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An Iterative Algorithm to Approximate the Solution of Fractional Two-Dimensional Nonlinear Functional Integral Equations in a Banach Space

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Abstract

This study investigates the solvability and approximation of fractional two-dimensional nonlinear functional integral equations in a Banach space. Motivated by the increasing relevance of fractional models in nonlinear sciencesincluding diffusion, viscoelasticity, and nonlinear wave propagationwe establish new existence results using a generalized Darbo fixed-point theorem combined with the measure of noncompactness. To validate the theory, an iterative Sinc-interpolation algorithm is developed, achieving exponential convergence for numerical approximation. The proposed approach not only generalizes existing one-dimensional results to higher dimensions but also provides a practical framework for analyzing soliton-type and other localized nonlinear structures in fractional systems. Numerical experiments confirm the accuracy and effectiveness of the method.

Keywords. Measure of non-compactness, Fixed point theorem, Two dimensional non-linear integral equation, Fractional order, Sinc interpolation.
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1. Introduction and Preliminaries

In numerous fields, it is important to investigate two-dimensional fractional nonlinear functional integral equations. These complex and nonlinear equations are crucial for understanding real-world phenomena in physics, engineering, and applied mathematics. Examples include the conductor-like screening model, quantum chemistry, kinetic theory of gases, bioengineering, and free-electron lasers, among others [11, 17, 22, 25, 28, 33]. As demonstrated in [8, 16, 19, 32, 34, 37, 39, 41, 42], integral equations with singular kernels or fractional forms constitute a significant part of linear and nonlinear analysis. Addressing these problems not only contributes to the advancement of mathematical analysis but also provides valuable insights applicable to a wide range of scientific and industrial fields. The solution of nonlinear integral equations depends largely on the application of fixed-point theory. Consequently, the Schauder and Darbo fixed-point theorems, as well as the measure of noncompactness, have found numerous applications to this type of integral equation, leading to a variety of solvability results for functional equations [2, 5, 7, 13, 20, 21, 29, 30, 35, 36, 43].

Recent studies on soliton-type and nonlinear wave solutions in fractional and integral systems demonstrate the growing interest in capturing localized and stable structures governed by nonlocal nonlinearities. However, most existing works focus on one-dimensional or integer-order models, and the general theory for two-dimensional fractional functional integral equations remains underdeveloped. This gap motivates the current study, which establishes existence results and a constructive algorithm applicable to higher-dimensional fractional systems. In particular, the works [3, 4, 18, 23, 24, 31, 40] have presented analytical and computational approaches for fractional nonlinear evolution equations and soliton solutions, providing valuable foundations for extending these ideas to two-dimensional fractional functional equations.

In this article, we investigate the existence of solutions to two-dimensional fractional nonlinear functional integral equations (which are a kind of singular nonlinear Volterra integral equation of fractional order) in a Banach space,

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using the measure of noncompactness, in the following form:

$$u(\tau,\varpi) = g(\tau,\varpi) + h(\tau,\varpi,u(\tau,\varpi)) \times \int_0^{\varpi}\int_0^{\tau} \frac{(\tau^{\beta}-\xi^{\beta})^{\gamma-1}(\varpi^{\eta}-\zeta^{\eta})^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau,\varpi,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta, \tag{1.1}$$

The superposition operator H generates the function $h(\tau,\varpi,u)$, where $0<\gamma,\theta\leq 1$, $\tau,\varpi\in[0,1]$, and $\beta,\eta>0$. The function $\Gamma(\gamma)$ is defined on the interval $\varpi^{\gamma-1}e^{-\varpi}$ from zero to infinity. In particular, the function $(Hu)(\tau,\varpi)$ is defined as $h(\tau,\varpi,u)$ on the interval $([0,1]\times[0,1])\longrightarrow\Re$. We prove the existence of nontrivial (non-reducing) solutions of Eq. (1.1) in the set of all continuous functions on $([0,1]\times[0,1])\longrightarrow\Re$. In this article, we assume that A is a non-empty subset of $\mathbb{B}$, where $(\mathbb{B},\|\cdot\|)$ denotes a real Banach space in both the one- and two-dimensional cases. In addition, we denote by $\mathfrak{M}\mathbb{B}$ a non-empty family of bounded subsets of $\mathbb{B}$, and by $\mathfrak{N}\mathbb{B}$ a subfamily that includes all relatively compact sets.

Definition 1.1. [13] A mapping $\omega:\mathfrak{M}_{\mathbb{B}}\to\mathbb{R}^{+}$ is considered a measure of non-compactness in $\mathbb{B}$ if it fulfills the following conditions:

(1^0) The family $ker\omega=\{A\in\mathfrak{M}_{\mathbb{B}}:\omega(A)=0\}$ is nonempty and $ker\omega\subset\mathfrak{N}_{\mathbb{B}}$,
(2^0) $A\subset B\Rightarrow\omega(A)\leq\omega(B)$,
(3^0) $\omega(\bar{A})=\omega(A)$,
(4^0) $\omega(ConvA)=\omega(A)$,
(5^0) $\omega(\lambda A+(1-\lambda)B)\leq\lambda\omega(A)+(1-\lambda)\omega(B)$ for $\lambda\in[0,1]$,
(6^0) If $\{A_n\}$ be a sequence of closed sets from m_E such that $A_{n+1}\subset A_n$ for $n\in\mathbb{N}$ and if $\lim_{n\longrightarrow\infty}\omega(A_n)=0$, then the set $A_\infty=\bigcap_{n=1}^{\infty}A_n$ is nonempty.

Fractional nonlinear functional integral equations, particularly those involving ErdlyiKober and RiemannLiouville type operators, play a central role in modeling complex nonlinear phenomena with memory and nonlocal effects. Such equations arise in various areas of nonlinear science and engineering, including anomalous diffusion, viscoelastic materials, quantum mechanics, and nonlinear wave propagation, where fractional orders describe hereditary or spatially distributed interactions. Although fixed-point techniques and measures of noncompactness are well-established tools for proving solvability, the present study extends these ideas to a more general setting of two-dimensional fractional nonlinear functional integral equations in a Banach space.

The main novelty of this work lies in combining the generalized $(\Psi,G,\pounds)$-Darbo fixed point framework with a two-dimensional Sinc-interpolationbased iterative algorithm that achieves exponential convergence. This hybrid analyticalnumerical approach avoids transforming the nonlinear problem into large algebraic systems and provides a highly accurate means of approximating the solution. Beyond establishing solvability, the proposed methodology can be applied to study soliton-type behaviors and other localized nonlinear structures, contributing to the broader development of fractional models in nonlinear sciences.

2. A generalization of Darbo fixed point theorem

To handle a generalization of the Darbo fixed-point theorem [5], we apply a type of contraction utilized in [35]. From now on, we assume that the functions G,Ψ, and $\pounds:[0,+\infty)\to[0,+\infty)$ satisfy the following conditions:

(i) $G\in C[0,+\infty)$ and $G(0)=0<G(\tau), for\ all\ \tau>0$;
(ii) $\pounds(\tau)<\Psi(\tau), for\ all\ \tau>0$ and $\pounds(0)=\Psi(0)=0$;
(iii) $\pounds(\tau),\Psi(\tau)\in C[0,+\infty)$;
(iv) Ψ is increasing.

Furthermore, let $\mathbb{G}=\{G:G$ satisfies condition (i)$\}$ and $\Sigma=\{(\Psi,\pounds):\Psi$ and $\pounds$ satisfy situations (ii),(iii), and (iv)$\}$. By the following definition and theorem, we present a generalization of the $(\Psi,G,\pounds)$-contractive mapping using the measure of noncompactness and its application [35].

