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On theoretical and numerical analysis of Time-Fractional Fornberg-Whitham Equation

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Abstract

In this study, the Atangana-Baleanu fractional operator was employed with the Fornberg-Witham equation to investigate The computational solutions to this equation. The existence and uniqueness of the solution were proven using precise mathematical conditions that ensure their validity. Furthermore, the convergence of the approximated solutions derived through the use of the Yang-Daftardar-Jafari Method (YDJM) to the exact solutions was verified. The effectiveness of the proposed method was tested by applying it to two illustrative examples. The results demonstrated that the solutions obtained through this method exhibit high accuracy and align well with the analytical solutions under certain conditions, which highlights the efficiency of the method in handling nonlinear fractional differential equations.

Keywords. Daftardar-Jafari iteration method, Yang Transform, Fornberg–Witham equation, Atangana–Baleanu fractional derivative.

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1. Introduction

In recent decades, fractional calculus (FC) has emerged as a powerful mathematical tool for modeling complex phenomena across a wide range of disciplines, including natural sciences, engineering, fluid dynamics, and biological systems. Unlike classical calculus, FC effectively captures hereditary and memory-dependent behaviors in various materials and processes, making it particularly suitable for real-world applications [2, 3, 29, 37].

Fractional derivatives (FDs) have been applied to model systems involving viscoelastic behavior, anomalous diffusion, signal processing, and damping phenomena. Among the many fractional partial differential equations (FPDEs) studied, the Fornberg–Whitham equation (FWE) has received significant attention due to its ability to describe wave breaking and nonlinear dispersive wave propagation.

The classical FWE, introduced by Fornberg and Whitham [16, 45], admits peaked solutions (peakons), which provide valuable insight into wave height limitations and wave-breaking phenomena. Its fractional counterpart, the fractional Fornberg–Whitham equation (FFWE), introduces a time-fractional derivative that enables more accurate modeling of memory effects in physical systems:

$$D_t^{\vartheta} u - u_{\varkappa\varkappa t} + u_{\varkappa} + uu_{\varkappa} = 3u_{\varkappa}u_{\varkappa\varkappa} + uu_{\varkappa\varkappa\varkappa}, \tag{1.1}$$

where $0 < \vartheta \leq 1$.

Numerous analytical and semi-analytical methods have been proposed to solve the FWE and its fractional variants, including the Laplace decomposition method [27], variational iteration method [29], and other iterative and transformation techniques [1, 4, 5, 9, 11–13, 18–25, 28, 30, 32–36, 38–41, 43, 44]. Additionally, recent studies have contributed further to the development of numerical and analytical approaches for fractional models, offering new perspectives and results in the field [6, 7].

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In this paper, we focus on solving the time-fractional FWE using the Yang–Daftardar–Jafari Method (YDJM) in the sense of the Atangana–Baleanu fractional derivative. The Atangana–Baleanu operator [8, 42] was chosen due to its non-singular and non-local kernel, characterized by the Mittag–Leffler function. Unlike classical FDs such as Caputo and Riemann–Liouville [15], this operator provides a more realistic modeling framework for systems with smooth memory effects and avoids the numerical instabilities associated with singular kernels.

The main objective of this study is to develop an efficient and accurate semi-analytical method for obtaining approximate solutions to the FFWE. The proposed approach combines the Yang transform [46, 47] with the Daftardar–Jafari iterative method to yield series-form solutions that converge rapidly with minimal computational cost. This work not only provides insights into the qualitative behavior of the FFWE but also offers a versatile framework for tackling other nonlinear FPDEs in applied sciences. Moreover, the analytical approximate solutions derived in this study allow for explicit expressions that facilitate deeper analysis, reduce computational efforts, and enhance understanding of the influence of fractional parameters on system dynamics, making them particularly valuable in engineering and physical applications.

