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The cotangent fractional Cauchy problem: analysis and numerical method

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Abstract

This article will address the resolution of a new Cauchy problem: The cotangent fractional Cauchy problem (CF-CP). First, we will conduct a theoretical study on 'existence and uniqueness' using the fractional Gronwall inequality. Next, we will formulate a decomposition formula for the CF-CP. Moreover, we present a numerical method for solving the CF-CP. Finally, we will present numerical results to verify the reliability of the proposed numerical method and validate the stability.

Keywords. Cotangent fractional derivative, Fractional Cauchy problem, Existence and uniqueness theorem, Numerical technique, Convergence analysis.

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1. Introduction

An equation featuring a fractional derivative (FD) that is non-integer in nature. can be used to simulate a fractional order system (FOS) [25, 42]. Numerous systems, including dynamical systems in electrochemistry, physics, viscoelasticity, biology, and chaotic systems, can be studied with the help of FOS [25, 40]. Moreover, the application of fractional calculus in numerous theories of control, including stability [6, 26, 27, 41], finite-time stability [7, 18, 29], stabilization [28, 35], observer design [15, 28], controllability [33, 34], and fault estimation [16, 17, 19], has been greatly boosted by the progress of scientific and technical systems. On the other hand, fractional calculus has a wide range of uses in disciplines such as fluid mechanics, physics, and more (see to [4, 23, 30, 31]).

Recent research highlights the importance of fractional Cauchy problems in modeling memory-dependent and complex dynamic systems. In Luchko, 2023 [8], explicit solutions to fractional Cauchy problems using Dzherbashyan–Caputo derivatives are developed, with connections to probabilistic frameworks. The abstract fractional Cauchy problem is addressed in Kostić et al ., 2021 [13], where existence and weak differentiability of solutions are studied in Banach spaces. The work of Li et al., 2023 [14] introduces variable-order fractal derivatives to better model nonlocal and evolving systems. In Ashyralyev et al., 2024 [2], the fractional Cauchy problem is explored in the context of the multidimensional heat equation, using integral methods. Numerical solutions for the Helmholtz equation are proposed in Boudjella, 2024 [20], where a regularized approach to the ill-posed Cauchy problem is presented. Further, Torres & Trujillo, 2021 [21] study generalized initial and internal Cauchy-type conditions using Riemann–Liouville derivatives. Lastly, El Kinani & Hbid, 2022[1] present a broader perspective on the application of fractional calculus in solving inverse and direct problems related to Cauchy-type formulations. Together, these works deepen both the theoretical understanding and numerical treatment of fractional Cauchy problems across various mathematical and physical contexts.

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Within this broad context, the Cotangent Fractional Cauchy Problem (CF-CP) presents a powerful and flexible modeling tool. By incorporating a cotangent kernel, this formulation captures complex dynamical behaviors and memory effects that classical integer-order derivatives cannot. The CF-CP framework is particularly suited for systems exhibiting nonlocality or history dependence, offering enhanced modeling capabilities for phenomena where traditional models fall short. It provides new avenues for both theoretical development and practical application of fractional calculus, enriching its utility across scientific and engineering disciplines.

Integral inequalities serve as a powerful tool for analyzing both the quantitative and qualitative properties of differential equations. With the growing demand across various applications, there has been a notable resurgence of interest in researching these inequalities. Several studies have explored different methodologies to propose and investigate these inequalities [9, 12, 24, 32, 44]. Among them. The Gronwall inequality is widely acknowledged as a highly influential and consequential outcome in the literature.

Concurrently with the increasing interest in fractional differential equations theory, numerous studies have expanded these mathematical inequalities to encompass fractional differential equations with both nonsingular and singular kernels. Notably, Sadek [36] has introduced Riemann-Liouville (RL) and Caputo cotangent fractional derivatives, incorporating exponential cotangent functions in their kernels. These derivatives offer advantages, including possessing a semi-group property in their cotangent fractional integrals (FI), thereby generalizing existing RL and Caputo FI and FD. The Laplace transform of cotangent fractional integrals and derivatives has been computed and applied to solve linear CF-CP. Sadek et al. [38] have presented a new version of the Gronwall inequality within the context of the cotangent FD, outlining its characteristics.

On a separate note, Ricardo et al. [3] have introduced a CF-CP dependent on the Caputo Katugampola derivative, along with a numerical method for solving such equations.

Building upon the contributions of [36], [38] and [3], we propose a novel numerical technique for solving the CF-CP. Our approach involves developing a decomposition formula for the cotangent Caputo derivative, from which we derive theorems guaranteeing the solvability and singularity of the given CF-CP.

The organization of the paper is structured as: section 2 introduces foundational principles, section 3 delineates the CF-CP, section 4 discusses the numerical method for addressing it, with a convergence analysis presented in two stages in Theorems 4.1 and 4.2, and section 5 focuses on practical applications or case studies.

2. Preliminarily

In this portion, we introduce the conceptual framework and symbols for the cotangent FI and FD, as outlined in prior literature [36–38].

Definition 2.1. Let $\Xi \in L^1([a,b]), \sigma > 0$. The FI of Ξ of order σ is defined as:

$$I_a^\sigma \Xi(\ell) = \frac{1}{\Gamma(\sigma)} \int_a^\ell (\ell - s)^{\sigma-1} \Xi(s) ds.$$

Definition 2.2. Let $\sigma \in (n-1, n)$ and $\Xi \in AC^n[a,b]$. The Caputo F.Dof Ξ of order σ, become

$${}^C D_a^\sigma \Xi(\ell) = \frac{1}{\Gamma(n-\sigma)} \int_a^\ell (\ell - \tau)^{n-\sigma-1} \Xi^{(n-1)}(\tau) d\tau,$$

where $AC^m[c,d] := \left\{\Xi : [c,d] \to \mathbb{R}, \Xi \in C^{m-1}[c,d] \ , \ \Xi^{(m-1)} \in AC[c,d]\right\}$

Definition 2.3. Let $\Xi \in L^1([a,b]), \sigma \in \mathbb{R}_*^+$, $0 < r \leq 1$. The cotangent FI of Ξ of order σ is defined as:

$$\begin{aligned} I_a^{\sigma,r} \Xi(\ell) &= \frac{1}{\sin(\frac{\pi}{2} r)^\sigma \Gamma(\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2} r)(\ell - s)} (\ell - s)^{\sigma-1} \Xi(s) ds \\ &= \sin(\frac{\pi}{2} r)^{-\sigma} e^{-\cot(\frac{\pi}{2} r)\ell} \left(I_{a^+}^\sigma \left(e^{\cot(\frac{\pi}{2} r)\ell} \Xi(\ell) \right) \right). \end{aligned}$$

Definition 2.4. Consider $\sigma \in (n-1, n)$ and $r \in (0,1]$. The cotangent FD of RL type with order σ for a function x is expressed as:

$$\begin{aligned} D_a^{\sigma,r}\Xi(\ell) &= \frac{D_\ell^{n,r}}{\sin(\frac{\pi}{2}r)^{n-\sigma}\Gamma(n-\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{n-\sigma-1}\Xi(\tau)d\tau \\ &= D^{n,r} I_a^{n-\sigma,r}\Xi(\ell), \end{aligned}$$

where, $\left(D^{1,r}\Xi\right)(\ell) = \cos(\frac{\pi}{2}r)\Xi(\ell) + \sin(\frac{\pi}{2}r)\Xi'(\ell)$ and $\left(D^{n,r}\Xi\right)(\ell) = D^{1,r}\left(D^{n-1,r}\Xi\right)(\ell)$.

Remark 2.5. When $r = 1$, the result yields the RL-FD case [22, 39].

Definition 2.6. Consider $\sigma \in (p-1, p)$ and $r \in (0,1]$. The cotangent FD of Caputo type with order σ for a function Ξ is expressed as:

$$^{C}D_a^{\sigma,r}\Xi(\ell) = D_a^{\sigma,r}\left(\Xi(\ell) - \sum_{k=0}^{p-1} \frac{\left(D^{k,r}\Xi\right)(a)}{\sin(\frac{\pi}{2}r)^k k!}(\ell-a)^k e^{-\cot(\frac{\pi}{2}r)(\ell-a)}\right).$$

Remark 2.7. When $r = 1$, the result yields the Caputo FD [22, 39].

Lemma 2.8. *[36] Let $\sigma > 0, \beta > 0$ and $0 < r \leq 1$. Then,*

$$I_a^{\sigma,r}\left(e^{-\cot(\frac{\pi}{2}r)s}(s-a)^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta+\sigma)\sin(\frac{\pi}{2}r)^\sigma} e^{-\cot(\frac{\pi}{2}r)s}(s-a)^{\beta+\sigma-1}.$$

Lemma 2.9. *[36] Let $\beta > 0, \sigma > 0$. Then, for Ξ is continuous, we have*

$$\left(I_a^{\sigma,r} I_a^{\beta,r}\Xi\right)(\ell) = \left(I_a^{\sigma+\beta,r}\Xi\right)(\ell).$$

Lemma 2.10. *[36] Let $\sigma > 0$ and Ξ be integrable on $\ell \geq a$. Then, we have*

$$\left(D_a^{\sigma,r} I_a^{\sigma,r}\Xi\right)(\ell) = \Xi(\ell).$$

Lemma 2.11. *Let $r \in (0,1], \sigma > 0, n = [\sigma]+1$, and $\Xi \in AC^n[a,b]$, the Caputo cotangent FD of Ξ of order σ, become*

$$\begin{aligned} {}^{C}D_a^{\sigma,r}\Xi(\ell) &= \left(I_a^{n-\sigma,r} D^{n,r}\Xi\right)(\ell) \\ &= \frac{1}{\sin(\frac{\pi}{2}r)^{n-\sigma}\Gamma(n-\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{n-\sigma-1}\left(D^{n,r}\Xi\right)(\tau)d\tau, \end{aligned}$$

Especially for $\sigma \in (0,1)$ we find

$$\begin{aligned} {}^{C}D_a^{\sigma,r}\Xi(\ell) &= \left(I_a^{1-\sigma,r} D^{1,r}\Xi\right)(\ell) \\ &= \frac{1}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(1-\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{-\sigma}\left(D^{1,r}\Xi\right)(\tau)d\tau \\ &= \frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(1-\sigma)} \int_a^\ell e^{\cot(\frac{\pi}{2}r)\tau}(\ell-\tau)^{-\sigma}\left(\cos(\frac{\pi}{2}r)\Xi(\tau) + \sin(\frac{\pi}{2}r)\Xi'(\tau)\right)d\tau. \end{aligned}$$

Remark 2.12. When $r = 1$, the result yields the Caputo FD in Definition 2.2.