Definition 2.1. Let $\varrho\neq\varnothing$ is a subset of $\mathbb{B}$, and $\varsigma:\varrho\to\varrho$ be a mapping. We define ς as a generalized $(\Psi,G,\pounds)$-contractive mapping if, for any $0<a<b<\infty$, there exist $0<\rho_{ab}<1$, $G\in\mathbb{G}$, and $(\Psi,\pounds)\in\Sigma$, where for all $A\subseteq\varrho$

and arbitrary measure of non-compactness ω, the following holds:

$$a \leq G(\omega(A)) \leq b \Longrightarrow \Psi(G(\omega(\varsigma A))) \leq \rho_{ab}\pounds(G(\omega(A))). \tag{2.1}$$

Theorem 2.2. *Regard $\varrho \neq \varnothing$, a closed, bounded, convex subset of $\mathbb{B}$, and $\varsigma : \varrho \to \varrho$ as a generalized $(\Psi, G, \pounds)$-contractive continuous mapping. It follows that ς has at least one fixed point in ϱ.*

Proof. In [35] the proof is done in Banach space $\mathbb{B} = C([0,1])$ with standard norm. Because it is similar to proof in Banach space $\mathbb{B} = C([0,1] \times [0,1])$ equipped to standard norm, thus we leave it.

As an application of a generalization of Darbo fixed point theorem, we prove the existence of solution of two-dimensional fractional non-linear functional integral equations in Banach space $\mathbb{B} = C([0,1] \times [0,1])$ with the standard norm:

$$||u|| = max\{|u(\tau,\varpi)| : \tau,\varpi \geq 0\}.$$

Thus, let $A \neq \varnothing$ be a bounded subset of $C([0,1] \times [0,1])$, and for $u \in A$ and $\epsilon \geq 0$, we define

$$\Pi(u,\epsilon) := sup\{|u(\tau,\varpi) - u(\xi,\zeta)| : \tau,\varpi,\xi\,,\zeta\ in[0,1], |\tau - \xi| \leq \epsilon\,, |\varpi - \zeta| \leq \epsilon\}, \tag{2.2}$$

$$\Pi(A,\epsilon) := sup\{\Pi(u,\epsilon) : u \in A\}, \tag{2.3}$$
$$\Pi_0(A) := \lim_{\epsilon\to 0} \Pi(A,\epsilon),$$

and

$$J(u) := sup\{|u(\xi,\zeta)) - u(\tau,\varpi)| - [u(\xi,\zeta)) - u(\tau,\varpi)] : \tau,\varpi,\xi\,,\zeta \in [0,1], \tau \leq \xi\ , \varpi \leq \zeta\}, \tag{2.4}$$
$$J(A) := sup\{J(u) : u \in A\}.$$

It is clear that all functions within A are non-decreasing on $[0,1] \times [0,1]$ if and only if $J(A) = 0$. Now let us introduce ω on $\mathfrak{M}_C([0,1] \times [0,1])$ by:

$$\omega(A) := \Pi_0(A) + J(A). \tag{2.5}$$

Following [12], it is easy to indicate that the function ω is a measure of noncompactness on $C([0,1] \times [0,1])$. Now, let's survey Eq. (1.1) due to following hypotheses:

- (b_1) $g : [0,1] \times [0,1] \to \mathbb{R}^+$ is continuous, non-decreasing and nonnegative function;
- (b_2) $h : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function in τ, ϖ, u such that $h([0,1] \times [0,1] \times \mathbb{R}^+) \subseteq \mathbb{R}^+$. Moreover, there was a continuous non-decreasing function $\pounds : \mathbb{R}^+ \to \mathbb{R}^+$ with $\pounds(0) = 0$. For every $\tau > 0$, it holds that $\pounds(\tau) < \tau$, such that

  $$|h(\tau,\varpi,u) - h(\tau,\varpi,z)| \leq \pounds(|u - z|), \forall\tau,\varpi \in [0,1], \forall u, z \in \mathbb{R},$$

  also $\pounds$ is superadditive, $\pounds(\tau) + \pounds(\xi) \leq \pounds(\tau + \xi)$ for all $\tau, \xi \in \mathbb{R}^+$;
- (b_3) $P : \text{Im}u \to \mathbb{R}^+$ is non-decreasing continuous function on the compact set $\text{Im}u$.

  $$||P(u(\xi,\zeta))|| \leq ||u(\xi,\zeta)||,\ ||P(u_1) - P(u_2)|| \leq L||u_1 - u_2||,$$
- (b_4) In Eq. (1.1), the operator H satisfies the condition $J(Hu) \leq \pounds(J(u))$ for any nonnegative function u, where $\pounds$ is introduced in (b_2).
- (b_5) $f : [0,1] \times [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ is continuous yet non-decreasing with respect to every variable separately;
- (b_6) $k : \text{Im}f \to \mathbb{R}^+$ is non-decreasing continuous function on the compact set $\text{Im}f$.
- (b_7) By hypotheses $M_1 = max\{|g(\tau,\varpi)| : \tau,\varpi \in [0,1]\}$ and $M_2 = max\{|h(\tau,\varpi,0)| : \tau,\varpi \in [0,1]\}$, the bellow inequality

  $$M_1\Gamma(\gamma + 1)\Gamma(\theta + 1) + (\pounds(r) + M_2)||k||||u|| \leq \Gamma(\gamma + 1)\Gamma(\theta + 1)r,$$

has a solution as $r_0 > 0$, where $\lambda = \dfrac{||k||r_0}{\Gamma(\gamma+1)\Gamma(\theta+1)} < 1$.

Lemma 2.3. *[14] If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by $\phi(\tau) = \tau^\beta$ and $I = [0,\ 1]$ then*

(1) *If $0 < \beta < 1$ and $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$ then $\tau_2^\beta - \tau_1^\beta \le (\tau_2 - \tau_1)^\beta$.*
(2) *If $\beta \ge 1$ and $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$ then $\tau_2^\beta - \tau_1^\beta \le \beta(\tau_2 - \tau_1)$.*

Lemma 2.4. *Let $\tau_1, \tau_2, \varpi_1, \varpi_2 \in [0,1]$, $\tau_2 \ge \tau_1, \varpi_2 \ge \varpi_1$ and $|\tau_2 - \tau_1| \le \epsilon, |\varpi_2 - \varpi_1| \le \epsilon$, $0 < \gamma, \theta < 1$ and $\beta, \eta > 0$ Then,*

$$(\tau_2^\beta - \tau_1^\beta)^\gamma (\varpi_2^\eta - \varpi_1^\eta)^\theta = \begin{cases} (\tau_2-\tau_1)^{\beta\gamma}(\varpi_2-\varpi_1)^{\eta\theta}\ ,\ 0<\beta,\ \eta<1 \\ (\tau_2-\tau_1)^{\beta\gamma}(\varpi_2-\varpi_1)^{\theta}\eta^{\theta}\ ,\ 0<\beta<1,\ \eta\ge 1 \\ \beta^\gamma(\tau_2-\tau_1)^{\gamma}(\varpi_2-\varpi_1)^{\eta\theta}\ ,\ \beta\ge 1,\ 0<\eta<1, \\ \beta^\gamma(\tau_2-\tau_1)^{\gamma}(\varpi_2-\varpi_1)^{\theta}\eta^\theta\ ,\ \beta,\ \eta\ge 1., \end{cases} \le \begin{cases} (\epsilon)^{\beta\gamma}(\epsilon)^{\eta\theta}\ ,\ 0<\beta,\ \eta<1 \\ (\epsilon)^{\beta\gamma}(\epsilon\eta)^{\theta}\ ,\ 0<\beta<1,\ \eta\ge 1 \\ (\beta\epsilon)^{\gamma}(\epsilon)^{\eta\theta}\ ,\ \beta\ge 1,\ 0<\eta<1, \\ (\beta\epsilon)^{\gamma}(\epsilon\eta)^{\theta}\ ,\ \beta,\ \eta\ge 1. \end{cases}$$

Proof. It is similar to proof of Lemma 2.3 and we neglect it.

We use Lemma 2.4 to prove the following theorem.