2. Preliminaries

Definition 2.1. For a function $u(\varkappa)$ that is sufficiently smooth, the Caputo fractional derivative of order $k-1<\vartheta\leq k$ is specified by, [15]:

$$ {}^{c}D_{\varkappa}^{\vartheta}u(\varkappa)=\begin{cases}\frac{1}{\Gamma(k-\vartheta)}\int_0^{\varkappa}(\varkappa-t)^{k-\vartheta-1}u^{(k)}(t)\,dt, & k-1<\vartheta\leq k\in\mathbb{N},\\ \frac{d^k}{d\varkappa^k}u(\varkappa), & \vartheta=k\in\mathbb{N}.\end{cases} \tag{2.1}$$

Remark 2.2. From **Definition 2.1**, The resulting outcome is as follows:

$$ {}^{c}D_{t}^{\vartheta}t^{\mathfrak{B}}=\begin{cases}\frac{\Gamma(\mathfrak{B}+1)}{\Gamma(\mathfrak{B}-\vartheta+1)}t^{\mathfrak{B}-\vartheta}, & k-1<\vartheta\leq k,\ \mathfrak{B}>k-1,\ \mathfrak{B}\in\mathbb{R},\\ 0, & k-1<\vartheta\leq k,\ \mathfrak{B}>k,\ \mathfrak{B}\in\mathbb{N}.\end{cases} \tag{2.2}$$

Definition 2.3. The Atangana–Baleanu fractional derivative (ABFD) is expressed as [8, 42]:

$$ {}^{AB}D_t^{\vartheta}u(t)=\frac{M(\vartheta)}{1-\vartheta}\int_a^t E_{\vartheta}\left(-\frac{\vartheta(t-\varkappa)^{\vartheta}}{1-\vartheta}\right)u'(\varkappa)\,d\varkappa, \tag{2.3}$$

where $0<\vartheta<1$ and $M(\vartheta)$ is a scaling function and $M(0)=1$, $M(1)=1$.

Definition 2.4. The Atangana–Baleanu fractional integral (ABFI) of order ϑ is expressed as follows [8]:

$$ {}^{AB}I_t^{\vartheta}u(t)=\frac{1-\vartheta}{M(\vartheta)}u(t)+\frac{\vartheta}{M(\vartheta)\Gamma(\vartheta)}\int_a^t(t-\varkappa)^{\vartheta-1}u(\varkappa)\,d\varkappa, \tag{2.4}$$

where $0<\vartheta<1$ and $M(\vartheta)$ is a scaling function.

Definition 2.5. The Yang transform (YT) is expressed as [26, 31, 46]:

$$ Y\{u(t)\}=\int_0^{\infty}e^{-t/v}u(t)\,dt,\quad t>0, \tag{2.5}$$

with v representing the transform variable.

Few properties:

$$ Y\{1\}=v, \tag{2.6}$$

$$ Y\{t\}=v^2, \tag{2.7}$$

$$ Y\{t^n\}=v^{n+1}n!, \tag{2.8}$$

$$ Y\{t^{\vartheta}\}=v^{\vartheta+1}\Gamma(\vartheta+1),\quad \vartheta\in\mathbb{R}. \tag{2.9}$$

Theorem 2.6. *The YT of fractional order derivative is formulated by:*

$$Y\left\{{}^{c}D_t^{\vartheta}u(\varkappa,t)\right\} = \frac{Y\{u(t)\}}{v^{\vartheta}} - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{v^{\vartheta-k-1}}, \quad n-1<\vartheta\leq n, \tag{2.10}$$

$$Y\left\{{}^{AB}D_t^{\vartheta}u(t)\right\} = \frac{M(\vartheta)}{1-\vartheta+\vartheta v^{\vartheta}}\left(Y\{u(t)\} - vu(0)\right), \quad 0<\vartheta\leq 1. \tag{2.11}$$

Proof. The proof of part (a) is as in [31]. To prove part (b), we take the transformation into Equation (2.3) and then utilize the convolution property to obtain the desired result after simplification. Thus, the proof is completed. □

Definition 2.7. The Mittag-Leffler function with two parameters is outlined as [17, 27]:

$$E_{\vartheta,p}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\vartheta+p)}, \quad \vartheta, p, z\in\mathbb{C},\ \mathrm{Re}(\vartheta)>0,\ \mathrm{Re}(p)>0. \tag{2.12}$$

Remark 2.8. From **Definition 2.12**, the following results are obtained:

$$E_{2,1}(\varkappa^2) = \cosh(\varkappa), \tag{2.13}$$

$$E_{2,2}(\varkappa^2) = \frac{\sinh(\varkappa)}{\varkappa}, \tag{2.14}$$

$$E_{2,3}(\varkappa^2) = \frac{1}{\varkappa^2}\left[-1+\cosh(\varkappa)\right]. \tag{2.15}$$

3. Methodology of YDJM

The Yang-Daftardar-Jafari Method (YDJM) is a semi-analytical technique that combines the Yang integral transform with the iterative Daftardar–Jafari method (DJM). This hybrid approach leverages the simplification power of the Yang transform to reduce the complexity of fractional differential equations, while the DJM iteratively refines the solution without requiring linearization or discretization. YDJM offers rapid convergence and high accuracy, making it suitable for solving a wide range of nonlinear fractional differential equations.

