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# A novel formula for conjugate gradient method impulse noise reduction from images

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**Abstract**

The main emphasis in conjugate gradient methods is usually on the conjugate formula. This research introduces a novel coefficient for the conjugate gradient approach, based on Perry's conjugacy condition and a quadratic model, aimed at addressing image restoration difficulties. The algorithms demonstrate global convergence and possess the essential property of descent. The new approach showed a considerable enhancement through numerical experimentation. It has been proven that the inventive conjugate gradient method performs superiorly compared to the traditional FR conjugate gradient technique. The new method displayed a significant improvement through numerical testing. It has been illustrated that the innovative conjugate gradient technique outperforms the conventional FR conjugate gradient method.

**Keywords.** Advanced Parameter gradient, Global convergence, Image restoration problems.
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## 1. Introduction

Several large-scale nonlinear optimization problems have been shown to be effectively addressed by gradient approaches, a type of first-order methodology. Utilizing these methods is common in image processing applications, [1].

Adaptive median filter and variational approach advantages are incorporated in a two-phase strategy in [2, 3]. For salt-and-pepper noise, an adaptive median filter is used in the first stage [22]. Assume that $A = \{1, 2, 3, \dots, M\} \times \{1, 2, 3, \dots, N\}$ is the $\mathbf{X}$ index set and that $\mathbf{X}$ is the actual image. The set of noise pixel indices discovered during the first stage should be represented by $N \subset A$. Reducing the functional as much as feasible is now the difficulty.

$$f_\alpha(u) = \sum_{(i,j)\in \mathrm{N}} \left[ |u_{i,j} - y_{i,j}| + \frac{\beta}{2}(2 \times S^1_{i,j} + S^2_{i,j}) \right] \tag{1.1}$$

An edge-preserving potential function $\phi_\alpha = \sqrt{\alpha + x^2}$, $\alpha > 0$ , a regularization parameter $\beta$, and

$$S^1_{i,j} = 2 \sum_{(m,n)\in \mathrm{P}_{i,j}\cap \mathrm{N}^c} \phi_\alpha(u_{i,j} - y_{m,n}),$$

$$S^2_{i,j} = \sum_{(m,n)\in \mathrm{P}_{i,j}\cap \mathrm{N}} \phi_\alpha(u_{i,j} - y_{m,n}),$$

are used to increase a system's accuracy. Let $y_{i,j}$ stand for the observed pixel value of the image at position $(i, j)$, $u_{i,j} = [u_{i,j}]_{(i,j)\in \mathrm{N}}$ for a lexicographically organized column vector of length $c$ and $P_{i,j}$ for the collection of the four closest neighbors of the pixel at location $(i, j) \in$ A. $c$ provides the number of elements in N. As shown in [3, 22], the term U-Y in Equation (1.1) allows for the identification of noisy pixels while adding a little bias to the repair of

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damaged pixels. It suggests that the term should be excluded from the computation and that the only consideration should be the functional of the next form.

$$f_\alpha(u) = \sum_{(i,j)\in \mathrm{N}} \left[(2 \times S^1_{i,j} + S^2_{i,j})\right]. \tag{1.2}$$

The conjugate gradient (CG) method is a useful iterative approach for removing impulse noise:

$$f(u^*) = \min_{x\in R^N} f(u), \tag{1.3}$$

One type of iterative technique that creates a series with the following format is the conjugate gradient approach:

$$u_{k+1} = u_k + \alpha_k d_k, \tag{1.4}$$

where $d_k$ represents the search direction and $\alpha_k$ represents the step size obtained from a dependable precise line search. The is expressed as:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k}. \tag{1.5}$$

See [12]. Step length is specified as follows in the Wolfe scenario:

$$f(u_k + \alpha_k d_k) \leqslant f(u_k) + \delta \alpha_k g_k^T d_k, \tag{1.6}$$

$$d_k^T g(u_k + \alpha_k d_k) \geqslant \sigma\, d_k^T g_k, \tag{1.7}$$

where $0 < \delta < \sigma < 1$. More details can be found in [19]. Using the conjugate gradient approach, the search direction may be established using the following formula:

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \tag{1.8}$$

where $\beta_k$ is a scalar. There are two types of formulas available, namely the Dai-Yuan (DY) method [4] and the Fletcher-Reeves (FR) method [6]. These formulas can be expressed as follows:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}\ , \quad \beta_{k+1}^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k}. \tag{1.9}$$