Theorem 2.5. *Under hypotheses (b_1)-(b_6), the Eq. (1.1) has at least one non-decreasing solution as $u = u(\tau, \varpi) \in C([0,1]\times[0,1])$.*

Proof. By considering Eq. (1.1) operators G and ς on $C([0,1]\times[0,1])$ are defined as follows,

$$(Gu)(\tau,\varpi) = \int_0^\varpi \int_0^\tau \frac{(\tau^\beta - \xi^\beta)^{\gamma-1}(\varpi^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau,\varpi,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta, \tag{2.6}$$
$$(\varsigma u)(\tau,\varpi) = g(\tau,\varpi) + h(\tau,\varpi,u(\tau,\varpi))(Gu)(\tau,\varpi).$$

Initially, it's shown that G is self-map on $C([0,1]\times[0,1])$. Let $\epsilon > 0$ is given and assume $u \in C([0,1]\times[0,1])$ and $\tau_1, \tau_2, \varpi_1, \varpi_2 \in [0,1]$ (without loss of generality) let $\tau_2 \ge \tau_1, \varpi_2 \ge \varpi_1$ and $|\tau_2 - \tau_1| \le \epsilon, |\varpi_2 - \varpi_1| \le \epsilon$. Then,

$$\begin{aligned}
|(Gu)(\tau_2,\varpi_2) - (Gu)(\tau_1,\varpi_1)\,| = &\Big| \int_0^{\varpi_2}\int_0^{\tau_2} \frac{(\tau_2^\beta - \xi^\beta)^{\gamma-1}(\varpi_2^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_2,\varpi_2,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \\
&- \int_0^{\varpi_1}\int_0^{\tau_1} \frac{(\tau_1^\beta - \xi^\beta)^{\gamma-1}(\varpi_1^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_1,\varpi_1,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \Big| \\
\le &\Big| \int_0^{\varpi_2}\int_0^{\tau_2} \frac{(\tau_2^\beta - \xi^\beta)^{\gamma-1}(\varpi_2^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_2,\varpi_2,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \\
&- \int_0^{\varpi_2}\int_0^{\tau_2} \frac{(\tau_2^\beta - \xi^\beta)^{\gamma-1}(\varpi_2^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma),\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_1,\varpi_1,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \Big| \\
&+ \Big| \int_0^{\varpi_2}\int_0^{\tau_2} \frac{(\tau_2^\beta - \xi^\beta)^{\gamma-1}(\varpi_2^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_1,\varpi_1,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \\
&- \int_0^{\varpi_1}\int_0^{\tau_1} \frac{(\tau_2^\beta - \xi^\beta)^{\gamma-1}(\varpi_2^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma),\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_1,\varpi_1,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \Big| \\
&+ \Big| \int_0^{\varpi_1}\int_0^{\tau_1} \frac{(\tau_2^\beta - \xi^\beta)^{\gamma-1}(\varpi_2^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_1,\varpi_1,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \\
&- \int_0^{\varpi_1}\int_0^{\tau_1} \frac{(\tau_1^\beta - \xi^\beta)^{\gamma-1}(\varpi_1^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau_1,\varpi_1,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \Big| \\
\le &\int_0^{\varpi_2}\int_0^{\tau_2} \frac{(\tau_2^\beta - \xi^\beta)^{\gamma-1}(\varpi_2^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}|k(f(\tau_2,\varpi_2,\xi,\zeta)) - k(f(\tau_1,\varpi_1,\xi,\zeta))|
\end{aligned}$$

$$
\begin{aligned}
&|P(u(\xi,\zeta))|d\xi d\zeta\\
&+\int_{\varpi_1}^{\varpi_2}\int_{\tau_1}^{\tau_2}\frac{(\tau_2^\beta-\xi^\beta)^{\gamma-1}(\varpi_2^\eta-\zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,\beta\xi^{\beta-1}\eta\zeta^{\eta-1}|k(f(\tau_1,\varpi_1,\xi,\zeta))||P(u(\xi,\zeta))|d\xi d\zeta\\
&+\int_0^{\varpi_1}\int_0^{\tau_1}\frac{|(\tau_2^\beta-\xi^\beta)^{\gamma-1}(\varpi_2^\eta-\zeta^\eta)^{\theta-1}-(\tau_1^\beta-\xi^\beta)^{\gamma-1}(\varpi_1^\eta-\zeta^\eta)^{\theta-1}|}{\Gamma(\gamma)\Gamma(\theta)}\\
&\beta\xi^{\beta-1}\eta\zeta^{\eta-1}|k(f(\tau_1,\varpi_1,\xi,\zeta))||P(u(\xi,\zeta))|d\xi d\zeta.
\end{aligned}
$$

Thus, if we put

$$\Pi_{kof}(\epsilon,.)=sup\{|k(f(\tau,\varpi,\xi,\zeta))-k(f(\tau',\varpi',\xi,\zeta))|:\tau,\tau',\varpi,\varpi',\xi\ ,\zeta\ in\ [0,1]\ and\ |\tau-\tau'|\le\epsilon\ ,|\varpi-\varpi'|\le\epsilon\},$$

As a result, by applying hypothesis (b_3) we can write,

$$
\begin{aligned}
|(Gu)(\tau_2,\varpi_2)-(Gu)(\tau_1,\varpi_1)|\le&\frac{\|P(u)\|\Pi_{kof}(\epsilon,.)}{\Gamma(\gamma)\Gamma(\theta)}\int_0^{\varpi_2}\int_0^{\tau_2}(\tau_2^\beta-\xi^\beta)^{\gamma-1}(\varpi_2^\eta-\zeta^\eta)^{\theta-1}\beta\xi^{\beta-1}\eta\zeta^{\eta-1}d\xi d\zeta\\
&+\frac{\|P(u)\|||k||}{\Gamma(\gamma)\Gamma(\theta)}\int_{\varpi_1}^{\varpi_2}\int_{\tau_1}^{\tau_2}(\tau_2^\beta-\xi^\beta)^{\gamma-1}(\varpi_2^\eta-\zeta^\eta)^{\theta-1}\beta\xi^{\beta-1}\eta\zeta^{\eta-1}d\xi d\zeta\\
&+\frac{\|P(u)\|||k||}{\Gamma(\gamma)\Gamma(\theta)}\int_0^{\varpi_1}\int_0^{\tau_1}[(\tau_1^\beta-\xi^\beta)^{\gamma-1}(\varpi_1^\eta-\zeta^\eta)^{\theta-1}-(\tau_2^\beta\\
&-\xi^\beta)^{\gamma-1}(\varpi_2^\eta-\zeta^\eta)^{\theta-1}]\beta\xi^{\beta-1}\eta\zeta^{\eta-1}d\xi d\zeta\\
\le&\frac{||u|\Pi_{kof}(\epsilon,.)}{\Gamma(\gamma)\Gamma(\theta)}\frac{\tau_2^{\beta\gamma}}{\gamma}\frac{\varpi_2^{\eta\theta}}{\theta}+\frac{||u|||k||}{\Gamma(\gamma)\Gamma(\theta)}\frac{(\tau_2^\beta-\tau_1^\beta)^\gamma}{\gamma}\frac{(\varpi_2^\eta-\varpi_1^\eta)^\theta}{\theta}\\
&+\frac{||u||||k||}{\Gamma(\gamma)\Gamma(\theta)}[\frac{(\tau_2^\beta-\tau_1^\beta)^\gamma(\varpi_2^\eta-\varpi_1^\eta)^\theta}{\gamma\theta}+\frac{\tau_1^{\beta\gamma}}{\gamma}\frac{\varpi_1^{\eta\theta}}{\theta}-\frac{\tau_2^{\beta\gamma}}{\gamma}\frac{\varpi_2^{\eta\theta}}{\theta}]\\
\le&\frac{\|u\|\Pi_{kof}(\epsilon,.)}{\Gamma(\theta+1)\Gamma(\gamma+1)}+\frac{2\|u\|||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}(\tau_2^\beta-\tau_1^\beta)^\gamma(\varpi_2^\eta-\varpi_1^\eta)^\theta \qquad (2.7)
\end{aligned}
$$