Consider the fractional Fornberg–Whitham equation:

$${}^{AB}D_t^{\vartheta}u - u_{\varkappa\varkappa t} + u_{\varkappa} + uu_{\varkappa} = 3u_{\varkappa}u_{\varkappa\varkappa} + uu_{\varkappa\varkappa\varkappa}, \quad 0<\vartheta\leq 1, \tag{3.1}$$

subject to

$$u(\varkappa,0) = g(\varkappa). \tag{3.2}$$

By using the Yang transform on Equation (3.1), we derive:

$$\frac{1}{1-\vartheta+\vartheta v^{\vartheta}}\left[Y\{u\} - vu(\varkappa,0)\right] = Y\{u_{\varkappa\varkappa t} - u_{\varkappa} - uu_{\varkappa} + 3u_{\varkappa}u_{\varkappa\varkappa} + uu_{\varkappa\varkappa\varkappa}\}. \tag{3.3}$$

Rewriting, we have:

$$Y\{u\} = vu(\varkappa,0) + (1-\vartheta+\vartheta v^{\vartheta})Y\{u_{\varkappa\varkappa t} - u_{\varkappa} - uu_{\varkappa} + 3u_{\varkappa}u_{\varkappa\varkappa} + uu_{\varkappa\varkappa\varkappa}\}. \tag{3.4}$$

By employing the inverse Yang transform, the following result is obtained:

$$u(\varkappa,t) = u(\varkappa,0) + Y^{-1}\left[(1-\vartheta+\vartheta v^{\vartheta})Y\{u_{\varkappa\varkappa t} - u_{\varkappa} - uu_{\varkappa} + 3u_{\varkappa}u_{\varkappa\varkappa} + uu_{\varkappa\varkappa\varkappa}\}\right]. \tag{3.5}$$

Assuming:

$$u(\varkappa,t) = \sum_{i=0}^{\infty}u_i, \tag{3.6}$$

where the nonlinear terms are decomposed as follows:

$$uu_{\varkappa} = u_0u_{0\varkappa} + \sum_{i=1}^{\infty}\left[\left(\sum_{k=0}^{i}u_k\right)\left(\sum_{k=0}^{i}u_k\right)_{\varkappa} - \left(\sum_{k=0}^{i-1}u_k\right)\left(\sum_{k=0}^{i-1}u_k\right)_{\varkappa}\right]$$

$$= \sum_{i=0}^{\infty} G_i, \tag{3.7}$$

where $G_0 = u_0 u_{0\varkappa}$.

Similarly, we have:

$$u_\varkappa u_{\varkappa\varkappa} = u_{0\varkappa} u_{0\varkappa\varkappa} + \sum_{i=1}^{\infty} \left[\left(\sum_{k=0}^{i} u_k \right)_\varkappa \left(\sum_{k=0}^{i} u_k \right)_{\varkappa\varkappa} - \left(\sum_{k=0}^{i-1} u_k \right)_\varkappa \left(\sum_{k=0}^{i-1} u_k \right)_{\varkappa\varkappa} \right]$$
$$= \sum_{i=0}^{\infty} H_i, \tag{3.8}$$

where $H_0 = u_{0\varkappa} u_{0\varkappa\varkappa}$.

$$u u_{\varkappa\varkappa\varkappa} = u_0 u_{0\varkappa\varkappa\varkappa} + \sum_{i=1}^{\infty} \left[\left(\sum_{k=0}^{i} u_k \right) \left(\sum_{k=0}^{i} u_k \right)_{\varkappa\varkappa\varkappa} - \left(\sum_{k=0}^{i-1} u_k \right) \left(\sum_{k=0}^{i-1} u_k \right)_{\varkappa\varkappa\varkappa} \right]$$
$$= \sum_{i=0}^{\infty} K_i, \tag{3.9}$$

where $K_0 = u_0 u_{0\varkappa\varkappa\varkappa}$.

