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# Analytical wave solutions of $(n+1)$-dimensional generalized KP equation

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**Abstract**

The KP equation was un folded and developed by Kadomtsev and Petviashvili in the 1970, as a two spatial dimensional analogue of classical KdV equation, which can well replicate nonlinear incident in plasma physics, fluid dynamics, optics and so on. Because of the mathematical and physical outstanding nature, KP equation has attracted the curiosity of contemporary scholars. The purpose of this study is to provide exact solitary wave solutions to the $(n+1)$-dimensional generalized KP equation using the improved Sardar sub-equation approach and the improved generalized Riccatii equation mapping approach. With these two methods, we were able to investigate the rationale, exponential, trigonometric, and trigonometric hyperbolic solutions of the KP equation. The suggested methods are efficient, and straightforward in computing novel soliton solutions to many types of NLPDEs in applied sciences and engineering. We finally provided the graphical display for some of the obtained exact soliton solutions.

**Keywords.** IMSSEM, IGREMM, $(n+1)$-Dimensional Generalized KP Equation, Solitary Wave Solutions, Nonlinear PDEs.
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## 1. Introduction

In various fields of the society, we apply nonlinear partial differential equations (NLPDEs), for example, optical fibres [12, 32], fluid mechanics [15], plasma physics [2], physics of solid state [8, 39] and some realm of engineering. A monumental progress has been made in the formation of solving NLPDEs, different approaches continue to be revealed all over the globe. Among them are the method of exponential-function [45, 49, 51], the sine-cosine approach [4, 6, 44], the Jacobi-elliptic function exp-method [22, 52], the method of homogenous balancing [11, 16, 41], the generalized Riccatii equation mapping approach [25], the unified method [1, 26, 27, 42] and many more. As such, looking for exact solutions of NLPDEs becomes a crucial topic.

Contemporary scholars have studied NLDPEs which include Schrdinger equations [7, 13, 34, 35, 37], Fokas/Lenells equation [19, 20], Fornberge/Whitham equation [5], Ablowitze/Ladike equation [10], Swift/ Hohenberge model [40], Boussinesque equation [29] and Phi-4 model [21]. Moreover, scholars have also analized the properties and dynamic behaviors of other models [30, 31, 33, 48]. The KP equation was unfolded and developed by Kadomtsev and Petviashvili in the 1970 [17], as a two-spatial dimensional analogue of classical Korteweg-de Vries equation, which can well replicate nonlinear incident in fluid physics, plasma physics, Bose/Einstein condensates, optics etc. Due to the mathematical and physical importance, the KP equation have attracted the curiosity of contemporary scholars. The integrability of its features and numerous exact solutions have been investigated by using various approaches [9, 14, 24, 28, 38, 43, 50, 53]. Considering that the real situation in (3+1)-dimensions, various higher order dimensional extensions of KP model have

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been analyzed from contrary point of views [3, 18, 23, 36, 47, 54]. Furthermore, of recent, the new integrability and localized solutions for the new (n+1)-dimensional generalized Kadomtsev-Petviashvili equation have been provided:

$$(u_t + \rho_1 u u_{x_1} + \rho_2 u_{x_1 x_1 x_1})_{x_1} + \rho_3 u_{x_2 x_2} + \sum_{j=0}^{N} \phi_j u_{x_1 x_j} = 0, \tag{1.1}$$

where $N \geq 2$, $u$ is a differentiable function with respect to spatial variables $x_1, x_2, ..., x_N$ and time variable $t$, the subscripts represent the partial derivatives, and and are constant parameters, given by Xu and Wazwaz [46]. However, in this study, new exact solution to the (n+1)-dimensional generalized Kadomtsev-Petviashvili equation will be established for the first time by employing improved modified Sardar Sub-equation method (IMSSEM) and improved generalized Ricatti equation mapping method (IGREMM). The remaining part of this paper is structured as follows: In Section two, we discuss the methodology of improved modified Sardar sub-equation method and improved generalized Riccati equation mapping approach briefly. In section three, we apply the IMSSEM and IGREMM to provide new exact, accurate solitary wave solutions for the $(n+1)$-dimensional generalized KP model Equation (1.1). In section four, we portray the graphs for some of the obtained exact solutions and in section five, we provide the concluding remark of the study.

## 2. Description of the Proposed Techniques

Here, from the nonlinear PDEs

$$F(u, u_t, u_x, u_{xx}, \ldots), \tag{2.1}$$

where $u = u(x,t)$ is an unknown function. We now introduce the transformation of the wave

$$u = u(\xi), \quad \xi = x - ct, \tag{2.2}$$

where $c \neq 0$, By using the transformation of Equation (2.2) in Equation (2.1) to reduce the NLPDE to NLODE.

$$N(u', u'', u''', \ldots). \tag{2.3}$$

We shall solve the NLODE equation by using IMSSEM and it has the following standard forms:

$$u(\xi) = a_0 + \sum_{i=1}^{N} a_i \phi^i(\xi) a_N \neq 0. \tag{2.4}$$

The value of $N$ can be obtain by considering the highest order derivative term and the highest order nonlinear term in Equation (2.3) by balancing them. Therefore, the highest degree of $\frac{d^p u}{d\xi^p}$ is classified as:

$$O\left(\frac{d^p u}{d\xi^p}\right) = n + p, \quad p = 1, 2, 3, \ldots, \tag{2.5}$$

$$O\left(u^q \frac{d^p u}{d\xi^p}\right) = (q+1)n + p, \quad q = 0, 1, 2, \ldots, \quad p = 1, 2, 3, \ldots. \tag{2.6}$$

2.1. **The analysis of the improved Sardar sub-equation method.** The function $\phi(\xi)$ in Equation (2.4) suffices the equation below:

$$(\phi')^2(\xi) = \delta_2 \phi^4(\xi) + \delta_1 \phi^2(\xi) + \delta_0, \tag{2.7}$$

where $\delta_j, \quad j = 0, 1, 2$ are constants. The solution that validate Equation (2.7) with $C$ as the constant of integration after collection are as follows:

1. For $\delta_0 = \delta_1 = 0$ and $\delta_2 > 0$, we have the rational form solution as

$$\phi_1^{\pm}(\xi) = \pm \frac{1}{\sqrt{\delta_2}(\xi + C)}. \tag{2.8}$$

2. For $\delta_0 = 0$ and $\delta_1 > 0$, we have the exponential form solution as

$$\phi_2^{\pm}(\xi) = \frac{4\delta_1 e^{\pm\sqrt{\delta_1}(\xi+C)}}{e^{\pm 2\sqrt{\delta_1}(\xi+C)} - 4\delta_1\delta_2}, \tag{2.9}$$

$$\phi_3^{\pm}(\xi) = \frac{\pm 4\delta_1 e^{\pm\sqrt{\delta_1}(\xi+C)}}{1 - 4\delta_1\delta_2 e^{\pm 2\sqrt{\delta_1}(\xi+C)}}. \tag{2.10}$$