Several studies have been conducted to examine the properties of convergence exhibited by conjugate gradient methods. This research was initiated by Zoutendijk [23], who demonstrated that the FR approach achieves global convergence when accurate line searching is performed. Several researchers have developed new equations for conjugate gradient coefficients that have shown excellent numerical performance and can lead to a global solution. Although this is just a prototype, the conjugate gradient technique has evolved with more advanced adaptations. Various methods have been employed to develop conjugate gradient algorithms with the essential property of adequate descent. Wu and Chen [21] provide examples of CG approaches, which are as follows:

$$\beta_k^{WC} = \frac{y_{k+1}^T g_{k+1}}{d_k^T y_k} + \frac{2(f_k - f_{k+1}) + g_k^T s_k}{d_k^T y_k}. \tag{1.10}$$

Both in their theoretical and practical applications, these strategies are outstanding. Conjugate gradient technique and Wu and Chen algorithm differ primarily in how the search direction is computed. Regarding the optimization techniques and references, please see [7, 12, 15, 17] for more details.

Building on the Taylor series, we develop a new class of formula and analyze and report on their theoretical characteristics and numerical performance.

## 2. A New Parameter For $\beta_k$ .

Using the Taylor series, we obtain the new conjugate gradient formula, which as:

$$f(x) = f(x_{k+1}) - g_{k+1}^T s_k + \frac{1}{2} s_k^T Q(u_{k+1}) s_k, \tag{2.1}$$

where $Q$ is Hessian matrix. The derivative can be determined through the following calculation:

$$g_{k+1} = g_k + Q(u_{k+1}) s_k. \tag{2.2}$$

By utilizing (2.2) in (2.1), yield:

$$s_k^T Q(u_k) s_k = 2( f_k - f_{k+1}) + 2\, y_k^T s_k + 2\, g_k^T s_k. \tag{2.3}$$

As a result, after certain algebraic alteration, implying:

$$s_k^T Q(u_k) s_k = 1/2\, y_k^T s_k + ( f_{k+1} - f_k) - g_k^T s_k. \tag{2.4}$$

The parameter derived through Perry's conjugecy condition will be defined as:

$$d_{k+1}^T y_k = -s_k^T g_{k+1}. \tag{2.5}$$

We obtained results by utilizing (1.8), (2.3), and (2.5), yield:

$$\beta_k d_k^T y_k = g_{k+1}^T y_k - g_k^T s_k - 1/2 y_k^T s_k - ( f_{k+1} - f_k) + g_k^T s_k. \tag{2.6}$$

As resulted:

$$\beta_k = \frac{g_{k+1}^T y_k}{d_k^T y_k} + \frac{-1/2 y_k^T s_k + ( f_k - f_{k+1}) + g_k^T s_k}{d_k^T y_k} - \frac{g_k^T s_k}{d_k^T y_k}. \tag{2.7}$$

The methods created by the parameter previously stated are called the BBF.

**Algorithm 1** BBF algorithm.

1: Initialization. Given $x_0 \in R^n$, set $k = 0$, $d_0 = -g_0$
2: Should $\|g_k\| \leqslant \varepsilon$ then stop.
3: Calculate $\alpha_k$ by (1.6) and (1.7).
4: Assume $x_{k+1} = x_k + \alpha_k d_k$ , and calculate $\beta_k$ by (2.7).
5: Calculate $d_{k+1} = -g_{k+1} + \beta_k d_k$.
6: Set $k = k + 1$ and go to stage 2.

## 3. Global convergence:

Examining the methods global convergence properties is the goal of this section. First, we take the following action.

**Hypotheses**

(1) The $\Omega = \{u : u \in R^n, f(u) \leqslant f(u_1)\}$ has a boundary in the level set.
(2) The Lipchitz condition is satisfied by the gradient $g(u)$ in a neighborhood $\Lambda$ in $\Omega$ in the following way:

$$\|g(t_1) - g(t_2)\| \leqslant L\, \|t_1 - t_2\| \ , \ \forall t_1, t_2 \in \Lambda. \tag{3.1}$$

Based on Assumption 1 above, there has to be a $\mu\ > 0$ such that:

$$(\nabla f(r_1) - \nabla f(r_2))^T \geqslant \mu \|r_1 - r_2\|^2 \ , \ \forall\ r_1,\ r_2\ \in R^n. \tag{3.2}$$

See [13, 14].