Substituting Equations (3.6), (3.7), (3.8), and (3.9) into Equation (3.5) yields the following result:

$$\sum_{i=0}^{\infty} u_i = u(\varkappa, 0) + Y^{-1} \left[(1 - \vartheta + \vartheta v^{\vartheta}) Y \left\{ \left(\sum_{k=0}^{\infty} u_k \right)_{\varkappa\varkappa t} - \left(\sum_{k=0}^{\infty} u_k \right)_\varkappa - \sum_{i=0}^{\infty} G_i + 3 \sum_{i=0}^{\infty} H_i + \sum_{i=0}^{\infty} K_i \right\} \right]. \tag{3.10}$$

The recurrence relation is given by:

$$u_0 = u(\varkappa, 0),$$
$$u_1 = Y^{-1} \left[(1 - \vartheta + \vartheta v^{\vartheta}) Y \{ u_{0\varkappa\varkappa t} - u_{0\varkappa} - G_0 + 3H_0 + K_0 \} \right],$$
$$u_2 = Y^{-1} \left[(1 - \vartheta + \vartheta v^{\vartheta}) Y \{ u_{1\varkappa\varkappa t} - u_{1\varkappa} - G_1 + 3H_1 + K_1 \} \right],$$
$$u_{i+1} = Y^{-1} \left[(1 - \vartheta + \vartheta v^{\vartheta}) Y \{ u_{i\varkappa\varkappa t} - u_{i\varkappa} - G_i + 3H_i + K_i \} \right]. \tag{3.11}$$

The solution can then be expressed in series form as:

$$u(\varkappa, t) = u_0 + u_1 + u_2 + \cdots. \tag{3.12}$$

4. Convergence

Theorem 4.1 (Banach Fixed Point Theorem). *Let κ be a Banach space and $\mathfrak{T} : \kappa \to \kappa$ be a nonlinear mapping. Assume that the following condition holds:*

$$\| \mathfrak{T}(u) - \mathfrak{T}(\omega) \| \leq \epsilon \| u - \omega \|, \quad u, \omega \in \kappa, \ 0 < \epsilon < 1. \tag{4.1}$$

It is asserted that $\mathfrak{T}$ possesses a fixed point, and the sequence produced by the YDJM method is defined as $u_{n+1} = \mathfrak{T}(u_n)$, beginning with an arbitrary initial value $u_0 \in \kappa$. Additionally, the subsequent inequality is satisfied:

$$\| u_r - u_t \| \leq \| u_1 - u_0 \| \sum_{k=t-1}^{r-2} \epsilon^k. \tag{4.2}$$

This theorem acts as a cornerstone for the subsequent analysis, which is elucidated using the Banach fixed point theorem.

Theorem 4.2. *Let $u(\varkappa, t) \in H$ and $\vartheta \in (0, 1)$, where H denotes a Hilbert space, and assume $u(\varkappa, t)$ is the exact solution to Equation (3.1). The computed results $\sum_{r=0}^{\infty} u_r$ converge to $u(\varkappa, t)$ if $\| u_r \| \leq \| u_{r-1} \|$.*

Proof. Let $\sum_{r=0}^{\infty} u_r$ and the sequence defined as:

$$\begin{aligned}
&\mathfrak{T}_0 = u_0,\\
&\mathfrak{T}_1 = u_0 + u_1,\\
&\mathfrak{T}_2 = u_0 + u_1 + u_2,\\
&\mathfrak{T}_3 = u_0 + u_1 + u_2 + u_3, \quad \ldots,\\
&\mathfrak{T}_r = u_0 + u_1 + \cdots + u_r.
\end{aligned}$$

We aim to show that the sequence $\{\mathfrak{T}_r\}_{r=0}^{\infty}$ forms a Cauchy sequence under the given conditions.
Additionally, consider:

$$\|\mathfrak{T}_r - \mathfrak{T}_{r+1}\| = \|u_{r+1}\| \leq \epsilon \|u_r\| \leq \epsilon^2 \|u_{r-1}\| \leq \epsilon^3 \|u_{r-2}\| \leq \cdots \leq \epsilon^{r+1} \|u_0\|. \tag{4.3}$$

