



A study on approximate controllability of Hilfer fractional stochastic evolution equations of order $1 < \beta < 2$

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Abstract

In this paper, we investigate the approximate controllability of Hilfer fractional stochastic evolution equations of order $\beta \in (1, 2)$. The main findings are carried out by using fractional calculus, stochastic analysis theory, measure of noncompactness, and the fixed point theorem. At first, we prove the existence of a mild solution for Hilfer fractional stochastic evolution equations, and then we establish the concept of approximate controllability. Finally, we provide an example to illustrate our theoretical results.

Keywords. Approximate controllability, Hilfer fractional derivative, Stochastic evolution equation.

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1. INTRODUCTION

The study of fractional differential equations and fractional calculus have emerged as crucial components of mathematics in recent decades. Currently, the idea of fractional calculus has been extensively investigated in many fields, including engineering, image processing, viscoelasticity, biological diffusion, control theory, fluid dynamics, and porous media. For various practical applications, fractional order models perform better than integer-order models. For more information, refer these books [11, 29] and the research articles [10, 14, 19, 21]. Hilfer [8] introduced the Hilfer fractional derivative, which is a generalized Riemann-Liouville fractional derivative. It includes both Riemann-Liouville and Caputo fractional derivatives. Further, the significance and applications of the Hilfer fractional derivative have emerged in the theoretical simulation of dielectric relaxation in glass-forming materials, engineering, polymer science, and so on. For more information on the Hilfer fractional derivative, refer [5, 6, 9, 26, 31, 32].

Controllability is a fundamental concept in mathematical control theory and plays an important role in both deterministic and stochastic control systems. The idea of controllability plays important role in both finite and infinite dimensional spaces. It has many applications in different fields such as physics, engineering, robotics, electronics, chemistry, biology, economics, power systems, and space technology. In recent decades, various works has been discussed on the concept of approximate controllability for differential equations, integro-differential equations, impulsive functional inclusions, stochastic equations, neutral differential equations, and semilinear functional equations, evolution equations and references therein [7, 14, 20, 25].

In many cases, deterministic models often fluctuate due to environmental noise, which is random or at least appears to be so. Stochastic differential equations can be used for escape and jump issues of Brownian particles in physics, along with option pricing problems in economics. Consequently, the significance of stochastic differential equations is crucial in several scientific fields, including chemistry, physics, economics, etc. In recent decades, existence, stability, uniqueness, controllability and other qualitative and quantitative attributes of solutions of stochastic evolution equations received a lot of attention by researchers. An overview of stochastic differential equations and their applications in [1, 4, 16, 22].

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We observe that recent researchers doing research on the topic of fractional differential equations of order $0 < \beta < 1$. For example, [2, 12, 30] discussed the existence of mild solutions for stochastic evolution equations with Caputo fractional derivative of order $\beta \in (0, 1)$. Authors of [21, 22, 24] investigated the existence of mild solutions for Hilfer fractional derivative by using Mönch fixed point theorem. While numerous research results have been conducted on fractional stochastic differential equations with order $\beta \in (1, 2)$. While most research has focused on Caputo fractional stochastic differential equations of order $\beta \in (1, 2)$. In [27], authors discussed the approximate controllability for stochastic impulsive evolution system by using Mönch fixed point theorem. In [18], Li and Zhou investigated the existence of mild solutions for Hilfer fractional stochastic evolution equations with order $\mu \in (1, 2)$.

To the best of our knowledge, no work has been reported in the literature about the approximate controllability of Hilfer fractional stochastic evolution equations of order $\beta \in (1, 2)$. Inspired by the above-mentioned work, this paper aims to fill this gap. Let us consider the Hilfer fractional stochastic evolution equations of order $\beta \in (1, 2)$ and type $\mathfrak{S} \in [0, 1]$ of the form

$$\begin{cases} {}^H D_{0+}^{\beta, \mathfrak{S}} \varkappa(\mathfrak{t}) &= A\varkappa(\mathfrak{t}) + Bu(\mathfrak{t}) + f(\mathfrak{t}, \varkappa(\mathfrak{t})) + h(\mathfrak{t}, \varkappa(\mathfrak{t})) \frac{dW(\mathfrak{t})}{dt}, \quad \mathfrak{t} \in \mathfrak{P}' = (0, b], \\ (I_{0+}^{2-\alpha} \varkappa)(0) &= \varkappa_0, \quad (I_{0+}^{2-\alpha} \varkappa)'(0) = \varkappa_1, \end{cases} \quad (1.1)$$

where ${}^H D_{0+}^{\beta, \mathfrak{S}}$ stands for Hilfer fractional derivative of order $\beta \in (1, 2)$ and type $\mathfrak{S} \in [0, 1]$, the Riemann-Liouville integral operator of order $(2 - \alpha)$ is denoted by $I_{0+}^{2-\alpha}$, where $\alpha = \beta + \mathfrak{S}(2 - \beta)$. The state variable $\varkappa(\cdot)$ takes the value in a separable Hilbert space $\mathcal{Z}$. $A : D(A) \subset \mathcal{Z} \rightarrow \mathcal{Z}$ is the infinitesimal generator of a cosine family $\{\mathcal{C}(\mathfrak{t})\}_{\mathfrak{t} \geq 0}$, which consist of strongly continuous and uniformly bounded linear operators. Here, $\mathfrak{P} = [0, b]$ and control function $u(\mathfrak{t}) \in L^2(\mathfrak{P}, \mathbb{U})$ of admissible control functions for a separable Hilbert space $\mathbb{U}$. Furthermore, $B : \mathbb{U} \rightarrow \mathcal{Z}$ is a bounded linear operator. If $\{W(\mathfrak{t})\}_{\mathfrak{t} \geq 0}$ is a given $\mathfrak{J}$ -valued Wiener process having a finite trace nuclear covariance operator $Q \geq 0$ specified on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\{\mathcal{F}_{\mathfrak{t}}\}_{\mathfrak{t} \geq 0}$ and $\mathfrak{J}$ represents another separable Hilbert space. Finally, we consider that $f : \mathfrak{P} \times \mathcal{Z} \rightarrow \mathcal{Z}$ and $h : \mathfrak{P} \times \mathcal{Z} \rightarrow L(\mathfrak{J}, \mathcal{Z})$ are appropriate functions and that $\varkappa_0, \varkappa_1 \in L^2(\Omega, \mathcal{Z})$.

The following are the main findings of our manuscript:

- (i) A study on the approximate controllability of Hilfer fractional stochastic evolution equations of order $\beta \in (1, 2)$ in the form of Eq. (1.1) is an unexplored topic in this literature that gives motivation for writing this manuscript.
- (ii) In this paper, using the Mönch fixed point theorem, we establish the existence of mild solutions for Hilfer fractional stochastic evolution equations of order $\beta \in (1, 2)$.
- (iii) The approximate controllability of Eq. (1.1) is illustrated, considering that the corresponding linear system is approximately controllable.
- (iv) The main findings are demonstrated by an example.

The structure of this paper is organized as follows: Some fundamental definitions, theorems, and lemmas that are utilized in this study are given in section 2. In section 3, we demonstrate the approximate controllability for the system Eq. (1.1). Finally, in section 4, we illustrate an example to highlight our main findings.

2. PRELIMINARIES

In this section, we provide some basic definitions, lemmas, theorems, and preliminary results that will be useful for understanding this article. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_{\mathfrak{t}}\}_{\mathfrak{t} \geq 0}, \mathbb{P})$ be a complete probability space equipped with a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets. $L_0^2(\Omega, \mathcal{Z}) = \{\varkappa \in L^2(\Omega, \mathcal{Z}); \varkappa \text{ is } \mathcal{F}_0 - \text{measurable}\}$ is a significant subspace of $L^2(\Omega, \mathcal{Z})$. The Banach space of all continuous mappings from $[0, b]$ into $L^2(\Omega, \mathcal{Z})$ is denoted by $C([0, b], L^2(\Omega, \mathcal{Z}))$, whose norm is $\|\varkappa(\cdot)\|_C = (\sup_{\mathfrak{t} \in [0, b]} E\|\varkappa(\mathfrak{t})\|^2)^{\frac{1}{2}} < \infty$.

Given a norm $\|\cdot\|$, let $L(\mathfrak{J}, \mathcal{Z})$ represent the space containing all bounded linear operators from $\mathfrak{J}$ into $\mathcal{Z}$. Considered that there exists a complete orthonormal basis $\{e_k\}_{k \geq 1}$ for $\mathfrak{J}$. For $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$, let $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$ be indicated. According to proposition 2.9 in [3], given that $\varpi(\mathfrak{t}) \in L(\mathfrak{J}, \mathcal{Z})$, and that $\varpi(\mathfrak{t})$ is measurable with regard to



$\mathcal{F}_t$ for $t \in [0, b]$, and fulfills $\int_0^t E\|\varpi(s)\|^2 ds < \infty$, consequently, we possess the following property:

$$E\left\|\int_0^t \varpi(s) dW(s)\right\|^2 \leq Tr(Q) \int_0^t E\|\varpi(s)\|^2 ds. \quad (2.1)$$

Another space is now introduced:

$$C_1([0, b], L^2(\Omega, \mathcal{Z})) := \left\{ \varkappa \in C((0, b], L^2(\Omega, \mathcal{Z})) : \lim_{t \rightarrow 0^+} t^{2-\alpha} \varkappa(t) \text{ exists and finite} \right\}.$$

Define the symbol $\mathcal{C} = C_1([0, b], L^2(\Omega, \mathcal{Z}))$ with the norm $\|\varkappa(\cdot)\|_{\mathcal{C}} = \left(\sup_{t \in (0, b]} E\|t^{2-\alpha} \varkappa(t)\|^2 \right)^{\frac{1}{2}}$, the space is obviously a Banach space.

In this study, we present the function $g_\alpha(\cdot)$, which is specified as follows for calculating convenience:

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\alpha > 0$, and the gamma function $\Gamma(\cdot)$ fulfills $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$. In case $\alpha = 0$, we represent $g_0(t) = \delta(t)$; concentration of the Dirac measure occurs at the origin.

The basics of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative, the Caputo fractional derivative and the relationship between the Riemann-Liouville and the Caputo fractional derivative, one can refer [11].

Definition 2.1. [11] The Riemann-Liouville fractional integral is defined as follows:

$$I_{0+}^\beta \varkappa(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varkappa(s) ds = (g_\beta * \varkappa)(t), \quad t > 0, \beta > 0,$$

where $*$ is the convolution.

Definition 2.2. [11] The Riemann-Liouville fractional derivative is defined as follows:

$$\begin{aligned} {}^{RL}D_{0+}^\beta \varkappa(t) &= \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \left(\int_0^t (t-s)^{n-\beta-1} \varkappa(s) ds \right) \\ &= \frac{d^n}{dt^n} (g_{n-\beta} * \varkappa)(t), \quad t > 0, \quad n-1 < \beta < n, \end{aligned}$$

in particular, its Laplace transform is as follows:

$$L({}^{RL}D_{0+}^\beta \varkappa(t)) = \lambda^{\beta-1} L(\varkappa(t))(\lambda) - \sum_{k=0}^n (g_{n-\beta} * e)(0) \lambda^{n-1-k}. \quad (2.2)$$

Definition 2.3. [11] The Caputo fractional derivative is defined as follows:

$${}^CD_{0+}^\beta \varkappa(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} \varkappa'(s) ds := g_{n-\beta} * \left(\frac{d^n}{dt^n} \varkappa \right)(t), \quad t > 0, \quad n-1 < \beta < n,$$

where the function $\varkappa(t)$ is $n-1$ times continuously differentiable, and is absolutely continuous.

