



Numerical approaches of the time-fractional Burgers' equation in the sense of conformable fractional derivative

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Abstract

In the present study, we develop two exponential finite difference methods that are effectively applied to obtain numerical solutions of the time-fractional nonlinear Burgers' equation involving a conformable derivative. To evaluate the effectiveness of the proposed methods, three test problems are considered, with the fractional order parameter α chosen in the range $0 < \alpha \leq 1$. Furthermore, to verify the reliability of the methods, we set $\alpha = 1$ to recover the classical first-order derivative and compare the resulting approximate solutions with the corresponding exact solutions. The numerical results demonstrate that the proposed methods are both efficient and easy to implement for solving time-fractional Burgers-type differential equations.

Keywords. Time-fractional Burgers' equation, Exponential finite difference method, Conformable fractional derivative.

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1. INTRODUCTION

The time-fractional Burgers' equation is one of the partial differential equations that has attracted significant attention from researchers in recent years. This equation arises in the modeling of various physical phenomena encountered in both scientific and engineering applications. Understanding these phenomena often requires solving the mathematical models derived from them. In this study, we propose explicit and implicit exponential finite difference methods to numerically solve the time-conformable fractional Burgers' equation of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, L_1 \leq x \leq L_2, t > 0, \quad (1.1)$$

with boundary conditions

$$u(L_1, t) = f_1(t) \text{ and } u(L_2, t) = f_2(t), \quad (1.2)$$

and initial condition

$$u(x, t_0) = g(x). \quad (1.3)$$

Fractional derivatives have garnered the interest of mathematicians for centuries. Over time, many distinct definitions have been introduced, such as the Caputo, Riemann–Liouville, Jumarie, He's fractal, and Yang's local fractional derivatives. These fractional operators have found increasing use in the formulation of differential equations modeling complex phenomena in fields like fluid dynamics, rheology, viscoelasticity, quantum mechanics, acoustics, and heat transfer. They are particularly effective in characterizing memory and hereditary properties of various materials and processes, as demonstrated in electromagnetics, electrochemistry, and material science applications [14]. The fractional derivative has attracted considerable interest in recent years [6, 8, 12, 13, 22, 25, 26, 33].

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Among the various definitions, the conformable fractional derivative, introduced by Khalil et al. [21], offers a natural generalization of the classical derivative and is particularly advantageous in the numerical treatment of fractional differential equations. Compared to the Caputo and Riemann–Liouville derivatives, the conformable derivative maintains a simpler and more local structure, closely aligned with the rules of classical calculus—such as the product and chain rules. This facilitates its integration into numerical schemes like the finite difference method and makes it well-suited for initial value problems. Therefore, it is a practical and efficient tool for modeling and solving fractional differential equations.

Burgers originally introduced the classical Burgers' equation [5], building on earlier work by Bateman [3], as a mathematical model for turbulence. The equation has since become a fundamental model in nonlinear wave propagation and fluid dynamics, particularly in the study of shock wave formation, viscous flows, and acoustic wave propagation [1]. The time-fractional Burgers' equation extends this model by incorporating memory effects through fractional derivatives, capturing more realistic dynamics. It has been applied to describe phenomena such as the decay of non-planar shock waves, unidirectional propagation of nonlinear acoustic waves in gas-filled pipes, and other nonlinear physical processes [10].

Numerous numerical and analytical approaches have been proposed to solve the time-fractional Burgers' equation with various definitions of fractional derivatives, including conformable derivatives [7, 9, 11, 14, 20, 23, 24, 29–32].

In parallel, the exponential finite difference method, first proposed by Bhattacharya [4] for the heat equation, has been successfully applied to other nonlinear equations. Bahadır applied this method to the KdV equation [2], and subsequently, various exponential schemes have been developed for solving the Burgers' equation [15, 16], the Huxley equation, and the Burgers–Huxley equation [17–19]. Diaz et al. [27] used explicit exponential methods for the Burgers–Fisher equation, while Macías-Díaz and Inan [28] proposed a modified exponential finite difference method and analyzed its consistency, stability, and convergence.

The structure of this paper is organized as follows. Section 2 provides an overview of the conformable fractional derivative. Section 3 introduces the numerical methods for solving the one-dimensional time-fractional Burgers' equation. Section 4 presents the stability analysis of the proposed methods. Section 5 discusses numerical results based on three benchmark problems to assess the accuracy and effectiveness of the schemes. Finally, conclusions and suggestions for future research are given in Section 6.

2. ON THE CONFORMABLE DERIVATIVE

Definition 2.1. Provided that y satisfies the $y : [0, \infty) \rightarrow \mathbb{R}$ condition, the “conformable fractional derivative” of y of order α is defined by Khalil et al. [21] as following:

For all $t > 0$, $\alpha \in (0, 1)$

$$X_\alpha(y)(t) = \lim_{\epsilon \rightarrow 0} \frac{y(t + \epsilon t^{1-\alpha}) - y(t)}{\epsilon}. \quad (2.1)$$

If y is α -differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0^+} y^{(\alpha)}(t)$ exists, then define

$$y^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} y^{(\alpha)}(t). \quad (2.2)$$

As a consequence of the the definition , Khalil et al. presented that X_α satisfies all the properties in the following properties[21]:

Theorem 2.2. Let $\alpha \in (0, 1]$ and y, z be α -differentiable at a point $t > 0$. Then

- (i) $X_\alpha(ay + bz) = aX_\alpha(y) + bX_\alpha(z)$, for all $a, b \in \mathbb{R}$.
- (ii) $X_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
- (iii) $X_\alpha(\lambda) = 0$, for all $p \in \mathbb{R}$.
- (iv) $X_\alpha(yz) = yX_\alpha(z) + zX_\alpha(y)$.
- (v) $X_\alpha\left(\frac{y}{z}\right) = \frac{zX_\alpha(y) - yX_\alpha(z)}{z^2}$.
- (vi) If y is differentiable, then $X_\alpha(y)(t) = t^{1-\alpha} \frac{dy}{dt}(t)$.



3. METHODS OF SOLUTION

In this section, we first present some finite difference operators to approximate the solution and derive numerical algorithms based on finite difference formulas for solving the time-conformable fractional Burgers' equation. Next, we define the explicit exponential finite difference method (Method 1) and the implicit exponential finite difference method (Method 2) for obtaining numerical solutions to the equation. These methods are referred to as Method 1 and Method 2 throughout the remainder of this paper.