3. The trigonometric-hyperbolic form solutions are as follows:
   i) For $\delta_0 = 0, \delta_1 > 0$ and $\delta_2 \neq 0$, we have

$$\phi_4^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{\delta_2}}\,\mathrm{sech}\left(\sqrt{\delta_1}(\xi+C)\right), \tag{2.11}$$

$$\phi_5^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{\delta_2}}\,\mathrm{csch}\left(\sqrt{\delta_1}(\xi+C)\right). \tag{2.12}$$

   ii) For $\delta_0 = \frac{\delta_1^2}{4\delta_2}, \quad \delta_1 < 0$ and $\delta_2 > 0$, we have

$$\phi_6^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{2\delta_2}}\,\tanh\left(\sqrt{-\frac{\delta_1}{2}}(\xi+C)\right), \tag{2.13}$$

$$\phi_7^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{2\delta_2}}\,\coth\left(\sqrt{-\frac{\delta_1}{2}}(\xi+C)\right), \tag{2.14}$$

$$\phi_8^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{2\delta_2}}\left(\tanh\left(\sqrt{-2\delta_1}(\xi+C)\right) \pm i\,\mathrm{sech}\left(\sqrt{-2\delta_1}(\xi+C)\right)\right), \tag{2.15}$$

$$\phi_9^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{2\delta_2}}\left(\coth\left(\sqrt{-2\delta_1}(\xi+C)\right) \pm \mathrm{csch}\left(\sqrt{-2\delta_1}(\xi+C)\right)\right), \tag{2.16}$$

$$\phi_{10}(\xi) = \pm\sqrt{-\frac{\delta_1}{8\delta_2}}\left(\tanh\left(\sqrt{-\frac{\delta_1}{8}}(\xi+C)\right) + \coth\left(\sqrt{-\frac{\delta_1}{8}}(\xi+C)\right)\right). \tag{2.17}$$

4. For the trigonometric form, the solutions are:
   i) For $\delta_0 = 0, \delta_1 < 0$ and $\delta_2 \neq 0$, we have

$$\phi_{11}^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{\delta_2}}\,\sec\left(\sqrt{-\delta_1}(\xi+C)\right), \tag{2.18}$$

$$\phi_{12}^{\pm}(\xi) = \pm\sqrt{-\frac{\delta_1}{\delta_2}}\,\csc\left(\sqrt{-\delta_1}(\xi+C)\right). \tag{2.19}$$

   ii) For $\delta_0 = \frac{\delta_1^2}{4\delta_2}, \quad \delta_1 > 0$ and $\delta_2 > 0$, we have

$$\phi_{13}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}\,\tan\left(\sqrt{\frac{\delta_1}{2}}(\xi+C)\right), \tag{2.20}$$

$$\phi_{14}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}\,\cot\left(\sqrt{\frac{\delta_1}{2}}(\xi+C)\right), \tag{2.21}$$

$$\phi_{15}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}\left(\tan\left(\sqrt{2\delta_1}(\xi+C)\right) \pm \sec\left(\sqrt{2\delta_1}(\xi+C)\right)\right), \tag{2.22}$$

$$\phi_{16}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{2\delta_2}}\left(\cot\left(\sqrt{2\delta_1}(\xi+C)\right) \pm \csc\left(\sqrt{2\delta_1}(\xi+C)\right)\right), \tag{2.23}$$

$$\phi_{17}^{\pm}(\xi) = \pm\sqrt{\frac{\delta_1}{8\delta_2}}\left(\tan\left(\sqrt{\frac{\delta_1}{8}}(\xi+C)\right) - \cot\left(\sqrt{\frac{\delta_1}{8}}(\xi+C)\right)\right). \tag{2.24}$$

Substituting Equations (2.4) and (2.7) into Equation (2.3), and therefore making all coefficients of each power of $\phi(\xi)$, to zero, solving the obtained system of algebraic equation with the help of Mathematica and incorporating these constants into Equation (2.4), then considering Equations (2.8)–(2.24), we can get different solutions for NPDEs.

2.2. **The analysis of the improved generalized Riccatii equation mapping method.** Consider the function $\phi(\xi)$ in Equation (2.4) which suffices the equation

$$\phi'(\xi) = \beta_2\phi^2(\xi) + \beta_1\phi(\xi) + \beta_0, \tag{2.25}$$

where $\beta_j, \quad j = 0, 1, 2,$ are constant. The solution that validate Equation (2.7) with $C$ as the constant of integration after collection are as follows:

- For $\beta_0 = \beta_1 = 0$ and $\beta_2 \neq 0$, we have the rational form of the solution as follows

$$\phi_1^{\pm}(\xi) = \pm\frac{1}{\beta_2(\xi+C)}. \tag{2.26}$$

- For $\beta_0 = 0$, we have the exponential forms of solutions as

$$\phi_2(\xi) = -\frac{\beta_1\phi}{\beta_1 e^{-\beta_1(\xi+C)} + \varphi}, \tag{2.27}$$

$$\phi_3(\xi) = -\frac{\beta_1 e^{\beta_1(\xi+C)}}{\beta_2(e^{\beta_1(\xi+C)} + \varphi)}. \tag{2.28}$$

- For $\Omega = \beta_1^2 - 4\beta_0\beta_1 > 0, \quad \beta_1\beta_2 \neq 0$ or $\beta_0\beta_2 \neq 0$, we have the trigonometric-hyperbolic form of solutions as follows:

$$\phi_4(\xi) = -\frac{\sqrt{\Omega}}{2\beta_2}\tanh\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right) - \frac{\beta_1}{2\beta_2}, \tag{2.29}$$

$$\phi_5(\xi) = -\frac{\sqrt{\Omega}}{2\beta_2}\coth\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right) - \frac{\beta_1}{2\beta_2}, \tag{2.30}$$

$$\phi_6^{\pm}(\xi) = -\frac{\sqrt{\Omega}}{2\beta_2}\left(\tanh\left(\sqrt{\Omega}(\xi+C)\right) \pm i\,\mathrm{sech}\left(\sqrt{\Omega}(\xi+C)\right)\right) - \frac{\beta_1}{2\beta_2}, \tag{2.31}$$

$$\phi_7^{\pm}(\xi) = -\frac{\sqrt{\Omega}}{2\beta_2}\left(\coth\left(\sqrt{\Omega}(\xi+C)\right) \pm \mathrm{csch}\left(\sqrt{\Omega}(\xi+C)\right)\right) - \frac{\beta_1}{2\beta_2}, \tag{2.32}$$

$$\phi_8(\xi) = -\frac{\sqrt{\Omega}}{4\beta_2}\left(\tanh\left(\frac{\sqrt{\Omega}}{4}(\xi+C)\right) + \coth\left(\frac{\sqrt{\Omega}}{4}(\xi+C)\right)\right) - \frac{\beta_1}{2\beta_2}, \tag{2.33}$$

$$\phi_9^{\pm}(\xi) = \frac{\pm\sqrt{\Omega(p^2+q^2)} - p\sqrt{\Omega}\cosh\left(\sqrt{\Omega}(\xi+C)\right)}{2\beta_2\left(p\sinh\left(\sqrt{\Omega}(\xi+C)\right) + q\right)} - \frac{\beta_1}{2\beta_2}, \tag{2.34}$$

$$\phi_{10}(\xi) = \frac{2\beta_0\cosh\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right)}{\sqrt{\Omega}\sinh\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right) - \beta_1\cosh\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right)}, \tag{2.35}$$

$$\phi_{11}(\xi) = \frac{2\beta_0\sinh\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right)}{\sqrt{\Omega}\cosh\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right) - \beta_1\sinh\left(\frac{\sqrt{\Omega}}{2}(\xi+C)\right)}, \tag{2.36}$$

$$\phi_{12}^{\pm}(\xi)=\frac{2\beta_0\cosh\left(\sqrt{\Omega}(\xi+C)\right)}{\sqrt{\Omega}\,\sinh\left(\sqrt{\Omega}(\xi+C)\right)-\beta_1\cosh\left(\sqrt{\Omega}(\xi+C)\right)\pm i\sqrt{\Omega}}, \tag{2.37}$$

$$\phi_{13}^{\pm}(\xi)=\frac{2\beta_0\sinh\left(\sqrt{\Omega}(\xi+C)\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}(\xi+C)\right)-\beta_1\sinh\left(\sqrt{\Omega}(\xi+C)\right)\pm\sqrt{\Omega}}, \tag{2.38}$$

$$\phi_{14}(\xi)=\frac{2\beta_0\sinh\left(\frac{\sqrt{\Omega}}{4}(\xi+C)\right)\cosh\left(\frac{\sqrt{\Omega}}{4}(\xi+C)\right)}{2\sqrt{\Omega}\cosh^2\left(\frac{\sqrt{\Omega}}{4}(\xi+C)\right)-2\beta_1\sinh\left(\frac{\sqrt{\Omega}}{4}(\xi+C)\right)\cosh\left(\frac{\sqrt{\Omega}}{4}(\xi+C)\right)-\sqrt{\Omega}}. \tag{2.39}$$