Remark 2.4. [18] Relationship between the Riemann-Liouville and the Caputo fractional derivatives, we get

$${}^CD_{0+}^{1-\zeta} (t^{\ell-1} Q_\ell(t) \varkappa) = {}^{RL}D_{0+}^{1-\zeta} (t^{\ell-1} Q_\ell(t) \varkappa), \quad \zeta \in (0, 1), \quad \varkappa \in \mathcal{Z}.$$

Definition 2.5. [8] The Hilfer fractional derivative of order $n-1 < \beta < n$ and type $0 \leq \mathfrak{S} \leq 1$ is defined by

$${}^HD_{0+}^{\beta, \mathfrak{S}} \varkappa(t) = \left(I_{0+}^{\mathfrak{S}(n-\beta)} \frac{d^n}{dt^n} (I_{0+}^{(1-\mathfrak{S})(n-\beta)} \varkappa) \right)(t), \quad t > 0.$$



Remark 2.6. (i) If $\Im = 0$ and $n - 1 < \beta < n$, then $D_{0+}^{\beta, \Im}$ becomes a Riemann-Liouville fractional derivative:

$$D_{0+}^{\beta, 0} \varkappa(\mathfrak{t}) = \frac{d^n}{d\mathfrak{t}^n} I_{0+}^{(n-\beta)} \varkappa(\mathfrak{t}) = {}^{RL}D_{0+}^{\beta} \varkappa(\mathfrak{t}).$$

(ii) If $\Im = 1$ and $n - 1 < \beta < n$, then $D_{0+}^{\beta, \Im}$ becomes a Caputo fractional derivative:

$$D_{0+}^{\beta, 1} \varkappa(\mathfrak{t}) = I_{0+}^{(n-\beta)} \frac{d^n}{d\mathfrak{t}^n} \varkappa(\mathfrak{t}) = {}^CD_{0+}^{\beta} \varkappa(\mathfrak{t}).$$

Definition 2.7. [29] If $(I_{0+}^{2-\alpha} \varkappa)(\mathfrak{t})$ is continuous and $(I_{0+}^{2-\alpha} \varkappa)'(\mathfrak{t})$ is absolutely continuous, then

$$I_{0+}^{\alpha} ({}^{RL}D_{0+}^{\alpha} \varkappa(\mathfrak{t})) = \varkappa(\mathfrak{t}) - \frac{(I_{0+}^{2-\alpha} \varkappa)(0)}{\Gamma(\alpha - 1)} \mathfrak{t}^{\alpha-2} - \frac{(I_{0+}^{2-\alpha} \varkappa)'(0)}{\Gamma(\alpha)} \mathfrak{t}^{\alpha-1},$$

where $1 < \alpha < 2$.

Lemma 2.8. [13] Given a sequence of Bochner integrable function $\{\varkappa_m(\mathfrak{t})\}_{m=1}^{\infty} : [0, b] \rightarrow \mathbb{Y}$, if there exists $\xi \in L([0, b], \mathbb{R}^+)$ such that

$$\|\varkappa_m(\mathfrak{t})\|_{\mathbb{Y}} \leq \xi(\mathfrak{t}), \quad \mathfrak{t} \in [0, b].$$

Following that, $\chi(\{\varkappa_m(\mathfrak{t})\}_{m=1}^{\infty}) \in L([0, b], \mathbb{R}^+)$, and it fulfills

$$\chi(\{\int_0^{\mathfrak{t}} \varkappa_m(\mathfrak{s}) d\mathfrak{s} : m = 1, 2, \dots\}) \leq 2 \int_0^{\mathfrak{t}} \chi(\{\varkappa_m(\mathfrak{s}) : m = 1, 2, \dots\}) d\mathfrak{s}.$$

Definition 2.9. [15] The Wright function M_{κ} is defined as follows:

$$M_{\kappa}(\vartheta) = \sum_{m=1}^{\infty} \frac{(-\vartheta)^{m-1}}{(m-1)! \Gamma(1 - \kappa m)}, \quad 0 < \kappa < 1, \quad \vartheta \in \mathbb{C},$$

which fulfills

$$\int_0^{\infty} \vartheta^{\delta} M_{\kappa}(\vartheta) d\vartheta = \frac{\Gamma(1 + \delta)}{\Gamma(1 + \kappa \delta)}, \quad \text{for } \delta \geq 0.$$

Definition 2.10. [23] If $\mathbb{Y}$ is a Banach space, then bounded linear operators mapping $\{\mathcal{C}(\mathfrak{t})\}_{\mathfrak{t} \in \mathbb{R}} : \mathbb{Y} \rightarrow \mathbb{Y}$ are called a strongly continuous cosine family iff

- (i) $\mathcal{C}(\mathfrak{t}'' + \mathfrak{t}') + \mathcal{C}(\mathfrak{t}'' - \mathfrak{t}') = 2\mathcal{C}(\mathfrak{t}'')\mathcal{C}(\mathfrak{t}')$ for all $\mathfrak{t}'', \mathfrak{t}' \in \mathbb{R}$;
- (ii) $\mathcal{C}(0)$ is the identity operator I ;
- (iii) $\mathcal{C}(\mathfrak{t})y$ is continuous for $\mathfrak{t} \in \mathbb{R}$ and $y \in \mathbb{Y}$.

One parameter family, $\{\mathcal{S}(\mathfrak{t})\}_{\mathfrak{t} \in \mathbb{R}}$ is defined by

$$\mathcal{S}(\mathfrak{t})\varkappa = \int_0^{\mathfrak{t}} \mathcal{C}(\mathfrak{s})\varkappa d\mathfrak{s}, \quad \mathfrak{t} \in \mathbb{R}, \quad \varkappa \in \mathbb{Y},$$

where the strongly continuous cosine family in $\mathbb{Y}$ is represented by $\{\mathcal{C}(\mathfrak{t})\}_{\mathfrak{t} \in \mathbb{R}}$. An operator $A : \mathbb{Y} \rightarrow \mathbb{Y}$ is an infinitesimal generator for a strongly continuous cosine family $\{\mathcal{C}(\mathfrak{t})\}_{\mathfrak{t} \in \mathbb{R}}$, which is specified by

$$A\varkappa = \left(\frac{d^2}{d\mathfrak{t}^2} \mathcal{C}(\mathfrak{t})\varkappa \right)_{\mathfrak{t}=0}, \quad \varkappa \in D(A),$$

here $D(A) = \{\varkappa \in \mathbb{Y} : \mathcal{C}(\mathfrak{t})\varkappa \text{ is a twice continuously differentiable with regard to } \mathfrak{t}\}$.

Lemma 2.11. [23] Strongly continuous cosine family $\{\mathcal{C}(\mathfrak{t})\}_{\mathfrak{t} \in \mathbb{R}}$ fulfilling $\|\mathcal{C}(\mathfrak{t})\|_{\mathbb{Y}} \leq M_0 e^{\omega|\mathfrak{t}|}$ in $\mathbb{Y}$, for all $\mathfrak{t} \geq 0$ and some $\omega \geq 0$, $M_0 \geq 1$, and $\{\mathcal{C}(\mathfrak{t})\}_{\mathfrak{t} \in \mathbb{R}}$ has an infinitesimal generator denoted by A . Afterwards, for $\text{Re} \lambda > \omega$, $\lambda^2 \in \rho(A)$, and

$$\lambda R(\lambda^2; A)\varkappa = \int_0^{\infty} e^{-\lambda \mathfrak{t}} \mathcal{C}(\mathfrak{t})\varkappa d\mathfrak{t}, \quad R(\lambda^2; A)\varkappa = \int_0^{\infty} e^{-\lambda \mathfrak{t}} \mathcal{S}(\mathfrak{t})\varkappa d\mathfrak{t}, \quad \text{for } \varkappa \in \mathbb{Y}.$$



In this work, A represents the infinitesimal generator of a strongly continuous cosine family of uniformly bounded linear operators $\{\mathcal{C}(\mathbf{t})\}_{\mathbf{t} \geq 0}$ in $\mathcal{Z}$, if $\mathbf{t} \geq 0$, then $\|\mathcal{C}(\mathbf{t})\|_{L(\mathcal{Z})} \leq M$, there exists a constant $M \geq 1$.

Lemma 2.12. [18] *The Hilfer fractional stochastic differential system Eq. (1.1) is equivalent to the integral equations as the form:*

$$\varkappa(\mathbf{t}) = g_{\alpha-1}(\mathbf{t})\varkappa_0 + g_{\alpha}(\mathbf{t})\varkappa_1 + \int_0^{\mathbf{t}} g_{\beta}(\mathbf{t} - \mathbf{s}) \left[A\varkappa(\mathbf{s}) + Bu(\mathbf{s}) + f(\mathbf{s}, \varkappa(\mathbf{s})) \right] d\mathbf{s} + \int_0^{\mathbf{t}} g_{\beta}(\mathbf{t} - \mathbf{s}) h(\mathbf{s}, \varkappa(\mathbf{s})) dW(\mathbf{s}), \quad (2.3)$$

where $\alpha = \beta + \Im(2 - \beta)$.

Lemma 2.13. [18] *If $\varkappa(\mathbf{t})$ fulfills the integral Eq. (2.3), then*

$$\begin{aligned} \varkappa(\mathbf{t}) = & {}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}^{\ell-1}Q_{\ell}(\mathbf{t})\varkappa_0) + I_{0+}^{\zeta}(\mathbf{t}^{\ell-1}Q_{\ell}(\mathbf{t})\varkappa_1) + \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1}Q_{\ell}(\mathbf{t} - \mathbf{s})Bu(\mathbf{s})d\mathbf{s} \\ & + \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1}Q_{\ell}(\mathbf{t} - \mathbf{s})f(\mathbf{s}, \varkappa(\mathbf{s}))d\mathbf{s} + \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1}Q_{\ell}(\mathbf{t} - \mathbf{s})h(\mathbf{s}, \varkappa(\mathbf{s}))dW(\mathbf{s}), \end{aligned} \quad (2.4)$$

for $\mathbf{t} \in (0, b]$, $\zeta = \Im(2 - \beta) \in (0, 1)$, $\beta = 2\ell$, where

$$Q_{\ell}(\mathbf{t}) = \int_0^{\infty} \ell \vartheta M_{\ell}(\vartheta) S(\mathbf{t}^{\ell} \vartheta) d\vartheta.$$

Definition 2.14. [18] An $\mathcal{F}_{\mathbf{t}}$ -adapted stochastic process $\varkappa \in \mathcal{C}$ is a mild solution for the Cauchy problem Eq. (1.1), if $\varkappa_0, \varkappa_1 \in L_0^2(\Omega, \mathcal{Z})$, $u(\cdot) \in L_{\mathcal{F}}^2(\mathfrak{P}, \mathbb{U})$ and

$$\begin{aligned} \varkappa(\mathbf{t}) = & {}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}^{\ell-1}Q_{\ell}(\mathbf{t})\varkappa_0) + I_{0+}^{\zeta}(\mathbf{t}^{\ell-1}Q_{\ell}(\mathbf{t})\varkappa_1) + \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1}Q_{\ell}(\mathbf{t} - \mathbf{s})Bu(\mathbf{s})d\mathbf{s} \\ & + \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1}Q_{\ell}(\mathbf{t} - \mathbf{s})f(\mathbf{s}, \varkappa(\mathbf{s}))d\mathbf{s} + \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1}Q_{\ell}(\mathbf{t} - \mathbf{s})h(\mathbf{s}, \varkappa(\mathbf{s}))dW(\mathbf{s}), \quad \mathbf{t} \in (0, b]. \end{aligned} \quad (2.5)$$

Lemma 2.15. [6] *Given any $\varkappa \in \mathcal{Z}$, the following inequality holds:*

$$\|Q_{\ell}(\mathbf{t})\varkappa\| \leq \frac{M\mathbf{t}^{\ell}}{\Gamma(2\ell)} \|\varkappa\|, \quad \mathbf{t} \geq 0.$$

Furthermore, $Q_{\ell}(\mathbf{t})$ is uniformly continuous: that is, for every $\mathbf{t}_1, \mathbf{t}_2 \geq 0$,

$$\|Q_{\ell}(\mathbf{t}_2) - Q_{\ell}(\mathbf{t}_1)\| \rightarrow 0, \quad \text{as } \mathbf{t}_2 \rightarrow \mathbf{t}_1.$$

Lemma 2.16. [18] *The following formula is accurate if Eq. (2.3) holds for every $\mathbf{t} > 0$ and $\varkappa \in \mathcal{Z}$:*

$$\frac{d}{d\mathbf{t}}(\mathbf{t}^{\ell-1}Q_{\ell}(\mathbf{t})\varkappa) = (\ell - 1)\mathbf{t}^{\ell-2}Q_{\ell}(\mathbf{t})\varkappa + \mathbf{t}^{2\ell-2} \int_0^{\infty} \ell^2 \vartheta^2 M_{\ell}(\vartheta) \mathcal{C}(\mathbf{t}^{\ell} \vartheta) \varkappa d\vartheta.$$

Moreover,

$$\left\| \frac{d}{d\mathbf{t}}(\mathbf{t}^{\ell-1}Q_{\ell}(\mathbf{t})\varkappa) \right\| \leq \frac{M\mathbf{t}^{2\ell-2}}{\Gamma(2\ell)} \|\varkappa\|, \quad \mathbf{t} > 0.$$

Lemma 2.17. [28] *Suppose*

- (i) $\nu > 0$, $0 < P \leq \infty$.
- (ii) positive function $a(\mathbf{t})$ and $e(\mathbf{t})$ are locally integrable on $0 \leq \mathbf{t} < P$.
- (iii) Continuous function $g(\mathbf{t})$ is positive, increasing and bounded, $0 \leq \mathbf{t} < P$. If

$$e(\mathbf{t}) \leq a(\mathbf{t}) + g(\mathbf{t}) \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\nu-1} e(\mathbf{s}) d\mathbf{s},$$

then

$$e(\mathbf{t}) \leq a(\mathbf{t}) + \int_0^{\mathbf{t}} \left[\sum_{n=1}^{\infty} \frac{(g(\mathbf{t})\Gamma(\nu))^n}{\Gamma(n\nu)} (\mathbf{t} - \mathbf{s})^{\nu-1} a(\mathbf{s}) \right] d\mathbf{s}, \quad \text{for } 0 \leq \mathbf{t} < P,$$

in particular, if $a(\mathbf{t}) = 0$, then $e(\mathbf{t}) = 0$ for each $0 \leq \mathbf{t} < P$.