Now, for $r, n \in \mathbb{N}$ with $r > n$, we have:

$$\begin{aligned}
\|\mathfrak{T}_r - \mathfrak{T}_n\| &= \|\mathfrak{T}_r - \mathfrak{T}_{r-1} + \mathfrak{T}_{r-1} - \mathfrak{T}_{r-2} + \cdots + \mathfrak{T}_{n+1} - \mathfrak{T}_n\| && (4.4)\\
&\leq \sum_{k=n+1}^{r} \|\mathfrak{T}_k - \mathfrak{T}_{k-1}\|\\
&\leq \sum_{k=n+1}^{r} \epsilon^k \|u_0\|\\
&= \|u_0\| \left(\epsilon^r + \epsilon^{r-1} + \cdots + \epsilon^{n+1}\right) && (4.5)\\
&= \|u_0\| \frac{\epsilon^{n+1}(1-\epsilon^{r-n})}{1-\epsilon}.
\end{aligned}$$

Since $u_0(\varkappa, t)$ and $0 < \epsilon < 1$ are bounded, and as $r \to \infty$, we get $\frac{\epsilon^{n+1}(1-\epsilon^{r-n})}{1-\epsilon} \to 0$.
Hence, $\{\mathfrak{T}_r(\varkappa, t)\}_{r=0}^{\infty}$ forms a Cauchy sequence in H and converges to:

$$\lim_{r\to\infty} u_r(\varkappa, t) = u(\varkappa, t), \quad \text{for some } u(\varkappa, t) \in H.$$

□

Theorem 4.3. *Suppose $u(\varkappa, t)$ represents the obtained series solution and $\sum_{k=0}^{r} u_k(\varkappa, t)$ is finite. For $\epsilon > 0$ and $\|u_k\| \geq \|u_{k+1}\|$, the maximum absolute error is given by:*

$$\|\mathfrak{T}_r - \mathfrak{T}_n\| < \frac{\epsilon^{n+1}}{1-\epsilon}\|u_0\|. \tag{4.6}$$

Proof. Assume $\sum_{k=0}^{r} u_k(\varkappa, t)$ is bounded such that $\sum_{k=0}^{r} \|u_k\| < \infty$. Then:

$$\begin{aligned}
\|\mathfrak{T}_r - \mathfrak{T}_n\| &= \left\| \sum_{k=n+1}^{r} u_k \right\| && (4.7)\\
&\leq \sum_{k=n+1}^{r} \|u_k\| && (4.8)\\
&\leq \sum_{k=n+1}^{r} \epsilon^k \|u_0\| && (4.9)\\
&\leq \epsilon^{n+1} \left(1 + \epsilon + \epsilon^2 + \cdots\right) \|u_0\| && (4.10)\\
&= \frac{\epsilon^{n+1}}{1-\epsilon}\|u_0\|. && (4.11)
\end{aligned}$$

Thus, the error is bounded by:

$$\|\mathfrak{T}_r(\varkappa,t)-\mathfrak{T}_n(\varkappa,t)\| = A_R\|u_0(\varkappa,t)\|.$$

□

Remark 4.4. The component A_R denotes the maximum truncation error of $u(\varkappa,t)$.

5. Uniqueness

Let the analytical solution of the fractional Fornberg–Whitham equation (FWE) obtained using the Yang-Dafterdar-Jafari Method (YDJM) be unique whenever $0<\gamma<1$. Specifically, consider the equation:

$$^{AB}D_t^{\vartheta}u = \mathfrak{L}(u)+\mathfrak{N}(u), \quad 0<\vartheta\le 1, \tag{5.1}$$

where $\mathfrak{L}(u)=u_{xxt}-u_x$ and $\mathfrak{N}(u)=3u_xu_{xx}+uu_{xxx}-uu_x$ represent the linear and nonlinear operators, respectively. Assuming that the function u is bounded and its derivatives are continuous, it follows that these operators satisfy the Lipschitz condition. For further details, refer to [10, 14].