- For $\Omega=\beta_1^2-4\beta_0\beta_2<0,\quad \beta_1\beta_2\neq 0$ or $\beta_0\beta_2\neq 0$, the trigonometric form solutions are as follows:

$$\phi_{15}(\xi)=\frac{\sqrt{-\Omega}}{2\beta_2}\tan\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)-\frac{\beta_1}{2\beta_2}, \tag{2.40}$$

$$\phi_{16}(\xi)=-\frac{\sqrt{-\Omega}}{2\beta_2}\cot\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)-\frac{\beta_1}{2\beta_2}, \tag{2.41}$$

$$\phi_{17}^{\pm}(\xi)=\frac{\sqrt{-\Omega}}{2\beta_2}\left(\tan\left(\sqrt{-\Omega}(\xi+C)\right)\pm\sec\left(\sqrt{-\Omega}(\xi+C)\right)\right)-\frac{\beta_1}{2\beta_2}, \tag{2.42}$$

$$\phi_{18}^{\pm}(\xi)=-\frac{\sqrt{-\Omega}}{2\beta_2}\left(\cot\left(\sqrt{-\Omega}(\xi+C)\right)\pm\csc\left(\sqrt{-\Omega}(\xi+C)\right)\right)-\frac{\beta_1}{2\beta_2}, \tag{2.43}$$

$$\phi_{19}(\xi)=\frac{\sqrt{-\Omega}}{4\beta_2}\left(\tan\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{-\Omega}}{4}(\xi+C)\right)\right)-\frac{\beta_1}{2\beta_2}, \tag{2.44}$$

$$\phi_{20}^{\pm}(\xi)=\frac{\pm\sqrt{-\Omega(p^2-q^2)}-p\sqrt{-\Omega}\,\cos\left(\sqrt{-\Omega}(\xi+C)\right)}{2\beta_2\left(p\sin\left(\sqrt{-\Omega}(\xi+C)\right)+q\right)}-\frac{\beta_1}{2\beta_2}, \tag{2.45}$$

$$\phi_{21}(\xi)=-\frac{2\beta_0\cos\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)}{\sqrt{-\Omega}\,\sin\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)+\beta_1\cos\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)}, \tag{2.46}$$

$$\phi_{22}(\xi)=\frac{2\beta_0\sin\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)}{\sqrt{-\Omega}\,\cos\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)-\beta_1\sin\left(\frac{\sqrt{-\Omega}}{2}(\xi+C)\right)}, \tag{2.47}$$

$$\phi_{23}^{\pm}(\xi)=-\frac{2\beta_0\cos\left(\sqrt{-\Omega}(\xi+C)\right)}{\beta_1\cos\left(\sqrt{-\Omega}(\xi+C)\right)+\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}(\xi+C)\right)\pm\sqrt{-\Omega}}, \tag{2.48}$$

$$\phi_{24}^{\pm}(\xi)=\frac{2\beta_0\sin\left(\sqrt{-\Omega}(\xi+C)\right)}{\beta_1\sin\left(\sqrt{-\Omega}(\xi+C)\right)-\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}(\xi+C)\right)\pm\sqrt{-\Omega}}, \tag{2.49}$$

$$\phi_{25}(\xi)=\frac{4\beta_0\sin\left(\frac{\sqrt{-\Omega}}{4}(\xi+C)\right)\cos\left(\frac{\sqrt{-\Omega}}{4}(\xi+C)\right)}{2\sqrt{-\Omega}\cos^2\left(\frac{\sqrt{-\Omega}}{4}(\xi+C)\right)-2\beta_1\sin\left(\frac{\sqrt{-\Omega}}{4}(\xi+C)\right)\cos\left(\frac{\sqrt{-\Omega}}{4}(\xi+C)\right)-\sqrt{-\Omega}}. \tag{2.50}$$

Substituting Equations (2.4) and (2.25) into Equation (2.3), therefore set all the coefficients of $\phi^i(\xi)$ to zero. Solving the resultant system of algebraic equations with the help of Mathematica and fusing these constants into Equation (2.4), then considering Equations (2.26)–(2.50), we can get different solutions for NLPDEs.

## 3. $(n+1)$-Dimensional Generalized KP Equation and its Solutions

In this section, IMSSEM and IGREMM are used to find the new exact solitary wave solutions of the $(n+1)$-dimensional generalized PKP Equation (1.1). Consider the wave variable

$$u = u(\xi), \quad \xi = \sum_{j=1}^{n} x_j - rt. \tag{3.1}$$

We use the wave variable $\xi = \sum_{j=1}^{n} x_j - rt$ where $r \neq 0$, which transform the equation into ODE:

$$\left(-ru' + \rho_1 \left(u^2\right)' + \rho_2 u'''\right)' + \rho_3 u'' + \sum_{j=0}^{N} \phi_j u'' = 0. \tag{3.2}$$

Integrating Equation (3.2) two times with respect to $\xi$ and denote the constant of integration as zero, the above equation becomes

$$\left(\rho_3 - r + \sum_{j=0}^{N} \phi_j\right) u + \rho_1 u^2 + \rho_2 u'' = 0. \tag{3.3}$$

Now, by balancing between $v^2$ and $v''$, we obtain $N = 2$. Thus, Equation (??) offers a solution of the form

$$U(\xi) = a_0 + a_1 \Phi(\xi) + a_2 \Phi(\xi)^2. \tag{3.4}$$

For simplicity, let $k = \sum_{j=0}^{N} \phi_j$ in 3.3, thus we get

$$(\rho_3 - r + k)\, u + \rho_1 u^2 + \rho_2 u'' = 0. \tag{3.5}$$

3.1. **New exact solutions of equation 1.1 by improved Sardar sub-equation method.** Now, plugging Equations (2.7) and (3.4) into Equation (3.5) and making the coefficient of each power of $\Phi(\xi)$ to zero, we obtain a system of algebraic equation with the help of Mathematica as follows:

$$\begin{aligned}
ka_0 - ra_0 + a_0^2\rho_1 + 2a_2\delta_0\rho_2 + a_0\rho_3 &= 0,\\
ka_1 - ra_1 + 2a_0a_1\rho_1 + a_1\delta_1\rho_2 + a_1\rho_3 &= 0,\\
ka_2 - ra_2 + a_1^2\rho_1 + 2a_0a_2\rho_1 + 4a_2\delta_1\rho_2 + a_2\rho_3 &= 0,\\
2a_1a_2\rho_1 + 2a_1\delta_2\rho_2 &= 0,\\
a_2^2\rho_1 + 6a_2\delta_2\rho_2 &= 0.
\end{aligned}$$

Solving the above system we obtain

$$a_1 = 0, \qquad \delta_0 = -\frac{a_0(k - r + a_0\rho_1 + \rho_3)}{2a_2\rho_2}, \qquad \delta_1 = \frac{-k + r - 2a_0\rho_1 - \rho_3}{4\rho_2}, \qquad \delta_2 = -\frac{a_2\rho_1}{6\rho_2}. \tag{3.6}$$