Definition 2.18. [20] Indicate that at terminal time b , the reachable set Eq. (1.1) is

$$\mathcal{R}(b, \varkappa_0, \varkappa_1) = \{\varkappa(b, \varkappa_0, \varkappa_1, u) : u(\cdot) \in L^2(\mathfrak{P}, \mathbb{U})\}.$$

If $\overline{\mathcal{R}(b, \varkappa_0, \varkappa_1)} = \mathcal{Z}$, then Eq. (1.1) is considered to be approximately controllable on $\mathfrak{P}$.

Consider the following linear control system

$$\begin{cases} {}^H D_{0+}^{\beta, \mathfrak{S}} \varkappa(\mathfrak{t}) &= A\varkappa(\mathfrak{t}) + Bu(\mathfrak{t}), \quad \mathfrak{t} \in (0, b], \\ (I_{0+}^{2-\alpha} \varkappa)(0) &= \varkappa_0, \quad (I_{0+}^{2-\alpha} \varkappa)'(0) = \varkappa_1. \end{cases} \quad (2.6)$$

Given Eq. (2.6), define its corresponding operator as

$$\begin{aligned} \Gamma_0^b &= \int_0^b (b-s)^{\ell-1} Q_\ell(b-s) B B^* Q_\ell^*(b-s) ds, \\ R(\epsilon, \Gamma_0^b) &= (\epsilon I + \Gamma_0^b)^{-1}, \text{ for } \epsilon > 0, \end{aligned}$$

where B^* and $Q_\ell^*(\mathfrak{t})$ represents the adjoint of B and $Q_\ell(\mathfrak{t})$. Then, Γ_0^b becomes a linear bounded operator.

Lemma 2.19. [19] The linear system Eq. (2.6) is approximately controllable in $\mathfrak{P}$ iff $\varkappa(\varkappa I + \Gamma_0^b)^{-1} \rightarrow 0$ as $\varkappa \rightarrow 0^+$ in the strong operator topology.

Lemma 2.20. [17] Let V be a closed, convex subset of a Banach space $\mathbb{Y}$ and $0 \in V$. Suppose that $\Psi : V \rightarrow \mathbb{Y}$ is continuous map that fulfills Mönch condition, i.e., $V_1 \subset V$ is countable $V_1 \subset \overline{\text{co}(\{0\} \cup \Psi(V_1))} \Rightarrow \overline{V_1}$ is compact. Then Ψ has a fixed point in V .

3. MAIN RESULTS

The following hypotheses are required in order to illustrate the primary result of this paper:

(A1) The function $f : \mathfrak{P} \times \mathcal{Z} \rightarrow \mathcal{Z}$ fulfills the following conditions:

- (i) The function $f(\mathfrak{t}, \cdot) : \mathcal{Z} \rightarrow \mathcal{Z}$ is continuous, and for each $\varkappa \in \mathcal{Z}$, $f(\cdot, \varkappa) : \mathfrak{P} \rightarrow \mathcal{Z}$ is strongly measurable.
- (ii) There exist a function $m_2 \in L^1(\mathfrak{P}, \mathbb{R}^+)$, and non-decreasing continuous function $\Theta_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that

$$E\|f(\mathfrak{t}, \varkappa)\|^2 \leq m_1(\mathfrak{t})\Theta_1(\|\varkappa\|_{\mathcal{Z}}^2),$$

for arbitrary $(\mathfrak{t}, \varkappa) \in \mathfrak{P} \times \mathcal{Z}$, and the function Θ_1 fulfilling

$$\liminf_{\tau \rightarrow \infty} \frac{\Theta_1(\tau)}{\tau} = \Lambda_f < \infty.$$

- (iii) There exists a function $\eta_1 \in L^1(\mathfrak{P}, \mathbb{R}^+)$ such that for bounded and countable subset $D \subset \mathcal{Z}$,

$$\chi(f(\mathfrak{t}, D)) \leq \eta_1 \mathfrak{t}^{2-\alpha} \chi(D).$$

(A2) The function $h : \mathfrak{P} \times \mathcal{Z} \rightarrow L_2^0(\mathfrak{J}, \mathcal{Z})$ fulfills the following conditions:

- (i) For each $\mathfrak{t} \in \mathfrak{P}$, the function $h(\mathfrak{t}, \cdot) : \mathcal{Z} \rightarrow L_2^0(\mathfrak{J}, \mathcal{Z})$ is continuous, and for each $\varkappa \in \mathcal{Z}$, the function $h(\cdot, \varkappa) : \mathfrak{P} \rightarrow L_2^0(\mathfrak{J}, \mathcal{Z})$ is strongly measurable.
- (ii) There exist a function $m_2 \in L^1(\mathfrak{P}, \mathbb{R}^+)$, and non-decreasing continuous function $\Theta_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that

$$E\|h(\mathfrak{t}, \varkappa)\|^2 \leq m_2(\mathfrak{t})\Theta_2(\|\varkappa\|_{\mathcal{Z}}^2),$$

for arbitrary $(\mathfrak{t}, \varkappa) \in \mathfrak{P} \times \mathcal{Z}$, and the function Θ_2 fulfilling

$$\liminf_{\tau \rightarrow \infty} \frac{\Theta_2(\tau)}{\tau} = \Lambda_h < \infty.$$

- (iii) There exists a function $\eta_2 \in L^1(\mathfrak{P}, \mathbb{R}^+)$ such that for bounded and countable subset $D \subset \mathcal{Z}$,

$$\chi(h(\mathfrak{t}, D)) \leq \eta_2 \mathfrak{t}^{2-\alpha} \chi(D).$$



(A3)

$$K^* = \left(\frac{2b^{3\ell}}{3\ell} \left(\frac{MM_B}{\epsilon^{\frac{1}{2}}\Gamma(\beta)} \right)^2 + 1 \right) \left[\frac{2Mb^{2-\alpha}}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} \eta_1(s) ds + \frac{Mb^{2-\alpha}}{\Gamma(\beta)} \times \left(2\text{Tr}(Q) \int_0^b (b-s)^{2(\beta-1)} \eta_2^2(s) ds \right)^{\frac{1}{2}} \right] < 1.$$

Lemma 3.1. [4] For any $\tilde{\kappa}_b \in L^2(\mathcal{F}_b, \mathcal{Z})$, there exists $\tilde{\phi} \in L^2_{\mathcal{F}}(\Omega, L^2([0, b], L^0_2))$, such that $\tilde{\kappa}_b = E\tilde{\kappa}_b + \int_0^b \tilde{\phi}(s) dW(s)$. For any $\epsilon > 0$ and $\tilde{\kappa}_b \in L^2(\mathcal{F}_b, \mathcal{Z})$, we define the control function

$$\begin{aligned} u_{\kappa}^{\epsilon}(t) = & B^* Q_p^*(b-t) R(\epsilon, \Gamma_0^b) \left\{ E\tilde{\kappa}_b + \int_0^b \tilde{\phi}(s) dW(s) - {}^{RL}D_{0+}^{1-\zeta} (b^{\ell-1} Q_{\ell}(b) \kappa_0) \right. \\ & - I_{0+}^{\zeta} (b^{\ell-1} Q_{\ell}(b) \kappa_1) - \int_0^b (b-s)^{\ell-1} Q_{\ell}(b-s) f(s, \kappa(s)) ds \\ & \left. - \int_0^b (b-s)^{\ell-1} Q_{\ell}(b-s) h(s, \kappa(s)) dW(s) \right\}. \end{aligned}$$

Theorem 3.2. If assumptions (A1)-(A2) are true, then it has a unique mild solution for the system Eq. (1.1) in $\mathfrak{P}$.

Proof. Assuming that the mapping $\Lambda : (\Lambda\kappa)(t) = (\Lambda_1\kappa)(t) + (\Lambda_2\kappa)(t)$, $\kappa \in C_1((0, b], L^2(\Omega, \mathcal{Z}))$, where

$$\begin{aligned} (\Lambda_1\kappa)(t) &= {}^{RL}D_{0+}^{1-\zeta} (t^{\ell-1} Q_{\ell}(t) \kappa_0) + I_{0+}^{1-\zeta} (t^{\ell-1} Q_{\ell}(t) \kappa_1), \text{ for } t \in (0, b], \\ (\Lambda_2\kappa)(t) &= \int_0^t (t-s)^{\ell-1} Q_{\ell}(t-s) B u_{\kappa}^{\epsilon}(s) ds + \int_0^t (t-s)^{\ell-1} Q_{\ell}(t-s) f(s, \kappa(s)) ds \\ &\quad + \int_0^t (t-s)^{\ell-1} Q_{\ell}(t-s) h(s, \kappa(s)) dW(s), \text{ for } t \in (0, b]. \end{aligned}$$

It is evident that, if Λ has a fixed point $\kappa^* \in C_1([0, b], L^2(\Omega, \mathcal{Z}))$, then (1.1) has a mild solution, $\kappa \in C_1([0, b], L^2(\Omega, \mathcal{Z}))$. For all $z \in C((0, b], L^2(\Omega, \mathcal{Z}))$, set

$$\kappa(t) = t^{\alpha-2} z(t), \quad t \in (0, b].$$

Define an operator Ψ :

$$(\Psi z)(t) = (\Psi_1 z)(t) + (\Psi_2 z)(t), \text{ for } t \in [0, b],$$

where

$$\begin{aligned} (\Psi_1 z)(t) &= \begin{cases} t^{2-\alpha} (\Lambda_1 \kappa)(t), & \text{for } t \in (0, b], \\ \frac{\kappa_0}{\Gamma(\zeta+2\ell-1)}, & \text{for } t = 0, \end{cases} \\ (\Psi_2 z)(t) &= \begin{cases} t^{2-\alpha} (\Lambda_2 \kappa)(t), & \text{for } t \in (0, b], \\ 0, & \text{for } t = 0. \end{cases} \end{aligned}$$

Let $B_q = \{z : z \in C([0, b], L^2(\Omega, \mathcal{Z})), \|z\|_C \leq q\}$, $\tilde{B}_q = \{\kappa : \kappa \in C_1([0, b], L^2(\Omega, \mathcal{Z})), \|\kappa\|_{\mathcal{C}} \leq q\}$. It appears that, B_q and $\tilde{B}_q$ are nonempty, convex, and closed subsets of $C([0, b], L^2(\Omega, \mathcal{Z}))$ and $C_1([0, b], L^2(\Omega, \mathcal{Z}))$, accordingly. Let $\mathcal{T} := \{e : e(t) = (\Psi z)(t), z \in B_q\}$.

Then we demonstrate the fixed point property of Ψ . The proof is broken down into many steps.

Step 1 : We demonstrate that there exists q such that Ψ maps $B_q(\mathfrak{P})$ into $B_q(\mathfrak{P})$.