Lemma 2.13. *[38] Suppose σ, r are positive constants. Let $h_1(\ell)$ and $h_2(\ell)$ be nonnegative functions locally integrable on the interval $[0, L)$, and let $d(\ell)$ be a nonnegative, nondecreasing, and continuous function defined on ℓ in the interval $[0, L)$ such that $d(\ell) \leq C$, with C being a constant. Furthermore, consider that $h_2(\ell)$ increases monotonically for ℓ in the interval $[0, T)$.*

$$h_1(\ell) \leq h_2(\ell) + \sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)d(\ell)\left({}_0I^{\sigma,r}f\right)(\ell),$$

then

$$h_1(\ell) \leq h_2(\ell)E_\sigma(d(\ell)\Gamma(\sigma), \ell), \quad \ell \in [0, L),$$

where

$$E_\sigma(x,y)=\sum_{k=0}^{\infty}\frac{x^k y^{k\sigma}}{\Gamma(\sigma k+1)},$$

denote the Mittag-Leffler function with a single variable.

Theorem 2.14. *[5] Consider Z as a Banach space where $T\subset Z$ is both closed and convex. Suppose $\mathcal{O}$ is a relatively open subset of T with $0\in\mathcal{O}$ and $\mathcal{N}:\mathcal{O}\to T$ is a compact map. Then, either:*

- *$\mathcal{N}$ possesses a fixed point within K;*
- *there is a point $u\in\partial\mathcal{O}$ and $0<\lambda<1$ with $u=\lambda\mathcal{N}(u)$.*

3. Caputo CF-CP

Consider the following Caputo CF-CP:

$$\begin{cases} {}^c_aD^{\sigma,r}x(\ell)=f(\ell,\Xi(\ell)), & 0<\sigma<1, r\in(0,1],\\ \Xi(a)=\Xi_a, & a\in\mathbb{R}, \end{cases} \tag{3.1}$$

where ${}^cD_a^{\sigma,r}$ is the Caputo cotangent derivative and $f\in C(J\times\mathbb{R},\mathbb{R})$ with $J=[a,L]$.

Lemma 3.1. *Let Ξ be a continuous function defined on the interval $[a,L]$ and taking values in $\mathbb{R}$.*

Ξ verifies (3.1) $\Longleftrightarrow$ Ξ verifies integral Equation (3.2).

Where integral Equation define by:

$$\Xi(\ell)=\Xi_a e^{-\cot(\frac{\pi}{2}r)(\ell-a)}+\frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-s)}(\ell-s)^{\sigma-1}f(s,\Xi(s))ds. \tag{3.2}$$

Proof. **Direct sens:** Ξ solution to Eq. (3.2), show that: Ξ satisfies problem (3.1):

$$\begin{aligned} {}^CD_a^{\sigma,r}\Xi(\ell)=&D_a^{\sigma,r}\left(\Xi(\ell)-\Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)}\right)\\ =&D_a^{\sigma,r}\left(\frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-s)}(\ell-s)^{\sigma-1}f(s,\Xi(s))ds\right)\\ =&\left(D_a^{\sigma,r}\left(I_a^{\sigma,r}f\right)\right)(\ell)\\ =&f(\ell,\Xi(\ell)). \end{aligned}$$

On the other hand:

$$\begin{aligned} \Xi(a)=&\Xi_a e^{-\cot(\frac{\pi}{2}r)(a-a)}+\frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\int_a^a e^{-\cot(\frac{\pi}{2}r)(\ell-s)}(\ell-s)^{\sigma-1}f(s,\Xi(s))ds\\ =&\Xi_a, \end{aligned}$$

then Ξ is a solution to Eq.(3.1).

Reciprocally show that Ξ satisfies problem (3.1) then, Ξ is a solution to Eq. (3.2) let

$$\begin{aligned} g(\ell)&=f(\ell,\Xi(\ell))\\ &=D_a^{\sigma,r}\left(\Xi(\ell)-\Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)}\right)\\ &=D^{1,r}I_a^{1-\sigma,r}\left(\Xi(\ell)-\Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)}\right). \end{aligned} \tag{3.3}$$

Applying the operator $I_a^{1,r}$ to (3.3) we get

$$I_a^{1,r}g(\ell)=I_a^{1-\sigma,r}\left(\Xi(\ell)-\Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)}\right). \tag{3.4}$$

Applying $D_a^{1-\sigma,r}$ to (3.4) we have

$$\begin{aligned}\Xi(\ell) &= \Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)} + D_a^{1-\sigma,r} I_a^{1-\sigma,r} g(\ell)\\ &= \Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)} + D_a^{1,r} I_a^{\sigma,r} I_a^{1,r} g(\ell)\\ &= \Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)} + D_a^{1,r} I_a^{\sigma,r} I_a^{1,r} g(\ell)\\ &= \Xi(a)e^{-\cot(\frac{\pi}{2}r)(\ell-a)} + I_a^{\sigma,r} g(\ell).\end{aligned}$$

□

Theorem 3.2. *Suppose there exist $g, h \in C(J, \mathbb{R}_+)$such that*

$$|f(\ell,\Xi)| \leq g(\ell) + h(\ell)|\Xi|, \qquad \forall \ell \in J \text{ and } \Xi \in \mathbb{R}, \qquad (H1)$$

the Eq. (3.1) possesses at least one solution.

Proof. Consider the operator $\mathcal{N} : \mathcal{C}([a,L],\mathbb{R}) \to \mathcal{C}([a,L],\mathbb{R})$ as defined in Eq. (3.2). Our objective is to demonstrate that $\mathcal{N}$ fulfills the assumption of Schauder's fixed point Theorem 2.14. The proof will be presented in multiple stages

$$\mathcal{N}(\Xi) = \Xi_a e^{-\cot(\frac{\pi}{2}r)(\ell-a)} + \frac{1}{\sin(\frac{\pi}{2}r)^\sigma \Gamma(\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-s)}(\ell-s)^{\sigma-1} f(s,\Xi(s))ds.$$

Claim 1 $\mathcal{N}$ is a continuous operator. Let Ξ_n be a sequence such that $\Xi_n \to \Xi \in \mathcal{C}([a,L],\mathbb{R})$, for all $\ell \in [a,L]$:

$$|\mathcal{N}(\Xi_n)(\ell) - \mathcal{N}(\Xi)(\ell)| \leq \frac{1}{\sin(\frac{\pi}{2}r)^\sigma \Gamma(\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-s)}(\ell-s)^{\sigma-1}|f(\tau,\Xi_n(\tau)) - f(\tau,\Xi(\tau))|d\tau.$$

Therefore, the continuity of the operator $\mathcal{N}$ follows from the continuity of f.

Claim 2 Show that $\mathcal{N}(B_b) \subset B_l$; avec $B_h = \{\Xi \in \mathcal{C}([a,L],\mathbb{R}), \|\Xi\|_\infty \leq h\}$
Let $\Xi \in B_b$ we show that $\mathcal{N}(\Xi) \in B_l$. For each $\ell \in [a,L]$, we have

$$|\mathcal{N}(\Xi)(\ell)| \leq |\Xi_a| e^{-\cot(\frac{\pi}{2}r)(\ell-a)} + \frac{1}{\sin(\frac{\pi}{2}r)^\sigma \Gamma(\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{\sigma-1}|f(\tau,\Xi(\tau))|d\tau.$$

From (H1), we find :

$$\begin{aligned}|\mathcal{N}(\Xi)(\ell)| &\leq |\Xi_a| e^{-\cot(\frac{\pi}{2}r)(\ell-a)} + \frac{1}{\sin(\frac{\pi}{2}r)^\sigma \Gamma(\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{\sigma-1}\left(g_1(\tau) + g_2(\tau)|\Xi(\tau)|\right)d\tau\\ &\leq |\Xi_a| + I_a^{\sigma,r}\left(g_1(L) + b g_2(L)\right).\end{aligned}$$

Then, for any $\Xi \in B_b$ there exists $l = |\Xi_a| + I_a^{\sigma,r}\left(g_1(L) + b g_2(L)\right) > 0$, such that $\|\mathcal{N}(\Xi)\|_\infty \leq l$.