Now, substituting Equation (3.6) into Equations (2.8)–(2.24), we obtain the following exact solutions:

- For rational form, we have

$$u_1^{\pm}(x,t) = a_0 - \frac{6\rho_2}{(C+\xi)^2\rho_1}.$$

- For exponential function, we have

$$u_2^{\pm}(x,t) = a_0 + \frac{e^{\pm(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}\, a_2\left(-k + r - 2a_0\rho_1 - \rho_3\right)^2}{\rho_2^2\left(e^{\pm(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}} + \frac{a_2\rho_1(-k+r-2a_0\rho_1-\rho_3)}{6\rho_2^2}\right)^2},$$

$$u_3^{\pm}(x,t)=a_0+\frac{e^{\pm(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}\,a_2\left(-k+r-2a_0\rho_1-\rho_3\right)^2}{\rho_2^2\left(1+\frac{e^{\pm(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}\,a_2\rho_1(-k+r-2a_0\rho_1-\rho_3)}{6\rho_2^2}\right)^2}.$$

- For trigonometric and hyperbolic form, we have

$$u_4^{\pm}(x,t)=a_0-\frac{3\sec^2\left[\pm\frac{1}{2}(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}\right](-k+r-2a_0\rho_1-\rho_3)}{2\rho_1},$$

$$u_5^{\pm}(x,t)=a_0-\frac{3\csc^2\left[\pm\frac{1}{2}(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}\right](-k+r-2a_0\rho_1-\rho_3)}{2\rho_1},$$

$$u_6^{\pm}(x,t)=a_0+\frac{3(-k+r-2a_0\rho_1-\rho_3)\tanh^2\left[\frac{\pm(C+\xi)\sqrt{-\frac{(-k+r-2a_0\rho_1-\rho_3)}{\rho_2}}}{2\sqrt{2}}\right]}{4\rho_1},$$

$$u_7^{\pm}(x,t)=a_0+\frac{3(-k+r-2a_0\rho_1-\rho_3)\coth^2\left[\frac{\pm(C+\xi)\sqrt{-\frac{(-k+r-2a_0\rho_1-\rho_3)}{\rho_2}}}{2\sqrt{2}}\right]}{4\rho_1},$$

$$u_8^{\pm}(x,t)=a_0+\frac{3(-k+r-2a_0\rho_1-\rho_3)\left(\tanh\left[\frac{\pm(C+\xi)\sqrt{\frac{a_2\rho_1}{\rho_2}}}{\sqrt{3}}\right]\pm i\,\mathrm{sech}\left[\frac{(C+\xi)\sqrt{\frac{a_2\rho_1}{\rho_2}}}{\sqrt{3}}\right]\right)^2}{4\rho_1},$$

$$u_9^{\pm}(x,t)=a_0+\frac{3(-k+r-2a_0\rho_1-\rho_3)\left(\tanh\left[\frac{\pm(C+\xi)\sqrt{\frac{a_2\rho_1}{\rho_2}}}{\sqrt{3}}\right]\pm i\,\mathrm{sech}\left[\frac{(C+\xi)\sqrt{\frac{a_2\rho_1}{\rho_2}}}{\sqrt{3}}\right]\right)^2}{4\rho_1},$$

$$u_{10}^{\pm}(x,t)=a_0+\frac{3(-k+r-2a_0\rho_1-\rho_3)\left(\coth\left[\frac{(C+\xi)\sqrt{-\frac{(-k+r-2a_0\rho_1-\rho_3)}{\rho_2}}}{4\sqrt{2}}\right]+\tanh\left[\frac{(C+\xi)\sqrt{-\frac{(-k+r-2a_0\rho_1-\rho_3)}{\rho_2}}}{4\sqrt{2}}\right]\right)^2}{16\rho_1}.$$

- For trigonometric function, we have

$$u_{11}^{\pm}(x,t)=a_0+\frac{3\sec^2\left[\pm\frac{1}{2}(C+\xi)\sqrt{-\frac{(-k+r-2a_0\rho_1-\rho_3)}{\rho_2}}\right](-k+r-2a_0\rho_1-\rho_3)}{2\rho_1},$$

$$u_{12}^{\pm}(x,t)=a_0+\frac{3\csc^2\left[\pm\frac{1}{2}(C+\xi)\sqrt{-\frac{(-k+r-2a_0\rho_1-\rho_3)}{\rho_2}}\right](-k+r-2a_0\rho_1-\rho_3)}{2\rho_1},$$

$$u_{13}^{\pm}(x,t)=a_0-\frac{3(-k+r-2a_0\rho_1-\rho_3)\tan^2\left[\pm\frac{(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{2\sqrt{2}}\right]}{4\rho_1},$$

$$u_{14}^{\pm}(x,t)=a_0-\frac{3(-k+r-2a_0\rho_1-\rho_3)\cot^2\left[\pm\frac{(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{2\sqrt{2}}\right]}{4\rho_1},$$

$$u_{15}^{\pm}(x,t)=a_0-\frac{3(-k+r-2a_0\rho_1-\rho_3)\left(\sec\left[\frac{(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{\sqrt{2}}\right]+\tan\left[\frac{\pm(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{\sqrt{2}}\right]\right)^2}{4\rho_1},$$

$$u_{16}^{\pm}(x,t)=a_0-\frac{3(-k+r-2a_0\rho_1-\rho_3)\left(\sec\left[\frac{(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{\sqrt{2}}\right]+\tan\left[\frac{\pm(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{\sqrt{2}}\right]\right)^2}{4\rho_1},$$

$$u_{17}^{\pm}(x,t)=a_0-\frac{3(-k+r-2a_0\rho_1-\rho_3)\left(\cot\left[\frac{(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{4\sqrt{2}}\right]+\tan\left[\frac{(C+\xi)\sqrt{\frac{-k+r-2a_0\rho_1-\rho_3}{\rho_2}}}{4\sqrt{2}}\right]\right)^2}{16\rho_1}.$$

3.2. **New exact solutions of Equation (1.1) by improved generalized Riccati equation mapping method.** Now, substituting Equations (2.25) and (3.4) into Equation (3.5) and making the coefficient of each power of $\Phi(\xi)$ to zero, we obtain a system of algebraic equation with the help of Mathematica as follows:

$$\begin{aligned}
ka_0-ra_0+a_0^2\rho_1+2a_2\beta_0^2\rho_2+a_1\beta_0\beta_1\rho_2+a_0\rho_3&=0,\\
ka_1-ra_1+2a_0a_1\rho_1+6a_2\beta_0\beta_1\rho_2+a_1\beta_1^2\rho_2+2a_1\beta_0\beta_2\rho_2+a_1\rho_3&=0,\\
ka_2-ra_2+a_1^2\rho_1+2a_0a_2\rho_1+4a_2\beta_1^2\rho_2+8a_2\beta_0\beta_2\rho_2+3a_1\beta_1\beta_2\rho_2+a_2\rho_3&=0,\\
2a_1a_2\rho_1+10a_2\beta_1\beta_2\rho_2+2a_1\beta_2^2\rho_2&=0,\\
a_2^2\rho_1+6a_2\beta_2^2\rho_2&=0.
\end{aligned}$$

Solving the above system, we obtain

$$a_0=\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$\beta_1=-\frac{ia_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}},$$

$$\beta_2=-\frac{i\sqrt{a_2}\sqrt{\rho_1}}{\sqrt{6}\sqrt{\rho_2}},$$

$$r=k-\frac{1}{6}\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}+\rho_3.$$

Now, substituting the above into Equations (2.26)–(2.50), we obtain the following exact solutions