**Theorem 3.1.** *Upon generating $\{x_k\}$ and $\{d_k\}$ using a new approach. Then:*

$$d_{k+1}^T g_{k+1} \leqslant -c\|g_{k+1}\|^2. \tag{3.3}$$

*Proof.* After that, $g_0^T d_0 = -\|g_0\|^2$ if $k = 0$. For every $k$ , let $d_k^T g_k < 0$. After multiplying (1.10) by $g_{k+1}$,, we get:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k s_k^T g_{k+1}. \tag{3.4}$$

Using (2.2), and changing (2.7) to (3.4), we get:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \left( \frac{g_{k+1}^T y_k}{s_k^T y_k} - \frac{s_k^T g_{k+1}}{s_k^T y_k} \right) s_k^T g_{k+1}. \tag{3.5}$$

It suggests:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \frac{g_{k+1}^T y_k s_k^T g_{k+1}}{s_k^T y_k} - \frac{(s_k^T g_{k+1})^2}{s_k^T y_k}. \tag{3.6}$$

Utilizing the Cauchy-Schwartz inequality $w^T v \leqslant \frac{1}{2}(\|w\|^2 + \|v\|^2$, where $w = (y_k^T s_k) g_{k+1}$ and $v = (s_k^T g_{k+1}) y_k$ we obtain:

$$\frac{g_{k+1}^T y_k s_k^T g_{k+1}}{s_k^T y_k} \leqslant \frac{\frac{1}{2}\left[ \|g_{k+1}\|^2 (y_k^T s_k)^2 + (s_k^T g_{k+1})^2 \|y_k\|^2 \right]}{(s_k^T y_k)^2}. \tag{3.7}$$

As we plug in (3.7) into (3.6) we get:

$$d_{k+1}^T g_{k+1} \leqslant -\|g_{k+1}\|^2 + \frac{1/2 \left[ \|g_{k+1}\|^2 (y_k^T s_k)^2 + (s_k^T g_{k+1})^2 \|y_k\|^2 \right]}{(s_k^T y_k)^2} - \frac{(s_k^T g_{k+1})^2}{s_k^T y_k}. \tag{3.8}$$

By utilizing (3.1) in (3.8), it guarantees:

$$d_{k+1}^T g_{k+1} \leqslant -\frac{1}{2}\|g_{k+1}\|^2 + \left[ \frac{1}{2} L - 1 \right] \frac{(s_k^T g_{k+1})^2}{s_k^T y_k}. \tag{3.9}$$

Thus, as follows:

$$d_{k+1}^T g_{k+1} \leqslant -c\|g_{k+1}\|^2. \tag{3.10}$$

Thus, it has been proved. □

Any conjugate gradient method combined with a Wolfe line search yields convergence. All it requires to be weak is for the Zoutendijk condition [5, 23].

**Lemma 3.2.** *Applying Wolfe conditions with descents search direction yields any iteration technique with $\alpha_k$. Then:*

$$\sum_{k \geqslant 0} \frac{1}{\|d_{k+1}\|^2} = \infty. \tag{3.11}$$

*Then*

$$\lim_{k \to \infty} \inf \| g_k \| = 0. \tag{3.12}$$

**Theorem 3.3.** *Global convergence of the BBF Algorithm occurs whenever our assumptions are true:*

$$\lim_{k \to \infty} \inf \| g_k \| = 0. \tag{3.13}$$

*Proof.* Yet it remains true from (1.10), that

$$\|d_{k+1}\| = \left\| -g_{k+1} + \beta_k^{\mathrm{BBF}} s_k \right\|. \tag{3.14}$$

Utilizing (2.2), to insert (2.7) into (3.14), suggests:

$$\|\mathrm{d}_{k+1}\| = \left\| -g_{k+1} + \frac{g_{k+1}^T y_k}{d_k^T y_k} s_k - \frac{s_k^T g_{k+1}}{d_k^T y_k} s_k \right\|. \tag{3.15}$$

After applying (3.1) and (3.2), it becomes as follows:

$$\begin{aligned}\|d_{k+1}\| &\leq \|g_{k+1}\| + \frac{\|g_{k+1}\| \, L \, \|s_k\|^2}{\mu \, \|s_k\|^2} + \frac{\|g_{k+1}\| \, \|s_k\|^2}{\mu \, \|s_k\|^2} \\ &\leq \left(1 + \frac{L}{\mu} + \frac{1}{\mu}\right) \|g_{k+1}\| \\ &\leq \left[\frac{\mu + L + 1}{\mu}\right] \|g_{k+1}\| .\end{aligned} \tag{3.16}$$