Claim 3 Establishing that the operator $\mathcal{N}$ transforms bounded sets into sets that are equicontinuous in terms of $\mathcal{C}([a,L],\mathbb{R})$. Let $\ell_1, \ell_2 \in [a,L]$, $\ell_1 < \ell_2$, and $\Xi \in B_b$. Then

$$\begin{aligned}|\mathcal{N}(\Xi)(\ell_1) - \mathcal{N}(\Xi)(\ell_2)| \leq |\Xi_a| &\left|e^{-\cot(\frac{\pi}{2}r)}(\ell_2-a) - e^{-\cot(\frac{\pi}{2}r)}(\ell_1-a)\right|\\ &+ \frac{1}{\sin(\frac{\pi}{2}r)^\sigma \Gamma(\sigma)} \int_a^{\ell_1} \left|e^{-\cot(\frac{\pi}{2}r)(\ell_2-\tau)}(\ell_2-\tau)^{\sigma-1}\right.\\ &\left.- e^{-\cot(\frac{\pi}{2}r)(\ell_1-\tau)}(\ell_1-\tau)^{\sigma-1}\right||f(\tau,\Xi(\tau))|d\tau\\ &+ \frac{1}{\sin(\frac{\pi}{2}r)^\sigma \Gamma(\sigma)} \int_{\ell_1}^{\ell_2} \left|e^{-\cot(\frac{\pi}{2}r)(\ell_2-\tau)}(\ell_2-\tau)^{\sigma-1}|f(\tau,\Xi(\tau))|\right|d\tau.\end{aligned}$$

From (H1), we get

$$|\mathcal{N}(\Xi)(\ell_1)-\mathcal{N}(\Xi)(\ell_2)| \leq |\Xi_a| \left|e^{-\cot(\frac{\pi}{2}r)}(\ell_2-a)-e^{-\cot(\frac{\pi}{2}r)}(\ell_1-a)\right| + \frac{G_1+bG_2}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\times$$
$$\int_a^{\ell_1}\left|e^{-\cot(\frac{\pi}{2}r)(\ell_2-\tau)}(\ell_2-\tau)^{\sigma-1}-e^{-\cot(\frac{\pi}{2}r)(\ell_1-\tau)}(\ell_1-\tau)^{\sigma-1}\right|$$
$$+\frac{G_1+bG_2}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\left|e^{-\cot(\frac{\pi}{2}r)(\ell_2-\tau)}(\ell_2-\tau)^{\sigma-1}\right|.$$

Where $G_1=\max\limits_{\ell\in[a,L]}(g_1(\ell))$ and $G_2=\max\limits_{\ell\in[a,L]}(g_2(\ell))$. As $\ell_1\to\ell_2$ the right-hand side of the above inequality tends to zero.

Claim 4 We now show there exists an open set $U\subset\mathcal{C}([a,L],\mathbb{R})$ with $\Xi=\lambda\mathcal{N}(\Xi)$, for $\lambda\in[0,1]$ and $\Xi\in\partial U$. Let $\Xi\in\mathcal{C}([a,L],\mathbb{R})$ such that $\Xi=\lambda\mathcal{N}(\Xi)$, for some $\lambda\in[0,1]$, Then, for any $\ell\in[a,L]$, we have

$$|\Xi(\ell)|\leq|\Xi_a|e^{-\cot(\frac{\pi}{2}r)(\ell-a)}+\frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{\sigma-1}|f(\tau,\Xi(\tau))|d\tau.$$

Then, by (H1) we find

$$\begin{aligned}|\Xi(\ell)|&\leq|\Xi_a|+\frac{G_1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{\sigma-1}d\tau\\&+\frac{G_2}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{\sigma-1}|\Xi(\tau)|d\tau\\&\leq|\Xi_a|\frac{G_1}{\sin(\frac{\pi}{2}r)^\sigma}e^{-\cot(\frac{\pi}{2}r)(\ell-a)}(\ell-a)^\sigma E_{1,\sigma+1}\left(-\cot(\frac{\pi}{2}r)(\ell-a)\right)\\&+\frac{G_2}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{\sigma-1}|\Xi(\tau)|d\tau,\end{aligned}\tag{3.5}$$

where

$$E_{1,\sigma+1}\left(\cot(\frac{\pi}{2}r)(\ell-a)\right)=\sum_{k=0}^{\infty}\frac{(\cot(\frac{\pi}{2}r)(\ell-a))^k}{\Gamma(k+\sigma+1)}.$$

Conversely, we possess:

$$\Gamma(j+\sigma+1)\geq\Gamma(j+1)\quad\forall j\geq1\quad and\qquad\Gamma(j+\sigma+1)\geq\Gamma\left(\frac{2}{3}\right)\Gamma(j+1),\ j=0,$$

thus:

$$\begin{aligned}E_{1,\sigma+1}\left(\cot(\frac{\pi}{2}r)(\ell-a)\right)&=\sum_{k=0}^{\infty}\frac{\left(\cot(\frac{\pi}{2}r)(\ell-a)\right)^k}{\Gamma(k+\sigma+1)}\\&\leq\frac{1}{\Gamma(\frac{2}{3})}\sum_{k=0}^{\infty}\frac{\left(\cot(\frac{\pi}{2}r)(\ell-a)\right)^k}{\Gamma(k+1)}\\&\leq\frac{1}{\Gamma(\frac{2}{3})}e^{\cot(\frac{\pi}{2}r)(\ell-a)}.\end{aligned}\tag{3.6}$$

From (3.5) and (3.6), we obtain:

$$|\Xi(\ell)|\leq|\Xi_a|+\frac{G_1(\ell-a)^\sigma}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\frac{3}{2})}+G_2I_a^{\sigma,r}(|\Xi(\ell)|).$$

From the Lemma 2.13, we get

$$\begin{aligned}|\Xi(\ell)| &\leq \left(|\Xi_a| + \frac{G_1(\ell-a)^\sigma}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\frac{2}{3})}\right) E_\sigma\left(\frac{G_2}{\sin(\frac{\pi}{2}r)^\sigma}, (\ell-a)\right)\\ &\leq \left(|\Xi_a| + \frac{G_1(L-a)^\sigma}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\frac{2}{3})}\right) E_\sigma\left(\frac{G_2}{\sin(\frac{\pi}{2}r)^\sigma}, (L-a)\right)\\ &\leq M',\end{aligned}$$

where

$$E_\sigma(\lambda,\ell) = \sum_{k=0}^{\infty} \frac{\lambda^k \ell^{k\sigma}}{\Gamma(k\sigma+1)}.$$

Let $U = \{\Xi \in \mathcal{C}([a,L],\mathbb{R}) : \|\Xi\|_\infty < M'+1\}$, $\forall \Xi \in \partial U \quad \Xi \neq \lambda\mathcal{N}(\Xi) \qquad \forall\lambda \in [0,1]$.
Based on Theorem 2.14, we conclude that N possesses a fixed point Ξ within the closure of $\bar{U}$, representing a solution to Eq. (3.1). □

Theorem 3.3. *Suppose there exists $\mathbb{L} > 0$ such that*

$$|f(\ell,t_1) - f(\ell,t_2)| \leq \mathbb{L}|t_1 - t_2| \text{ for each } \ell \in J \text{ and } \forall t_1,t_2 \in \mathbb{R}, \qquad (H2)$$

the problem (3.1) has a unique solution.

Proof. There is at least one solution to Eq. (3.1) based on (H2) and Theorem 3.2. We shall demonstrate the uniqueness of this solution.

Let Ξ_1 and Ξ_2 two solutions of the Eq. (3.1). Then, by (H2) we get

$$\begin{aligned}|\Xi_2(\ell) - \Xi_2(\ell)| &\leq \frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-s)}(\ell-s)^{\sigma-1}|f(s,\Xi_1(s)) - f(s,\Xi_2(s))|\,ds\\ &\leq \frac{\mathbb{L}}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(\sigma)} \int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-s)}(\ell-\tau)^{\sigma-1}|\Xi_1(s) - \Xi_2(s)|ds\\ &= g(\ell) + \sin\left(\frac{\pi}{2}r\right)^\sigma \Gamma(\sigma)w(\ell)\left[I_a^{\sigma,r}|\Xi(\ell) - y(\ell)|\right],\end{aligned}$$

where $g(\ell) = 0$ and $w(\ell) = \frac{\mathbb{L}}{\Gamma(\sigma)\sin\left(\frac{\pi}{2}r\right)}$. From Theorem 2.14, we get $\Xi = y$. □

4. Numerical method and convergence analysis

Theorem 4.1. *Let $m \in \mathbb{N}^*$, $\Xi \in AC^2([a,L],\mathbb{R})$. We assume:*

$$\begin{aligned}\mathcal{A}_m &= \frac{1}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)} \sum_{j=0}^{m} \frac{\Gamma(j+\sigma-1)}{\Gamma(\sigma-1)j!},\\ \mathcal{B}_{m,j} &= \frac{\Gamma(j+\sigma-1)}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)\Gamma(\sigma-1)(j-1)!}, j = 1,2,\ldots,m,\end{aligned} \tag{4.1}$$

and functions $\mathcal{V}_i : [a,L] \to \mathbb{R}$ by

$$\mathcal{V}_i(\ell) = \int_a^\ell (s-a)^{i-1} e^{\cot(\frac{\pi}{2}r)s}\left(\cos(\frac{\pi}{2}r)\Xi(s) + \sin(\frac{\pi}{2}r)\Xi'(s)\right)ds.$$

Then

$${}^C D_a^{\sigma,r}\Xi(\ell) = \mathcal{A}_m(\ell-a)^{1-\sigma}\left(\cos(\frac{\pi}{2}r)\Xi(\ell) + \sin(\frac{\pi}{2}r)\Xi'(\ell)\right) - e^{-\cot(\frac{\pi}{2}r)\ell}\sum_{i=1}^{m}\mathcal{B}_{m,i}(\ell-a)^{1-\sigma-i}\mathcal{V}_i(\ell) + \mathcal{E}_m(\ell),$$

with

$$\lim_{m\to\infty}\mathcal{E}_m(\ell) = 0, \quad \forall\ell \in (a,L].$$

Proof. We have

$$^{C}D_a^{\sigma,r}\Xi(\ell)=\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(1-\sigma)}\int_a^\ell z'(\tau)w(\tau)d\tau,$$

whrer $z'(\tau)=(\ell-\tau)^{-\sigma}$ and $w(\tau)=e^{-\cot(\frac{\pi}{2}r)\tau}(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))$, then integrating by parts, we get

$$\begin{aligned}^{C}D_a^{\sigma,r}\Xi(\ell)&=\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}e^{\cot(\frac{\pi}{2}r)a}(\ell-a)^{1-\sigma}\left(\cos(\frac{\pi}{2}r)\Xi(a)+\sin(\frac{\pi}{2}r)\Xi'(a)\right)\\&+\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}\int_a^\ell(\ell-\tau)^{1-\sigma}\frac{d}{ds}\left(e^{\cot(\frac{\pi}{2}r)\tau}(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))\right)d\tau.\end{aligned}$$