Throughout this work, we let $K, N \in \mathbb{N}$. We consider that $[0, T]$, and $[A, B]$ intervals are separated equal partitions consisting of K and N subintervals, respectively, with norms denoted by Δt and Δx . For $k \in \{0, 1, \dots, K\}$ and $n \in \{0, 1, \dots, N\}$ define $t_k = k\Delta t$ and $x_n = n\Delta x$ and let w_n^k represents an approximation solution of Eq. (1.1) at (x_n, t_k) . In this paper, we use the following computation constants for the sake of brevity $R_1 = \frac{\Delta t}{2\Delta x}$, $R_2 = \nu \frac{\Delta t}{(\Delta x)^2}$. To obtain the methods, we use the following difference operators:

$$\delta_t w_n^k = \frac{w_n^{k+1} - w_n^k}{\Delta t}, \quad (3.1)$$

$$\delta_x^{(1)} w_n^k = \frac{w_{n+1}^k - w_{n-1}^k}{2\Delta x}, \quad (3.2)$$

$$\delta_x^{(2)} w_n^k = \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{(\Delta x)^2}, \quad (3.3)$$

$$\delta_x^{(1)} w_n^{k+1} = \frac{w_{n+1}^{k+1} - w_{n-1}^{k+1}}{2\Delta x}, \quad (3.4)$$

$$\delta_x^{(2)} w_n^{k+1} = \frac{w_{n+1}^{k+1} - 2w_n^{k+1} + w_{n-1}^{k+1}}{(\Delta x)^2}, \quad (3.5)$$

for each $k \in \{0, 1, \dots, K-1\}$ and $n \in \{1, \dots, N-1\}$. In addition, we define the new discrete operator

$$\begin{aligned} \Lambda(w_n^k)^\alpha &= t^{1-\alpha} \Lambda w_n^k = t^{1-\alpha} \frac{\ln(w_n^{k+1}) - \ln(w_n^k)}{\Delta t} \\ &= \frac{1}{\Delta t} t^{1-\alpha} \ln\left(\frac{w_n^{k+1}}{w_n^k}\right). \end{aligned} \quad (3.6)$$

3.1. Explicit Method (Method 1). Eq. (1.1) is discretized by finite difference approximations and then is obtained the following difference equation:

$$\Lambda(w_n^k)^\alpha + \frac{1}{w_n^k} \left[w_n^k \delta_x^{(1)} w_n^k - \nu \delta_x^{(2)} w_n^k \right] = 0, \quad (3.7)$$

or

$$\Lambda w_n^k + t^{\alpha-1} \delta_x^{(1)} w_n^k - \nu t^{\alpha-1} \frac{1}{w_n^k} \delta_x^{(2)} w_n^k = 0. \quad (3.8)$$

Later, substituting Eq. (3.2), Eq. (3.3) and Eq. (3.6) in Eq. (3.8) and doing some algebraic simplifications, the explicit system is obtained;

$$\begin{cases} w_n^{k+1} = w_n^k \exp\{t^{\alpha-1} \Phi_n^k\}, \\ \Phi_n^k = -R_1 (w_{n+1}^k - w_{n-1}^k) + R_2 \frac{1}{w_n^k} (w_{n+1}^k - 2w_n^k + w_{n-1}^k). \end{cases} \quad (3.9)$$

Eq. (3.9) shows the explicit exponential finite difference method for the time conformable fractional Burgers' equation.



3.2. Implicit Method (Method 1). When Eq. (1.1) is discretized using implicit exponential finite difference approximations, we obtain the difference equations:

$$\Lambda (w_n^k)^\alpha + \frac{1}{w_n^k} \left[w_n^{k+1} \delta_x^{(1)} w_n^{k+1} - \nu \delta_x^{(2)} w_n^{k+1} \right] = 0. \quad (3.10)$$

Later, substituting Eq. (3.4), Eq. (3.5), and Eq. (3.6) in Eq. (3.10) and doing some algebraic simplifications, the implicit system is obtained;

$$\begin{cases} w_n^{k+1} = w_n^k \exp \{ t^{\alpha-1} \Psi_n^{k+1} \}, \\ \Psi_n^{k+1} = -R_1 (w_{n+1}^{k+1} - w_{n-1}^{k+1}) + R_2 \frac{1}{w_n^k} (w_{n+1}^{k+1} - 2w_n^{k+1} + w_{n-1}^{k+1}). \end{cases} \quad (3.11)$$

Eq. (3.11) presents the implicit exponential finite difference method for the time conformable fractional Burgers' equation. Eq. (3.11) comprises nonlinear difference equations. The nonlinear systems solve Newton's iteration at each time-step.

4. STABILITY ANALYSIS

In this section, the Fourier method is employed to perform the stability analysis of the proposed methods. For the purpose of analyzing stability, the nonlinear term is treated as a constant, allowing the analysis to be carried out in a linearized sense. The stability analysis presented here is based on von Neumann's theory using a typical Fourier mode:

$$w_n^k = e^{ak\Delta t} e^{ibn\Delta x} = \xi^k e^{ibn\Delta x}, \xi = e^{a\Delta t}, i^2 = -1, \quad (4.1)$$

where Δt is the time step and Δx is the spatial mesh size is determined from a linearization of the numerical method, so w_n^k is equal to local constant d , so that $uu_x = (\epsilon d) u_x$.