Hence, regarding the uniform continuity of the function kof on the set $[0,1]\times[0,1]\times[0,1]\times[0,1]$ we have $\zeta_{kof}(\epsilon,.)\to 0$ as $\epsilon\to 0$. Thus $Gu\in C([0,1]\times[0,1])$, and consequently, $\varsigma u\in C([0,1]\times[0,1])$. Moreover, from Lemma 2.4 the last team of (2.7) leads to zero as $\epsilon\to 0$. Therefore, we can get

$$
\begin{aligned}
|(Gu)(\tau,\varpi)|\le&\int_0^{\varpi}\int_0^{\tau}\frac{(\tau^\beta-\xi^\beta)^{\gamma-1}(\varpi^\eta-\zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,\beta\xi^{\beta-1}\eta\zeta^{\eta-1}|k(f(\tau,\varpi,\xi,\zeta))||P(u(\xi,\zeta))|d\xi d\zeta,\\
\le&\frac{||k||\ ||p(u)||}{\Gamma(\gamma)\Gamma(\theta)}\int_0^{\varpi}\int_0^{\tau}(\tau^\beta-\xi^\beta)^{\gamma-1}(\varpi^\eta-\zeta^\eta)^{\theta-1}\,\beta\xi^{\beta-1}\eta\zeta^{\eta-1}d\xi d\zeta\\
\le&\frac{||k||\ ||u||}{\Gamma(\gamma+1)\Gamma(\theta+1)}, \qquad (2.8)
\end{aligned}
$$

for all $\tau,\varpi\in[0,1]$. Thus

$$
\begin{aligned}
|(\varsigma u)(\tau,\varpi)|&\le|g(\tau,\varpi)|+|h(\tau,\varpi,u)||Gu(\tau,\varpi)|\\
&\le M_1+[|h(\tau,\varpi,u)-h(\tau,\varpi,0)|+|h(\tau,\varpi,0)|]\frac{||k||||u||}{\Gamma(\theta+1)\Gamma(\gamma+1)}\\
&\le M_1+(\pounds(||u||)+M_2)\frac{||k||||u||}{\Gamma(\theta+1)\Gamma(\gamma+1)}.
\end{aligned}
$$

For this reason,

$$||\varsigma u||\le M_1+(\pounds(||u||)+M_2)\frac{||k||||u||}{\Gamma(\theta+1)\Gamma(\gamma+1)}.$$

Hence, if $||u|| \leq r_0$ then by assumption (b_7) we have,

$$||\varsigma u|| \leq M_1 + (\pounds(r_0) + M_2)\frac{||k||||u||}{\Gamma(\gamma+1)\Gamma(\theta+1)} \leq r_0.$$

Therefore, the operator ς maps the ball $B_{r_0} \subset C([0,1]\times[0,1])$ into itself. To show the continuity of ς on B_{r_0}, assume that $\{u_n\}$ be a sequence in B_{r_0} as $u_n \to u$. We require to indicate that $\varsigma u_n \to \varsigma u$, specifically, for all $\tau, \varpi \in [0,1]$,

$$\begin{aligned}
|(Gu_n)(\tau,\varpi) - (Gu)(\tau,\varpi)| &= \Big| \int_0^{\varpi}\int_0^{\tau} \frac{(\tau^\beta - \xi^\beta)^{\gamma-1}(\varpi^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau,\varpi,\xi,\zeta))P(u_n(\xi,\zeta))d\xi d\zeta \\
&\quad - \int_0^{\varpi}\int_0^{\tau} \frac{(\tau^\beta - \xi^\beta)^{\gamma-1}(\varpi^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1}k(f(\tau,\varpi,\xi,\zeta))P(u(\xi,\zeta))d\xi d\zeta \Big| \\
&\leq \int_0^{\varpi}\int_0^{\tau} \frac{(\tau^\beta - \xi^\beta)^{\gamma-1}(\varpi^\eta - \zeta^\eta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)} \beta\xi^{\beta-1}\eta\zeta^{\eta-1} \mid k(f(\tau,\varpi,\xi,\zeta)) \mid \\
&\quad \mid P(u_n(\xi,\zeta)) - P(u(\xi,\zeta)) \mid d\xi d\zeta \\
&\leq \frac{1}{\Gamma(\gamma)\Gamma(\theta)}||k||||P(u_n) - P(u)||\frac{\tau^{\beta\gamma}\varpi^{\eta\theta}}{(\gamma)(\theta)},
\end{aligned}$$

Since $\tau^{\beta\gamma}\varpi^{\eta\theta} \leqslant 1$, then

$$||Gu_n - Gu|| \leq \frac{||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}||P(u_n) - P(u)||. \tag{2.9}$$

From (b_2) and (b_7) it is concluded that,

$$\begin{aligned}
|(\varsigma u_n)(\tau,\varpi) - (\varsigma u)(\tau,\varpi)| &= |h(\tau,\varpi,u_n(\tau,\varpi))(Gu_n)(\tau,\varpi) - h(\tau,\varpi,u(\tau,\varpi))(Gu)(\tau,\varpi)| \\
&\leq |h(\tau,\varpi,u_n(\tau,\varpi))(Gu_n)(\tau,\varpi) - h(\tau,\varpi,u(\tau,\varpi))(Gu_n)(\tau,\varpi)| \\
&\quad + |h(\tau,\varpi,u(\tau,\varpi))(Gu_n)(\tau,\varpi) - h(\tau,\varpi,u(\tau,\varpi))(Gu)(\tau,\varpi)| \\
&\leq |h(\tau,\varpi,u_n(\tau,\varpi)) - h(\tau,\varpi,u(\tau,\varpi))||(Gu_n)(\tau,\varpi)| \\
&\quad + |h(\tau,\varpi,u(\tau,\varpi)) - h(\tau,\varpi,0) + h(\tau,\varpi,0)||(Gu_n)(\tau,\varpi) - (Gu)(\tau,\varpi)| \\
&\leq \pounds(|u_n(\tau,\varpi) - u(\tau,\varpi)|)|(Gu_n)(\tau,\varpi)| \\
&\quad + (\pounds(|u(\tau,\varpi)|) + M_2)||Gu_n - Gu||,
\end{aligned}$$

regarding (2.8) and (2.9), it follows that

$$\begin{aligned}
||\varsigma u_n - \varsigma u|| &\leq \pounds(||u_n - u||)\frac{||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}||P(u_n)|| + (\pounds(||u||) + M_2)\frac{||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}||P(u_n) - P(u)|| \\
&\leq \pounds(||u_n - u||)\frac{||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}||u_n|| + (\pounds(||u||) + M_2)\frac{||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}L||u_n - u||
\end{aligned}$$

Thus ς is continuous on B_{r_0}. we show,

$$\widetilde{B}_{r_0} = \{u \in B_{r_0} : u(\tau,\varpi) \geq 0, \ for \ \tau,\varpi \in [0,1]\} \subseteqq B_{r_0},$$

clearly, $\widetilde{B}_{r_0} \neq \varnothing$ is closed, bounded and convex. By assumptions $(b_1), (b_2)$ and (b_5) if $u(\tau,\varpi) \geq 0$ then $(\varsigma u)(\tau,\varpi) \geq 0$ for all $\tau,\varpi \in [0,1]$, ς projects $\widetilde{B}_{r_0}$ into itself as a result. Thus, ς is continuous on $\widetilde{B}_{r_0}$. Suppose $A \neq \varnothing$ be a subset of $\widetilde{B}_{r_0}$, also $\epsilon > 0$ and

$$\tau_1, \ \tau_2, \ \varpi_1, \ \varpi_2 \in [0,1]; \quad |\tau_2 - \tau_1| \leq \epsilon; \quad |\varpi_2 - \varpi_1| \leq \epsilon,$$