Applying the theorem of generalized binomial

$$\begin{aligned}(\ell-\tau)^{1-\sigma}&=(\ell-a)^{1-\sigma}\left(1-\frac{\tau-a}{\ell-a}\right)^{1-\sigma}\\&=(\ell-a)^{1-\sigma}\sum_{k=0}^{\infty}\frac{\Gamma(k+\sigma-1)}{\Gamma(\sigma-1)k!}\left(\frac{s-a}{\ell-a}\right)^k.\end{aligned}$$

By substituting the previously mentioned equality into the FD expression, we arrive at

$$\begin{aligned}^{C}D_a^{\sigma,r}\Xi(\ell)&=\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}e^{\cot(\frac{\pi}{2}r)a}(\ell-a)^{1-\sigma}(\cos(\frac{\pi}{2}r)\Xi(a)+\sin(\frac{\pi}{2}r)\Xi'(a))\\&+\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}\int_a^\ell(\ell-a)^{1-\sigma}\sum_{k=0}^{m}\frac{\Gamma(k+\sigma-1)}{\Gamma(\sigma-1)k!}\left(\frac{\tau-a}{\ell-a}\right)^k\\&\times\frac{d}{d\tau}\left(e^{\cot(\frac{\pi}{2}r)\tau}(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))\right)d\tau+\mathcal{E}_m(\ell),\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_m(\ell)&=\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}\int_a^\ell(\ell-a)^{1-\sigma}\sum_{i=m+1}^{\infty}\frac{\Gamma(i+\sigma-1)}{\Gamma(\sigma-1)i!}\left(\frac{s-a}{\ell-a}\right)^i\\&\times\frac{d}{d\tau}\left(e^{\cot(\frac{\pi}{2}r)\tau}(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))\right)d\tau.\end{aligned}$$

Then

$$\begin{aligned}^{C}D_a^{\sigma,r}\Xi(\ell)&=\frac{1}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}(\ell-a)^{1-\sigma}(\cos(\frac{\pi}{2}r)\Xi(a)+\sin(\frac{\pi}{2}r)\Xi'(a))\\&+\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}(\ell-a)^{1-\sigma}\sum_{i=1}^{m}\frac{\Gamma(i+\sigma-1)}{\Gamma(\sigma-1)i!(\ell-a)^i}\int_a^\ell(\tau-a)^i\\&\times\frac{d}{d\tau}\left(e^{\cot(\frac{\pi}{2}r)\tau}(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))\right)d\tau+\mathcal{E}_m(\ell).\end{aligned}$$

Setting $u(\tau)=(\tau-a)^i$, and $g'(\tau)=\frac{d}{d\tau}\left(e^{\cot(\frac{\pi}{2}r)\tau}(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))\right)$, and by integrating by parts, we get

$$
\begin{aligned}
{}^{C}D_a^{\sigma,r}\Xi(\ell) &= \frac{1}{\sin\left(\frac{\pi}{2}r\right)^{1-\sigma}\Gamma(2-\sigma)}(\ell-a)^{1-\sigma}(\cos(\frac{\pi}{2}r)\Xi(a)+\sin(\frac{\pi}{2}r)\Xi'(a))\\
&\times\left(1+\sum_{i=1}^{m}\frac{\Gamma(i+\sigma-1)}{\Gamma(\sigma-1)i!}\right)-\frac{e^{-\cot(\frac{\pi}{2}r)\ell}}{\Gamma(2-\sigma)\sin(\frac{\pi}{2}r)^{\sigma}}\sum_{i=1}^{m}\frac{(\ell-a)^{1-\sigma-i}}{\Gamma(1-\sigma)(i-1)!}\int_a^{\ell}(\tau-a)^{i-1}e^{\cot(\frac{\pi}{2}r)\tau}\\
&\times(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))d\tau+\mathcal{E}_m(\ell).
\end{aligned}
$$

On the other side, we show that

$$\lim_{m\to+\infty}\mathcal{E}_m(\tau)=0,\quad \forall\tau\in[a,L].$$

We have

$$
\begin{aligned}
\sum_{i=m+1}^{\infty}\left|\frac{\Gamma(i+\sigma-1)}{\Gamma(\sigma-1)i!}\left(\frac{s-a}{\ell-a}\right)^i\right| &= \sum_{i=m+1}^{\infty}\left|\frac{\Gamma(\sigma-1)(\sigma-1)\sigma\times\ldots\times(\sigma+(i-2))}{\Gamma(\sigma-1)\times 1\times 2\times\ldots\times(i-1)\times i}\right|\times\left(\frac{s-a}{\ell-a}\right)^i\\
&\leq \sum_{i=m+1}^{\infty}\left|\left(\frac{\sigma-1}{1}\right)\times\left(\frac{\sigma+1}{2}\right)\times\left(\frac{\sigma+1}{3}\right)\times\ldots\left(\frac{\sigma+(i-2)}{i}\right)\right|\\
&\leq \sum_{i=m+1}^{\infty}(1-\sigma)\times\left|\prod_{k=2}^{i}\left(1-\frac{2-\sigma}{k}\right)\right|\\
&\leq \sum_{i=m+1}^{\infty}\left(e^{\ln 1-\sigma}\times e^{\left[\sum_{k=2}^{i}\ln\left(1-\frac{2-\sigma}{k}\right)\right]}\right)\\
&\leq \sum_{i=m+1}^{\infty}\left(e^{1-\sigma}\times e^{\left[-\sum_{k=2}^{i}\frac{2-\sigma}{k}\right]}\right)\\
&\leq \sum_{i=m+1}^{\infty}\left(e^{(2-\sigma)(1-\sigma)}\times e^{\left[-(2-\sigma)\sum_{k=2}^{i}\frac{1}{k}\right]}\right)\\
&\leq \sum_{i=m+1}^{\infty}\left(e^{(2-\sigma)(1-\sigma)}\times e^{[-(2-\sigma)\ln(i)]}\right)\\
&\leq \sum_{i=m+1}^{\infty}\frac{e^{(2-\sigma)(1-\sigma)}}{i^{2-\sigma}}\leq\sum_{i=m+1}^{\infty}\frac{e^{(1-\sigma)^2+(1-\sigma)}}{i^{2-\sigma}}\\
&\leq \int_m^{\infty}\frac{e^{(1-\sigma)^2+(1-\sigma)}}{s^{2-\sigma}}ds\\
&\leq \frac{e^{(1-\sigma)^2+(1-\sigma)}}{m^{1-\sigma}(1-\sigma)},
\end{aligned}
$$

then

$$|\mathcal{E}_m(\ell)| \leq \frac{e^{(1-\sigma)^2+(1-\sigma)} e^{-\cot(\frac{\pi}{2}r)\ell}}{m^{1-\sigma}(1-\sigma)\sin(\frac{\pi}{2}r)^{1-\sigma}\Gamma(2-\sigma)}(\ell-a)^{1-\sigma}$$
$$\times \int_a^\ell \left| \frac{d}{d\tau}\left(e^{\cot(\frac{\pi}{2}r)\tau}(\cos(\frac{\pi}{2}r)\Xi(\tau)+\sin(\frac{\pi}{2}r)\Xi'(\tau))\right)\right| d\tau. \tag{4.2}$$

The inequality above tends to 0 on the right side for any $\ell \in [a, L]$ as $m \to \infty$. □

Using Lemma 2.9, we can express Equation involving integrals (3.2) as follows:

$$\begin{aligned}
\Xi(\ell) &= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + \frac{1}{\sin(\frac{\pi}{2}r)^\sigma \Gamma(\sigma)}\int_c^\ell e^{-\cot(\frac{\pi}{2}r)\tau}(\ell-\tau)^{-\sigma} f(\tau,\Xi(\tau))d\tau \\
&= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + I_c^{\sigma,r} f(\ell,\Xi(\ell)) \\
&= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + I_c^{1-\widetilde{\sigma},r} f(\ell,\Xi(\ell));\ with\ \widetilde{\sigma} = 1-\sigma \\
&= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + \underbrace{I_c^{1-\tilde{\sigma},r}\mathcal{D}^{1,r}}_{Lemma\ 2.11}\underbrace{I_c^{1,r} f(\ell,\Xi(\ell))}_{note\ \widetilde{f(\ell)}} \\
&= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + {}^cD_c^{\widetilde{\sigma},r}\widetilde{f(\ell)} \\
&= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + {}^cD_c^{1-\sigma,r}\left(I_c^{1,r} f(\ell,\Xi(\ell))\right) \\
&= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + {}^cD_c^{1-\sigma,r}\left(\frac{1}{\sin(\frac{\pi}{2}r)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{-\sigma} f(\tau,\Xi(\tau))d\tau\right) \\
&= \Xi_a e^{-\cot(\frac{\pi}{2}r)\ell} + {}^cD_c^{1-\sigma,r}\left(\frac{1}{\sin(\frac{\pi}{2}r)}\int_a^\ell e^{-\cot(\frac{\pi}{2}r)(\ell-\tau)}(\ell-\tau)^{-\sigma} f(\tau,\Xi(\tau))d\tau\right).
\end{aligned}$$