As a result,

$$\sum_{k\geqslant 1} \frac{1}{\|d_k\|^2} \geqslant \left(\frac{\mu}{\mu + L + 1}\right) \frac{1}{\Gamma} \sum_{k\geqslant 1} 1 = \infty. \tag{3.17}$$

Based on Lemma 3.2, this study determines that $\lim_{k\to\infty} \inf \|g_k\| = 0$. □

## 4. Numerical Results

In this study, we present a set of numerical findings that demonstrate the effectiveness of new method in eliminating salt-and-pepper impulse noise. We compare the results obtained from the new approach to those obtained from the FR method in our experimental analysis [18]. The entire process is carried out using the MATLAB r2017a software. Subsequently, a computer executes the generated codes. The criteria used to determine the termination of both techniques are as follows:

$$\|f(u_k)\| \leqslant 10^{-4}(1 + |f(u_k)|) \text{ and } \frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \leqslant 10^{-4}. \tag{4.1}$$

The test photographs include Lena, House, the Cameraman, and Elaine. Along with the test text, we assess the restoration performance qualitatively using the PSNR (peak signal to noise ratio) in a manner that can be compared to previous studies [3, 22]. The definition of restoration performance is as follows:

$$PSNR = 10 \, \log_{10} \frac{255^2}{\frac{1}{MN} \sum_{i,j} (u^r_{i,j} - u^*_{i,j})^2}. \tag{4.2}$$

The original picture's pixel values and the restored image's pixel values are denoted by $u^r_{i,j}$ and $u^*_{i,j}$, respectively. This study examines the number of iterations (NI) and function evaluations (NF) required to complete the denoising process, as well as the resulting PSNR of the generated image. The FR technique takes a considerable amount of time to finish, while the novel technique is significantly faster. This can be seen from the data presented in Table 1. Additionally, the PSNR values achieved using both the new approach and the FR method are relatively similar. There are also several published studies in the field of optimization, such as [8–11, 16].

## 5. Conclusions

We not only discussed a newly developed conjugate gradient equation, but we also explored the conjugate gradient method known as BBF. By applying specific search parameters, we successfully identified the global convergence of the Wolfe line. It has been demonstrated that employing BBF can significantly reduce the amount of simulation iterations and function evaluations without compromising the quality of the image.

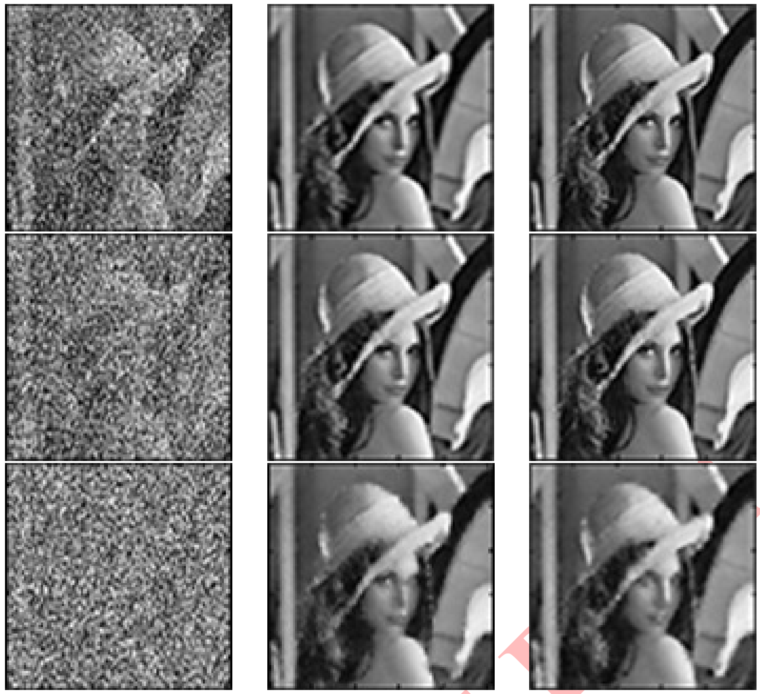

FIGURE 1. Demonstrates the results of algorithms FR and BBF of 256×256 Lena image.