Consider the Eq. (1.1)

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0. \quad (4.2)$$

If both sides of the equation are multiplied by $\frac{1}{u}$ and rearranged

$$\frac{1}{u} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{u} \left(-u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \right), \quad (4.3)$$

or

$$\frac{\partial \ln u^\alpha}{\partial t^\alpha} = \frac{1}{u} \left(-u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \right), \quad (4.4)$$

or

$$t^{1-\alpha} \frac{\partial \ln u}{\partial t} = \frac{1}{u} \left(-u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \right), \quad (4.5)$$

$$t^{1-\alpha} \frac{\partial \ln u}{\partial t} = t^{1-\alpha} \frac{\partial}{\partial t} \ln(e^{at} e^{ibx}) = t^{1-\alpha} a. \quad (4.6)$$

Substituting Eq. (4.6) in Eq. (4.5)

$$t^{1-\alpha} a = \frac{1}{u} \left(-u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \right), \quad (4.7)$$

and $w_n^k = e^{ak\Delta t} e^{ibn\Delta x}$ written in Eq. (3.11) gives



$$a = \frac{1}{e^{ak\Delta t} e^{ibn\Delta x}} t^{\alpha-1} \left[-R_1 (\epsilon d) (e^{a(k+1)\Delta t} e^{ib(n+1)\Delta x} - e^{a(k+1)\Delta t} e^{ib(n-1)\Delta x}) \right. \\ \left. + R_2 (e^{a(k+1)\Delta t} e^{ib(n+1)\Delta x} - 2e^{a(k+1)\Delta t} e^{ibn\Delta x} + e^{a(k+1)\Delta t} e^{ib(n-1)\Delta x}) \right], \quad (4.8)$$

and

$$a = t^{\alpha-1} e^{a\Delta t} \left[-R_1 (\epsilon d) 2i \sin(b\Delta x) - 2R_2 \sin^2 \frac{b\Delta x}{2} \right]. \quad (4.9)$$

$$\text{Real}(a) = t^{\alpha-1} e^{a\Delta t} \left[-2R_2 \sin^2 \left(\frac{b\Delta x}{2} \right) \right] \leq 0, \quad (4.10)$$

and this implies

$$\xi = e^{a\Delta t} \leq 1, \quad (4.11)$$

where Δx and Δt are usually a small quantity and positive. Hence, $|\xi| \leq 1$ is ensured for any problem. Since $|\xi| \leq 1$, the implicit exponential finite difference method is unconditionally stable. Similarly, stability analysis can be examined for the explicit exponential finite difference method. Also, it can similarly be shown to be stable for $R_2 = \nu \frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}$ in the explicit method.

5. NUMERICAL SOLUTIONS OF THE CONFORMABLE FRACTIONAL-ORDER EQUATION

In this section, several numerical problems are presented to investigate the effectiveness and efficiency of the proposed algorithms. It should be noted that the numerical algorithms were implemented in Fortran, while the graphical representations of the numerical solutions were generated using MATLAB R2021a. Three different problems are used to evaluate the performance of the methods. To assess the accuracy of the proposed schemes, the absolute error, relative error, and L_∞ error norms are considered and defined as follows. To further evaluate the performance and efficiency of the proposed methods, the rates of convergence and CPU time(in hours) values have also been calculated and included in the numerical results.

$$\text{Absolute Error} = |u_n^k - w_n^k|, \quad (5.1)$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{u_n^k}, \quad (5.2)$$

$$L_\infty = \|u - w\|_\infty = \max_{0 \leq n \leq N} |u_n - w_n|. \quad (5.3)$$

The accuracy of the method is measured in terms of the error norm defined by

$$E = \left[\frac{\sum_{i=0}^N |u_i - U_i|^2}{\sum_{i=0}^N |u_i|^2} \right]^{\frac{1}{2}}. \quad (5.4)$$

From comparisons of the numerical results with the exact solutions it is deduced that the proposed method gives highly accurate solutions. The rates of convergence of the method, computed using

$$\text{rate} = \frac{\log(E^h/E^{h/2})}{\log(2)}, \quad (5.5)$$

where E^h and $E^{h/2}$ are the errors defined in Eq. (3.9) and Eq. (3.11) with the grid size h and $h/2$ and $h = \Delta x$, respectively.



Example 5.1. We consider the time-fractional Burgers' equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, t \geq 0, \quad (5.6)$$

subject to boundary conditions

$$u(0, t) = u(1, t) = 0, \quad (5.7)$$

and initial condition

$$u(x, 0) = \sin(\pi x). \quad (5.8)$$

The exact solution of the problem is

$$u(x, t) = 2\nu\pi \frac{\sum_{n=1}^{\infty} a_n e^{-\frac{n^2 \pi^2 \nu t^\alpha}{\alpha}} n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2 \pi^2 \nu t^\alpha}{\alpha}} \cos(n\pi x)}, \quad (5.9)$$

where a_0 and a_n :

$$a_0 = \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\nu\pi}} dx, \quad (5.10)$$

and

$$a_n = \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\nu\pi}} \cos(n\pi x) dx, \quad n = 1, 2, 3, \dots \quad (5.11)$$

In this problem, we submit to show the comparison of approximate solutions obtained by Method 1, Method 2, and the exact solution for different values of α and $\nu = 0.1$. We choice $\Delta t = 0.000001$, $\Delta x = 0.05$ in Tables 1 and 2. Tables 1 and 2 present a comparison between the results obtained using Method 1 and Method 2 for different values of the fractional order parameter α at time $t = 1$. The exact solution is also provided for reference. The comparison is conducted for different values of α . As shown, both methods yield results that are in close agreement with the exact solution, demonstrating the accuracy and reliability of the proposed approaches. Particularly, Method 2 generally produces slightly more accurate results, as evidenced by the smaller deviations from the exact values. These comparisons validate the effectiveness of the proposed schemes for fractional orders. These findings further demonstrate the effectiveness and stability of the proposed numerical schemes across different fractional orders, including the classical derivative case.

In Table 3 presented L_∞ error norms for $\Delta t = 0.00001$, $\Delta x = 0.1$ and for various values of the time variable t , calculated using Method 1 and Method 2 for two different fractional orders, $\alpha = 0.5$ and $\alpha = 1$. This comparison illustrates how the error evolves over time and highlights the accuracy and long-term stability of the proposed methods. For both fractional orders, the errors decrease significantly as time increases, demonstrating that the methods remain accurate and stable for long-time simulations. Notably, for large time values such as $t = 10$ and $t = 100$, both methods yield extremely small error norms -approaching zero- which confirms the robustness and consistency of the numerical schemes over extended periods. In particular, Method 1 shows slightly better performance for $\alpha = 1$, whereas Method 2 appears to provide more consistent accuracy for $\alpha = 0.5$. These results verify the effectiveness of both methods for solving the time-fractional Burgers' equation under varying conditions of the fractional order and time.

Table 4 presents the L_∞ error norms and corresponding CPU time for both Method 1 and Method 2 at different time values for $\Delta t = 0.00001$, $\Delta x = 0.1$ and the fractional order $\alpha = 0.5$. Table 4 serves to evaluate not only the accuracy but also the computational efficiency of the proposed methods over varying temporal domains. As observed, both methods produce significantly decreasing error norms with increasing time t , which confirms the numerical stability and accuracy of the schemes. For instance, the L_∞ error norm drops from 1.902×10^{-3} at $t=1$ to 3.565×10^{-10} at $t = 100$ for Method 1, demonstrating high precision in long-time simulations.