Applying the decomposition formula provided in Theorem 4.1, we discover:

$$\Xi(s) = \Xi_a e^{-\cot(\frac{\pi}{2}r)(s-a)} + \tilde{\mathcal{A}}_m(s-a)^\sigma f(s,\Xi(s)) - e^{-\cot(\frac{\pi}{2}r)s}\sum_{k=1}^m \tilde{\mathcal{B}}_{m,k}(s-a)^{\sigma-k}\tilde{\mathcal{V}}_k(s) + \tilde{\mathcal{E}}_m(s), \tag{4.3}$$

where

$$\begin{aligned}
\tilde{\mathcal{A}}_m &= \frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)}\sum_{k=0}^m \frac{\Gamma(k-\sigma)}{\Gamma(-\sigma)k!}, \\
\tilde{\mathcal{B}}_{m,k} &= \frac{\Gamma(k-\sigma)}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)\Gamma(-\sigma)(k-1)!}, \quad k = 1,2,\ldots,m,
\end{aligned} \tag{4.4}$$

and

$$\tilde{\mathcal{V}}_k(s) = \int_a^s (\tau-a)^{k-1}e^{\cot(\frac{\pi}{2}r)\tau} f(\tau,\Xi(\tau))d\tau.$$

We consider the formula for the approximate solution, denoted as Ξ_m, by

$$\Xi_m(s) = \Xi_a e^{-\cot(\frac{\pi}{2}r)(s-a)} + \tilde{\mathcal{A}}_m(s-a)^\sigma f\left(s,\Xi_m(s)\right) - e^{-\cot(\frac{\pi}{2}r)s}\sum_{k=1}^m \tilde{\mathcal{B}}_{m,k}(s-a)^{\sigma-k}\tilde{\mathcal{V}}_{m,k}(s), \tag{4.5}$$

where

$$\tilde{\mathcal{V}}_{m,k}(s) = \int_a^s (\tau-a)^{k-1}e^{\cot(\frac{\pi}{2}r)\tau} f\left(\tau,\Xi_m(\tau)\right)d\tau.$$

The expression $\widetilde{\mathcal{E}_m}(s)$ is similar to $\mathcal{E}_m$, and furthermore, it satisfies inequalities (4.2)

$$|\widetilde{\mathcal{E}_m}(s)| \leq \frac{e^{\sigma^2+\sigma} e^{-\cot(\frac{\pi}{2}r)s}}{\sigma m^\sigma \sin(\frac{\pi}{2}r)^\sigma \Gamma(1+\sigma)}(s-a)^\sigma \int_a^\ell \left| \frac{d}{d\tau}\left(e^{\cot(\frac{\pi}{2}r)\tau}((1-r)\Xi(\tau)+r\Xi'(\tau))\right)\right| d\tau.$$

Theorem 4.2. *Let $f:[a,L]\times\mathbb{R}\to\mathbb{R}$ be a function that fulfills condition (H2). For $m\in\mathbb{N}$, let Ξ and Ξ_m as in Eq. (4.3) and Eq. (4.5). Suppose that*

$$a < L < a + \sin(\frac{\pi}{2}r)\left(\frac{\Gamma(1+\sigma)}{\mathbb{L}}\right)^{\frac{1}{\sigma}}. \tag{4.6}$$

Therefore, $\Xi_m(\ell)$ tends to $\Xi(\ell)$ as m tends to ∞.

Proof. Let

$$e_{\Xi_m} = \max_{\ell\in[a,L]} |\Xi_m(\ell)-\Xi(\ell)|.$$

It follows from Eq. (4.3) and Eq. (4.5) that

$$|\Xi_m(\ell)-\Xi(\ell)| \leq \widetilde{\mathcal{A}_m}(\ell-a)^\sigma |f(\ell,\Xi_m(\ell))-f(\ell,\Xi_(\ell))| + e^{-\cot(\frac{\pi}{2}r)\ell}\sum_{k=1}^m \left|\widetilde{\mathcal{B}_{m,k}}\right|(\ell-a)^{\sigma-k}\left|\widetilde{\mathcal{V}_{k,m}}(\ell)-\widetilde{\mathcal{V}_k}(\ell)\right| + |\widetilde{\mathcal{E}_m}(\ell)|.$$

As a result of the relations outlined below,

$$|f(\ell,\Xi_m(\ell))-f(\ell,\Xi(\ell))| \leq \mathbb{L}|\Xi_m(\ell)-\Xi(\ell)| \leq \mathbb{L}e_{\Xi_m},$$

$$\begin{aligned}\left|\widetilde{\mathcal{V}_{k,m}}(\ell)-\widetilde{\mathcal{V}_k}(\ell)\right| &\leq \int_a^\ell (\tau-a)^{k-1} e^{\cot(\frac{\pi}{2}r)\tau}|f(\tau,\Xi_m(\tau))-f(\tau,\Xi(\tau))|d\tau \\ &\leq \mathbb{L}e_{\Xi_m} e^{\cot(\frac{\pi}{2}r)\ell}\int_a^\ell (\tau-a)^{k-1}d\tau \\ &\leq \frac{\mathbb{L}e_{\Xi_m} e^{\cot(\frac{\pi}{2}r)\ell}}{k}(\ell-a)^k,\end{aligned}$$

and

$$\begin{aligned}|\tilde{\mathcal{A}}_m| &= \frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)}\left|\sum_{j=0}^m \frac{\Gamma(j-\sigma)}{\Gamma(-\sigma)j!}\right| \\ &\leq \frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)|\Gamma(-\sigma)|}\frac{\Gamma(m+1-\sigma)}{\sigma\Gamma(m+1)} \\ &\leq \frac{1}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)\Gamma(1-\sigma)}\frac{\Gamma(m+1-\sigma)}{\Gamma(m+1)} \\ &\leq \frac{1}{\sigma\pi\sin(\frac{\pi}{2}r)^\sigma}\frac{\Gamma(m+1-\sigma)}{\Gamma(m+1)}.\end{aligned}$$

By the Euler's reflection formula and Equation 3 in [11]

$$\begin{aligned}\sum_{k=1}^m \left|\widetilde{\mathcal{B}_{m,k}}\right|(\ell-a)^{\sigma-k}\left|\widetilde{\mathcal{V}_{k,m}}(\ell)-\widetilde{\mathcal{V}_k}(\ell)\right| &\leq \frac{\mathbb{L}e_{\Xi_m}(\ell-a)^\sigma}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)|\Gamma(-\sigma)|}\sum_{k=1}^m\frac{\Gamma(k-\sigma)}{k!} \\ &\leq \frac{\mathbb{L}e_{\Xi_m}(\ell-a)^\sigma}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)|\Gamma(-\sigma)|}\left[\frac{\Gamma(m+1-\sigma)}{\sigma\Gamma(m+1)}+|\Gamma(-\sigma)|\right] \\ &\leq \frac{\mathbb{L}e_{\Xi_m}(\ell-a)^\sigma}{\sin(\frac{\pi}{2}r)^\sigma}\left[\frac{1}{\sigma\pi}\frac{\Gamma(m+1-\sigma)}{\Gamma(m+1)}+\frac{1}{\Gamma(1+\sigma)}\right],\end{aligned}$$

we conclude that

$$|\Xi_m(\ell)-\Xi(\ell)| \le \frac{\mathbb{L} e_{\Xi_m}}{\sin(\frac{\pi}{2}r)^\sigma}(\ell-a)^\sigma\left[\frac{2}{\sigma\pi}\frac{\Gamma(m+1-\sigma)}{\sigma\Gamma(m+1)}+\frac{1}{\sigma(1+\sigma)}\right]+|\widetilde{\mathcal{E}_m}(\ell)|$$

for all $\ell \in [a, L]$. Taking the maximum, over $\ell \in [a, L]$, on both sides of the inequality, we get

$$e_{\Xi_m} \le \frac{\mathbb{L} e_{\Xi_m}}{\sin(\frac{\pi}{2}r)^\sigma}(T-a)^\sigma\left[\frac{2}{\sigma\pi}\frac{\Gamma(m+1-\sigma)}{\Gamma(m+1)}+\frac{1}{\Gamma(1+\sigma)}\right]+\max_{\ell\in[a,L]}|\widetilde{\mathcal{E}_m}(\ell)|. \tag{4.7}$$

It is obvious that:

$$\lim_{m\to\infty}|\widetilde{\mathcal{E}_m}(\ell)|=0.$$

Furthermore, due to the application of Stirling's formula (refer to, [43]), we obtain

$$\lim_{m\to\infty}\frac{\Gamma(m+1-\sigma)}{\Gamma(m+1)}=0.$$

Therefore, setting $m\to\infty$ in (4.7) we get

$$\lim_{m\to\infty} e_{\Xi_m}\left[1-\frac{\mathbb{L}}{\sin(\frac{\pi}{2}r)^\sigma\Gamma(1+\sigma)}(L-a)^\sigma\right]\le 0,$$

and by the definition of $\mathbb{L}$, we must have $e_{\Xi_m}\to 0$ from where $\Xi_m(\ell)$ tends to $\Xi(\ell)$. ◻

Theorem 4.3. *Let $m\in\mathbb{N}$, $\mathcal{A}_m$ and $\mathcal{B}_{m,i}$, $i=1,2,...,m$ are defined in Eq. (4.1). Let*

$$\begin{aligned}
f_1(\ell,Y) &= \frac{1}{\mathcal{A}_m(\ell-a)^{1-\sigma}\sin\left(\frac{\pi}{2}r\right)}\Bigg[f(\ell,Y_1)+Y_1\left[\mathcal{B}_{m,1}\sin\left(\frac{\pi}{2}r\right)(\ell-a)^{-\sigma}-\mathcal{A}_m(\ell-a)^{1-\sigma}\cos\left(\frac{\pi}{2}r\right)\right]\\
&\quad+e^{-\cot(\frac{\pi}{2}r)\ell}\sum_{i=2}^{m}\mathcal{B}_{m,i}(\ell-a)^{1-\sigma-i}Y_i+\sin\left(\frac{\pi}{2}r\right)e^{-\cot(\frac{\pi}{2}r)(\ell-a)}\mathcal{B}_{m,1}(\ell-a)^{-\sigma}\Xi_a\Bigg],\\
f_k(\ell,Y) &= \frac{(\ell-a)^{k-1}}{\mathcal{A}_m(\ell-a)^{1-\sigma}}\times\Bigg[f(\ell,Y_1)e^{\cot(\frac{\pi}{2}r)\ell}+\mathcal{B}_{m,1}(\ell-a)^{-\sigma}\sin\left(\frac{\pi}{2}r\right)\left[Y_1e^{\cot(\frac{\pi}{2}r)\ell}-\Xi_a e_r(a)\right]\\
&\quad+\sum_{i=2}^{m}\mathcal{B}_{m,i}(\ell-a)^{1-\sigma-i}Y_i\Bigg],\quad \forall k\in\{2,...,m\}.
\end{aligned}$$