For simplicity, assuming $\tau_2 \geq \tau_1$ and $\varpi_2 \geq \varpi_1$, and moreover, using (2.2), (2.7), (2.8), (b_2) and (b_7) then we have,

$$|(\varsigma u)(\tau_2,\varpi_2) - (\varsigma u)(\tau_1,\varpi_1)| = |g(\tau_2,\varpi_2) + h(\tau_2,\varpi_2,u(\tau_2,\varpi_2))(Gu)(\tau_2,\varpi_2) - g(\tau_1,\varpi_1) - h(\tau_1,\varpi_1,u(\tau_1,\varpi_1))(Gu)(\tau_1,\varpi_1)|$$

$$
\begin{aligned}
&\leq |g(\tau_2,\varpi_2)-g(\tau_1,\varpi_1)| + |h(\tau_2,\varpi_2,u(\tau_2,\varpi_2))(Gu)(\tau_2,\varpi_2) - h(\tau_1,\varpi_1,u(\tau_2,\varpi_2))(Gu)(\tau_2,\varpi_2)| \\
&+ |h(\tau_1,\varpi_1,u(\tau_2,\varpi_2))(Gu)(\tau_2,\varpi_2) - h(\tau_1,\varpi_1,u(\tau_1,\varpi_1))(Gu)(\tau_2,\varpi_2)| \\
&+ |h(\tau_1,\varpi_1,u(\tau_1,\varpi_1))(Gu)(\tau_2,\varpi_2) - h(\tau_1,\varpi_1,u(\tau_1,\varpi_1))(Gu)(\tau_1,\varpi_1)| \\
&\leq |g(\tau_2,\varpi_2)-g(\tau_1,\varpi_1)| + |h(\tau_2,\varpi_2,u(\tau_2,\varpi_2)) - h(\tau_1,\varpi_1,u(\tau_2,\varpi_2))||(Gu)(\tau_2,\varpi_2)| \\
&+ |h(\tau_1,\varpi_1,u(\tau_2,\varpi_2)) - h(\tau_1,\varpi_1,u(\tau_1,\varpi_1))||(Gu)(\tau_2,\varpi_2)| \\
&+ |h(\tau_1,\varpi_1,u(\tau_1,\varpi_1)) - h(\tau_1,\varpi_1,0) + h(\tau_1,\varpi_1,0)||(Gu)(\tau_2,\varpi_2) - (Gu)(\tau_1,\varpi_1)| \\
&\leq \Pi(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{||u||||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)} + \pounds(|u(\tau_2,\varpi_2)-u(\tau_1,\varpi_1)|)\frac{||u||||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)} \\
&+ (\pounds(||u||)+M_2)\Big[\frac{||u||\Pi_{kof}(\epsilon,.)}{\Gamma(\theta+1)\Gamma(\gamma+1)} + \frac{2||u||||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}(\tau_2^{\beta}-\tau_1^{\beta})^{\gamma}(\varpi_2^{\eta}-\varpi_1^{\eta})^{\theta}\Big],
\end{aligned}
$$

where we introduce

$$\rho_{r_0}(h,\epsilon) = sup\{|h(\tau,\varpi,u)-h(\tau',\varpi',u)| : \tau,\tau',\varpi,\varpi' \in [0,1], u \in [0,r_0]\times[0,r_0], |\tau-\tau'| \leq \epsilon\ , |\varpi-\varpi'| \leq \epsilon\},$$

in the last inequality, we drive that,

$$
\begin{aligned}
|(\varsigma u)(\tau_2,\varpi_2)-(\varsigma u)(\tau_1,\varpi_1)| &\leq \Pi(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{||u||||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)} + \pounds(|u(\tau_2,\varpi_2)-u(\tau_1,\varpi_1)|)\frac{||u||||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)} \\
&+ (\pounds(||u||)+M_2)\Big[\frac{||u||\Pi_{kof}(\epsilon,.)}{\Gamma(\gamma+1)\Gamma(\theta+1)} + \frac{2||u||||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}(\tau_2^{\beta}-\tau_1^{\beta})^{\gamma}(\varpi_2^{\eta}-\varpi_1^{\eta})^{\theta}\Big], \quad (2.10)
\end{aligned}
$$

with regard to (2.2) and (2.10) indicate that,

$$
\begin{aligned}
\Pi(\varsigma u,\epsilon) &\leq \Pi(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)} + \pounds(\Pi(u,\epsilon))\frac{r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)} \\
&+ (\pounds(r_0)+M_2)[\frac{r_0\Pi_{kof}(\epsilon,.)}{\Gamma(\theta+1)\Gamma(\gamma+1)} + \frac{2r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}(\tau_2^{\beta}-\tau_1^{\beta})^{\gamma}(\varpi_2^{\eta}-\varpi_1^{\eta})^{\theta}].
\end{aligned}
$$

Considering supremum on $u \in A$, concludes that,

$$
\begin{aligned}
\Pi(\varsigma A,\epsilon) &\leq \Pi(g,\epsilon) + \rho_{r_0}(h,\epsilon)\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)} + \pounds(\Pi(A,\epsilon))\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)} \\
&+ (\pounds(r_0)+M_2)[\frac{r_0\Pi_{kof}(\epsilon,.)}{\Gamma(\theta+1)\Gamma(\gamma+1)} + \frac{2r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}(\tau_2^{\beta}-\tau_1^{\beta})^{\gamma}(\varpi_2^{\eta}-\varpi_1^{\eta})^{\theta}].
\end{aligned}
$$

As g is continuous on $[0,1]\times[0,1]$, and both h and $k\circ f$ are uniformly continuous on $[0,1]\times[0,1]\times[0,r_0]$ and $[0,1]\times[0,1]\times[0,1]\times[0,1]$ respectively, hence when $\epsilon \to 0$ then $\Pi(g,\epsilon)\to 0$, $\rho_{r_0}(h,\epsilon)\to 0$, $\Pi_{kof}(\epsilon,.)\to 0$ and considering Lemma 2.4, $(\tau_2^{\beta}-\tau_1^{\beta})^{\gamma}(\varpi_2^{\eta}-\varpi_1^{\eta})^{\theta}\to 0$ as $\epsilon\to 0$. Moreover, regarding (2.3) we can get,

$$\Pi_0(\varsigma A) \leq \frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}\ \pounds(\Pi_0(A)). \quad (2.11)$$

Supposing $u \in A$ and $\tau_1,\tau_2,\varpi_1,\varpi_2 \in [0,1]$ which $\tau_1 < \tau_2, \varpi_1 < \varpi_2$ and considering (2.4), (2.7), (b_2), (b_7) and non-decreasing functions g, Gu then,

$$
\begin{aligned}
&|(\varsigma u)(s_2,\tau_2)-(\varsigma u)(s_1,\tau_1)| - [(\varsigma u)(s_2,\tau_2)-(\varsigma u)(s_1,\tau_1)] \\
&= |g(s_2,\tau_2)+h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2) - g(s_1,\tau_1) - h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)| \\
&- \Big[g(s_2,\tau_2)+h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2) - g(s_1,\tau_1) - h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)\Big] \\
&\leq \{|g(s_2,\tau_2)-g(s_1,\tau_1)| - [g(s_2,\tau_2)-g(s_1,\tau_1)]\} \\
&+ |h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2) - h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)| \\
&+ |h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2) - h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)| \\
&- \Big[h(s_2,\tau_2,u(s_2,\tau_2))(Gu)(s_2,\tau_2) - h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)\Big]
\end{aligned}
$$

$$+\Big[h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_2,\tau_2)-h(s_1,\tau_1,u(s_1,\tau_1))(Gu)(s_1,\tau_1)\Big]$$
$$\leq J(g)+\Big\{|h(s_2,\tau_2,u(s_2,\tau_2))-h(s_1,\tau_1,u(s_1,\tau_1))|-\Big[h(s_2,\tau_2,u(s_2,\tau_2))-h(s_1,\tau_1,u(s_1,\tau_1))\Big]\Big\}(Gu)(s_2,\tau_2)$$
$$+h(s_1,\tau_1,u(s_1,\tau_1))\Big\{|(Gu)(s_2,\tau_2)-(Gu)(s_1,\tau_1)|-\Big[(Gu)(s_2,\tau_2)-(Gu)(s_1,\tau_1)\Big]\Big\}$$
$$\leq J(g)+J(Hu)\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}+(\pounds(||u||)+M_2)J(Gu)$$
$$=J(Hu)\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}.$$