TABLE 1. Comparison of solutions for different values of α at $t = 1$.

x	$\alpha = 0.25$			$\alpha = 0.5$		
	Method 1	Method 2	Exact Solution	Method 1	Method 2	Exact Solution
0.1	0.005071	0.005095	0.004528	0.028885	0.028958	0.028758
0.2	0.009681	0.009728	0.008642	0.056103	0.056252	0.055845
0.3	0.013403	0.013469	0.011955	0.079851	0.080079	0.079455
0.4	0.015874	0.015953	0.014147	0.098086	0.098394	0.097553
0.5	0.016830	0.016914	0.014985	0.108542	0.108924	0.107890
0.6	0.016141	0.016223	0.014357	0.108946	0.109382	0.108226
0.7	0.013835	0.013906	0.012295	0.097545	0.097994	0.096850
0.8	0.010112	0.010166	0.008981	0.073938	0.074339	0.073398
0.9	0.005336	0.005366	0.004738	0.039942	0.040223	0.039686

TABLE 2. Comparison of solutions for different values of α at $t = 1$.

x	$\alpha = 0.75$			$\alpha = 1$		
	Method 1	Method 2	Exact Solution	Method 1	Method 2	Exact Solution
0.1	0.050257	0.050315	0.050165	0.066401	0.066437	0.066316
0.2	0.098863	0.098991	0.098663	0.131396	0.131483	0.131210
0.3	0.143765	0.143988	0.143429	0.193109	0.193279	0.192786
0.4	0.182057	0.182413	0.181544	0.248560	0.248873	0.248041
0.5	0.209497	0.210032	0.208769	0.292707	0.293259	0.291916
0.6	0.220190	0.217911	0.219254	0.317182	0.318103	0.316068
0.7	0.207068	0.208030	0.206038	0.309431	0.310842	0.308089
0.8	0.164201	0.165269	0.163360	0.254848	0.256723	0.253718
0.9	0.091526	0.092468	0.091258	0.146196	0.148166	0.146065

TABLE 3. L_∞ error norms for different values of α and t .

t	$\alpha = 0.5$		$\alpha = 1$	
	Method 1	Method 2	Method 1	Method 2
0.1	9.689x10 ⁻³	2.119x10 ⁻²	2.682x10 ⁻²	2.327x10 ⁻³
0.5	3.079x10 ⁻³	8.608x10 ⁻³	1.764x10 ⁻²	1.859x10 ⁻²
1.0	1.902x10 ⁻³	4.612x10 ⁻³	3.339x10 ⁻³	1.259x10 ⁻²
2.0	1.023x10 ⁻³	2.241x10 ⁻³	1.376x10 ⁻³	4.083x10 ⁻³
3.0	6.562x10 ⁻⁴	1.314x10 ⁻³	7.189x10 ⁻⁴	1.742x10 ⁻³
4.0	4.480x10 ⁻⁴	8.434x10 ⁻⁴	3.782x10 ⁻⁴	7.712x10 ⁻⁴
5.0	3.163x10 ⁻⁴	5.675x10 ⁻⁴	1.856x10 ⁻⁴	3.353x10 ⁻⁴
10.0	7.397x10 ⁻⁵	1.154x10 ⁻⁴	3.039x10 ⁻⁶	4.172x10 ⁻⁶
100.0	3.565x10 ⁻¹⁰	4.204x10 ⁻¹⁰	1.307x10 ⁻⁴³	1.371x10 ⁻⁴³

Additionally, the CPU time increases moderately with larger time values, which is expected due to the greater computational demand. Method 1 consistently achieves lower CPU times compared to Method 2 while maintaining comparable or even better accuracy, especially for large t . This suggests that Method 1 is not only reliable but also computationally more efficient for solving the time-fractional Burgers' equation.



TABLE 4. L_∞ error norms and CPU time for $\alpha = 0.5$ at different values of t .

t	Method 1		Method 2	
	L_∞	CPU Time	L_∞	CPU Time
1.0	1.902×10^{-3}	0.0010	4.612×10^{-3}	0.0016
5.0	3.163×10^{-4}	0.0071	5.675×10^{-4}	0.0087
10.0	7.397×10^{-5}	0.0146	1.154×10^{-4}	0.0173
100.0	3.565×10^{-10}	0.0934	4.204×10^{-10}	0.1360

Table 5 reports the relative errors of the proposed numerical methods for $\Delta t = 0.00001$, $\Delta x = 0.1$ and different values of $\alpha = 0.5$ and $\alpha = 1$, evaluated at various time levels and spatial locations $x = 0.1, 0.5, 0.9$. The comparison is made between Method 1 and Method 2. From the results, it is evident that both methods perform well at early time levels with small relative errors. As the time increases, especially at $t = 100$, the relative errors also increase, which is typical due to accumulation of numerical approximation errors over time. Despite this, the relative errors remain within acceptable limits for practical applications. Notably, Method 1 generally yields lower relative errors compared to Method 2, especially at lower values of t , indicating better numerical performance and stability in short-term computations. For long-term simulations (e.g., $t = 100$), although the relative error increases, both methods still exhibit consistency across different values of x , confirming the robustness of the schemes for larger time horizons. These results validate the accuracy, reliability, and applicability of the proposed methods to solve the generalized time-fractional Burgers' equation.