The numerical solution of the problem (3.1) is equivalent to solve the following ordinary differential equation:

$$\begin{cases}\dot{Y}(\ell)=F(\ell,Y(\ell)),\\ Y(a)=\begin{pmatrix}\Xi_a\\0\\0\\\vdots\\0\end{pmatrix}\in\mathbb{R}^{m\times1},\end{cases} \tag{4.8}$$

where

$$F(\ell,Y(\ell))=\begin{pmatrix}f_1(\ell,Y(\ell))\\f_2(\ell,Y(\ell))\\\vdots\\f_m(\ell,Y(\ell))\end{pmatrix}, \text{ and numerical solution is } \Xi=Y_1.$$

Proof. Let $Z_{r,\Xi}$ and e_r be two functions defined by:

$$Z_{r,\Xi}(s) = \cos(\frac{\pi}{2}r)\Xi(s) + \sin(\frac{\pi}{2}r)\Xi'(s),$$
$$e_r(\ell) = e^{\cot(\frac{\pi}{2}r)\ell},$$

according to the Theorem 4.1, we have

$$\begin{aligned}{}^{c}D_c^{\sigma,r}\Xi(\ell) &= \mathcal{A}_m(\ell-a)^{1-\sigma}Z_{r,\Xi}(\ell) - e_r(-\ell)\sum_{k=0}^{m}\mathcal{B}_{m,k}(\ell-a)^{1-\sigma-k}\mathcal{V}_k(\ell) + \mathcal{E}_m(\ell)\\ &\approx \mathcal{A}_m(\ell-a)^{1-\sigma}Z_{r,\Xi}(\ell) - e_r(-\ell)\sum_{k=0}^{m}\mathcal{B}_{m,k}(\ell-a)^{1-\sigma-k}\mathcal{V}_k(\ell), \end{aligned} \tag{4.9}$$

and $\mathcal{V}_i$ is defined as being the solution of the equation

$$\mathcal{V}_i'(s) = (s-a)^{i-1}e_r(s)Z_{r,\Xi}(s), \tag{4.10}$$

$$\mathcal{V}_i(a) = 0, \quad i = 1, ..., m. \tag{4.11}$$

According to problem (3.1) and approximation Eq. (4.9), we obtain

$$\mathcal{A}_m(\ell-a)^{1-\sigma}Z_{r,\Xi}(\ell) - e_r(-\ell)\sum_{k=1}^{m}\mathcal{B}_{m,k}(\ell-a)^{1-\sigma-k}\mathcal{V}_k(\ell) = f(\ell,\Xi(\ell)), \tag{4.12}$$

on the other hand Eq. (4.10), we have

$$Z_{r,\Xi}(\ell) = \frac{\mathcal{V}_i'(\ell)}{(\ell-a)^{i-1}e_r(\ell)}, \qquad i \in \{1, ..., m\}, \tag{4.13}$$

from Eqs. (4.12) and (4.13) we establish

$$\mathcal{A}_m(\ell-a)^{1-\sigma}\frac{\mathcal{V}_i'(\ell)}{(\ell-a)^{i-1}e_r(\ell)} - e_r(-\ell)\sum_{k=1}^{m}\mathcal{B}_{m,k}(\ell-a)^{1-\sigma-k}\mathcal{V}_k(\ell) = f(\ell,\Xi(\ell)),$$

so

$$\begin{aligned}\mathcal{A}_m(\ell-a)^{1-\sigma}\frac{\mathcal{V}_i'(\ell)}{(\ell-a)^{i-1}e_r(\ell)} &- e_r(-\ell)\mathcal{B}_{m,1}(\ell-a)^{-\sigma}\sin(\frac{\pi}{2}r)\left[\Xi(\ell)e_r(\ell) - \Xi_a e_r(a)\right]\\ &- e_r(-\ell)\sum_{k=2}^{m}\mathcal{B}_{m,k}(\ell-a)^{1-\sigma-k}\mathcal{V}_k(\ell) = f(\ell,\Xi(\ell)),\end{aligned}$$

then

$$\begin{aligned}\mathcal{V}_i'(\ell) = &\frac{(\ell-a)^{i-1}}{\mathcal{A}_m(\ell-a)^{1-\sigma}} \times \Big[f(\ell,\Xi(\ell))e_r(\ell) + \mathcal{B}_{m,1}(\ell-a)^{-\sigma}\sin(\frac{\pi}{2}r)\left[\Xi(\ell)e_r(\ell) - \Xi_a e_r(a)\right]\\ &+ \sum_{k=2}^{m}\mathcal{B}_{m,k}(\ell-a)^{1-\sigma-k}\mathcal{V}_k(\ell)]. \quad \forall i \in \{1, ..., m\}.\end{aligned} \tag{4.14}$$

By replace $Z_{r,\Xi}$ to Eq. (4.12), we derive the following result

$$\begin{aligned}\Xi'(\ell) = &\frac{1}{\mathcal{A}_m(\ell-a)^{1-\sigma}\sin\left(\frac{\pi}{2}r\right)}\Bigg[f(\ell,\Xi(\ell)) + \Xi(\ell)\left[\mathcal{B}_{m,1}\sin\left(\frac{\pi}{2}r\right)(\ell-a)^{-\sigma} - \mathcal{A}_m(\ell-a)^{1-\sigma}\cos\left(\frac{\pi}{2}r\right)\right]\\ &+ e_r(-\ell)\sum_{k=2}^{m}\mathcal{B}_{m,k}(\ell-a)^{1-\sigma-k}\mathcal{V}_k(\ell) + \sin\left(\frac{\pi}{2}r\right)e_r(a-\ell)\mathcal{B}_{m,1}(\ell-a)^{-\sigma}\Xi_a\Bigg],\end{aligned} \tag{4.15}$$

from Eqs. (4.14) and (4.15), we obtain the system (4.8). □

5. Numerical tests

In this section, we present two numerical illustrations aimed at validating the theoretical findings outlined in the preceding section 4.

Test 1

Let $0 < \sigma < 1$ and $r \in (0, 1]$. Let's examine Initial condition issue of Cauchy type outlined below.

$$\begin{cases} {}^{C}D_{1}^{\sigma,r}\Xi(\ell) = \frac{\sin(\frac{\pi}{2}r)^{\sigma}\Gamma(3)}{\Gamma(3-\sigma)} e^{-\cot(\frac{\pi}{2}r)\ell}(\ell-1)^{2-\sigma}, \quad \ell \in [1,2], \\ \Xi(1) = 0. \end{cases} \tag{5.1}$$

Now the existence of solution of Eq. (5.1), let $(\ell, X) \in [1,2] \times \mathbb{R}$:

$$|f(\ell, X)| \leq \quad \frac{\Gamma(3)}{\Gamma(3-\sigma)}(\ell-1)^{2-\sigma} \quad \leq p(\ell) + q(\ell)|X|.$$

So there exists $p, q \in \mathcal{C}([1,2], \mathbb{R}^{+})$ with $p(\ell) = \frac{\Gamma(3)}{\Gamma(3-\sigma)}(\ell-1)^{2-\sigma}$, $q(\ell) = 0$ such as:

$$|f(\ell, X)| \leq p(\ell) + q(\ell)|X|,$$

then from the Theorem 3.2, the problem (5.1) has at least one solution. Now for the uniqueness of solution of Eq. (5.1), let $(\ell, X, Y) \in [1,2] \times \mathbb{R} \times \mathbb{R}$

$$|f(\ell, X) - f(\ell, Y)| = 0 \leq \mathbb{L}|X - Y|, \quad \text{with} \quad \mathbb{L} = 1 > 0.$$

According to Theorem 3.3, the issue (5.1) possesses a singular solution. From the Theorem 3.3, then the problem (5.1) has a unique solution.

The system's precise solution is expressed as $\Xi(\ell) = e^{-\cot(\frac{\pi}{2}r)\ell}(\ell-1)^{2}$. Employing the Explicit Euler method in Matlab, we conduct the numerical solution. Below, you'll find two Figures 1 depicting the exact and approximate solutions across a range of m values: $m = \{2, 4, 6, 8, 16, 32\}$, within the interval $[1, 2]$. Evaluation of the maximum absolute error Em is performed using the subsequent formula:

$$E_m = \|\Xi - \Xi_m\|_{\infty},$$

and the order of convergence implies the following formula:

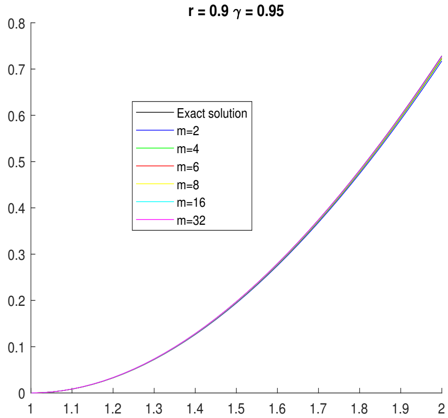
$$Order = \log_2\left(\frac{E_m}{E_{2m}}\right).$$

To assess the stability of our model described in Eq. (5.1) against variations in initial conditions, we introduced a minor yet representative perturbation to the initial condition of the Cauchy problem (5.1). This involved modifying the initial state. to $\Xi^{*}(1) = 0.0001$, thereby creating a disturbed system. We quantified the error magnitude $E_m = \|\Xi - \Xi_m\|_{\infty}$ by comparing Ξ and Ξ_m for the original (undisturbed) system. Additionally, we computed perturbed absolute error E_m^p for the perturbed system by contrasting the exact solution with its approximate counterpart Ξ_m^p. Additionally, in the final column of Table 2, we determined the absolute difference $AD = |E_m^p(\ell) - E_m(\ell)|$ between the approximate solutions of the unperturbed and perturbed systems, $\Xi_m(\ell)$ and $\Xi_m^p(\ell)$, respectively. The data within Table 2 unmistakably illustrates that the solutions from the perturbed system closely coincide with the exact solution, thereby validating the model's stability. Figure 2 provides a graphical depiction of The disparity between the accurate solution Ξ and the approximated solution Ξ_m for ℓ within the interval $[1, 2]$ associated with Eq. (5.1).