When $t > 0$, Lemma 2.16 and Remark 2.4 allow us to derive

$$\begin{aligned} \left\| {}^{RL}D_{0+}^{1-\zeta} (t^{\ell-1} Q_{\ell}(t) \kappa) \right\| &= \left\| {}^CD_{0+}^{1-\zeta} (t^{\ell-1} Q_{\ell}(t) \kappa) \right\| = \left\| I_{0+}^{\zeta} \left(\frac{d}{dt} t^{\ell-1} Q_{\ell}(t) \kappa \right) \right\| \\ &\leq \int_0^t g_{\zeta}(t-s) \left\| \frac{d}{ds} s^{\ell-1} Q_{\ell}(s) \kappa \right\| ds \leq \frac{M}{\Gamma(2\ell)} \int_0^t g_{\zeta}(t-s) s^{2\ell-2} \|\kappa\| ds \\ &= \frac{M t^{2\ell+\zeta-2}}{(2\ell-1)\Gamma(\zeta+2\ell-1)} \|\kappa\|, \end{aligned}$$



which implies that

$$\left\| {}^{RL}D_{0+}^{1-\zeta}(\mathfrak{t}^{\ell-1}Q_{\ell}(\mathfrak{t})\varkappa) \right\| \leq \frac{M\mathfrak{t}^{2\ell+\zeta-2}}{(2\ell-1)\Gamma(\zeta+2\ell-1)}\|\varkappa\|, \quad \mathfrak{t} > 0, \quad \varkappa \in \mathcal{Z}. \quad (3.1)$$

Similarly, by using Lemma 2.15, we get

$$\left\| I_{0+}^{\zeta}(\mathfrak{t}^{\ell-1}Q_{\ell}(\mathfrak{t})\varkappa) \right\| \leq \frac{M\mathfrak{t}^{2\ell+\zeta-1}}{\Gamma(\zeta+2\ell)}\|\varkappa\|, \quad \mathfrak{t} > 0, \quad \varkappa \in \mathcal{Z}, \quad (3.2)$$

and

$$\begin{aligned} E\|u_{\varkappa}^{\epsilon}(\mathfrak{t})\|^2 &\leq 5\left(\frac{MM_B(b-\mathfrak{t})^{\ell}}{\epsilon\Gamma(2\ell)}\right)^2 \left\{ 2\|E\tilde{\varkappa}_b\|^2 + 2\left\|\int_0^b \tilde{\phi}(\mathfrak{s})d\mathbf{W}(\mathfrak{s})\right\|^2 \right. \\ &\quad + \left(\frac{Mb^{2\ell+\zeta-2}}{(2\ell-1)\Gamma(\zeta+2\ell-1)}\right)^2 E\|\varkappa_0\|^2 + \left(\frac{Mb^{2\ell+\zeta-1}}{\Gamma(\zeta+2\ell)}\right)^2 E\|\varkappa_1\|^2 \\ &\quad + \left(\frac{Mb^{\ell}}{\Gamma(2\ell)}\right)^2 \frac{1}{2\ell} \int_0^b (b-\mathfrak{s})^{2\ell-1} m_1(\mathfrak{s})d\mathfrak{s} \cdot \Theta_1(\mathfrak{q}) \\ &\quad \left. + \left(\frac{M}{\Gamma(2\ell)}\right)^2 Tr(Q) \int_0^b (b-\mathfrak{s})^{2(2\ell-1)} m_2(\mathfrak{s})d\mathfrak{s} \cdot \Theta_2(\mathfrak{q}) \right\}, \end{aligned}$$

where $\|B\| \leq M_B$. If we assume $\Psi(B_{\mathfrak{q}}) \not\subseteq B_{\mathfrak{q}}$, then for every positive constant $\mathfrak{q}$ and $\mathfrak{t} \in \mathfrak{P}$, there exists $z^{\mathfrak{q}}(\cdot) \in B_{\mathfrak{q}}$, such that $E\|(\Psi z^{\mathfrak{q}})(\mathfrak{t})\|^2 > \mathfrak{q}$, we have

$$\begin{aligned} &\mathfrak{q} < E\|(\Psi z^{\mathfrak{q}})(\mathfrak{t})\|^2 \\ &\leq 5 \sup_{\mathfrak{t} \in \mathfrak{P}} \mathfrak{t}^{2(2-\alpha)} \left\{ E\left\| {}^{RL}D_{0+}^{1-\zeta}(\mathfrak{t}^{\ell-1}Q_{\ell}(\mathfrak{t})\varkappa_0) \right\|^2 + E\left\| I_{0+}^{\zeta}(\mathfrak{t}^{\ell-1}Q_{\ell}(\mathfrak{t})\varkappa_1) \right\|^2 \right. \\ &\quad + E\left\| \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\ell-1}Q_{\ell}(\mathfrak{t}-\mathfrak{s})Bu_{\varkappa}^{\epsilon}(\mathfrak{s})d\mathfrak{s} \right\|^2 + E\left\| \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\ell-1}Q_{\ell}(\mathfrak{t}-\mathfrak{s})f(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} \right\|^2 \\ &\quad \left. + E\left\| \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\ell-1}Q_{\ell}(\mathfrak{t}-\mathfrak{s})h(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathbf{W}(\mathfrak{s}) \right\|^2 \right\} \\ &\leq \sup_{\mathfrak{t} \in \mathfrak{P}} \left\{ 5\left(\frac{M}{(2\ell-1)\Gamma(\zeta+2\ell-1)}\right)^2 E\|\varkappa_0\|^2 + 5\left(\frac{\mathfrak{t}}{\Gamma(\zeta+2\ell)}\right)^2 E\|\varkappa_1\|^2 \right. \\ &\quad + 5\left(\frac{M\mathfrak{t}^{2-\alpha+\ell}}{\Gamma(2\ell)}\right)^2 \frac{1}{2\ell} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{2\ell-1} E\|f(\mathfrak{s},\varkappa(\mathfrak{s}))\|^2 d\mathfrak{s} \\ &\quad + 5\left(\frac{M\mathfrak{t}^{2-\alpha}}{\Gamma(2\ell)}\right)^2 Tr(Q) \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{2(2\ell-1)} E\|h(\mathfrak{s},\varkappa(\mathfrak{s}))\|^2 d\mathfrak{s} \\ &\quad \left. + 5\left(\frac{MM_B\mathfrak{t}^{2-\alpha+\ell}}{\Gamma(2\ell)}\right)^2 \frac{1}{2\ell} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{2\ell-1} E\|u_{\varkappa}^{\epsilon}(\mathfrak{s})\|^2 d\mathfrak{s} \right\} \\ &\leq 5\left(\frac{M}{(2\ell-1)\Gamma(\zeta+2\ell-1)}\right)^2 E\|\varkappa_0\|^2 + 5\left(\frac{b}{\Gamma(\zeta+2\ell)}\right)^2 E\|\varkappa_1\|^2 \\ &\quad + 5\left(\frac{Mb^{2-\alpha+\ell}}{\Gamma(2\ell)}\right)^2 \frac{1}{2\ell} \int_0^b (b-\mathfrak{s})^{2\ell-1} m_1(\mathfrak{s})d\mathfrak{s} \cdot \Theta_1(\mathfrak{q}) \\ &\quad + 5\left(\frac{Mb^{2-\alpha}}{\Gamma(2\ell)}\right)^2 Tr(Q) \int_0^b (b-\mathfrak{s})^{2(2\ell-1)} m_2(\mathfrak{s})d\mathfrak{s} \cdot \Theta_2(\mathfrak{q}) \\ &\quad + 25\left(\frac{MM_B}{\Gamma(2\ell)}\right)^4 \left(\frac{b^{6\ell-2\alpha+4}}{8\ell^2\epsilon^2}\right) \left\{ 2\|E\tilde{\varkappa}_b\|^2 + 2\left\|\int_0^b \tilde{\phi}(\mathfrak{s})d\mathbf{W}(\mathfrak{s})\right\|^2 \right\} \end{aligned}$$



$$\begin{aligned}
& + \left(\frac{Mb^{2\ell+\zeta-2}}{(2\ell-1)\Gamma(\zeta+2\ell-1)} \right)^2 E\|\kappa_0\|^2 + \left(\frac{Mb^{2\ell+\zeta-1}}{\Gamma(\zeta+2\ell)} \right)^2 E\|\kappa_1\|^2 \\
& + \left(\frac{Mb^\ell}{\Gamma(2\ell)} \right)^2 \frac{1}{2\ell} \int_0^b (b-s)^{2\ell-1} m_1(s) ds \cdot \Theta_1(\mathbf{q}) \\
& + \left(\frac{M}{\Gamma(2\ell)} \right)^2 Tr(Q) \int_0^b (b-s)^{2(2\ell-1)} m_2(s) ds \cdot \Theta_2(\mathbf{q}) \Big\}.
\end{aligned}$$

Dividing both sides by $\mathbf{q}$ and taking the limit as $\mathbf{q} \rightarrow \infty$, we get

$$\begin{aligned}
& \left(1 + 5 \left(\frac{MM_B}{\Gamma(2\ell)} \right)^4 \left(\frac{b^{6\ell}}{8\ell\epsilon^2} \right) \right) \left(\frac{5}{2\ell} \left(\frac{Mb^{2-\alpha+\ell}}{\Gamma(2\ell)} \right)^2 \int_0^b (b-s)^{2\ell-1} m_1(s) ds \cdot \Lambda_f \right. \\
& \left. + 5 \left(\frac{Mb^{2-\alpha}}{\Gamma(2\ell)} \right)^2 Tr(Q) \int_0^b (b-s)^{2(2\ell-1)} m_2(s) ds \cdot \Lambda_h \right) \geq 1,
\end{aligned}$$

which is a contradiction to our assumption. Thus there exists a $\mathbf{q} > 0$ such that $\Psi(B_{\mathbf{q}}) \subset B_{\mathbf{q}}$.

Step 2 : We demonstrate that Ψ is continuous on $B_{\mathbf{q}}(\mathfrak{P})$.

Assume that the sequence $\{z_n\}_{n=1}^\infty$ in $B_{\mathbf{q}}$ converges to z . Next,

$$\lim_{n \rightarrow \infty} z_n(\mathbf{t}) = z(\mathbf{t}) \text{ and } \lim_{n \rightarrow \infty} \mathbf{t}^{\alpha-2} z_n(\mathbf{t}) = \mathbf{t}^{\alpha-2} z(\mathbf{t}), \text{ for } \mathbf{t} \in (0, b].$$

As $y(\mathbf{t}) = \mathbf{t}^{\alpha-2} z(\mathbf{t})$, $\mathbf{t} \in (0, b]$, according to **(A1)** and **(A2)**, we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} E\|f(\mathbf{t}, \kappa_n(\mathbf{t}))\|^2 &= \lim_{n \rightarrow \infty} E\|f(\mathbf{t}, \mathbf{t}^{\alpha-2} z_n(\mathbf{t}))\|^2 = E\|f(\mathbf{t}, \mathbf{t}^{\alpha-2} z(\mathbf{t}))\|^2 = E\|f(\mathbf{t}, \kappa(\mathbf{t}))\|^2, \\
\lim_{n \rightarrow \infty} E\|h(\mathbf{t}, \kappa_n(\mathbf{t}))\|^2 &= \lim_{n \rightarrow \infty} E\|h(\mathbf{t}, \mathbf{t}^{\alpha-2} z_n(\mathbf{t}))\|^2 = E\|h(\mathbf{t}, \mathbf{t}^{\alpha-2} z(\mathbf{t}))\|^2 = E\|h(\mathbf{t}, \kappa(\mathbf{t}))\|^2.
\end{aligned}$$

By employing **(A1)**, we can obtain

$$(\mathbf{t} - \mathbf{s})^{2\ell-1} E\|f(\mathbf{s}, \kappa_n(\mathbf{s})) - f(\mathbf{s}, \kappa(\mathbf{s}))\|^2 \leq 4(\mathbf{t} - \mathbf{s})^{2\ell-1} m_1(\mathbf{s}) \Theta_1(\|\kappa\|_{\mathcal{C}}^2), \mathbf{t} \in (0, b].$$

As $\mathbf{s} \rightarrow 4(\mathbf{t} - \mathbf{s})^{2\ell-1} m_1(\mathbf{s}) \Theta_1(\|\kappa\|_{\mathcal{C}}^2)$ is integrable for $\mathbf{s} \in [0, \mathbf{t}]$, the Lebesgue's dominated convergence theorem allows us to determine

$$E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{2\ell-1} [f(\mathbf{s}, \kappa_n(\mathbf{s})) - f(\mathbf{s}, \kappa(\mathbf{s}))] d\mathbf{s} \right\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Similarly, we obtain

$$E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{2\ell-1} [h(\mathbf{s}, \kappa_n(\mathbf{s})) - h(\mathbf{s}, \kappa(\mathbf{s}))] d\mathbf{W}(\mathbf{s}) \right\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Consequently, we have for any $\mathbf{t} \in [0, b]$

$$\begin{aligned}
E \left\| (\Psi_2 z_n)(\mathbf{t}) - (\Psi_2 z)(\mathbf{t}) \right\|^2 &\leq 3\mathbf{t}^{2(2-\alpha)} E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1} Q_\ell(\mathbf{t} - \mathbf{s}) B[u_{\kappa_n}^\epsilon(\mathbf{s}) - u_\kappa^\epsilon(\mathbf{s})] d\mathbf{s} \right\|^2 \\
&+ 3\mathbf{t}^{2(2-\alpha)} E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1} Q_\ell(\mathbf{t} - \mathbf{s}) [f(\mathbf{s}, \kappa_n(\mathbf{s})) - f(\mathbf{s}, \kappa(\mathbf{s}))] d\mathbf{s} \right\|^2 \\
&+ 3\mathbf{t}^{2(2-\alpha)} E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\ell-1} Q_\ell(\mathbf{t} - \mathbf{s}) [h(\mathbf{s}, \kappa_n(\mathbf{s})) - h(\mathbf{s}, \kappa(\mathbf{s}))] d\mathbf{W}(\mathbf{s}) \right\|^2 \\
&\leq 3 \left(\frac{MM_B \mathbf{t}^{2-\alpha}}{\Gamma(2\ell)} \right)^2 E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{2\ell-1} [u_{\kappa_n}^\epsilon(\mathbf{s}) - u_\kappa^\epsilon(\mathbf{s})] d\mathbf{s} \right\|^2 \\
&+ 3 \left(\frac{M \mathbf{t}^{2-\alpha}}{\Gamma(2\ell)} \right)^2 E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{2\ell-1} [f(\mathbf{s}, \kappa_n(\mathbf{s})) - f(\mathbf{s}, \kappa(\mathbf{s}))] d\mathbf{s} \right\|^2
\end{aligned}$$



$$+ 3 \left(\frac{M \mathbf{t}^{2-\alpha}}{\Gamma(2\ell)} \right)^2 E \left\| \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{2\ell-1} [h(\mathbf{s}, \varkappa_n(\mathbf{s})) - h(\mathbf{s}, \varkappa(\mathbf{s}))] d\mathbf{W}(\mathbf{s}) \right\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

As a result, Ψ is continuous.