TABLE 5. Relative errors for different values of α and t .

t	x	$\alpha = 0.5$		$\alpha = 1$	
		Method 1	Method 2	Method 1	Method 2
0.1	0.1	8.886×10^{-3}	9.563×10^{-3}	2.599×10^{-3}	2.610×10^{-3}
	0.5	1.183×10^{-2}	1.632×10^{-2}	2.634×10^{-3}	2.645×10^{-3}
	0.9	3.949×10^{-2}	6.371×10^{-2}	7.914×10^{-2}	1.868×10^{-3}
1.0	0.1	1.167×10^{-2}	2.635×10^{-2}	4.097×10^{-3}	7.250×10^{-3}
	0.5	1.594×10^{-2}	3.657×10^{-2}	7.413×10^{-3}	1.814×10^{-2}
	0.9	8.454×10^{-3}	5.210×10^{-2}	2.339×10^{-2}	5.767×10^{-2}
10.0	0.1	4.651×10^{-2}	7.070×10^{-2}	7.034×10^{-2}	9.408×10^{-2}
	0.5	4.672×10^{-2}	7.096×10^{-2}	7.035×10^{-2}	9.409×10^{-2}
	0.9	4.666×10^{-2}	7.122×10^{-2}	7.035×10^{-2}	9.410×10^{-2}
100.0	0.1	1.465×10^{-1}	1.683×10^{-1}	5.512×10^{-1}	5.630×10^{-1}
	0.5	1.465×10^{-1}	1.683×10^{-1}	5.512×10^{-1}	5.630×10^{-1}
	0.9	1.465×10^{-1}	1.683×10^{-1}	5.512×10^{-1}	5.630×10^{-1}

Table 6 present rate of convergence for $\alpha = 0.75$, $\Delta t = 0.000001$ at $t = 1$. The convergence rates are computed based on successive mesh refinements, using standard log-based error ratios. For Method 2, the rate of convergence values are consistently around 2.0, indicating second-order convergence, which is expected for a well-constructed finite difference method. This demonstrates the method's stability and accuracy as the mesh is refined.

On the other hand, Method 1 exhibits extremely large negative values, which indicate that the error is increasing or that the method may be unstable or diverging for this particular fractional order and time level. Such behavior suggests that Method 1 may not be suitable or well-conditioned for $\alpha = 0.75$ under the current discretization strategy. These results clearly show that Method 2 is more robust and reliable in terms of convergence behavior for this problem setup.

From all tables for Example 5.1 can be seen that we obtain better results while α approaching to 1. Also, again from these tables, we observe that Method 1 gives better results than the Method 2. Besides the tables, we have



drawn graphs to examine how the solution behaves for Example 5.1. We choice $\Delta t = 0.00001$, $\Delta x = 0.05$ in Figures 1 and 2. The behavior of computational results obtained by using explicit exponential finite difference method for $\alpha = 0.25, 0.5, 0.75, 1$ when $0 \leq x \leq 1$ are demonstrated in Figures 1 and 2. We can see from Figures 1 and 2, how the solution changes as the α value changes. Figure 3 presents absolute errors obtained by using Method 1 and Method 2 for $\alpha = 0.5$, $\Delta t = 0.000001$, $\Delta x = 0.01$ when $0 \leq x \leq 1$, respectively. As a result of the tables and figures, it has been observed that the results are good for different values of α , x and t .

TABLE 6. Rate of convergence for $\alpha = 0.75$ at $t = 1$.

	Method 1	Method 2
2	—	—
4	-4.6698	2.4502
8	-9.4852	2.3053
16	-10.8767	2.0716
32	-12.1817	2.0074
64	-13.8141	1.9649
128	-15.5185	1.8531

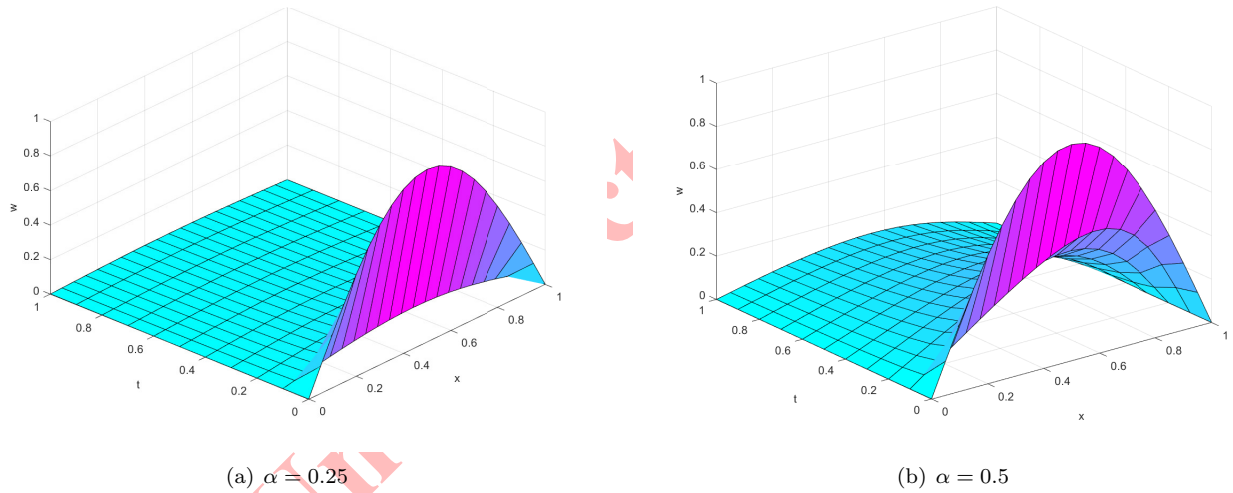


FIGURE 1. Numerical solutions are obtained by using Method 1 for different values of α .

Example 5.2. In this problem we regard the time-fractional Burgers' equation [14]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad L_1 \leq x \leq L_2, \alpha = 1, t \geq 0, \quad (5.12)$$

and initial condition

$$u(x, 0) = \frac{a + b + (b - a) \exp \left[\frac{a}{\nu} (x - c) \right]}{1 + \exp \left[\frac{a}{\nu} (x - c) \right]}. \quad (5.13)$$

The exact solution to this problem is

$$u(x, t) = \frac{a + b + (b - a) \exp \left[\frac{a}{\nu} (x - bt - c) \right]}{1 + \exp \left[\frac{a}{\nu} (x - bt - c) \right]}. \quad (5.14)$$