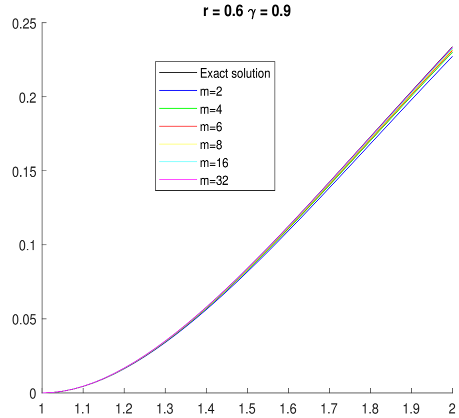
Test 2

Let $0 < \sigma < 1$ and $r \in (0, 1]$. Let's examine Initial condition issue of Cauchy type outlined below:

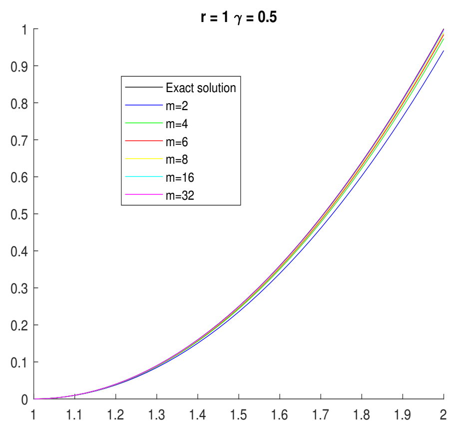
$$\begin{cases} {}^{C}D_{0}^{\sigma,r}\Xi(\ell) = \Xi(\ell) + \frac{\sin(\frac{\pi}{2}r)^{\sigma}\Gamma(3)}{\Gamma(3-\sigma)} e^{-\cot(\frac{\pi}{2}r)\ell}\ell^{2-\sigma} - e^{-\cot(\frac{\pi}{2}r)\ell}\ell^{2}, \ell \in [0,1], \\ \Xi(0) = 0. \end{cases} \tag{5.2}$$



(a) $r = 0.9;\ y = 0.95$



(b) $r = 0.6;\ y = 0.9$



(c) $r = 1;\ y = 0.5$

FIGURE 1. The approximate solution for various values of $m = \{2, 4, 6, 8, 16, 32\}$, within the interval $[1, 2]$ for Test 1.

TABLE 1. The error E_m and the convergence orders for Eq. (5.1) With σ and r varying.

m	$\sigma = 0.7, r = 0.5$ E_m	$Order$	$\sigma = 0.9, r = 0.85$ E_m	$Order$	$\sigma = 0.5, r = 1$ E_m	$Order$
2	0.0076		0.0171		0.0588	
4	0.0039	0.9563	0.0098	0.8060	0.0266	1.1439
8	0.0018	1.1058	0.0051	0.9420	0.0107	1.3015
16	0.0008	1.1563	0.0024	1.0823	0.0040	1.4043
32	0.0003	1.0299	0.0009	1.3705	0.0015	1.3821

Now the existence of solution of Eq. (5.2), lets $(\ell, X) \in [0, 1] \times \mathbb{R}$:

$$\begin{aligned} |f(\ell, X)| \leq \ & |X| + \frac{\Gamma(3)}{\Gamma(3-\sigma)}\ell^{2-\sigma} + \ell^2 \\ & \leq p(\ell) + q(\ell)|X|. \end{aligned}$$

TABLE 2. Absolute error for different values of $\ell \in [1,2]$ with $m = 6$, $r = 0.5$, and $\sigma = 0.7$ for Test 1.

ℓ	Ξ	E_m	Ξ_m^p	E_m^p	AD
1.1	0.0032	0.0001	0.0035	0.0001	0.0002
1.2	0.0118	0.0003	0.0120	0.0001	0.0002
1.3	0.0241	0.0005	0.0242	0.0004	0.0001
1.4	0.0388	0.0008	0.0389	0.0007	0.0001
1.5	0.0548	0.0011	0.0549	0.0010	0.0001
1.6	0.0714	0.0014	0.0714	0.0013	0.0001
1.7	0.0879	0.0017	0.0879	0.0016	0.0016
1.8	0.1038	0.0020	0.1039	0.0019	0.0001
1.9	0.1189	0.0023	0.1189	0.0022	0.00004
2.0	0.13280	0.0025	0.1328	0.0024	0.00004

TABLE 3. The error E_m and the convergence orders for Eq (5.2) with σ and r varying.

	$\sigma = 0.65, r = 0.45$		$\sigma = 0.8, r = 0.7$		$\sigma = 0.5, r = 1$	
m	E_m	*Order*	E_m	*Order*	E_m	*Order*
2	0.0391		0.0450		0.1181	
4	0.0207	0.9195	0.0250	0.8468	0.0566	1.0616
8	0.0097	1.0866	0.0126	0.9893	0.0238	1.2498
16	0.0044	1.1463	0.0060	1.0457	0.0094	1.3290
32	0.0024	0.8387	0.0034	0.81845	0.0038	1.2943

So there exists $p, q \in \mathcal{C}([0,1], \mathbb{R}^+)$ with $p(\ell) = \frac{\Gamma(3)}{\Gamma(3-\sigma)}\ell^{2-\sigma} + \ell^2$ and $q(\ell) = 1$ such as:

$$|f(\ell, X)| \leq p(\ell) + q(\ell)|X|,$$

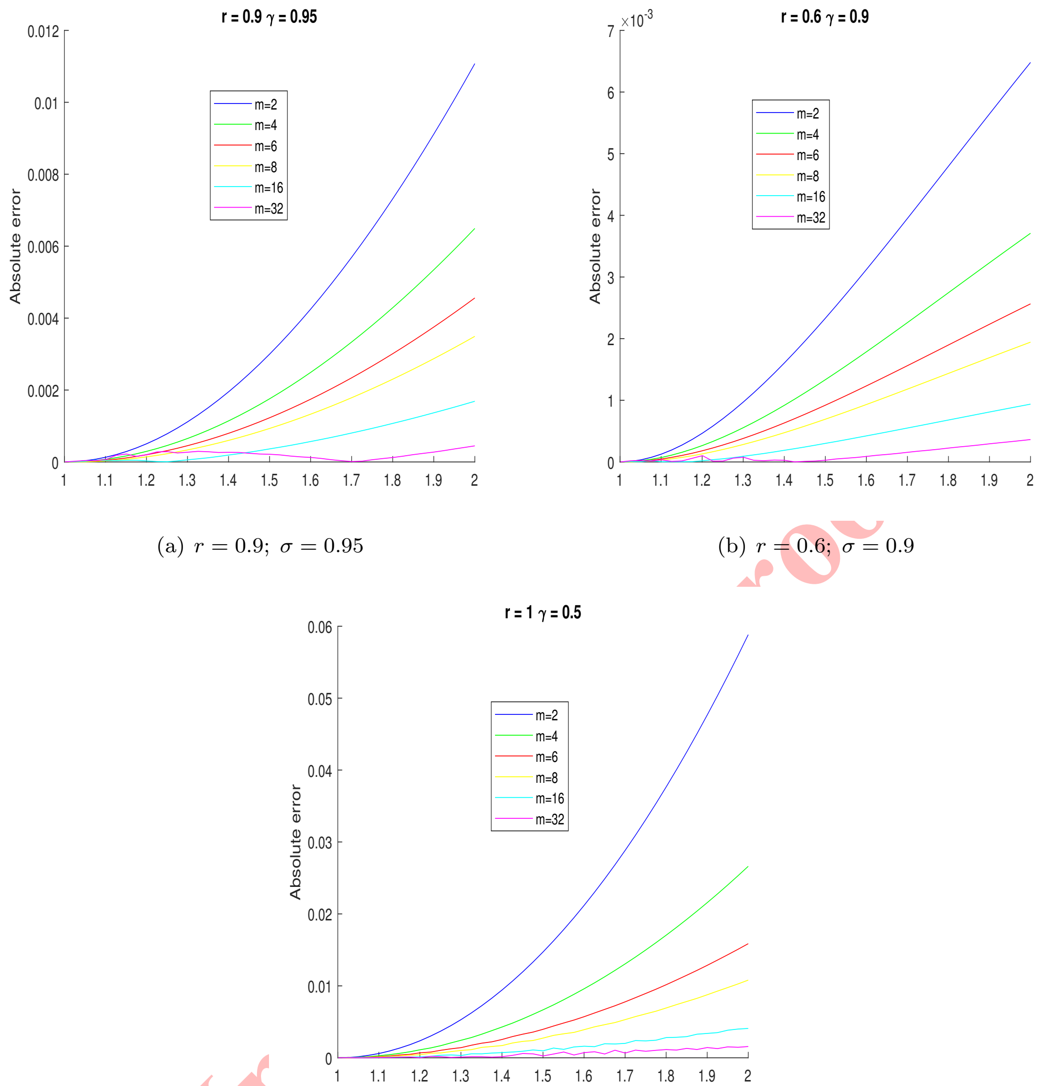
then from the Theorem 3.2, the problem (5.2) has at least one solution. Now for the uniqueness, let $(\ell, X, Y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$

$$|f(\ell, X) - f(\ell, Y)| = |X - Y| \leq L|X - Y|, \quad \text{with } \mathbb{L} = 1 > 0.$$