Step 3 : If **(A1)** – **(A2)** true, then the set $\mathcal{T}$ is equicontinuous.

In order to demonstrate that $\mathcal{T}$ is equicontinuous, we have to initially show that, given $\lim_{\mathbf{t}_2 \rightarrow \mathbf{t}_1} E\|(\Psi_2 z)(\mathbf{t}_2) - (\Psi_1 z)(\mathbf{t}_1)\|^2 \rightarrow 0$ for $\mathbf{t}_2, \mathbf{t}_1 \in \mathfrak{B}$. The two sections of this step are as follows:

Part 1 : $\left\{ e : e(\mathbf{t}) = (\Psi_1 z)(\mathbf{t}), z \in B_q \right\}$ is equicontinuous.

By using Definition 2.9 and $\mathcal{C}(0) = I$, we can get

$$\begin{aligned} & \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \mathbf{t}^{\zeta+\ell-2} (\ell-1) \int_0^1 \int_0^\infty g_\zeta(1-\mathbf{s}) \mathbf{s}^{\ell-2} \ell \vartheta M_\ell(\vartheta) \mathcal{S}((\mathbf{t}\mathbf{s})^\ell \vartheta) \varkappa_0 d\vartheta d\mathbf{s} \\ &= (\ell-1) \int_0^1 \int_0^\infty g_\zeta(1-\mathbf{s}) \mathbf{s}^{\ell-2} \ell \vartheta M_\ell(\vartheta) \lim_{\mathbf{t} \rightarrow 0} \frac{\mathcal{S}((\mathbf{t}\mathbf{s})^\ell \vartheta)}{\mathbf{t}^\ell} \varkappa_0 d\vartheta d\mathbf{s} \\ &= (\ell-1) \ell \int_0^1 g_\zeta(1-\mathbf{s}) \mathbf{s}^{\ell-2} d\mathbf{s} \left(\int_0^\infty \vartheta^2 M_\ell(\vartheta) \varkappa_0 d\vartheta \right) = \frac{(\ell-1) \varkappa_0}{(2\ell-1)\Gamma(\zeta+2\ell-1)}. \end{aligned} \quad (3.3)$$

Similarly, we obtain

$$\lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \mathbf{t}^{\zeta+\ell-1} \int_0^1 \int_0^\infty g_\zeta(1-\mathbf{s}) \mathbf{s}^{\ell-2} \ell \vartheta M_\ell(\vartheta) \mathcal{S}((\mathbf{t}\mathbf{s})^\ell \vartheta) \varkappa_1 d\vartheta d\mathbf{s} = 0 \quad (3.4)$$

$$\lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \mathbf{t}^{\zeta+\ell-2} \int_0^1 \int_0^\infty g_\zeta(1-\mathbf{s}) \mathbf{s}^{2\ell-2} \ell^2 \vartheta^2 M_\ell(\vartheta) \mathcal{C}((\mathbf{t}\mathbf{s})^\ell \vartheta) \varkappa_0 d\vartheta d\mathbf{s} = \frac{\ell \varkappa_0}{(2\ell-1)\Gamma(\zeta+2\ell-1)}. \quad (3.5)$$

By using Eq. (3.3)-(3.5) and Remark 2.4, as well as Lemma 2.3 and 2.16, We can determine the following findings:

$$\begin{aligned} \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} (\Lambda_1 \varkappa)(\mathbf{t}) &= \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \left({}^{RL}D_{0+}^{1-\zeta} (\mathbf{t}^{\ell-1} Q_\ell(\mathbf{t}) \varkappa_0) + I_{0+}^\zeta (\mathbf{t}^{\ell-1} Q_\ell(\mathbf{t}) \varkappa_1) \right) \\ &= \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \left({}^C D_{0+}^{1-\zeta} (\mathbf{t}^{\ell-1} Q_\ell(\mathbf{t}) \varkappa_0) + I_{0+}^\zeta (\mathbf{t}^{\ell-1} Q_\ell(\mathbf{t}) \varkappa_1) \right) \\ &= \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \left(I_{0+}^\zeta \left(\frac{d}{d\mathbf{t}} (\mathbf{t}^{\ell-1} Q_\ell(\mathbf{t}) \varkappa_0) \right) + I_{0+}^\zeta (\mathbf{t}^{\ell-1} Q_\ell(\mathbf{t}) \varkappa_1) \right) \\ &= \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} I_{0+}^\zeta \left((\ell-1) \mathbf{t}^{\ell-2} Q_\ell(\mathbf{t}) \varkappa_0 + \mathbf{t}^{2\ell-2} \int_0^\infty \ell^2 \vartheta^2 M_\ell(\vartheta) \mathcal{C}(\mathbf{t}^\ell \vartheta) \varkappa_0 d\vartheta + \mathbf{t}^{\ell-1} Q_\ell(\mathbf{t}) \varkappa_1 \right) \\ &= \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} (\ell-1) \int_0^{\mathbf{t}} \int_0^\infty g_\zeta(\mathbf{t}-\mathbf{s}) \mathbf{s}^{\ell-2} \ell \vartheta M_\ell(\vartheta) \mathcal{S}(\mathbf{s}^\ell \vartheta) \varkappa_0 d\vartheta d\mathbf{s} \\ &\quad + \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \int_0^{\mathbf{t}} \int_0^\infty g_\zeta(\mathbf{t}-\mathbf{s}) \mathbf{s}^{2\ell-2} \ell^2 \vartheta^2 M_\ell(\vartheta) \mathcal{C}(\mathbf{s}^\ell \vartheta) \varkappa_0 d\vartheta d\mathbf{s} \\ &\quad + \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \int_0^{\mathbf{t}} \int_0^\infty g_\zeta(\mathbf{t}-\mathbf{s}) \mathbf{s}^{\ell-1} \ell \vartheta M_\ell(\vartheta) \mathcal{S}(\mathbf{s}^\ell \vartheta) \varkappa_1 d\vartheta d\mathbf{s} \\ &= \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \mathbf{t}^{\zeta+\ell-2} (\ell-1) \int_0^1 \int_0^\infty g_\zeta(1-\mathbf{s}) \mathbf{s}^{\ell-2} \ell \vartheta M_\ell(\vartheta) \mathcal{S}((\mathbf{t}\mathbf{s})^\ell \vartheta) \varkappa_0 d\vartheta d\mathbf{s} \\ &\quad + \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \mathbf{t}^{\zeta+\ell-2} \int_0^1 \int_0^\infty g_\zeta(1-\mathbf{s}) \mathbf{s}^{2\ell-2} \ell^2 \vartheta^2 M_\ell(\vartheta) \mathcal{C}((\mathbf{t}\mathbf{s})^\ell \vartheta) \varkappa_0 d\vartheta d\mathbf{s} \\ &\quad + \lim_{\mathbf{t} \rightarrow 0} \mathbf{t}^{2-\alpha} \mathbf{t}^{\zeta+\ell-1} \int_0^1 \int_0^\infty g_\zeta(1-\mathbf{s}) \mathbf{s}^{\ell-1} \ell \vartheta M_\ell(\vartheta) \mathcal{S}((\mathbf{t}\mathbf{s})^\ell \vartheta) \varkappa_1 d\vartheta d\mathbf{s} \\ &= \frac{\varkappa_0}{\Gamma(\zeta+2\ell-1)}. \end{aligned}$$



Employing the previously given equation and Lemma 2.17, we can get the following findings, when $\mathfrak{t}_1 = 0$ and $\mathfrak{t}_2 \in (0, b]$:

$$\begin{aligned} E \left\| (\Psi_1 z)(\mathfrak{t}_2) - (\Psi_1 z)(0) \right\|^2 &= E \left\| \mathfrak{t}_2^{2-\alpha} (\Lambda_1 \mathfrak{x})(\mathfrak{t}_2) - \frac{\mathfrak{x}_0}{\Gamma(\zeta + 2\ell - 1)} \right\|^2 \\ &\leq 2E \left\| \mathfrak{t}_2^{2-\alpha} \mathfrak{t}_2^{\zeta+\ell-2} (\ell-1) \int_0^1 \int_0^\infty g_\zeta(1-s) \mathfrak{s}^{\ell-2} \ell \vartheta M_\ell(\vartheta) \mathcal{S}((\mathfrak{t}_2 \mathfrak{s})^\ell \vartheta) \mathfrak{x}_0 d\vartheta ds \right. \\ &\quad \left. + \mathfrak{t}_2^{2-\alpha} \mathfrak{t}_2^{\zeta+\ell-2} \int_0^1 \int_0^\infty g_\zeta(1-s) \mathfrak{s}^{2\ell-2} \ell^2 \vartheta^2 M_\ell(\vartheta) \mathcal{C}((\mathfrak{t}_2 \mathfrak{s})^\ell \vartheta) \mathfrak{x}_0 d\vartheta ds - \frac{\mathfrak{x}_0}{\Gamma(\zeta + 2\ell - 1)} \right\|^2 \\ &\quad + 2E \left\| \mathfrak{t}_2^{2-\alpha} \mathfrak{t}_2^{\zeta+\ell-1} \int_0^1 \int_0^\infty g_\zeta(1-s) \mathfrak{s}^{\ell-1} \ell \vartheta M_\ell(\vartheta) \mathcal{S}((\mathfrak{t}_2 \mathfrak{s})^\ell \vartheta) \mathfrak{x}_1 d\vartheta ds \right\|^2 \\ &\rightarrow 0, \text{ as } \mathfrak{t}_2 \rightarrow 0. \end{aligned}$$

For any $0 < \mathfrak{t}_1 < \mathfrak{t}_2 \leq b$, we can apply C_r -inequality, we obtain

$$\begin{aligned} E \left\| (\Psi_1 z)(\mathfrak{t}_2) - (\Psi_1 z)(\mathfrak{t}_1) \right\|^2 &= E \left\| \mathfrak{t}_2^{2-\alpha} (\Lambda_1 \mathfrak{x})(\mathfrak{t}_2) - \mathfrak{t}_1^{2-\alpha} (\Lambda_1 \mathfrak{x})(\mathfrak{t}_1) \right\|^2 \\ &= E \left\| \mathfrak{t}_2^{2-\alpha} \left[{}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0) + I_{0+}^\zeta (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_1) \right] \right. \\ &\quad \left. - \mathfrak{t}_1^{2-\alpha} \left[{}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_0) + I_{0+}^\zeta (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_1) \right] \right\|^2 \\ &\leq 2E \left\| \mathfrak{t}_2^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0)) - \mathfrak{t}_1^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_0)) \right\|^2 \\ &\quad + 2E \left\| \mathfrak{t}_2^{2-\alpha} I_{0+}^\zeta (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_1) - \mathfrak{t}_1^{2-\alpha} I_{0+}^\zeta (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_1) \right\|^2 \\ &= 2I_1 + 2I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= E \left\| \mathfrak{t}_2^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0)) - \mathfrak{t}_1^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_0)) \right\|^2, \\ I_2 &= E \left\| \mathfrak{t}_2^{2-\alpha} I_{0+}^\zeta (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_1) - \mathfrak{t}_1^{2-\alpha} I_{0+}^\zeta (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_1) \right\|^2. \end{aligned}$$

By applying the Lemma 2.17, we obtain

$$\begin{aligned} I_1 &\leq 2E \left\| \mathfrak{t}_2^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0)) - \mathfrak{t}_1^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0)) \right\|^2 \\ &\quad + 2E \left\| \mathfrak{t}_1^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0)) - \mathfrak{t}_1^{2-\alpha} ({}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_0)) \right\|^2 \\ &= 2 \left| \mathfrak{t}_2^{2-\alpha} - \mathfrak{t}_1^{2-\alpha} \right| E \left\| {}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0) \right\|^2 \\ &\quad + 2\mathfrak{t}_1^{2(2-\alpha)} E \left\| {}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_2^{\ell-1} Q_\ell(\mathfrak{t}_2) \mathfrak{x}_0) - {}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}_1^{\ell-1} Q_\ell(\mathfrak{t}_1) \mathfrak{x}_0) \right\|^2 \\ &= I_{11} + 2\mathfrak{t}_1^{2(2-\alpha)} I_{12}. \end{aligned}$$