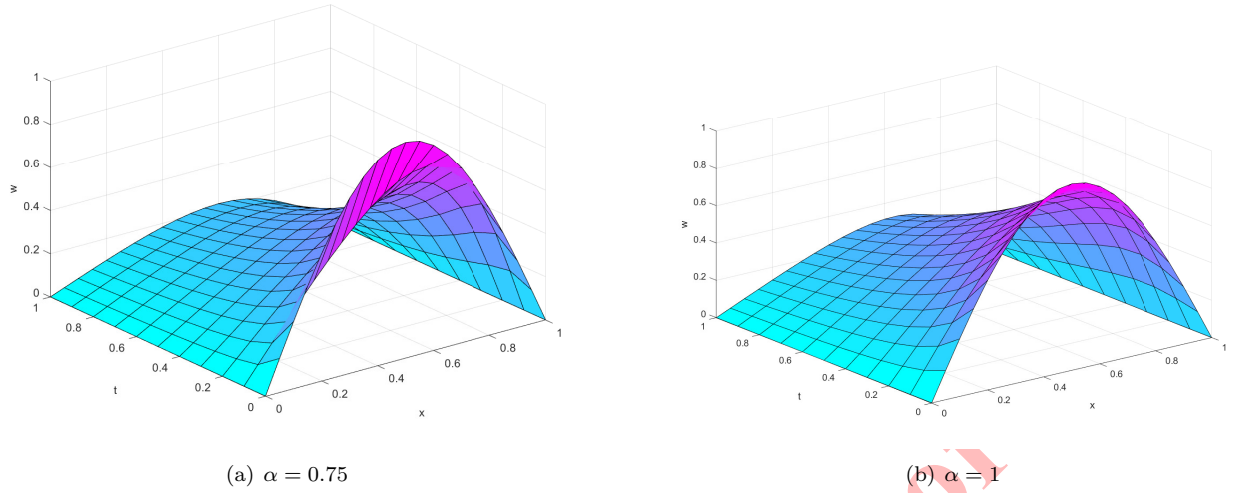


FIGURE 2. Numerical solutions are obtained by using Method 1 for different values of α .

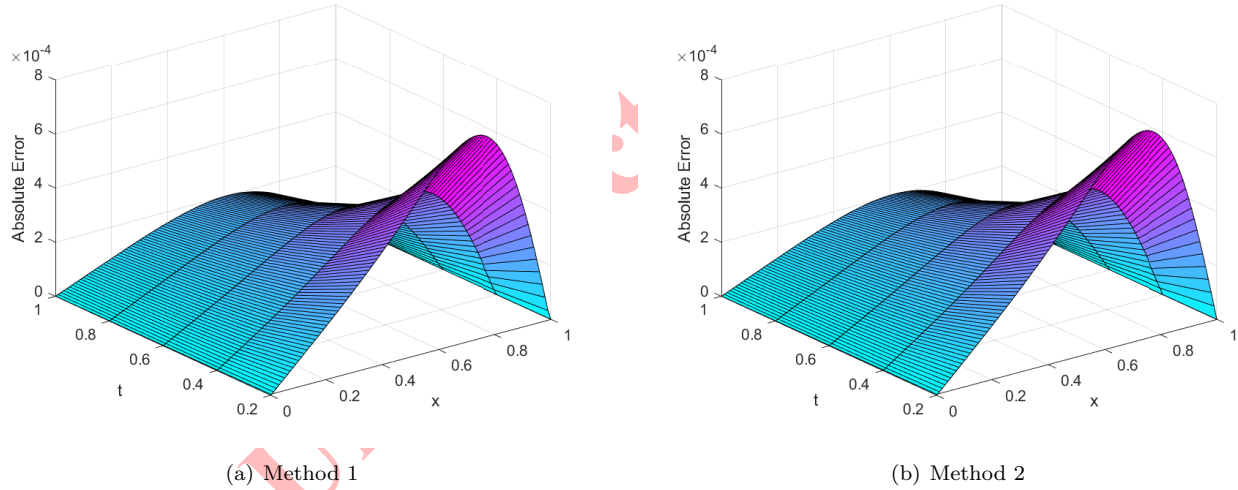


FIGURE 3. Absolute errors are obtained by using present methods.

Boundary conditions for the problem are extracted from exact solution. Comparison of exact solutions and present solutions and attained by Method 1 and Method 2 for $a = 0.3$, $b = 0.4$, $c = 0.8$, $\nu = 0.1$, $\Delta t = 0.0001$, $\Delta x = 0.1$ at $t = 1$ and $t = 3$ when $-3 \leq x \leq 3$ shown in Tables 7 and 8, respectively.

Tables 7 and 8 demonstrate that both Method 1 and Method 2 yield numerical solutions that are in excellent agreement with the exact solution at $t = 1$ and $t = 3$, respectively. Across most of the spatial domain, the errors are minimal, and both methods show nearly identical results, indicating high accuracy and consistency. These results confirm the reliability of both methods for solving the fractional Burgers-type equation.

Table 9 shows L_∞ error norms and CPU time for $a = 0.3$, $b = 0.4$, $c = 0.8$, $\nu = 0.1$, $\Delta t = 0.0001$, $\Delta x = 0.1$ at different values of t when $-3 \leq x \leq 3$. Relative errors for $a = 0.3$, $b = 0.4$, $c = 0.8$, $\nu = 0.1$, $\Delta t = 0.0001$, $\Delta x = 0.1$

TABLE 7. Comparison of the solutions at $t = 1$.

x	Method 1	Method 2	Exact Solution
-1.8	0.6999278	0.6999278	0.6999260
-1.5	0.6998224	0.6998224	0.6998179
-1.0	0.6992046	0.6992045	0.6991849
-0.5	0.6964471	0.6964465	0.6963641
0.0	0.68433201	0.6843298	0.6840418
0.5	0.6350534	0.6350463	0.6345419
1.0	0.4869642	0.4439782	0.4873938
1.5	0.2727753	0.2727758	0.2734303
1.8	0.1849891	0.1849929	0.1851106

TABLE 8. Comparison of the solutions at $t = 3$.

x	Method 1	Method 2	Exact Solution
-1.8	0.6999937	0.6999937	0.6999933
-1.5	0.6999846	0.6999846	0.6999835
-1.0	0.6999312	0.6999312	0.6999260
-0.5	0.6996912	0.6996911	0.6996683
0.0	0.6986101	0.6986095	0.6985164
0.5	0.6937414	0.6937388	0.6934078
1.0	0.6723835	0.6723741	0.6715445
1.5	0.5911160	0.5910928	0.5905447
1.8	0.4866624	0.4866398	0.4873938

at different values of t when $-3 \leq x \leq 3$ in Table 10. Table 9 demonstrates that both methods yield highly accurate results as indicated by the decreasing L_∞ error norms with increasing time t . Particularly at $t = 100$, the error drops to the order of 10^{-12} , confirming the long-time stability and precision of the methods. Additionally, while Method 2 consistently requires more CPU time than Method 1, it provides slightly better accuracy at certain time points, reflecting a trade-off between computational cost and precision.

