Table 2.

Table 2. Comparison of the approximate solutions obtained using the proposed method with Taylor wavelet collocation method ([5]) for $M=8, K=2$, and $J=4$ in Example 5.2.

	$\eta=1\quad\xi=0.1$		$\eta=0.5\quad\xi=0.2$	
(x,t)	The present method	TWCM [5]	The present method	TWCM [5]
$(\frac{1}{32},\frac{1}{32})$	2.7042e-17	1.4365e-16	3.8097e-15	7.0654e-14
$(\frac{7}{32},\frac{7}{32})$	3.2224e-17	2.0481e-16	3.8542e-14	6.831e-13
$(\frac{13}{32},\frac{13}{32})$	2.7023e-17	1.8351e-16	2.3451e-14	3.6372e-13
$(\frac{19}{32},\frac{19}{32})$	2.0219e-17	1.8794e-16	9.0998e-15	1.8324e-13
$(\frac{25}{32},\frac{25}{32})$	3.1099e-17	1.0114e-16	8.2340e-15	1.0011e-13
$(\frac{31}{32},\frac{31}{32})$	0.3984e-17	2.2379e-17	1.6539e-14	9.8046e-14

6. Conclusion

This paper has successfully developed and implemented a numerical scheme for solving the generalized time-fractional Burgers-Fisher equation. The proposed method combines the an iteration technique with a Haar wavelet collocation approach. The key strength of this methodology lies in its ability to handle the inherent nonlinearity of the problem. The numerical experiment presented demonstrate the exceptional performance of the proposed method. In summary, the Haar wavelet iteration method is a highly effective, accurate, and reliable technique for tackling this important class of nonlinear fractional partial differential equations. Its success suggests significant potential for future work. The algorithm can be directly extended to solve other nonlinear FPDEs, including higher-dimensional problems and systems of equations.

References

[1] N. Aghazadeh, A. Mohammadi, and G. Tanoglu, *Taylor wavelets collocation technique for solving fractional nonlinear singular PDEs*, Math. Sci., *18*(1) (2024), 41–54.

[2] N. Aghazadeh, *A Chebyshev wavelet approach to the generalized time-fractional Burgers-Fisher equation*, Comput. Methods Differ. Equ., *13*(4) (2025), 1135–1147.

[3] N. Aghazadeh, G. Ahmadnezhad, and S. Rezapour, *On time fractional modified Camassa-Holm and Degasperis-Procesi equations by using the Haar wavelet iteration method*, Iran. J. Math. Sci. Inform., *18*(1) (2023), 55–71.

[4] Gh. Ahmadnezhad, N. Aghazadeh, and Sh. Rezapour, *Haar wavelet iteration method for solving time fractional Fishers equation*, Comput. Methods Differ. Equ., *8*(3) (2020), 505–522.

[5] Z. Alkhafaji, A. Khani, and N. Aghazadeh, *Treatment of fractional Burgers-Fisher equation using Taylor wavelets*, Comput. Methods Differ. Equ., (2025).

[6] L. M. Angadi, *Wavelet based lifting schemes for the numerical solution of Burgers-Fisher equations*, Electron. J. Math. Anal. Appl., *13*(1) (2025), 7.

[7] A. K. Gupta and S. Saha Ray, *On the solutions of fractional Burgers-Fisher and generalized Fishers equations using two reliable methods*, Int. J. Math. Math. Sci., *2014* (2014), 682910.

[8] M. Hajipour and Y. Lakpour, *A highly accurate numerical technique for solving variable-order fractional Burgers-Huxley equation*, Comput. Methods Differ. Equ., (2025).

[9] F. diz, G. Tanolu, and N. Aghazadeh, *A numerical method based on Legendre wavelet and quasilinearization technique for fractional Lane-Emden type equations*, Numer. Algorithms, *95* (2024), 181–206.

[10] H. Ismail, K. Raslan, and A. Abd Rabboh, *Adomian decomposition method for Burger'sHuxley and Burger'sFisher equations*, Appl. Math. Comput., *159* (2004), 291–301.

[11] S. Kumar and S. Saha Ray, *Numerical treatment for BurgersFisher and generalized BurgersFisher equations*, Math. Sci., *15* (2021), 21–28.

[12] M. Lakestani, R. Ghasemkhani, and T. Allahviranloo, *Solving a system of fractional Volterra integro-differential equations using cubic Hermit spline functions*, Comput. Methods Differ. Equ., *13*(3) (2025), 980–994.

[13] M. Lakestani and R. Tunta, *Efficient Solution for Multi-Delay Fractional Optimal Control Problems via Cubic B-Splines*, Optim. Control Appl. Methods, (2025).

[14] R. Mittal and R. Jiwari, *Differential quadrature method for two-dimensional Burgers' equations*, Int. J. Comput. Methods Eng. Sci. Mech., *10* (2009), 450–459.

[15] R. Mohammadi, *Spline solution of the generalized Burgers'-Fisher equation*, Appl. Anal., *91* (2012), 2189–2215.

[16] Y. F. Patel and J. M. Dhodiya, *A Robust Analytical Approach to Study the Impact of Fuzzy Initial Conditions on Time-Fractional Fisher's Equation*, Comput. Methods Differ. Equ., (2025).

[17] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[18] M. Rashidi, D. Ganji, and S. Dinarvand, *Explicit analytical solutions of the generalized Burger and BurgerFisher equations by homotopy perturbation method*, Numerical Methods For Partial Differential Equations, *25* (2009), 409-417.

[19] U. Saeed and M. Rehman, *Haar wavelet Picard method for fractional nonlinear partial differential equations*, Appl. Math. Comput., *264* (2015), 310–322.

[20] U. Saeed and K. Gilani, *CAS wavelet quasi-linearization technique for the generalized BurgerFisher equation*, Math. Sci., *12* (2018), 61–69.

[21] B. N. Saray, M. Lakestani, and M. Dehghan, *On the sparse multiscale representation of 2-D Burgers equations by an efficient algorithm based on multiwavelets*, Numer. Methods Partial Differential Equations, *39*(3) (2023), 1938–1961.

[22] A. Singh, S. Dahiya, and S. Singh, *A fourth-order B-spline collocation method for nonlinear BurgersFisher equation*, Math. Sci., *14* (2020), 75–85.