We can observe that $I_{11} \rightarrow 0$ as $\mathbf{t}_2 \rightarrow \mathbf{t}_1$. According to Remark 2.16, we can derive

$$\begin{aligned}
& E \left\| {}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}_2^{\ell-1}Q_\ell(\mathbf{t}_2)\mathbf{x}_0) - {}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}_1^{\ell-1}Q_\ell(\mathbf{t}_1)\mathbf{x}_0) \right\|^2 \\
&= E \left\| {}^CD_{0+}^{1-\zeta}(\mathbf{t}_2^{\ell-1}Q_\ell(\mathbf{t}_2)\mathbf{x}_0) - {}^CD_{0+}^{1-\zeta}(\mathbf{t}_1^{\ell-1}Q_\ell(\mathbf{t}_1)\mathbf{x}_0) \right\|^2 \\
&= E \left\| I_{0+}^\zeta \left(\frac{d}{d\mathbf{t}_2}(\mathbf{t}_2^{\ell-1}Q_\ell(\mathbf{t}_2)\mathbf{x}_0) \right) - I_{0+}^\zeta \left(\frac{d}{d\mathbf{t}_1}(\mathbf{t}_1^{\ell-1}Q_\ell(\mathbf{t}_1)\mathbf{x}_0) \right) \right\|^2 \\
&= E \left\| \int_0^{\mathbf{t}_2} g_\zeta(\mathbf{t}_2 - \mathbf{s}) \frac{d}{d\mathbf{s}}(\mathbf{s}^{\ell-1}Q_\ell(\mathbf{s})\mathbf{x}_0) d\mathbf{s} - \int_0^{\mathbf{t}_1} g_\zeta(\mathbf{t}_1 - \mathbf{s}) \frac{d}{d\mathbf{s}}(\mathbf{s}^{\ell-1}Q_\ell(\mathbf{s})\mathbf{x}_0) d\mathbf{s} \right\|^2 \\
&\leq 2E \left\| \int_{\mathbf{t}_1}^{\mathbf{t}_2} g_\zeta(\mathbf{t}_2 - \mathbf{s}) \frac{d}{d\mathbf{s}}(\mathbf{s}^{\ell-1}Q_\ell(\mathbf{s})\mathbf{x}_0) d\mathbf{s} \right\|^2 \\
&\quad + 2E \left\| \int_0^{\mathbf{t}_1} (g_\zeta(\mathbf{t}_2 - \mathbf{s}) - g_\zeta(\mathbf{t}_1 - \mathbf{s})) \frac{d}{d\mathbf{s}}(\mathbf{s}^{\ell-1}Q_\ell(\mathbf{s})\mathbf{x}_0) d\mathbf{s} \right\|^2.
\end{aligned}$$

These results could be obtained by using Lemma 2.16:

$$\begin{aligned}
E \left\| \int_{\mathbf{t}_1}^{\mathbf{t}_2} g_\zeta(\mathbf{t}_2 - \mathbf{s}) \frac{d}{d\mathbf{s}}(\mathbf{s}^{\ell-1}Q_\ell(\mathbf{s})\mathbf{x}_0) d\mathbf{s} \right\|^2 &\leq \left(\frac{M}{\Gamma(2\ell)} \right)^2 E \left\| \int_{\mathbf{t}_1}^{\mathbf{t}_2} g_\zeta(\mathbf{t}_2 - \mathbf{s}) \mathbf{s}^{2\ell-2} \mathbf{x}_0 d\mathbf{s} \right\|^2 \\
&\leq \left(\frac{M\mathbf{t}_1^{\ell-2}}{\Gamma(2\ell)\Gamma(\zeta)} \right)^2 \left(\int_{\mathbf{t}_1}^{\mathbf{t}_2} (\mathbf{t}_2 - \mathbf{s})^{\zeta-1} d\mathbf{s} \right)^2 E \|\mathbf{x}_0\|^2 \\
&= \left(\frac{M\mathbf{t}_1^{\ell-2}}{\Gamma(2\ell)\Gamma(\zeta+1)} \right)^2 (\mathbf{t}_2 - \mathbf{t}_1)^{2\zeta} E \|\mathbf{x}_0\|^2 \rightarrow 0, \text{ as } \mathbf{t}_2 \rightarrow \mathbf{t}_1.
\end{aligned}$$

Noting that

$$((\mathbf{t}_2 - \mathbf{s})^{\zeta-1} - (\mathbf{t}_1 - \mathbf{s})^{\zeta-1}) \mathbf{s}^{2\ell-2} \leq (\mathbf{t}_2 - \mathbf{s})^{\zeta-1} \mathbf{s}^{2\ell-2}, \text{ for a.e. } \mathbf{s} \in [0, \mathbf{t}_1].$$

Afterwards, applying Lemma 2.16 and Lebesgue's dominated convergence theorem, we demonstrate

$$\begin{aligned}
& E \left\| \int_0^{\mathbf{t}_1} (g_\zeta(\mathbf{t}_2 - \mathbf{s}) - g_\zeta(\mathbf{t}_1 - \mathbf{s})) \frac{d}{d\mathbf{s}}(\mathbf{s}^{\ell-1}Q_\ell(\mathbf{s})\mathbf{x}_0) d\mathbf{s} \right\|^2 \\
&\leq \left(\frac{M}{\Gamma(\zeta)\Gamma(2\ell)} \right)^2 E \left\| \int_0^{\mathbf{t}_1} ((\mathbf{t}_2 - \mathbf{s})^{\zeta-1} - (\mathbf{t}_1 - \mathbf{s})^{\zeta-1}) \mathbf{s}^{2\ell-2} \mathbf{x}_0 d\mathbf{s} \right\|^2 \rightarrow 0, \text{ as } \mathbf{t}_2 \rightarrow \mathbf{t}_1.
\end{aligned}$$

This implies that

$$I_{12} = E \left\| {}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}_2^{\ell-1}Q_\ell(\mathbf{t}_2)\mathbf{x}_0) - {}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}_1^{\ell-1}Q_\ell(\mathbf{t}_1)\mathbf{x}_0) \right\|^2 \rightarrow 0, \text{ as } \mathbf{t}_2 \rightarrow \mathbf{t}_1.$$

Thus, by $I_{11} \rightarrow 0$ and $I_{12} \rightarrow 0$ as $\mathbf{t}_2 \rightarrow \mathbf{t}_1$, we derive

$$I_1 = E \left\| \mathbf{t}_2^{2-\alpha}({}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}_2^{\ell-1}Q_\ell(\mathbf{t}_2)\mathbf{x}_0)) - \mathbf{t}_1^{2-\alpha}({}^{RL}D_{0+}^{1-\zeta}(\mathbf{t}_1^{\ell-1}Q_\ell(\mathbf{t}_1)\mathbf{x}_0)) \right\|^2 \rightarrow 0, \text{ as } \mathbf{t}_2 \rightarrow \mathbf{t}_1.$$

Using similar methods for $I_1 \rightarrow 0$ as $\mathbf{t}_2 \rightarrow \mathbf{t}_1$, we can obtain the following result:

$$I_2 = E \left\| \mathbf{t}_2^{2-\alpha} I_{0+}^\zeta(\mathbf{t}_2^{\ell-1}Q_\ell(\mathbf{t}_2)\mathbf{x}_1) - \mathbf{t}_1^{2-\alpha} I_{0+}^\zeta(\mathbf{t}_1^{\ell-1}Q_\ell(\mathbf{t}_1)\mathbf{x}_1) \right\|^2 \rightarrow 0, \text{ as } \mathbf{t}_2 \rightarrow \mathbf{t}_1.$$

Therefore, we get

$$E \left\| (\Psi_1 z)(\mathbf{t}_2) - (\Psi_1 z)(\mathbf{t}_1) \right\|^2 \rightarrow 0, \text{ as } \mathbf{t}_2 \rightarrow \mathbf{t}_1.$$

Following the above described analysis, we may infer that the set $\{e : e(\mathbf{t}) = (\Psi_1 z)(\mathbf{t}) : z \in B_q\}$ is equicontinuous.



Part 2 : $\{e : e(t) = (\Psi_2 z)(t) : z \in B_q\}$ is equicontinuous.

Based on Lemma 2.15, Equation (2.1), (A1) – (A2), and Hölder's inequality, given that $t_1 = 0$, $0 < t_2 \leq b$, we acquire

$$\begin{aligned}
 E\|(\Psi_2 z)(t_2) - (\Psi_2 z)(0)\|^2 &\leq 3E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) B u_\varkappa^\varepsilon(s) ds\right\|^2 \\
 &\quad + 3E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) f(s, \varkappa(s)) ds\right\|^2 \\
 &\quad + 3E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) h(s, \varkappa(s)) dW(s)\right\|^2 \\
 &\leq 3\left(\frac{M M_B t_2^{2-\alpha+\ell}}{\Gamma(2\ell)}\right)^2 \frac{1}{2\ell} \int_0^{t_2} (t_2 - s)^{2\ell-1} E\|u_\varkappa^\varepsilon(s)\|^2 ds \\
 &\quad + 3\left(\frac{M t_2^{2-\alpha+\ell}}{\Gamma(2\ell)}\right)^2 \frac{1}{2\ell} \int_0^{t_2} (t_2 - s)^{2\ell-1} m_1(s) \Theta_1(\|\varkappa\|_\mathcal{C}^2) ds \\
 &\quad + 3\left(\frac{M t_2^{2-\alpha}}{\Gamma(2\ell)}\right)^2 Tr(Q) \int_0^{t_2} (t_2 - s)^{2(2\ell-1)} m_2(s) \Theta_2(\|\varkappa\|_\mathcal{C}^2) ds \rightarrow 0, \text{ as } t_2 \rightarrow 0.
 \end{aligned}$$

When $0 < t_1 < t_2 \leq b$, we obtain

$$\begin{aligned}
 E\|(\Psi_2 z)(t_2) - (\Psi_2 z)(t_1)\|^2 &\leq 3E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) B u_\varkappa^\varepsilon(s) ds - \int_0^{t_1} (t_1 - s)^{\ell-1} Q_\ell(t_1 - s) B u_\varkappa^\varepsilon(s) ds\right\|^2 \\
 &\quad + 3E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) f(s, \varkappa(s)) ds - \int_0^{t_1} (t_1 - s)^{\ell-1} Q_\ell(t_1 - s) f(s, \varkappa(s)) ds\right\|^2 \\
 &\quad + 3E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) h(s, \varkappa(s)) dW(s) - \int_0^{t_1} (t_1 - s)^{\ell-1} Q_\ell(t_1 - s) h(s, \varkappa(s)) dW(s)\right\|^2 \\
 &= 3J_1 + 3J_2 + 3J_3,
 \end{aligned}$$

where

$$\begin{aligned}
 J_1 &= E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) B u_\varkappa^\varepsilon(s) ds - \int_0^{t_1} (t_1 - s)^{\ell-1} Q_\ell(t_1 - s) B u_\varkappa^\varepsilon(s) ds\right\|^2, \\
 J_2 &= E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) f(s, \varkappa(s)) ds - \int_0^{t_1} (t_1 - s)^{\ell-1} Q_\ell(t_1 - s) f(s, \varkappa(s)) ds\right\|^2, \\
 J_3 &= E\left\|\int_0^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) h(s, \varkappa(s)) dW(s) - \int_0^{t_1} (t_1 - s)^{\ell-1} Q_\ell(t_1 - s) h(s, \varkappa(s)) dW(s)\right\|^2.
 \end{aligned}$$

Next, we prove $J_3 \rightarrow 0$ as $t_2 \rightarrow t_1$, according to Lemma 2.17 and 2.15, obtaining

$$J_3 \leq 3E\left\|\int_{t_1}^{t_2} (t_2 - s)^{\ell-1} Q_\ell(t_2 - s) h(s, \varkappa(s)) dW(s)\right\|^2$$



$$\begin{aligned}
& + 3E \left\| \mathfrak{t}_1^{2-\alpha} \int_0^{\mathfrak{t}_1} \left((\mathfrak{t}_2 - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t}_2 - \mathfrak{s}) - (\mathfrak{t}_1 - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t}_1 - \mathfrak{s}) \right) h(\mathfrak{s}, \varkappa(\mathfrak{s})) d\mathfrak{W}(\mathfrak{s}) \right\|^2 \\
& + 3 \left(\mathfrak{t}_2^{2-\alpha} - \mathfrak{t}_1^{2-\alpha} \right)^2 E \left\| \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2 - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t}_2 - \mathfrak{s}) h(\mathfrak{s}, \varkappa(\mathfrak{s})) d\mathfrak{W}(\mathfrak{s}) \right\|^2 \\
& \leq 3 \sum_{i=1}^3 J_{3i},
\end{aligned}$$

where

$$\begin{aligned}
J_{31} &= \left(\frac{M \mathfrak{t}_1^{2-\alpha}}{\Gamma(2\ell)} \right)^2 \text{Tr}(Q) \left(\int_0^{\mathfrak{t}_2} (\mathfrak{t}_2 - \mathfrak{s})^{2(2\ell-1)} m_2(\mathfrak{s}) \Theta_2(\|\varkappa\|_{\mathcal{C}}^2) d\mathfrak{s} - \int_0^{\mathfrak{t}_1} (\mathfrak{t}_2 - \mathfrak{s})^{2(2\ell-1)} m_2(\mathfrak{s}) \Theta_2(\|\varkappa\|_{\mathcal{C}}^2) d\mathfrak{s} \right), \\
J_{32} &= \mathfrak{t}_1^{2(2-\alpha)} E \left\| \int_0^{\mathfrak{t}_1} \left((\mathfrak{t}_2 - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t}_2 - \mathfrak{s}) - (\mathfrak{t}_1 - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t}_1 - \mathfrak{s}) \right) h(\mathfrak{s}, \varkappa(\mathfrak{s})) d\mathfrak{W}(\mathfrak{s}) \right\|^2, \\
J_{33} &= \left(\mathfrak{t}_2^{2-\alpha} - \mathfrak{t}_1^{2-\alpha} \right)^2 E \left\| \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2 - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t}_2 - \mathfrak{s}) h(\mathfrak{s}, \varkappa(\mathfrak{s})) d\mathfrak{W}(\mathfrak{s}) \right\|^2.
\end{aligned}$$