TABLE 9. L_∞ error norms and CPU time at different values of t .

t	Method 1		Method 2	
	L_∞	CPU Time	L_∞	CPU Time
1.0	8.819×10^{-4}	0.0004	8.862×10^{-4}	0.0032
5.0	6.916×10^{-3}	0.0007	1.142×10^{-3}	0.0141
10.0	1.411×10^{-4}	0.0012	8.583×10^{-5}	0.0285
100.0	4.213×10^{-12}	0.0095	4.222×10^{-12}	0.3342

Table 10 illustrates the relative errors of both methods at different spatial points and time levels. The results clearly demonstrate that the relative errors remain extremely small, especially as time increases. At $t = 100$, the relative errors are on the order of 10^{-12} , indicating excellent long-time numerical stability and high accuracy of the methods. Method 1 and Method 2 yield nearly identical results, suggesting both are robust and reliable for solving the problem across various time levels and spatial positions.

Rate of convergence for $a = 0.3$, $b = 0.4$, $c = 0.8$, $\nu = 0.1$, $\Delta t = 0.0001$ at $t = 1$ when $-3 \leq x \leq 3$ given by Table 11. As observed in the Table 11, after an initial anomaly, the convergence rate stabilizes around 2.00 for both Method 1 and Method 2, indicating that the proposed method achieves second-order accuracy asymptotically. The initial drop in convergence may be attributed to the pre-asymptotic behavior of the method or possible inconsistencies in the

ordering of error values. Nonetheless, the results strongly validate the effectiveness and consistency of the scheme for solving the target fractional differential equations.

Figure 4 shows absolute errors obtained by using Method 1 and Method 2 for $a = 1$, $b = 0.9$, $c = 0.4$, $\nu = 0.1$, $\Delta t = 0.0001$, $\Delta x = 0.01$ when $0 \leq x \leq 0.5$ at different values of t , respectively. From Figure 4, it can be seen that absolute errors decrease with t increasing for both methods.

TABLE 10. Relative errors at different values of t .

t	x	Method 1	Method 2
0.1	-1.5	1.898×10^{-6}	1.886×10^{-6}
	0	9.996×10^{-5}	9.908×10^{-5}
	1.5	3.498×10^{-5}	3.796×10^{-5}
1.0	-1.5	6.382×10^{-6}	6.344×10^{-6}
	0	4.241×10^{-4}	4.209×10^{-4}
	1.5	2.401×10^{-3}	2.399×10^{-3}
10.0	-1.5	8.637×10^{-10}	8.594×10^{-10}
	0	9.843×10^{-8}	9.791×10^{-8}
	1.5	7.569×10^{-6}	7.681×10^{-6}
100.0	-1.5	1.696×10^{-12}	1.692×10^{-12}
	0	3.392×10^{-12}	3.383×10^{-12}
	1.5	5.089×10^{-12}	5.076×10^{-12}

TABLE 11. Rate of convergence at $t = 1$.

	Method 1	Method 2
2	-	-
4	1.7386	1.2031
8	1.9457	1.5402
16	1.6183	1.6214
32	2.0053	2.0066
64	2.0010	2.0010
128	2.0011	1.9994

Example 5.3. Consider the time-fractional Burgers' equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad (5.15)$$

and initial condition

$$u(x, 0) = \frac{1}{-1 + 5 \left(\cosh \frac{x}{2} - \sinh \frac{x}{2} \right)}. \quad (5.16)$$

The exact solution to this problem is

$$u(x, t) = \frac{1}{-1 + 5 \left[\cosh \left(\frac{x}{2} + \frac{t^\alpha}{4\Gamma(1+\alpha)} \right) - \sinh \left(\frac{x}{2} + \frac{t^\alpha}{4\Gamma(1+\alpha)} \right) \right]}. \quad (5.17)$$

In this problem, we consider numerical experiments that we have obtained by using the data follows: $\nu = 1$, $0 \leq x \leq 1$ and $0 \leq t < 1$. In Table 12, we present a comparison of absolute errors obtained by Method 1 and Method 2, and with another numerical method which is the classical finite difference method with the Caputo formula



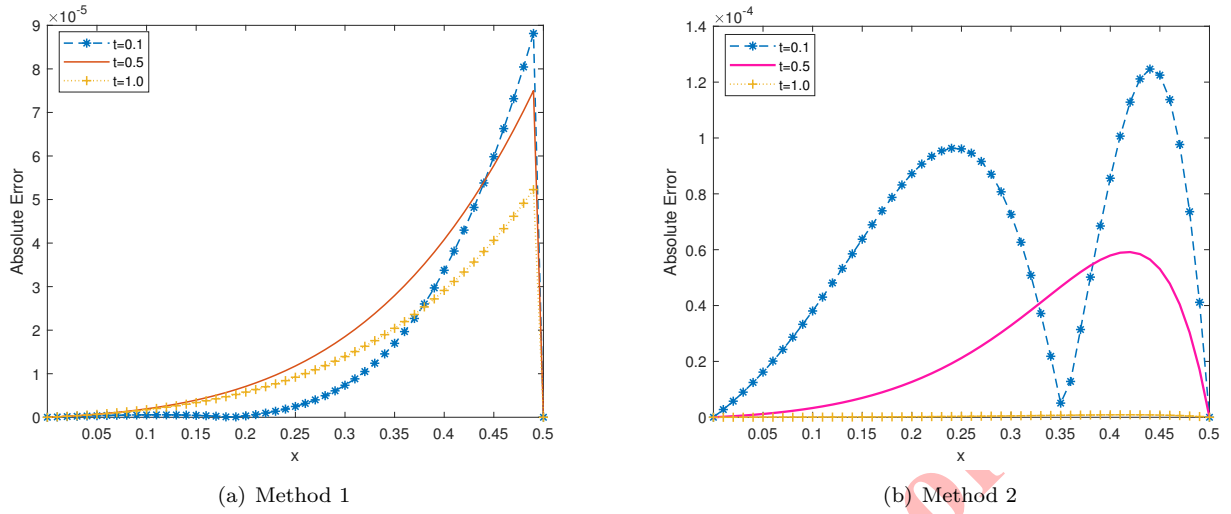


FIGURE 4. Absolute errors are obtained by using present methods $\Delta t = 0.0001$ and $\Delta x = 0.01$.

introduced by Yokuş and Kaya[31] for $\Delta t = 0.0001$, $\Delta x = 0.02$, at $t = 0.02$. The results clearly indicate that both methods significantly reduce the absolute error compared to the reference solution, with Method 2 consistently achieving slightly lower error values. The observed improvement in accuracy highlights the effectiveness of the proposed schemes, especially when a smaller time step size ($k = 0.00001$) is employed in Method 1. These findings suggest that both methods are capable of producing high-precision numerical results and outperform existing approaches in the literature for the considered fractional order.