Hence, with the help of (b_4) it concludes that,

$$J(\varsigma u)\leq J(Hu)\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}\leq \pounds(J(u))\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)},$$

and from (2.4) we drive a result,

$$J(\varsigma A)\leq \frac{r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}\pounds(J(A)). \tag{2.12}$$

Considering the definition of ω in (2.5), and (2.11) and (2.12) we have,

$$\begin{aligned}\omega(\varsigma A)&=\Pi_0(\varsigma A)+J(\varsigma A)\\
&\leq \frac{r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}\ \pounds(\Pi_0(A))+\frac{r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}\pounds(J(A))\\
&\leq \frac{r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}(\pounds(\Pi_0(A))+\pounds(J(A)))\\
&\leq \frac{r_0||k||}{\Gamma(\theta+1)\Gamma(\gamma+1)}(\pounds(\Pi_0(A)+J(A)))\\
&\leq \lambda\pounds(\omega(A)).\end{aligned}$$

We finish the proof by using Theorem 2.2 for the case of $G(\tau)=\Psi(\tau)=\tau$, aforementioned inequality and the condition $\lambda=\frac{r_0||k||}{\Gamma(\gamma+1)\Gamma(\theta+1)}<1$.

Now, we will look at some of the corollaries are obtained from this research.

Alamo and Rodríguez [6] defined Erdélyi-Kober fractional integral of a continuous function f as follows,

$$I^{\gamma}_{\beta}f(t)=\frac{\beta}{\Gamma(\gamma)}\int_0^t\frac{s^{\beta-1}f(s)}{(t^{\beta}-s^{\beta})^{1-\gamma}}ds,\ \beta>0,\ 0<\gamma<1.$$

Also, Erdélyi-Kober two-dimensional fractional integral is introduced by [14, 15] for a continuous function f on $\mathbb{R}\times\mathbb{R}$,

$$I^{\gamma,\theta}_{\beta}f(x,y)=\frac{\beta^2}{\Gamma(\gamma)\Gamma(\theta)}\int_0^x\int_0^y\frac{s^{\beta-1}t^{\beta-1}f(t,s)}{(y^{\beta}-t^{\beta})^{1-\gamma}(x^{\beta}-s^{\beta})^{1-\theta}}dtds,\ 0<\gamma,\ \theta<1,\ \beta>0, \tag{2.13}$$

where $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\ z>0$.

According to Theorem 2.5 and Erdélyi-Kober fractional integral (2.13) we can give the bellow corollary.

Corollary 2.6. *If Theorem 2.5's conditions are hold, then there is at least one solution for integral equations with fractional order in $C([0,1]\times[0,1])$, instances of this would be (i), (ii), and (iii);*
i) for $\beta=\eta$, It indicates, two dimensional Erdélyi-Kober fractional non-linear integral equations with fractional orders γ and θ as follows,

$$u(\tau,\varpi)=g(\tau,\varpi)+h(\tau,\varpi,u(\tau,\varpi))\int_0^{\varpi}\int_0^{\tau}\frac{(\tau^{\beta}-\xi^{\beta})^{\gamma-1}(\varpi^{\beta}-\zeta^{\beta})^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\beta^2\xi^{\beta-1}\zeta^{\beta-1}k(f(\tau,,\varpi,\xi,\zeta))P(u(\xi,\zeta))\,d\xi d\zeta,$$

ii) for $\beta=\eta$ and $h(\tau,\varpi,u(\tau,\varpi))=1$, *we provide, two dimensional Erdélyi-Kober nonlinear integral equations of the second kind with fractional orders* γ *and* θ*,*

$$u(\tau,\varpi)=g(\tau,\varpi)+\int_0^{\varpi}\int_0^{\tau}\frac{(\tau^{\beta}-\xi^{\beta})^{\gamma-1}(\varpi^{\beta}-\zeta^{\beta})^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,\beta^2\xi^{\beta-1}\zeta^{\beta-1}k(f(\tau,\varpi,\xi,\zeta))P(u(\xi,\zeta))\,d\xi d\zeta,$$

iii) for $\beta=\eta, k=I, h(\tau,\varpi,u(\tau,\varpi))=1$ *and* $g(\tau,\varpi)=0$*,*

introduced, two dimensional Erdélyi-Kober nonlinear integral equations of the first kind with fractional orders γ *and* θ*,*

$$u(\tau,\varpi)=\int_0^{\varpi}\int_0^{\tau}\frac{(\tau^{\beta}-\xi^{\beta})^{\gamma-1}(\varpi^{\beta}-\zeta^{\beta})^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,\beta^2\xi^{\beta-1}\zeta^{\beta-1}f(\tau,\varpi,\xi,\zeta)P(u(\xi,\zeta))\,d\xi d\zeta.$$

Based on the above process, integral equation (1.1) is a general form of some Erdélyi-Kober two-dimensional fractional integral equations.
Moreover, conforming with [1], we consider Riemann-Liouville two-dimensional fractional integral for function $f\in L^1(I), I=[0,1]\times[0,1]$ as follows,

$$I^{\gamma,\theta}f(\tau,\varpi)=\frac{1}{\Gamma(\gamma)\Gamma(\theta)}\int_0^{\varpi}\int_0^{\tau}\frac{f(\xi,\zeta)}{(\tau-\xi)^{1-\gamma}(\varpi-\zeta)^{1-\theta}}\,d\xi d\zeta,\ 0<\gamma,\theta<1. \tag{2.14}$$

Regarding Theorem 2.5 and Riemann-Liouville fractional integral (2.14) the following corollary is given.

Corollary 2.7. *If Theorem 2.5's conditions are pleased, then at least one solution for any integral equations with fractional order may be found in* $C([0,1]\times[0,1])$*, an instance of this would be (i), (ii), and (iii);*
i) for $\beta=1,\eta=1$*,*

It indicates, two dimensional Riemann-Liouville fractional non-linear integral equations with fractional orders γ *and* θ *as follows,*

$$u(\tau,\varpi)=g(\tau,\varpi)+h(\tau,\varpi,u(\tau,\varpi))\int_0^{\varpi}\int_0^{\tau}\frac{(\tau-\xi)^{\gamma-1}(\varpi-\zeta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,k(f(\tau,,\varpi,\xi,\zeta))P(u(\xi,\zeta))\,d\xi d\zeta,$$

ii) for $\beta=1,\eta=1$ *and* $h(\tau,\varpi,u(\tau,\varpi))=1$*,*

we provide, two dimensional Rimann-Liouville nonlinear integral equations of the second kind with fractional orders γ *and* θ*,*

$$u(\tau,\varpi)=g(\tau,\varpi)+\int_0^{\varpi}\int_0^{\tau}\frac{(\tau-\xi)^{\gamma-1}(\varpi-\zeta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,k(f(\tau,\varpi,\xi,\zeta))P(u(\xi,\zeta))\,d\xi d\zeta,$$

iii) for $\beta=1,\eta=1,k=I,h(\tau,\varpi,u(\tau,\varpi))=1$ *and* $g(\tau,\varpi)=0$*,*

introduced, two dimensional Riemann-Liouville nonlinear integral equations of the first kind with fractional orders γ *and* θ*,*

$$u(\tau,\varpi)=\int_0^{\varpi}\int_0^{\tau}\frac{(\tau-\xi)^{\gamma-1}(\varpi-\zeta)^{\theta-1}}{\Gamma(\gamma)\Gamma(\theta)}\,f(\tau,\varpi,\xi,\zeta)P(u(\xi,\zeta))\,d\xi d\zeta.$$

Thus, integral Eq. (1.1) is a general form of some Riemann-Liouville two dimensional fractional integral equations.

3. Application

Now, let's survey an example by employing Theorem 2.5.