We can deduce that $\lim_{\mathfrak{t}_2 \rightarrow \mathfrak{t}_1} J_{31} = 0$ and $\lim_{\mathfrak{t}_2 \rightarrow \mathfrak{t}_1} J_{33} = 0$. Then,

$$J_{32} = \mathfrak{t}_1^{2(2-\alpha)} E \left\| \int_0^{\mathfrak{t}_1} \int_{\mathfrak{t}_1-\mathfrak{s}}^{\mathfrak{t}_2-\mathfrak{s}} \frac{d}{d\mathfrak{t}} \left\{ \mathfrak{t}^{\ell-1} Q_\ell(\mathfrak{t}) h(\mathfrak{s}, \varkappa(\mathfrak{s})) \right\} d\mathfrak{t} d\mathfrak{W}(\mathfrak{s}) \right\|^2.$$

Moreover, Lemma 2.15 and Equation (2.1) implies that

$$\begin{aligned}
J_{32} &\leq \mathfrak{t}_1^{2(2-\alpha)} E \left\| \frac{M}{\Gamma(2\ell)} \int_0^{\mathfrak{t}_1} \int_{\mathfrak{t}_1-\mathfrak{s}}^{\mathfrak{t}_2-\mathfrak{s}} \mathfrak{t}^{2\ell-2} h(\mathfrak{s}, \varkappa(\mathfrak{s})) d\mathfrak{W}(\mathfrak{s}) \right\|^2 \\
&\leq \left(\frac{M \mathfrak{t}^{2-\alpha}}{(2\ell-1)\Gamma(2\ell)} \right)^2 \text{Tr}(Q) \int_0^{\mathfrak{t}_1} \left((\mathfrak{t}_2 - \mathfrak{s})^{2\ell-1} - (\mathfrak{t}_1 - \mathfrak{s})^{2\ell-1} \right)^2 m_2(\mathfrak{s}) \Theta_2(\|\varkappa\|_{\mathcal{C}}^2) d\mathfrak{s} \rightarrow 0, \text{ as } \mathfrak{t}_2 \rightarrow \mathfrak{t}_1.
\end{aligned}$$

Hence $J_3 \rightarrow 0$ as $\mathfrak{t}_2 \rightarrow \mathfrak{t}_1$. Thus, we can prove $J_1 \rightarrow 0$ and $J_2 \rightarrow 0$ as $\mathfrak{t}_2 \rightarrow \mathfrak{t}_1$ in a way similar to $J_3 \rightarrow 0$. Consequently,

$$E\|(\Psi_2 z)(\mathfrak{t}_2) - (\Psi_2 z)(\mathfrak{t}_1)\|^2 \rightarrow 0, \text{ as } \mathfrak{t}_2 \rightarrow \mathfrak{t}_1.$$

According to the above analysis, $\lim_{\mathfrak{t}_2 \rightarrow \mathfrak{t}_1} \|(\Psi_2 z)(\mathfrak{t}_2) - (\Psi_2 z)(\mathfrak{t}_1)\| \rightarrow 0$ for $\mathfrak{t}_1, \mathfrak{t}_2 \in [0, b]$; therefore, $\mathcal{T} = \{e : e(\mathfrak{t}) = (\Psi_2 z)(\mathfrak{t}), z \in B_q\}$ is equicontinuous.

Step 4 : We demonstrate that the Mönch's condition is satisfied.

Let $\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$, where

$$\Psi_1 z(\mathfrak{t}) = \mathfrak{t}^{2-\alpha} {}^{RL}D_{0+}^{1-\zeta} (\mathfrak{t}^{\ell-1} Q_\ell(\mathfrak{t}) \varkappa_0) + \mathfrak{t}^{2-\alpha} I_{0+}^\zeta (\mathfrak{t}^{\ell-1} Q_\ell(\mathfrak{t}) \varkappa_1),$$

$$\Psi_2 z(\mathfrak{t}) = \mathfrak{t}^{2-\alpha} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t} - \mathfrak{s}) B u_{\varkappa_m}^\varepsilon(\mathfrak{s}) d\mathfrak{s},$$

$$\Psi_3 z(\mathfrak{t}) = \mathfrak{t}^{2-\alpha} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t} - \mathfrak{s}) f(\mathfrak{s}, \varkappa^m(\mathfrak{s})) d\mathfrak{s},$$

$$\Psi_4 z(\mathfrak{t}) = \mathfrak{t}^{2-\alpha} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\ell-1} Q_\ell(\mathfrak{t} - \mathfrak{s}) h(\mathfrak{s}, \varkappa^m(\mathfrak{s})) d\mathfrak{W}(\mathfrak{s}).$$

Assume that $\mathcal{M} \subset B_q$ is countable and $\mathcal{M} \subset \overline{\text{co}}(\{0\} \cup \Psi(\mathcal{M}))$. We show that $\chi(\mathcal{M}) = 0$, where χ is the Kuratowski measure of noncompactness. We could say that $\mathcal{M} = \{z^m\}_{m=1}^\infty$ without losing generality. Therefore $\Psi(\mathcal{M}) = \{\Psi z^m\}_{m=1}^\infty$ and its relative compactness implies that $\mathcal{M}$ is also relatively compact. It is clear that $\Psi(\mathcal{M})$ is equicontinuous on $\mathfrak{B}$. Utilizing, Lemma 2.8 and 2.15, (A1)(iii), and (A2)(iii), we derive

$$\chi(\{z^m(\mathfrak{t})\}_{m=0}^\infty) = \chi(z^0(\mathfrak{t}) \cup \{z^m(\mathfrak{t})\}_{m=1}^\infty) = \chi(\{z^m(\mathfrak{t})\}_{m=1}^\infty).$$



$$\begin{aligned}
\chi(\{\Psi_2 z^m(\mathbf{t})\}_{m=1}^\infty) &\leq \chi\left\{\mathbf{t}^{2-\alpha} \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\ell-1} Q_\ell(\mathbf{t}-\mathbf{s}) B u_{\mathcal{K}^m}^\epsilon(\mathbf{s}) d\mathbf{s}\right\} \\
&\leq \frac{2MM_B \mathbf{t}^{2-\alpha}}{\Gamma(\beta)} \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{2\ell-1} \chi\left\{u_{\mathcal{K}^m}^\epsilon(\mathbf{s})\right\}_{m=1}^\infty d\mathbf{s} \\
&\leq \frac{2b^{2-\alpha+3\ell}}{3\ell} \left(\frac{MM_B}{\epsilon^{\frac{1}{2}}\Gamma(\beta)}\right)^2 \left[\frac{2M}{\Gamma(\beta)} \int_0^b (b-\mathbf{s})^{\beta-1} \eta_1(\mathbf{s}) d\mathbf{s}\right. \\
&\quad \left.+ \frac{M}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-\mathbf{s})^{2(\beta-1)} \eta_2^2(\mathbf{s}) d\mathbf{s}\right)^{\frac{1}{2}}\right] \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty),
\end{aligned}$$

where

$$\begin{aligned}
\chi\{u_{\mathcal{K}^m}^\epsilon(\mathbf{s})\}_{m=1}^\infty &\leq \frac{MM_B(b-\mathbf{t})^\ell}{\epsilon\Gamma(2\ell)} \chi\left\{E\tilde{\mathbf{x}}_b + \int_0^b \tilde{\phi}(\mathbf{s}) d\mathbb{W}(\mathbf{s}) - {}^{RL}D_{0+}^{1-\zeta}(b^{\ell-1}Q_\ell(b)\mathbf{x}_0) \right. \\
&\quad \left. - I_{0+}^\zeta(b^{\ell-1}Q_\ell(b)\mathbf{x}_1) - \int_0^b (b-\mathbf{s})^{\ell-1}Q_\ell(b-\mathbf{s})f(\mathbf{s}, \mathcal{K}^m(\mathbf{s}))d\mathbf{s} \right. \\
&\quad \left. - \int_0^b (b-\mathbf{s})^{\ell-1}Q_\ell(b-\mathbf{s})h(\mathbf{s}, \mathcal{K}^m(\mathbf{s}))d\mathbb{W}(\mathbf{s})\right\}_{m=1}^\infty \\
&\leq \frac{MM_B(b-\mathbf{t})^\ell}{\epsilon\Gamma(2\ell)} \left[\frac{2M}{\Gamma(\beta)} \int_0^b (b-\mathbf{s})^{\beta-1} \eta_1(\mathbf{s}) d\mathbf{s} \cdot \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \right. \\
&\quad \left. + \frac{M}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-\mathbf{s})^{2(\beta-1)} \eta_2^2(\mathbf{s}) d\mathbf{s}\right)^{\frac{1}{2}} \cdot \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \right] \\
&\leq \frac{MM_B(b-\mathbf{t})^\ell}{\epsilon\Gamma(2\ell)} \left[\frac{2M}{\Gamma(\beta)} \int_0^b (b-\mathbf{s})^{\beta-1} \eta_1(\mathbf{s}) d\mathbf{s} \right. \\
&\quad \left. + \frac{M}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-\mathbf{s})^{2(\beta-1)} \eta_2^2(\mathbf{s}) d\mathbf{s}\right)^{\frac{1}{2}}\right] \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty). \\
\chi(\{\Psi_3 z^m(\mathbf{t})\}_{m=1}^\infty) &\leq \chi\left\{\mathbf{t}^{2-\alpha} \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\ell-1} Q_\ell(\mathbf{t}-\mathbf{s}) f(\mathbf{s}, \mathcal{K}^m(\mathbf{s})) d\mathbf{s}\right\}_{m=1}^\infty \\
&\leq \mathbf{t}^{2-\alpha} \frac{2M}{\Gamma(\beta)} \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\beta-1} \chi(f(\mathbf{s}, \{\mathbf{s}^{\alpha-2} z^m(\mathbf{s})\}_{m=1}^\infty)) d\mathbf{s} \\
&\leq \mathbf{t}^{2-\alpha} \frac{2M}{\Gamma(\beta)} \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\beta-1} \eta_1(\mathbf{s}) \mathbf{s}^{2-\alpha} \chi(\{\mathbf{s}^{\alpha-2} z^m(\mathbf{s})\}_{m=1}^\infty) d\mathbf{s} \\
&\leq \frac{2Mb^{2-\alpha}}{\Gamma(\beta)} \int_0^b (b-\mathbf{s})^{\beta-1} \eta_1(\mathbf{s}) d\mathbf{s} \cdot \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty). \\
\chi(\Psi_4 z^m(\mathbf{t}))_{m=1}^\infty &\leq \chi\left\{\mathbf{t}^{2-\alpha} \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\ell-1} Q_\ell(\mathbf{t}-\mathbf{s}) h(\mathbf{s}, \mathcal{K}^m(\mathbf{s})) d\mathbb{W}(\mathbf{s})\right\}_{m=1}^\infty \\
&\leq \frac{M\mathbf{t}^{2-\alpha}}{\Gamma(\beta)} \left(2Tr(Q) \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{2(\beta-1)} [\chi(h(\mathbf{s}, \{\mathbf{s}^{\alpha-2} z^m(\mathbf{s})\}_{m=1}^\infty))]^2 d\mathbf{s}\right)^{\frac{1}{2}} \\
&\leq \frac{M\mathbf{t}^{2-\alpha}}{\Gamma(\beta)} \left(2Tr(Q) \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{2(\beta-1)} \eta_2^2(\mathbf{s}) \mathbf{s}^{2(2-\alpha)} [\chi(\{\mathbf{s}^{\alpha-2} z^m(\mathbf{s})\}_{m=1}^\infty)]^2 d\mathbf{s}\right)^{\frac{1}{2}} \\
&\leq \frac{Mb^{2-\alpha}}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-\mathbf{s})^{2(\beta-1)} \eta_2^2(\mathbf{s}) d\mathbf{s}\right)^{\frac{1}{2}} \cdot \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\chi(\{\Psi z^m(\mathbf{t})\}_{m=1}^\infty) &\leq \chi(\{\Psi_1 z^m(\mathbf{t})\}_{m=1}^\infty) + \chi(\{\Psi_2 z^m(\mathbf{t})\}_{m=1}^\infty) + \chi(\{\Psi_3 z^m(\mathbf{t})\}_{m=1}^\infty) + \chi(\{\Psi_4 z^m(\mathbf{t})\}_{m=1}^\infty) \\
&\leq \frac{2b^{2-\alpha+3\ell}}{3\ell} \left(\frac{MM_B}{\epsilon^{\frac{1}{2}}\Gamma(\beta)} \right)^2 \left[\frac{2M}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} \eta_1(s) ds \right. \\
&\quad + \frac{M}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-s)^{2(\beta-1)} \eta_2^2(s) ds \right)^{\frac{1}{2}} \left. \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \right] \\
&\quad + \frac{2Mb^{2-\alpha}}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} \eta_1(s) ds \cdot \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \\
&\quad + \frac{Mb^{2-\alpha}}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-s)^{2(\beta-1)} \eta_2^2(s) ds \right)^{\frac{1}{2}} \cdot \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \\
&\leq \frac{2b^{3\ell}}{3\ell} \left(\frac{MM_B}{\epsilon^{\frac{1}{2}}\Gamma(\beta)} \right)^2 \left[\frac{2Mb^{2-\alpha}}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} \eta_1(s) ds \right. \\
&\quad + \frac{Mb^{2-\alpha}}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-s)^{2(\beta-1)} \eta_2^2(s) ds \right)^{\frac{1}{2}} \left. \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \right] \\
&\quad + \left[\frac{2Mb^{2-\alpha}}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} \eta_1(s) ds \right. \\
&\quad + \frac{Mb^{2-\alpha}}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-s)^{2(\beta-1)} \eta_2^2(s) ds \right)^{\frac{1}{2}} \left. \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \right] \\
&\leq \left(\frac{2b^{3\ell}}{3\ell} \left(\frac{MM_B}{\epsilon^{\frac{1}{2}}\Gamma(\beta)} \right)^2 + 1 \right) \left[\frac{2Mb^{2-\alpha}}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} \eta_1(s) ds \right. \\
&\quad + \frac{Mb^{2-\alpha}}{\Gamma(\beta)} \left(2Tr(Q) \int_0^b (b-s)^{2(\beta-1)} \eta_2^2(s) ds \right)^{\frac{1}{2}} \left. \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty) \right] \\
&\leq K^* \sup_{\wp \in \mathfrak{P}} \chi(\{z^m(\wp)\}_{m=1}^\infty).
\end{aligned}$$