TABLE 12. Comparison of the absolute errors for $\alpha = 0.8$ at $t = 0.02$.

x	Method 1($k = 0.00001$)	Method 2	[31]
0.02	6.49203×10^{-5}	5.50992×10^{-5}	5.84566×10^{-4}
0.04	1.25895×10^{-4}	1.06350×10^{-4}	5.93809×10^{-4}
0.06	1.83122×10^{-4}	1.53980×10^{-4}	6.03243×10^{-4}
0.08	2.36801×10^{-4}	1.98219×10^{-4}	6.12874×10^{-4}
0.10	2.87126×10^{-4}	2.39289×10^{-4}	6.22707×10^{-4}

This comparison shows that the proposed methods present highly accurate numerical results than classical finite difference method. Table 13 presents relative errors for $\Delta t = 0.00001$, $\Delta x = 0.1$ and different values of α at $t = 0.5$. The results indicate that Method 2 consistently achieves slightly lower relative errors than Method 1 across all spatial points for both $\alpha = 0.5$ and $\alpha = 0.8$. Furthermore, as the fractional order α increases, the relative errors tend to decrease, suggesting enhanced accuracy for higher values of α . This behavior reflects the robustness and stability of both methods over varying fractional orders. Overall, the findings in this table confirm that both numerical methods are effective for solving the considered fractional PDE, with Method 2 demonstrating a marginal advantage in accuracy.

L_∞ error norms and CPU time for $\alpha = 0.5$, $\Delta t = 0.00001$, $\Delta x = 0.1$ at $t = 0.5$ given in Table 14. Although Method 2 takes slightly more CPU time, the difference is minimal and does not offset its advantage in accuracy. These results suggest that Method 2 offers a more efficient balance between computational cost and solution precision under the given conditions.

Table 15 presents rate of convergence for $\alpha = 0.5$ and $\Delta t = 0.000001$ at $t = 0.5$. The rate of convergence values remain close to 1 across different mesh refinements, suggesting that both methods achieve approximately first-order

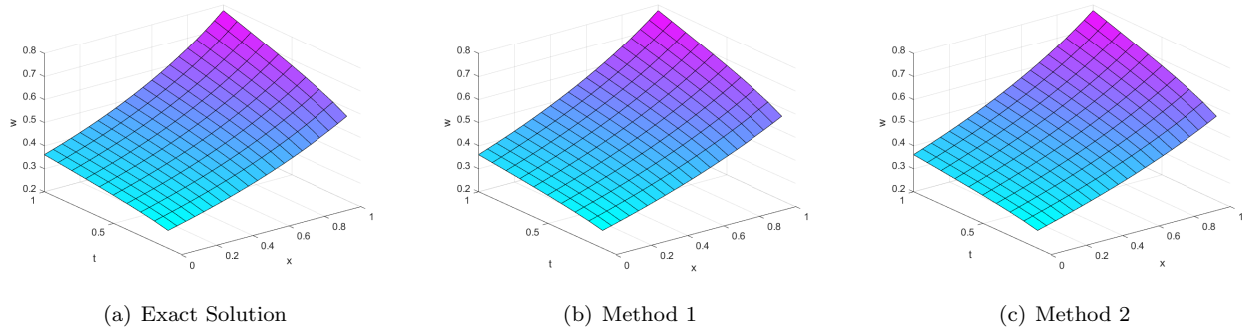


FIGURE 5. Exact and numerical solutions are obtained by using present methods.

accuracy. The similarity of the values between the two methods also implies comparable numerical performance in terms of convergence efficiency under the given conditions.

Graphical solutions which are exact and numerical solutions obtained by using Method 1 and Method 2 for $\Delta t = 0.000001$, $\Delta x = 0.05$, $\alpha = 0.5$ are plotted in Figure 5. From the figure can be seen that exact and approximate solutions are in good agreement.

TABLE 13. Relative errors for different values of α at $t = 0.5$.

x	$\alpha = 0.5$		$\alpha = 0.8$	
	Method 1	Method 2	Method 1	Method 2
0.1	7.669×10^{-3}	7.365×10^{-3}	2.441×10^{-3}	2.441×10^{-3}
0.5	1.883×10^{-2}	1.803×10^{-2}	6.026×10^{-3}	6.026×10^{-3}
0.9	6.277×10^{-3}	5.974×10^{-3}	1.992×10^{-3}	1.992×10^{-3}

TABLE 14. L_∞ error norms and CPU time for $\alpha = 0.5$ at $t = 0.5$.

t	Method 1		Method 2	
	L_∞	CPU Time	L_∞	CPU Time
0.5	8.836×10^{-3}	0.0006417	2.610×10^{-3}	0.0008417

TABLE 15. Rate of convergence for $\alpha = 0.5$ at $t = 0.5$.

	Method 1	Method 2
2	-	-
4	1.0395	1.0381
8	0.8193	0.8207
16	0.8901	0.8910
32	0.9402	0.9408

6. CONCLUSION

In this study, we successfully implemented explicit and implicit exponential finite difference methods to obtain approximate solutions of the time-fractional Burgers' equation with a conformable derivative for various values of α . The numerical results show good agreement with the exact solutions, demonstrating that the proposed methods maintain high accuracy even for small values of α . These findings indicate that the developed numerical schemes are both efficient and reliable, making them suitable for solving linear and nonlinear fractional equations encountered in applied sciences.

For future research, it would be of interest to investigate fractal modifications of the time-fractional Burgers' equation using alternative definitions of fractional and fractal derivatives. In particular, He's fractal derivative and other recent formulations may offer new insights into modeling complex phenomena. Such an extension could further enrich the theoretical framework and broaden the applicability of the proposed approach. We aim to pursue this direction in our upcoming work.

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DATA AVAILABILITY STATEMENT

Not applicable.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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