Example 3.1. Consider two dimensional non-linear Volterra fractional integral equation,

$$u(\tau,\varpi)=\frac{1}{7}\tau^2\varpi^2+\frac{3\tau\varpi}{7(1+\tau+\varpi)}\frac{|u(\tau,\varpi)|}{1+|u(\tau,\varpi)|}\int_0^{\varpi}\int_0^{\tau}\frac{4\xi\zeta[\frac{1}{50}(\tau+\xi)(\varpi+\zeta)+\frac{1}{5}]}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(\tau^2-\xi^2)^{\frac{1}{2}}(\varpi^2-\zeta^2)^{\frac{1}{2}}}\,u^2(\xi,\zeta)\,d\xi d\zeta, \tag{3.1}$$

where $\tau,\varpi \in [0,1]$. Thus $g(\tau,\varpi)=\frac{1}{7}\tau^2\varpi^2$ satisfies assumption (b_1) and $M_1=\frac{1}{7}$. Also function $h(\tau,\varpi,u)=\frac{3\tau\varpi}{7(1+\tau+\varpi)}\frac{|u(\tau,\varpi)|}{1+|u(\tau,\varpi)|}$ satisfies hypothesis (b_2) with assumption $\pounds(\tau)=\frac{3}{7}\tau$,

$$|h(\tau,\varpi,u)-h(\tau,\varpi,z)| \le \frac{3}{7}|u-z| = \pounds(|u-z|),\ \forall u,z\in\mathbb{R}, \quad \tau,\varpi\in[0,1].$$

In addition H satisfies in (b_4). Actually, consider an arbitrary nonnegative function $u \in C([0,1]\times[0,1])$ and $\tau_1,\tau_2,\varpi_1,\varpi_2 \in [0,1]$ ($\tau_1\le\tau_2$and $\varpi_1\le\varpi_2$), let write

$$\begin{aligned}
&|(Hu)(\tau_2,\varpi_2)-(Hu)(\tau_1,\varpi_1)|-[(Hu)(\tau_2,\varpi_2)-(Hu)(\tau_1,\varpi_1)]\\
&\quad= |h(\tau_2,\varpi_2,u(\tau_2,\varpi_2))-h(\tau_1,\varpi_1,u(\tau_1,\varpi_1))|-[h(\tau_2,\varpi_2,u(\tau_2,\varpi_2))-h(\tau_1,\varpi_1,u(\tau_1,\varpi_1))]\\
&\quad= |\frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}\frac{u(\tau_2,\varpi_2)}{1+u(\tau_2,\varpi_2)}-\frac{3\tau_1\varpi_1}{7(1+\tau_1+\varpi_1)}\frac{u(\tau_1,\varpi_1)}{1+u(\tau_1,\varpi_1)}|\\
&\quad- [\frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}\frac{u(\tau_2,\varpi_2)}{1+u(\tau_2,\varpi_2)}-\frac{3\tau_1\varpi_1}{7(1+\tau_1+\varpi_1)}\frac{u(\tau_1,\varpi_1)}{1+u(\tau_1,\varpi_1)}]\\
&\quad\le |\frac{3\tau_2\varpi_2u(\tau_2,\varpi_2)}{7(1+\tau_2+\varpi_2)(1+u(\tau_2,\varpi_2))}-\frac{3\tau_2\varpi_2u(\tau_1,\varpi_1)}{7(1+\tau_2+\varpi_2)(1+u(\tau_1,\varpi_1))}|\\
&\quad+ |\frac{3\tau_2\varpi_2u(\tau_1,\varpi_1)}{7(1+\tau_2+\varpi_2)(1+u(\tau_1,\varpi_1))}-\frac{3\tau_1\varpi_1u(\tau_1,\varpi_1)}{7(1+\tau_1+\varpi_1)(1+u(\tau_1,\varpi_1))}|\\
&\quad- [\frac{3\tau_2\varpi_2u(\tau_2,\varpi_2)}{7(1+\tau_2+\varpi_2)(1+u(\tau_2,\varpi_2))}-\frac{3\tau_2\varpi_2u(\tau_1,\varpi_1)}{7(1+\tau_2+\varpi_2)(1+u(\tau_1,\varpi_1))}]\\
&\quad+ [\frac{3\tau_2\varpi_2u(\tau_1,\varpi_1)}{7(1+\tau_2+\varpi_2)(1+u(\tau_1,\varpi_1))}-\frac{3\tau_1\varpi_1u(\tau_1,\varpi_1)}{7(1+\tau_1+\varpi_1)(1+u(\tau_1,\varpi_1))}]\\
&\quad\le \frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}|\frac{u(\tau_2,\varpi_2)}{1+u(\tau_2,\varpi_2)}-\frac{u(\tau_1,\varpi_1)}{1+u(\tau_1,\varpi_1)}|+|\frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}-\frac{3\tau_1\varpi_1}{7(1+\tau_1+\varpi_1)}|\frac{u(\tau_1,\varpi_1)}{1+u(\tau_1,\varpi_1)}\\
&\quad- \frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}[\frac{u(\tau_2,\varpi_2)}{1+u(\tau_2,\varpi_2)}-\frac{u(\tau_1,\varpi_1)}{1+u(\tau_1,\varpi_1)}]-[\frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}-\frac{3\tau_1\varpi_1}{7(1+\tau_1+\varpi_1)}]\frac{u(\tau_1,\varpi_1)}{1+u(\tau_1,\varpi_1)}\\
&\quad= \frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}\{|\frac{u(\tau_2,\varpi_2)-u(\tau_1,\varpi_1)}{(1+u(\tau_2+\varpi_2))(1+u(\tau_1,\varpi_1))}|-[\frac{u(\tau_2,\varpi_2)-u(\tau_1,\varpi_1)}{(1+u(\tau_2+\varpi_2))(1+u(\tau_1,\varpi_1))}]\}\\
&\quad+ \{|\frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}-\frac{3\tau_1\varpi_1}{7(1+\tau_1+\varpi_1)}|-[\frac{3\tau_2\varpi_2}{7(1+\tau_2+\varpi_2)}-\frac{3\tau_1\varpi_1}{7(1+\tau_1+\varpi_1)}]\}\frac{u(\tau_1,\varpi_1)}{1+u(\tau_1,\varpi_1)}\\
&\quad= \frac{3}{7}\{|u(\tau_2,\varpi_2)-u(\tau_1,\varpi_1)|-[u(\tau_2,\varpi_2)-u(\tau_1,\varpi_1)]\}\le\frac{3}{7}J(u)=\pounds(J(u)),
\end{aligned}$$

Hence, $J(Hu)\le\pounds(J(u))$. Furthermore, the function $f(\tau,\varpi,\xi,\zeta)=\frac{1}{10}(\tau+\xi)(\varpi+\zeta)$ satisfies assumption (b_5). Let $k:[0,\frac{2}{5}]\to\mathbb{R}^+$ be taken by $k(f)=\frac{1}{5}f+\frac{1}{5}$, then k pleases assumption (b_6) with $||k||=\frac{7}{25}$. Based on the instance, Eq. (2.14) in assumption (b_7) is converted as,

$$\frac{1}{7}\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})+\frac{3}{25}r^2\le\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})r,$$

moreover, a solution of the above inequality is $r_0=1$, and

$$\lambda=\frac{||k||\ r_0}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}=\frac{28}{25\pi}<1.$$

Therefore, Theorem 2.5 ensures that Eq. (3.1) possesses a non-decreasing solution.

4. Approximation of the solution Eq. (3.1) via Sinc iterative algorithm

In Section 2, the solvability of Eq. (1.1) was demonstrated, which was incorporated into equations such as (3.1). There are numerous methods for obtaining approximate solutions of integral equations with singular kernels based on the collocation method [27, 38, 45, 47] and projection methods such as the Galerkin multi-wavelet basis used in [10, 39]. In [9], a numerical method based on Bernstein polynomials was employed. We employ a technique based on two-dimensional Sinc interpolation to develop an iterative algorithm that approximates the solution of Eq. (3.1) with high accuracy. Compared with the above-mentioned methods: Firstly, our proposed method does not require converting the nonlinear problem into an algebraic system by expanding $u(s,t)$ in terms of Sinc functions with unknown coefficients. Secondly, this algorithm has an exponential convergence rate of order $O(\exp(-c\sqrt{N}))$ (see [26, 44]).