Therefore, from the Mönch's condition, we get

$$\chi(\mathcal{M}) \leq \chi(\overline{\text{co}}\{0\} \cup \Psi(\mathcal{M})) = \chi(\Psi\mathcal{M}) \leq K^* \chi(\mathcal{M}).$$

Since $K^* < 1$, we get $\chi(\mathcal{M}) = 0$. As a result, $\mathcal{M}$ is relatively compact. We may determine that Ψ has a fixed point z in $\mathcal{M}$ by using Lemma 2.20. The proof is concluded. $\square$

Theorem 3.3. Assume (A1) – (A2) are fulfilled. Furthermore, the control system Eq. (1.1) is approximately controllable on $\mathfrak{P}$ provided the functions f and h are uniformly bounded.

Proof. For each Ψ in B_q , consider $\varkappa^\epsilon(\cdot)$ to become a fixed point. Utilizing the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned}
\varkappa^\epsilon(b) &= \tilde{z}_b - \epsilon(\epsilon I + \Gamma_0^b)^{-1} \left\{ E\tilde{z}_b + \int_0^b \tilde{\phi}(s) d\mathbb{W}(s) - {}^{RL}D_{0+}^{1-\zeta} (b^{\ell-1} Q_\ell(b) \varkappa_0) \right. \\
&\quad - I_{0+}^\zeta (b^{\ell-1} Q_\ell(b) \varkappa_1) - \int_0^b (b-s)^{\ell-1} Q_\ell(b-s) f(s, \varkappa^\epsilon(s)) ds \\
&\quad \left. - \int_0^b (b-s)^{\ell-1} Q_\ell(b-s) h(s, \varkappa^\epsilon(s)) d\mathbb{W}(s) \right\}.
\end{aligned} \tag{3.6}$$



By the assumptions on f and h , we get the sequences $\{f(\mathbf{s}, \varkappa^\epsilon(\mathbf{s}))\}$ and $h(\mathbf{s}, \varkappa^\epsilon(\mathbf{s}))$ are uniformly bounded on $\mathfrak{P}$. The subsequences represented by $\{f(\mathbf{s}, \varkappa^\epsilon(\mathbf{s}))\}$ and $h(\mathbf{s}, \varkappa^\epsilon(\mathbf{s}))$ that weakly converge to $f(\mathbf{s})$ and $h(\mathbf{s})$. From the above equation, we have

$$\begin{aligned} E\|\varkappa^\epsilon(b) - \tilde{\varkappa}_b\|^2 &\leq 6E\left\|\epsilon(\epsilon I + \Gamma_0^b)^{-1}\left(E\tilde{\varkappa}_b - {}^{RL}D_{0+}^{1-\zeta}(b^{\ell-1}Q_\ell(b)\varkappa_0) - I_{0+}^\zeta(b^{\ell-1}Q_\ell(b)\varkappa_1)\right)\right\|^2 \\ &\quad + 6E\left(\int_0^b \left\|\epsilon(\epsilon I + \Gamma_0^b)^{-1}\tilde{\phi}(\mathbf{s})\right\|_{L_2^0}^2 d\mathbf{s}\right) \\ &\quad + 6E\left(\int_0^b (b-\mathbf{s})^{\ell-1} \left\|\epsilon(\epsilon I + \Gamma_0^b)^{-1}Q_\ell(b-\mathbf{s})[f(\mathbf{s}, \varkappa^\epsilon(\mathbf{s})) - f(\mathbf{s})]\right\| d\mathbf{s}\right)^2 \\ &\quad + 6E\left(\int_0^b (b-\mathbf{s})^{\ell-1} \left\|\epsilon(\epsilon I + \Gamma_0^b)^{-1}Q_\ell(b-\mathbf{s})f(\mathbf{s})\right\| d\mathbf{s}\right)^2 \\ &\quad + 6E\left(\int_0^b (b-\mathbf{s})^{\ell-1} \left\|\epsilon(\epsilon I + \Gamma_0^b)^{-1}Q_\ell(b-\mathbf{s})[h(\mathbf{s}, \varkappa^\epsilon(\mathbf{s})) - h(\mathbf{s})]\right\|_{L_2^0}^2 d\mathbf{s}\right) \\ &\quad + 6E\left(\int_0^b (b-\mathbf{s})^{\ell-1} \left\|\epsilon(\epsilon I + \Gamma_0^b)^{-1}Q_\ell(b-\mathbf{s})h(\mathbf{s})\right\|_{L_2^0}^2 d\mathbf{s}\right). \end{aligned}$$

For all $\mathbf{t} \in [0, b]$, the operator $\epsilon(\epsilon I + \Gamma_0^b)^{-1} \rightarrow 0$ strongly as $\epsilon \rightarrow 0^+$ and $\|\epsilon(\epsilon I + \Gamma_0^b)^{-1}\| \leq 1$. Thus, by the Lebesgue's dominated convergence theorem, we obtain $E\|\varkappa^\epsilon(b) - \tilde{\varkappa}_b\|^2 \rightarrow 0$ as $\epsilon \rightarrow 0^+$. This demonstrates that the system Eq. (1.1) is approximately controllable. $\square$

4. EXAMPLE

Consider the following equation

$$\begin{cases} \partial_{\mathbf{t}}^{\beta, \mathfrak{S}} \varkappa(\mathbf{t}, z) = \partial_z^2 \varkappa(\mathbf{t}, z) + Bu(\mathbf{t}, z) + f_1(\mathbf{t}, \varkappa(\mathbf{t}, z)) + h_1(\mathbf{t}, \varkappa(\mathbf{t}, z)) \frac{dw(\mathbf{t})}{d\mathbf{t}}, & \mathbf{t} \in (0, b], \quad z \in [0, \pi], \\ \varkappa(\mathbf{t}, 0) = \varkappa(\mathbf{t}, \pi) = 0, & \mathbf{t} \in (0, b], \\ (I_{0+}^{2-\gamma} \varkappa)(0, z) = \varkappa_0(z), \quad (I_{0+}^{2-\gamma} \varkappa)'(0, z) = \varkappa_1(z), & z \in [0, \pi], \end{cases} \quad (4.1)$$

where $\partial_{\mathbf{t}}^{\beta, \mathfrak{S}}$ represents the Hilfer fractional partial derivative of order $\beta \in (1, 2)$, $\mathfrak{S} \in [0, 1]$, $\alpha = \beta + \mathfrak{S}(2 - \beta)$, $f_1(\mathbf{t}, \varkappa(\mathbf{t}, z))$, $h_1(\mathbf{t}, \varkappa(\mathbf{t}, z))$ fulfill **(A1)** and **(A2)**, respectively.

Let $\mathcal{Z} = L^2([0, \pi])$ and defined A fulfills $A\varkappa = \frac{d^2}{dz^2} \varkappa$, $D(A) = \{\varkappa \in \mathcal{Z} : \varkappa(0) = \varkappa(\pi) = 0; \varkappa' \in \mathcal{Z}; \varkappa', \varkappa'' \text{ are absolutely continuous}\}$. Furthermore, A is an infinitesimal generator of a strongly continuous cosine family $\{\mathcal{C}(\mathbf{t})\}_{\mathbf{t} \leq 0}$ that is uniformly bounded.

Let $\Lambda_m(z) = \sqrt{\frac{2}{\pi}} \sin(m\pi z)$, indicating that $-m^2$, $m \in \mathbb{N}$ are eigenvalues of A , and that $\{\Lambda_m\}_{m=1}^\infty$ is an orthonormal basis of $\mathcal{Z}$. Then,

$$A\varkappa = - \sum_{m=1}^{\infty} m^2 \langle \varkappa, \Lambda_m \rangle \Lambda_m, \quad \varkappa \in D(A);$$

where the inner product in $\mathcal{Z}$ is denoted by $\langle \cdot, \cdot \rangle$. Based on [23], we may get

$$\mathcal{C}(\mathbf{t})\varkappa = \sum_{m=1}^{\infty} \cos(m\pi\mathbf{t}) \langle \varkappa, \Lambda_m \rangle \Lambda_m, \quad \mathcal{S}(\mathbf{t})\varkappa = \sum_{m=1}^{\infty} \frac{1}{m} \sin(m\pi\mathbf{t}) \langle \varkappa, \Lambda_m \rangle \Lambda_m, \quad \varkappa \in \mathcal{Z}.$$

Describe an infinite-dimensional space $\mathbb{U}$ by

$$\mathbb{U} = \left\{ u = \sum_{m=2}^{\infty} u_m \Lambda_m(z) \mid \sum_{m=2}^{\infty} u_m^2 < \infty \right\}.$$



Then, norm in $\mathbb{U}$ is specified by $\|u\| = \left(\sum_{m=2}^{\infty} u_m^2 \right)^{\frac{1}{2}}$. Define a linear operator $B : \mathbb{U} \rightarrow \mathcal{Z}$ by

$$(Bu)(z) = 2u_2\Lambda_1(z) + \sum_{m=2}^{\infty} u_m\Lambda_m(z), \text{ for } u = \sum_{m=2}^{\infty} u_m\Lambda_m(z) \in \mathbb{U}.$$

According to [32], we get

$$Q_\ell(\mathfrak{t})\varkappa = \sum_{m=1}^{\infty} \mathfrak{t}^\ell E_{\beta,\beta}(-m^2\mathfrak{t}^\beta) \langle \varkappa, \Lambda_m \rangle \Lambda_m, \quad \ell = \frac{\beta}{2},$$

where $E_{\beta,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\beta(m+1))}$ represents the Mittag-Leffler function. Let $\varkappa(\mathfrak{t})z = \varkappa(\mathfrak{t}, z)$. Therefore, the problem Eq. (4.1) can be written in the abstract form of problem Eq. (1.1) in $\mathcal{Z}$. As a result, all the conditions of Theorem 3.3 are fulfilled. Therefore, Eq. (4.1) is approximately controllable on $\mathfrak{P}$.

5. CONCLUSION

In this work, we discussed the approximate controllability of Hilfer fractional stochastic evolution equations of order $\beta \in (1, 2)$ and type $[0, 1]$. The Kuratowski measure of noncompactness, the fixed point approach, and the results and concepts from fractional calculus are applied to establish the main conclusions. In particular, for $f(\mathfrak{t}, \cdot)$ and $h(\mathfrak{t}, \cdot)$, the Lipschitz condition does not need to be satisfied. To illustrate the importance of our main results, an example was given at the end. The approximate controllability of Sobolev-type Hilfer fractional stochastic evolution equations with infinite delay of order $\beta \in (1, 2)$ will be investigated in the future.

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