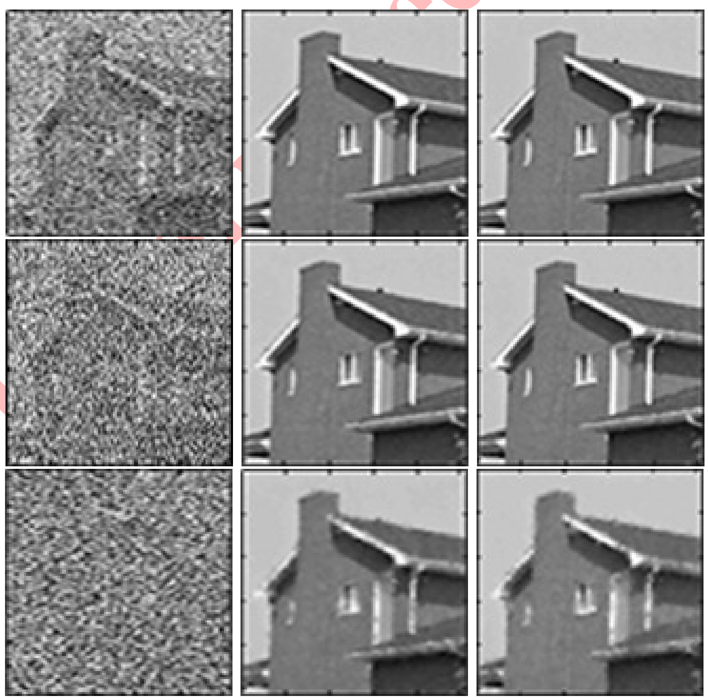

FIGURE 2. Demonstrates the results of algorithms FR and BBF of 256×256 House image.

FIGURE 3. Demonstrates the results of algorithms FR and BBF of 256×256 Elaine image.

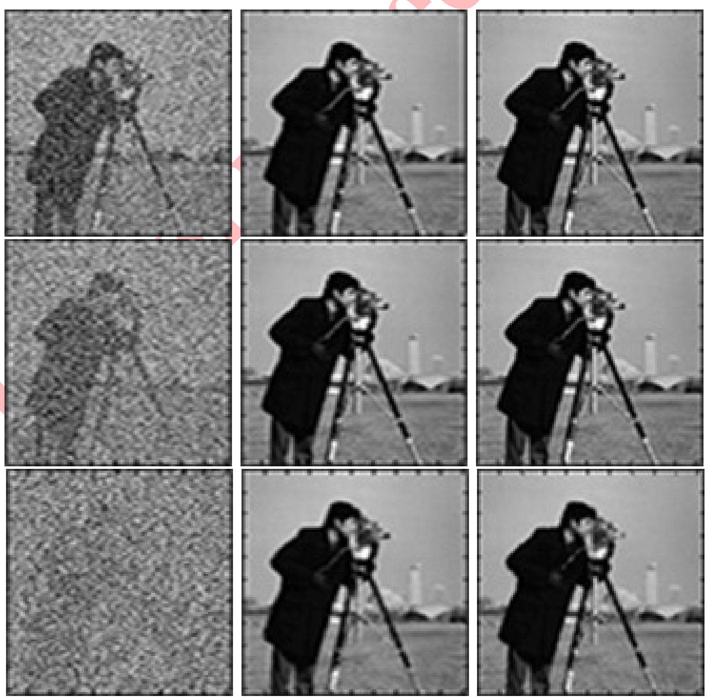

FIGURE 4. Demonstrates the results of algorithms FR and BBF of 256×256 Cameraman image.

Table 1. Numerical results of FR and BBF algorithms.

| Image | Noise level r (%) | FR-Method | | | BBF-Method | | |
|---|---|---|---|---|---|---|---|
| | | NI | NF | PSNR (dB) | NI | NF | PSNR (dB) |
| Le | 50 | 82 | 136 | 30.5529 | 67 | 101 | 30.3957 |
| | 70 | 101 | 155 | 27.4811 | 105 | 157 | 27.4181 |
| | 90 | 88 | 121 | 22.8583 | 84 | 122 | 22.7627 |
| Ho | 50 | 72 | 116 | 31.2044 | 57 | 84 | 30.5442 |
| | 70 | 90 | 133 | 27.9865 | 73 | 107 | 27.0004 |
| | 90 | 65 | 91 | 23.0052 | 63 | 89 | 25.0004 |
| El | 50 | 35 | 56 | 33.9192 | 41 | 61 | 33.8763 |
| | 70 | 38 | 61 | 30.2169 | 39 | 59 | 30.1604 |
| | 90 | 45 | 72 | 25.8098 | 46 | 72 | 25.4898 |
| c512 | 50 | 85 | 134 | 26.8395 | 45 | 64 | 26.3998 |
| | 70 | 102 | 152 | 25.3062 | 54 | 75 | 24.9998 |
| | 90 | 121 | 162 | 24.3962 | 56 | 77 | 24.8998 |

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