- For rational form, we have

$$\begin{aligned}
u_{18}^{\pm}(x,t)={}&\pm\frac{i\sqrt{6}\sqrt{\rho_2}}{(C+\xi)\sqrt{a_2}\sqrt{\rho_1}}a_1+\left(\pm\frac{i\sqrt{6}\sqrt{\rho_2}}{(C+\xi)\sqrt{a_2}\sqrt{\rho_1}}\right)^2a_2\\
&+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}.
\end{aligned}$$

- For exponential form

$$u_{19}(x,t) = -\frac{\phi a_1}{e^{\frac{i(C+\xi)a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}+\vartheta} + \frac{\phi^2 a_2}{\left(e^{\frac{i(C+\xi)a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}+\vartheta\right)^2}$$
$$+ \frac{\frac{a_1^2\rho_1}{a_2} + 8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2} - \sqrt{\frac{a_1^4\rho_1^2}{a_2^2} - \frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}} - 96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{20}(x,t) = \frac{e^{-\frac{i\sqrt{\frac{2}{3}}(C+\xi)a_1\sqrt{\rho_1}}{\sqrt{a_2}\sqrt{\rho_2}}}a_1^2}{\left(e^{-\frac{i(C+\xi)a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}+\vartheta\right)^2 a_2} - \frac{e^{-\frac{i(C+\xi)a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}a_1^2}{\left(e^{-\frac{i(C+\xi)a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}+\vartheta\right) a_2}$$
$$+ \frac{\frac{a_1^2\rho_1}{a_2} + 8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2} - \sqrt{\frac{a_1^4\rho_1^2}{a_2^2} - \frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}} - 96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}.$$

- For trigonometric and hyperbolic form, we have

$$u_{21}(x,t) = \frac{\frac{a_1^2\rho_1}{a_2} + 8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2} - \sqrt{\frac{a_1^4\rho_1^2}{a_2^2} - \frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}} - 96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}$$
$$+ a_1\left(-\frac{a_1}{2a_2} - \frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\sqrt{\rho_2}\tanh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]}{\sqrt{a_2}\sqrt{\rho_1}}\right)$$
$$+ a_2\left(-\frac{a_1}{2a_2} - \frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\sqrt{\rho_2}\tanh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2,$$

$$u_{22}(x,t) = a_1\left(-\frac{a_1}{2a_2} - \frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\coth\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)$$
$$+ a_2\left(-\frac{a_1}{2a_2} - \frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\coth\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2$$
$$+ \frac{\frac{a_1^2\rho_1}{a_2} + 8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2} - \sqrt{\frac{a_1^4\rho_1^2}{a_2^2} - \frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}} - 96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{23}^{\pm}(x,t) = a_1\left(-\frac{a_1}{2a_2} - \frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\left(\tanh\left[(C+\xi)\sqrt{\Omega}\right] \pm i\,\mathrm{sech}\left[(C+\xi)\sqrt{\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)$$

$$+ a_2\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\left(\tanh\left[(C+\xi)\sqrt{\Omega}\right]\pm i\,\mathrm{sech}\left[(C+\xi)\sqrt{\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{24}^{\pm}(x,t)=a_1\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\left(\coth\left[(C+\xi)\sqrt{\Omega}\right]\pm \mathrm{csch}\left[(C+\xi)\sqrt{\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)$$

$$+ a_2\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\left(\coth\left[(C+\xi)\sqrt{\Omega}\right]\pm \mathrm{csch}\left[(C+\xi)\sqrt{\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{25}(x,t)=\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}$$

$$+ a_1\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\sqrt{\rho_2}\left(\coth\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]+\tanh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]\right)}{2\sqrt{a_2}\sqrt{\rho_1}}\right)$$

$$+ a_2\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\sqrt{\rho_2}\left(\coth\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]+\tanh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]\right)}{2\sqrt{a_2}\sqrt{\rho_1}}\right)^2,$$

$$u_{26}^{\pm}(x,t)=a_1\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\left(-p\sqrt{\Omega}\cosh\left[(C+\xi)\sqrt{\Omega}\right]+\pm\sqrt{(p^2+q^2)\Omega}\right)\sqrt{\rho_2}}{\left(q+p\sinh\left[(C+\xi)\sqrt{\Omega}\right]\right)\sqrt{a_2}\sqrt{\rho_1}}\right)$$

$$+ a_2\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\left(-p\sqrt{\Omega}\cosh\left[(C+\xi)\sqrt{\Omega}\right]+\pm\sqrt{(p^2+q^2)\Omega}\right)\sqrt{\rho_2}}{\left(q+p\sinh\left[(C+\xi)\sqrt{\Omega}\right]\right)\sqrt{a_2}\sqrt{\rho_1}}\right)^2$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{27}(x,t)=\frac{4\cosh^2\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_2\beta_0^2}{\left(\sqrt{\Omega}\sinh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]+\frac{i\cosh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\right)^2}$$

$$+\frac{2\cosh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_1\beta_0}{\sqrt{\Omega}\sinh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]+\frac{i\cosh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{28}(x,t)=\frac{4\sinh^2\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_2\beta_0^2}{\left(\sqrt{\Omega}\cosh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]+\frac{i\sinh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\right)^2}$$

$$+\frac{2\sinh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_1\beta_0}{\sqrt{\Omega}\cosh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]+\frac{i\sinh\left[\frac{1}{2}(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{29}^{\pm}(x,t)=\frac{2\cosh\left[(C+\xi)\sqrt{\Omega}\right]a_1\beta_0}{-\left(-\frac{i\cosh\left[(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\pm i\sqrt{\Omega}\right)+\sqrt{\Omega}\sinh\left[(C+\xi)\sqrt{\Omega}\right]}$$

$$+\frac{4\cosh^2\left[(C+\xi)\sqrt{\Omega}\right]a_2\beta_0^2}{\left(-\left(-\frac{i\cosh\left[(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\pm i\sqrt{\Omega}\right)+\sqrt{\Omega}\sinh\left[(C+\xi)\sqrt{\Omega}\right]\right)^2}$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{30}^{\pm}(x,t)=\frac{2\sinh\left[(C+\xi)\sqrt{\Omega}\right]a_1\beta_0}{\sqrt{\Omega}\cosh\left[(C+\xi)\sqrt{\Omega}\right]-\left(-\frac{i\sinh\left[(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\pm\sqrt{\Omega}\right)}$$

$$+\frac{4\sinh^2\left[(C+\xi)\sqrt{\Omega}\right]a_2\beta_0^2}{\left(\sqrt{\Omega}\cosh\left[(C+\xi)\sqrt{\Omega}\right]-\left(-\frac{i\sinh\left[(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\pm\sqrt{\Omega}\right)\right)^2}$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$u_{31}(x,t)=\frac{4\cosh^2\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]\sinh^2\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]a_2\beta_0^2}{\left(-\sqrt{\Omega}+2\sqrt{\Omega}\cosh^2\left[(C+\xi)\sqrt{\Omega}\right]+\frac{i\sqrt{\frac{2}{3}}\cosh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]\sinh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{a_2}\sqrt{\rho_2}}\right)^2}$$

$$
+\frac{2\cosh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]\sinh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]a_1\beta_0}{-\sqrt{\Omega}+2\sqrt{\Omega}\cosh^2\left[(C+\xi)\sqrt{\Omega}\right]+\frac{i\sqrt{\frac{2}{3}}\cosh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]\sinh\left[\frac{1}{4}(C+\xi)\sqrt{\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{a_2}\sqrt{\rho_2}}}
+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}.
$$