We now recall some properties of the Sinc function [44]:

$$\mathrm{Sinc}(u) = \begin{cases} \dfrac{\sin(\pi u)}{\pi u}, & u \neq 0 \\ 1, & u = 0. \end{cases} \tag{4.1}$$

For integer k and $h > 0$, k'th Sinc function with step size h is presented as,

$$S(k,h)(u) = \mathrm{Sinc}\left(\frac{u-kh}{h}\right), \; S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases} \tag{4.2}$$

Definition 4.1. If u, be a function noted on real line; so for $h > 0$ the series,

$$C(u,h) = \sum_{k=-\infty}^{\infty} u(kh)S(k,h)(u). \tag{4.3}$$

is named the Whittaker cardinal expansion of u, wherever this series converges (see [44]). Clearly, by (4.2) and (4.3) the cardinal function interpolates function u at the points $\{kh\}_{k=-\infty}^{\infty}$.

In keeping with (4.2) and (4.3) a bounded error was introduced by Theorem18 in [26, 44].

Theorem 4.2. *Let $\mathcal{C}$ is a complex plane, $D_d = \{z \in \mathcal{C} : |Im(z)| < d\}$, where $d > 0$, and for $\alpha > 0$ $\;L_\alpha(D_d) = \{f : f \text{ is an analytic function in } D_d\}$ and $h = \sqrt{\frac{\pi d}{\alpha N}}$, then*

$$\|f(.+iy) - \sum_{k=-\infty}^{\infty} u(kh)S(k,h)(.+iy)\|_\infty \leq c_1 N^{\frac{1}{2}} exp((-\frac{\pi\alpha}{d})^{\frac{1}{2}}(d-|y|)N^{\frac{1}{2}}).$$

In this article we discuss in real line therefore, then,

$$\|f(.) - \sum_{k=-\infty}^{\infty} u(kh)S(k,h)(.)\|_\infty \leq c_1 N^{\frac{1}{2}} exp(-c_2 N^{\frac{1}{2}}). \tag{4.4}$$

As demonstrated, the upper bound of error exponentially converging to zero as N be large enough. In the case of two dimensional the error bound is similar to (4.4) and we neglect it.

The intervals integrating in (3.1) are $[0,\tau]$ and $[0,\varpi]$, where $\tau, \varpi \in [0,1]$, thus we show a conformal map as follows:

$$\begin{aligned} &\varphi : [0,1] \to (-\infty, \infty) \\ &\tau \to \ln(\frac{\tau}{1-\tau}). \end{aligned} \tag{4.5}$$

In fact, $\lim_{\tau\to 0}\varphi(\tau) = -\infty$ and $\lim_{\tau\to 1}\varphi(\tau) = \infty$. By (4.2) and (4.5) combination of functions $S(k,h)$ and φ in the case of two dimensional is $S(k,h).\varphi(\tau)S(k',h).\varphi(\varpi)$ interpolation. Let u be an integrated function, then by cardinal function (4.3), we have,

$$u_n(\tau,\varpi) = \sum_{k=-N}^{N}\sum_{k'=-N}^{N} u(kh,k'h)S(k,h).\varphi(\tau)S(k',h).\varphi(\varpi). \tag{4.6}$$

Following (4.6) and (4.2), if $\varphi(\tau) = kh$ and $\varphi(\varpi) = k'h$ for $k, k' = -N, ..., N$, then $u_n(kh, k'h) = u(kh, k'h)$.
On the other word, (4.6) is an interpolation of u such that the interpolating points can be taken by

$$\begin{cases} \tau_k = \varphi^{-1}(kh) = \frac{\exp^{kh}}{1+\exp^{kh}}, & k = -N+1, ..., N, \tau_{-N} = 0 \\ \varpi_{k'} = \varphi^{-1}(k'h) = \frac{\exp^{k'h}}{1+\exp^{k'h}}, & k' = -N+1, ..., N, \varpi_{-N} = 0. \end{cases} \tag{4.7}$$

By Eqs. (4.5)-(4.7) and similar to [15, 44], we compute two dimensional integral on $[0, \varpi] \times [0, \tau]$ for $\tau, \varpi \in [0, 1]$ as follows:

$$\int_0^{\varpi} \int_0^{\tau} u(\xi, \zeta) d\xi d\zeta \approx h^2 \sum_{k=-N}^{N} \sum_{k'=-N}^{N} \frac{u(\tau_k, \varpi_{k'})}{\varphi'(\tau_k)\varphi'(\varpi_{k'})},$$

where

$$\varphi'(\varsigma_k) = \frac{1}{\varsigma_k(1-\varsigma_k)}, \ \varsigma = \tau, \ \varpi.$$

Algorithm 1 Iterative algorithm based on two-dimensional sinc interpolation.

Initialize $u_0(\tau, \varpi) = 0$.
for $n = 1, 2, 3, \ldots$ **do**
    Compute

$$u_{n+1}(\tau, \varpi) = \frac{1}{7}\tau^2\varpi^2 + \frac{3\tau\varpi}{7(1+\tau+\varpi)} \frac{u_n(\tau, \varpi)}{(1+u_n(\tau, \varpi))\,\Gamma(\frac{1}{2})^2} h^2 \times \sum_{k=-N}^{N} \sum_{k'=-N}^{N} \frac{u_n(\varpi_k, \tau_{k'})\,\varpi_k \tau_{k'}(1-\varpi_k)(1-\tau_{k'})}{\sqrt{(\tau^2-\tau_{k'}^2)(\varpi^2-\varpi_k^2)}}.$$

end for
The collocation points ϖ_k and $\tau_{k'}$ for $k, k' = -N, \ldots, N$ are given by (4.7).

For $N = 5$ and $h = \frac{\pi}{\sqrt{2N}}$, we obtain approximate solution $u_i(\tau, \varpi)$ for $i = 1, 2$ by Algorithm 1 with start function $u_0(\tau, \varpi) = 0$. In Table 1, the absolute error for $u_2(\tau, \varpi)$ is presented using the Eq. (4.4).

TABLE 1. Absolute errors for Eq. (4.4) by sinc interpolation.

(τ, ϖ)	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0	0	0	0	0	0
0.2	0	5.3×10^{-10}	4.1×10^{-9}	1.5×10^{-8}	4.8×10^{-8}	1.5×10^{-7}
0.4	0	4.1×10^{-9}	3.2×10^{-8}	1.2×10^{-7}	3.9×10^{-7}	1.2×10^{-6}
0.6	0	1.5×10^{-8}	1.2×10^{-7}	5.0×10^{-7}	1.6×10^{-6}	$4.0\ \times10^{-6}$
0.8	0	4.8×10^{-8}	3.9×10^{-7}	1.6×10^{-6}	8.6×10^{-6}	$1.1\ \times10^{-5}$
1.0	0	1.5×10^{-7}	1.2×10^{-6}	4.0×10^{-6}	1.1×10^{-5}	7.5×10^{-5}

5. CONCLUSION

We have established existence results for a broad class of fractional two-dimensional nonlinear functional integral equations using a generalized Darbo fixed point theorem and the measure of noncompactness. An explicit example was analyzed, and a highly accurate Sinc-based iterative algorithm was implemented. The method yields an exponentially convergent approximation and is suitable for simulating soliton-type profiles or other localized wave structures in nonlinear fractional models.

Future work will focus on applying the present framework to fractional partial differential equations that generate explicit soliton solutions and extending the algorithm to variable-order and stochastic fractional operators.

We examined the solvability of a two-dimensional fractional nonlinear functional integral equation in a Banach space. To confirm the results obtained from our theorems, we presented an application in the form of a fractional nonlinear integral equation with two variables. The effectiveness and validity of the proposed algorithm were demonstrated through the process of determining an approximate solution to Example 3.1. In the example, we verified both the existence of the solution and a highly accurate approximation by employing a convergent iterative algorithm.

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