Based on Theorem 3.3, the problem (5.2) possesses a unique solution. The system's exact solution is given by $\Xi(\ell) = e^{-\cot(\frac{\pi}{2}r)\ell}\ell^2$. Herein, we depict two graphs (Figure 3) illustrating both the exact and approximate solutions across various values of $m = \{2, 4, 6, 8, 16, 32\}$ within the interval $[0,1]$. To gauge the stability of our model described by Eq. (5.2) against fluctuations in initial conditions, we introduced a minor yet representative perturbation to the initial condition of the Cauchy problem (5.2). By adjusting $\Xi^*(0) = 0.0001$, we induced a perturbed system. *AE* was determined by contrasting Ξ with Ξ_m for the original system, unperturbed. Additionally, we assessed E_m in the perturbed system by comparing its approximate solution Ξ_m^p with the exact solution Ξ. Furthermore, Table 4 encapsulates *AD* between the approximate solutions Ξ_m and Ξ_m^p of the unperturbed and perturbed systems, respectively. The data in Table 4 compellingly indicates that the solutions derived from the perturbed system closely approximate the exact solution, thereby affirming the model's stability. Figure 4 visually represents the absolute difference between the exact solution and the approximate solution for ℓ within the interval $[0,1]$ associated with the system (5.2) for Test 2.

6. CONCLUSION

In this work, we have introduced and investigated a novel class of fractional Cauchy problems involving the Caputo cotangent derivative. We established theoretical results on the existence and uniqueness of solutions using a tailored version of the fractional Gronwall inequality. Furthermore, a numerical scheme was developed and supported by a thorough convergence analysis. The numerical experiments conducted validated the efficiency, accuracy, and stability of the proposed method.



(a) $r = 0.9$; $\sigma = 0.95$

(b) $r = 0.6$; $\sigma = 0.9$

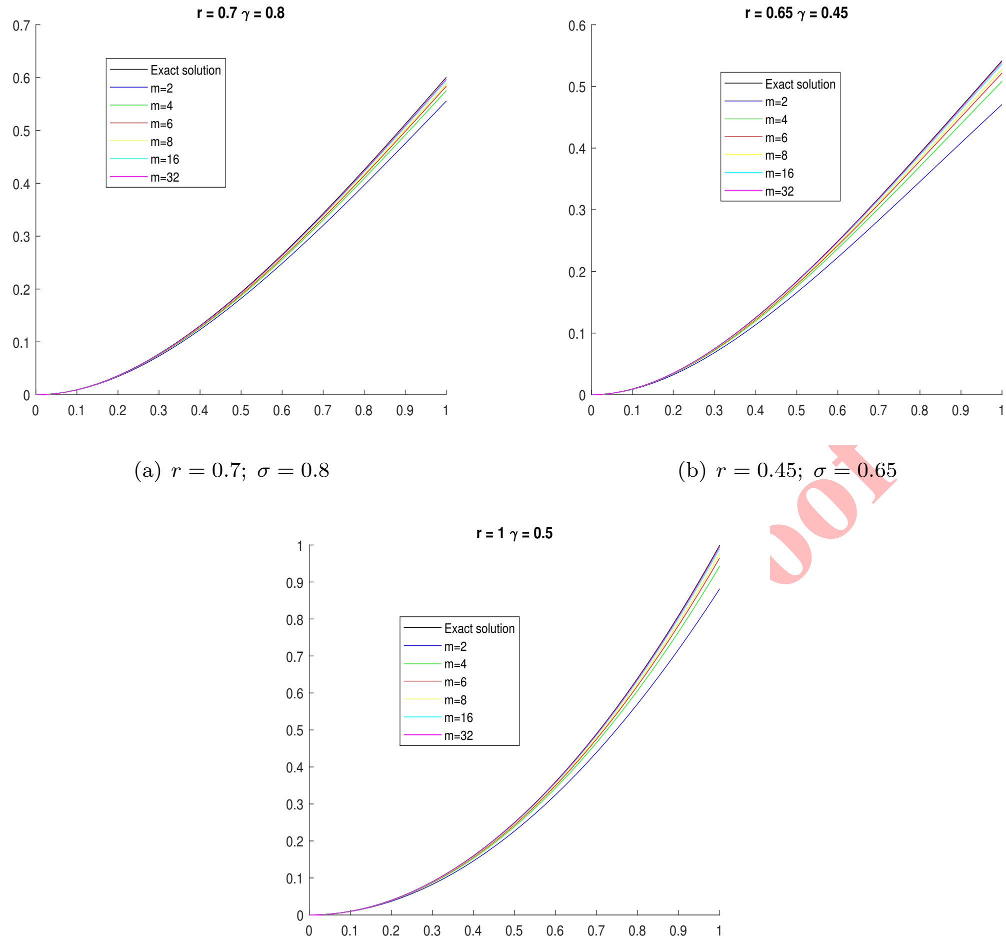
(c) $r = 1$; $\sigma = 0.5$

FIGURE 2. The absolute error for various values of $m = \{2, 4, 6, 8, 16, 32\}$, within the interval $[1, 2]$ for Test 1.

As a direction for future research, a promising extension would be to generalize the studied cotangent fractional Cauchy problem to systems of coupled fractional differential equations, particularly in multidimensional settings or with variable coefficients. Such a development would broaden the scope of applications, especially in modeling complex physical phenomena with memory effects, as encountered in viscoelastic materials, biomedical engineering, and delayed dynamical systems.

Data availability

Data sharing is not applicable.



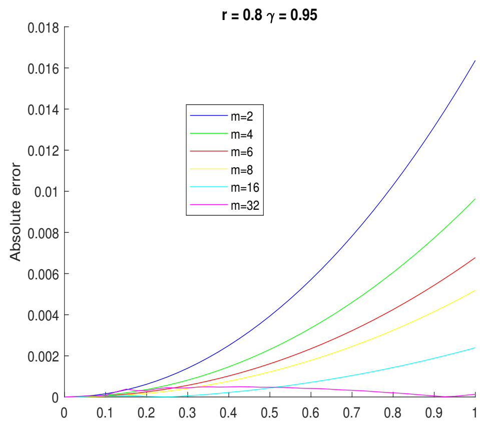
(a) $r = 0.7$; $\sigma = 0.8$ (b) $r = 0.45$; $\sigma = 0.65$

(c) $r = 1$; $\sigma = 0.5$

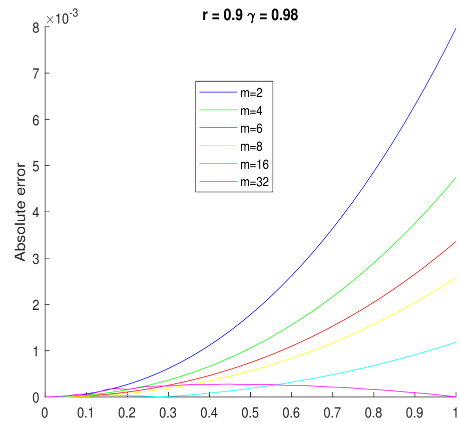
FIGURE 3. The approximate solution for various values of $m = \{2, 4, 6, 8, 16, 32\}$, within the interval $[0, 1]$ for Test 2.

TABLE 4. Absolute error for different values of $\ell \in [0, 1]$ with $m = 6$, $r = 0.7$, and $\sigma = 0.8$ for Test 2.

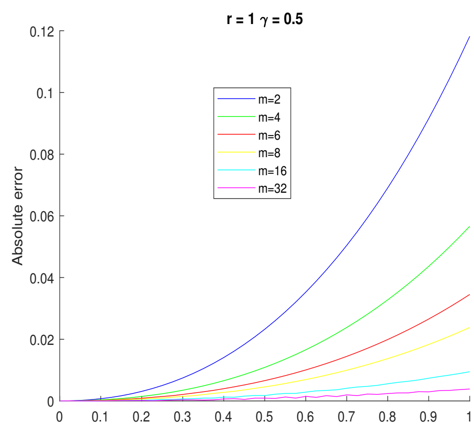
ℓ	Ξ_m	E_m	Ξ_m^p	E_m^p	AD
0.1	0.0094	0.0001	0.0098	0.0001	0.00036
0.2	0.0356	0.0007	0.03605	0.0003	0.00036
0.3	0.0760	0.0015	0.0763	0.0011	0.00037
0.4	0.1280	0.0027	0.1284	0.0023	0.00039
0.5	0.1898	0.0043	0.1902	0.0038	0.00041
0.6	0.2592	0.00615	0.2597	0.0057	0.00043
0.7	0.3348	0.0083	0.3353	0.0079	0.00045
0.8	0.4150	0.0109	0.4154	0.0104	0.00048
0.9	0.4984	0.0137	0.4989	0.0132	0.00051
1	0.5838	0.0169	0.5843	0.0163	0.00054



(a) $r = 0.8$; $\sigma = 0.95$



(b) $r = 0.9$; $\sigma = 0.98$



(c) $r = 1$; $\sigma = 0.5$

FIGURE 4. The absolute error for various values of $m = \{2, 4, 6, 8, 16, 32\}$, within the interval $[0, 1]$ for Test 2.

DECLARATIONS

Conflict of interest. The author declare that they have no conflict of interest.

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