- For trigonometric and hyperbolic form, we have

$$
u_{32}(x,t)=\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}
+a_1\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\sqrt{\rho_2}\tan\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]}{\sqrt{a_2}\sqrt{\rho_1}}\right)
+a_2\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\sqrt{\rho_2}\tan\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2,
$$

$$
u_{33}(x,t)=a_1\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\cot\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)
+a_2\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\cot\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2
+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},
$$

$$
u_{34}^{\pm}(x,t)=a_1\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\left(\tan\left[(C+\xi)\sqrt{-\Omega}\right]\pm\sec\left[(C+\xi)\sqrt{-\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)
+a_2\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\left(\tan\left[(C+\xi)\sqrt{-\Omega}\right]\pm\sec\left[(C+\xi)\sqrt{-\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2
+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},
$$

$$
u_{35}^{\pm}(x,t)=a_1\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\left(\cot\left[(C+\xi)\sqrt{-\Omega}\right]\pm\csc\left[(C+\xi)\sqrt{-\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)
+a_2\left(-\frac{a_1}{2a_2}-\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\left(\cot\left[(C+\xi)\sqrt{-\Omega}\right]\pm\csc\left[(C+\xi)\sqrt{-\Omega}\right]\right)\sqrt{\rho_2}}{\sqrt{a_2}\sqrt{\rho_1}}\right)^2
$$

$$+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},$$

$$\begin{aligned}u_{36}(x,t)=&\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}\\&+a_1\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\sqrt{\rho_2}\left(-\cot\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]+\tan\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]\right)}{2\sqrt{a_2}\sqrt{\rho_1}}\right)\\&+a_2\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\sqrt{-\Omega}\sqrt{\rho_2}\left(-\cot\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]+\tan\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]\right)}{2\sqrt{a_2}\sqrt{\rho_1}}\right)^2,\end{aligned}$$

$$\begin{aligned}u_{37}^{\pm}(x,t)=&a_1\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\left(-p\sqrt{-\Omega}\cos\left[(C+\xi)\sqrt{-\Omega}\right]+\pm\sqrt{-(p^2-q^2)\Omega}\right)\sqrt{\rho_2}}{\left(q+p\sin\left[(C+\xi)\sqrt{-\Omega}\right]\right)\sqrt{a_2}\sqrt{\rho_1}}\right)\\&+a_2\left(-\frac{a_1}{2a_2}+\frac{i\sqrt{\frac{3}{2}}\left(-p\sqrt{-\Omega}\cos\left[(C+\xi)\sqrt{-\Omega}\right]+\pm\sqrt{-(p^2-q^2)\Omega}\right)\sqrt{\rho_2}}{\left(q+p\sin\left[(C+\xi)\sqrt{-\Omega}\right]\right)\sqrt{a_2}\sqrt{\rho_1}}\right)^2\\&+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},\end{aligned}$$

$$\begin{aligned}u_{38}(x,t)=&\frac{4\cos^2\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_2\beta_0^2}{\left(\sqrt{-\Omega}\sin\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]-\frac{i\cos\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\right)^2}\\&-\frac{2\cos\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_1\beta_0}{\sqrt{-\Omega}\sin\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]-\frac{i\cos\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}\\&+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},\end{aligned}$$

$$\begin{aligned}u_{39}(x,t)=&\frac{4\sin^2\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_2\beta_0^2}{\left(\sqrt{-\Omega}\cos\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]+\frac{i\sin\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\right)^2}\\&+\frac{2\sin\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_1\beta_0}{\sqrt{-\Omega}\cos\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]+\frac{i\sin\left[\frac{1}{2}(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}}\\&+\frac{\frac{a_1^2\rho_1}{a_2}+8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2}-\sqrt{\frac{a_1^4\rho_1^2}{a_2^2}-\frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}}-96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},\end{aligned}$$

$$
\begin{aligned}
u_{40}^{\pm}(x,t) =& \frac{4\cos^2\left[(C+\xi)\sqrt{-\Omega}\right] a_2\beta_0^2}{\left(\sqrt{-\Omega}\sin\left[(C+\xi)\sqrt{-\Omega}\right] \pm \sqrt{-\Omega} - \frac{i\cos\left[(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\right)^2} \\
&+ \frac{2\cos\left[(C+\xi)\sqrt{-\Omega}\right] a_1\beta_0}{\sqrt{-\Omega}\sin\left[(C+\xi)\sqrt{-\Omega}\right] \pm \sqrt{-\Omega} - \frac{i\cos\left[(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}} \\
&+ \frac{\frac{a_1^2\rho_1}{a_2} + 8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2} - \sqrt{\frac{a_1^4\rho_1^2}{a_2^2} - \frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}} - 96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},
\end{aligned}
$$

$$
\begin{aligned}
u_{41}^{\pm}(x,t) =& \frac{4\sin^2\left[(C+\xi)\sqrt{-\Omega}\right] a_2\beta_0^2}{\left(-\left(\sqrt{-\Omega}\cos\left[(C+\xi)\sqrt{-\Omega}\right] \pm \sqrt{-\Omega}\right) - \frac{i\sin\left[(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}\right)^2} \\
&+ \frac{2\sin\left[(C+\xi)\sqrt{-\Omega}\right] a_1\beta_0}{-\left(\sqrt{-\Omega}\cos\left[(C+\xi)\sqrt{-\Omega}\right] \pm \sqrt{-\Omega}\right) - \frac{i\sin\left[(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{6}\sqrt{a_2}\sqrt{\rho_2}}} \\
&+ \frac{\frac{a_1^2\rho_1}{a_2} + 8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2} - \sqrt{\frac{a_1^4\rho_1^2}{a_2^2} - \frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}} - 96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1},
\end{aligned}
$$

$$
\begin{aligned}
u_{42}^{\pm}(x,t) =& \frac{16\cos^2\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]\sin^2\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right] a_2\beta_0^2}{\left(-\sqrt{-\Omega} + 2\sqrt{-\Omega}\cos^2\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right] + \frac{i\sqrt{\frac{2}{3}}\cos\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]\sin\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{a_2}\sqrt{\rho_2}}\right)^2} \\
&+ \frac{4\cos\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]\sin\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right] a_1\beta_0}{-\sqrt{-\Omega} + 2\sqrt{-\Omega}\cos^2\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right] + \frac{i\sqrt{\frac{2}{3}}\cos\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]\sin\left[\frac{1}{4}(C+\xi)\sqrt{-\Omega}\right]a_1\sqrt{\rho_1}}{\sqrt{a_2}\sqrt{\rho_2}}} \\
&+ \frac{\frac{a_1^2\rho_1}{a_2} + 8i\sqrt{6}\sqrt{a_2}\beta_0\sqrt{\rho_1}\sqrt{\rho_2} - \sqrt{\frac{a_1^4\rho_1^2}{a_2^2} - \frac{8i\sqrt{6}a_1^2\beta_0\rho_1^{3/2}\sqrt{\rho_2}}{\sqrt{a_2}} - 96a_2\beta_0^2\rho_1\rho_2}}{12\rho_1}.
\end{aligned}
$$

## 4. Figures of the Solutions

Here, some of the obtained results will be represented graphically. We derived variety of solutions which include, dark, singular, combined dark-singular soliton, singular periodic wave and rational function solutions for the governing model. The 2D, 3D and contour graphs for some of the exact soliton solutions by choosing suitable values for the parameters involved and with $(-15 \leq x \leq 15, 0 \leq t \leq 2)$ are as follows.

## 5. Conclusion

In this paper, we used the improved modified Sardar sub-equation approach and the improved generalized Riccati equation mapping approach to analyze the exact soliton solutions to the $(n+1)$-dimensional generalized KP equation. Moreover, we have found the exponential, trigonometric, rational and trigonometric hyperbolic solutions for the $(n+1)$-dimensional generalized KP equation. The methods are highly effective, and we used wave variable to transform the NLPDE into a nonlinear ODE with integer order and then compare the coefficients of equal powers in the resulting ODE to produce a system algebraic equation that were subsequently solved with the aid of a Mathematica software. Our findings has it that the approaches are strong, well-defined with algorithm that are exceedingly efficient. Therefore, these methods can be employed to solve many NLPDEs arising in the field of soliton theory and other related areas of study. In addition, it is our hope that the obtained solutions may be useful in all areas of engineering and mathematical

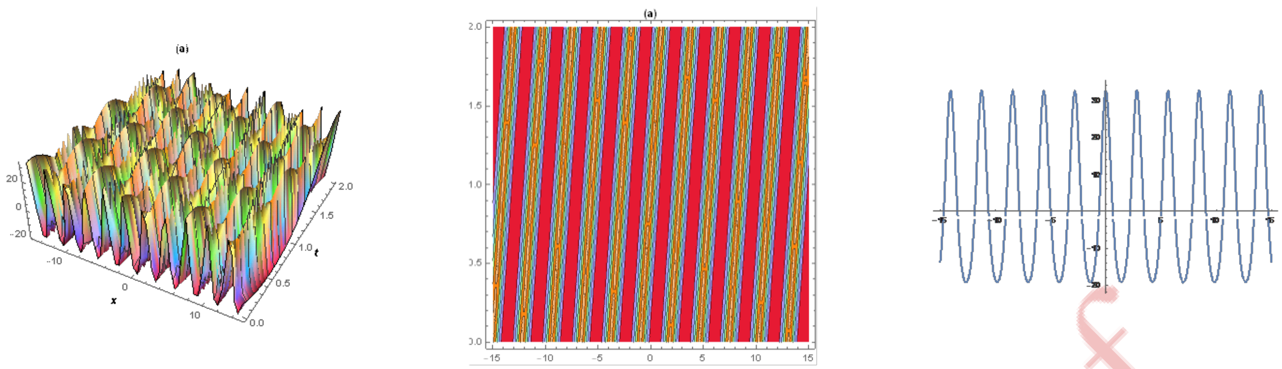

FIGURE 1. The 2D, 3D and contour graphs for the exact solution Re $u_2(x,t)$ with parameter values $\rho_2 = 1$, $\rho_1 = 1$, $\rho_3 = 1$, $a_0 = 1$, $k = 3$, $r = 1$, $a_2 = 1$; 2D plot at $t = 1$.

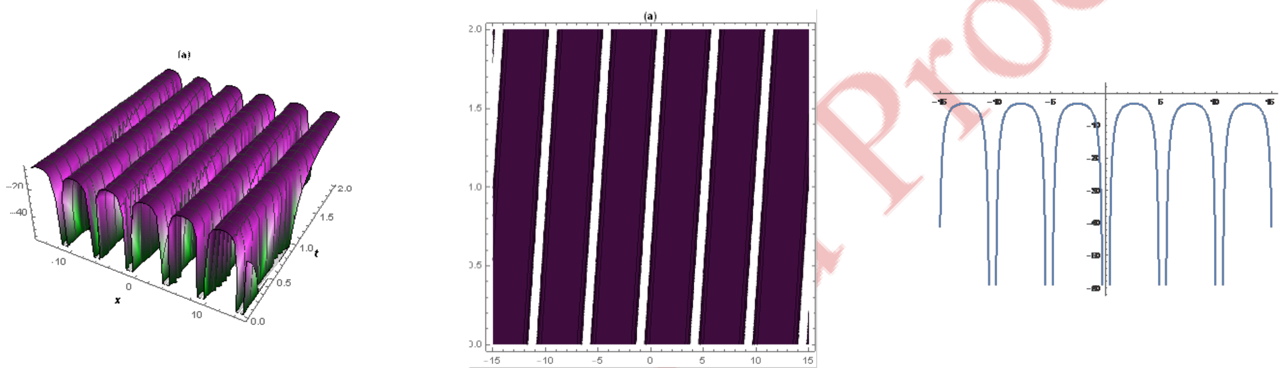

FIGURE 2. The 2D, 3D and contour graphs for the exact solution Re $u_{17}(x,t)$ by choosing these parameter values $\rho_2 = 1$, $\rho_1 = 1$, $\rho_3 = 1$, $a_0 = 1$, $k = 1$, $r = 1$, $a_1 = 1$; 2D plot at $t = 1$.

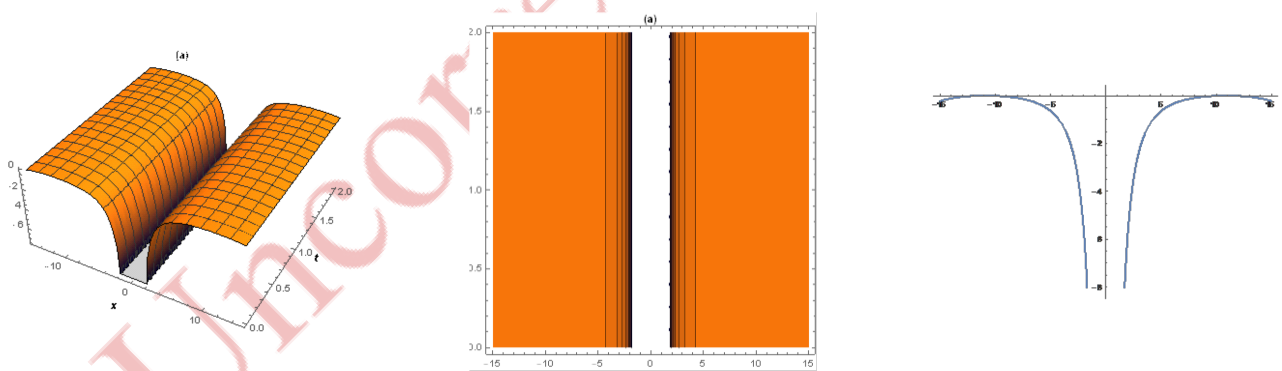

FIGURE 3. The 2D, 3D and contour graphs for the exact solution Re $u_{19}(x,t)$ by choosing these parameter values $\rho_2 = 1$; $\rho_1 = 1$; $\rho_3 = -1$; $a_0 = -1$; $k = 1$; $a_1 = -1$; $a_2 = 1$; $r = 1$; $\beta_0 = 3$; $\phi = 1$; $\vartheta = -1$; 2D plot at $t = 1$.

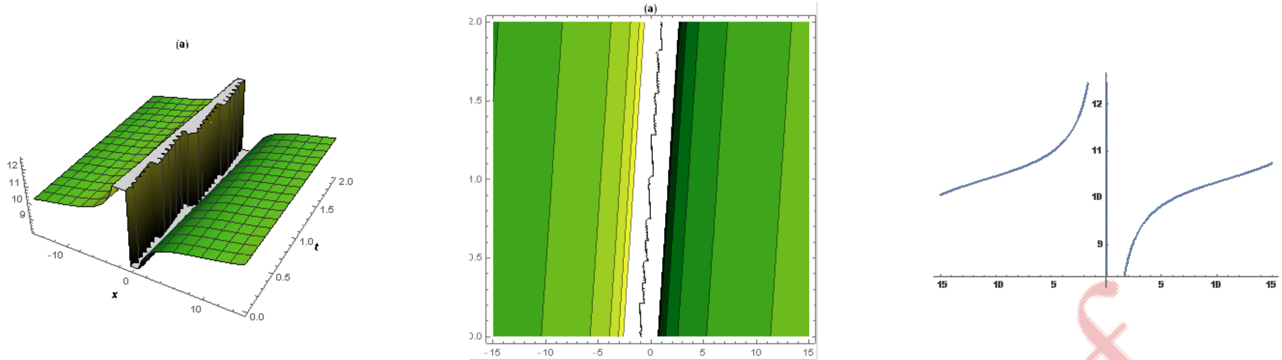


FIGURE 4. The 2D, 3D and contour graphs for the exact solution $\mathrm{Im}u_{19}(x,t)$ by choosing these parameter values $\rho_2 = 1;\ \rho_1 = 1;\ \rho_3 = -1;\ a_0 = -1;\ k = 1;\ a_1 = -1;\ a_2 = 1;\ r = 1;\ a_2 = 2;\ \beta_0 = 3;\ \phi = 1;\ \vartheta = -1;$ 2D plot at $t = 1$.

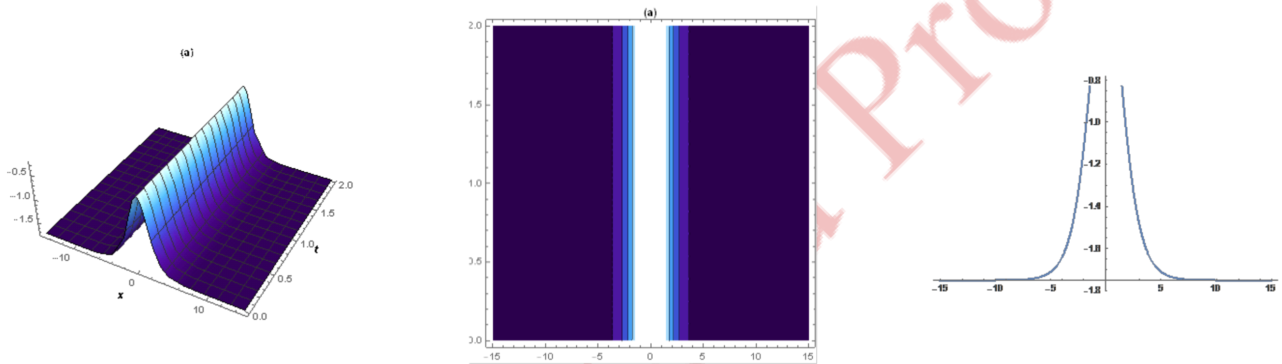


FIGURE 5. The 2D, 3D and contour graphs for the exact solution $\mathrm{Re}u_{21}(x,t)$ by choosing these parameter values $\rho_2 = 1;\ \rho_1 = 1;\ \rho_3 = -1;\ a_0 = -1;\ k = 1;\ a_1 = -1;\ a_2 = 1;\ r = 1;\ \beta_0 = 3;\ \Omega = 1;\ t = 1$.

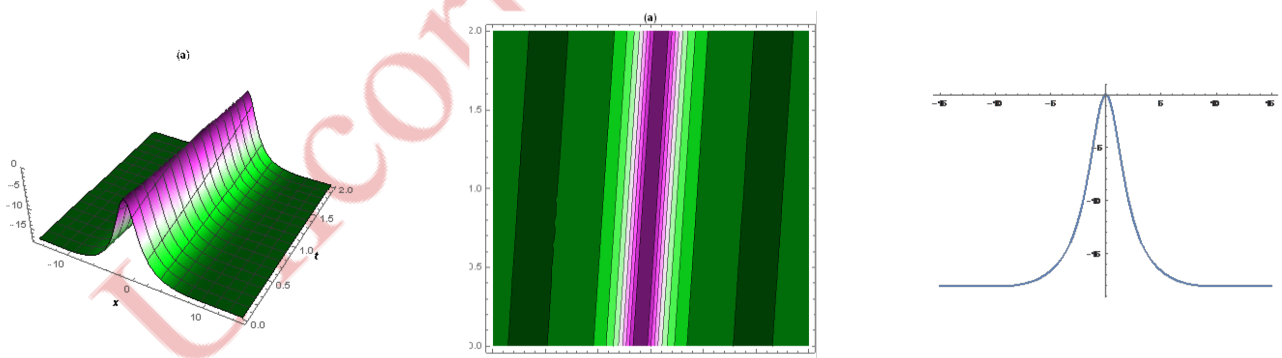


FIGURE 6. The 2D, 3D and contour graphs for the exact solution $\mathrm{Re}u_{31}(x,t)$ by choosing these parameter values $C = 1;\ \rho_2 = 1;\ \rho_1 = 1;\ \rho_3 = -1;\ a_0 = -1;\ k = 1;\ a_1 = -1;\ a_2 = 1;\ r = 1;\ \Omega = -1;\ \beta_0 = 3$ and its 2D plot at $t = 1$.

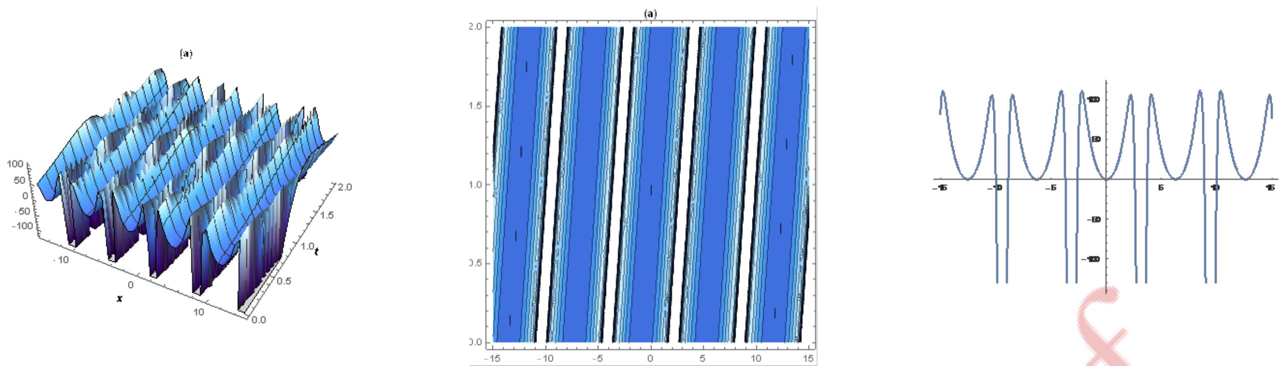


FIGURE 7. The 2D, 3D and contour graphs for the exact solution $\mathrm{Re}u_{42}(x,t)$ by choosing these parameter values $\rho_2 = 1;\ \rho_1 = 1;\ \rho_3 = -1;\ a_0 = -1;\ k = 1;\ a_1 = -1;\ a_2 = 1;\ r = 1;\ a_2 = 2;\ \Omega = -1;\ \beta_0 = 3$ and its 2D plot at $t = 1, C = 1$.

physics. We also provided the graphical representation for some of the obtained results, as was shown in the figures above. Lastly, future studies shall consider various intriguing results associated with the model in question.

## Declarations

**Ethical Approval.** The authors state that they did not conduct any experiments involving animals throughout the duration of this research.
**Funding.** The authors confirm that they did not receive any financial assistance, grants, or any other form of support while preparing this manuscript.
**Data Availability.** This study did not make use of any data sets.
**Conflict of Interest.** The authors assert that they have no conflicts of interest to disclose.
**Author Contributions.** A.D. made significant advancements in the methodology, while I.I.A. played a crucial role in enhancing, evaluating, and refining the concept. H.R. and M.A.H. skillfully led the investigation and managed the computational aspects. They also took responsibility for writing the paper. Additionally, S.S. assumed a prominent role in the development of the model.

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