Proof. Given the solution to the equation obtained using the YDJM:

$$^{AB}D_t^{\vartheta}u = \mathfrak{L}(u)+\mathfrak{N}(u), \quad 0<\vartheta\le 1, \tag{5.2}$$

and noting that $\mathfrak{L}$ and $\mathfrak{N}$ satisfy the Lipschitz conditions, we apply the Yang Transform (YT) to obtain:

$$\frac{1}{1-\vartheta+\vartheta v^{\vartheta}}\left[Y\{u\}-vu(\varkappa,0)\right]=Y\{\mathfrak{L}(u)+\mathfrak{N}(u)\}. \tag{5.3}$$

Rewriting this, we have:

$$Y\{u(\varkappa,t)\}=vu(\varkappa,0)+(1-\vartheta+\vartheta v^{\vartheta})Y\{\mathfrak{L}(u)+\mathfrak{N}(u)\}. \tag{5.4}$$

By applying the inverse Yang Transform:

$$u(\varkappa,t)=u(\varkappa,0)+Y^{-1}\left[(1-\vartheta+\vartheta v^{\vartheta})Y\{\mathfrak{L}(u)+\mathfrak{N}(u)\}\right]. \tag{5.5}$$

Suppose there are two potential solutions, $u(\varkappa,t)$ and $v(\varkappa,t)$, where $u(\varkappa,0)=v(\varkappa,0)$. Considering these functions, we obtain:

$$|u-v|=\left|u(\varkappa,0)-v(\varkappa,0)+Y^{-1}\left[(1-\vartheta+\vartheta v^{\vartheta})Y\{\mathfrak{L}(u)+\mathfrak{N}(u)-\mathfrak{L}(v)-\mathfrak{N}(v)\}\right]\right|. \tag{5.6}$$

Applying the triangle inequality, this becomes:

$$|u-v|\le|u(\varkappa,0)-v(\varkappa,0)|+\left|Y^{-1}\left[(1-\vartheta+\vartheta v^{\vartheta})Y\{\mathfrak{L}(u)+\mathfrak{N}(u)-\mathfrak{L}(v)-\mathfrak{N}(v)\}\right]\right|. \tag{5.7}$$

Simplifying, we find:

$$\begin{aligned}|u-v|&\le(1-\vartheta)\left|\mathfrak{L}(u)+\mathfrak{N}(u)-\mathfrak{L}(v)-\mathfrak{N}(v)\right|\\&\quad+\int_0^t\left|\mathfrak{L}(u)+\mathfrak{N}(u)-\mathfrak{L}(v)-\mathfrak{N}(v)\right|\left|\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}\right|d\tau.\end{aligned} \tag{5.8}$$

Since $\mathfrak{L}$ and $\mathfrak{N}$ satisfy the Lipschitz conditions, we know that $\mathfrak{L}$ is bounded such that $|\mathfrak{L}(u)-\mathfrak{L}(v)|\le\mu|u-v|$, where μ is a constant. Similarly, $\mathfrak{N}$ satisfies $|\mathfrak{N}(u)-\mathfrak{N}(v)|\le\epsilon|u-v|$ for some $\epsilon>0$. Substituting these bounds, we rewrite the inequality as:

$$|u-v|\le\int_0^t|u-v|(\epsilon+\mu)\left|\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}\right|d\tau. \tag{5.9}$$

We denote $\mathscr{M}=\max\left|\frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)}\right|$ over the interval $[0,t]$. This simplifies the inequality to:

$$|u-v|\le\int_0^t|u-v|(\epsilon+\mu)\mathscr{M}\,d\tau. \tag{5.10}$$

Letting $\gamma = (\epsilon + \mu)\mathscr{M}$, we obtain:

$$|u - v| \leq \int_0^t |u - v|\gamma \, d\tau. \tag{5.11}$$

Using Grönwall's inequality, we conclude that $u = v$ when $\gamma < 1$. Thus, the solution is unique for $0 < \gamma < 1$. □

Before introduce illustrative example, we have the following remakes

Remark 5.1. If $u_k = \sum_{\mathfrak{r}=0}^{\mathfrak{n}} a_{\mathfrak{r},k} e^{\varkappa/2}$, then $3H_i + K_i - G_i = 0$ for all $i \geq 1$.

Proof. We begin by expanding $3H_i + K_i - G_i$ as follows:

$$3H_i + K_i - G_i = \left(\sum_{k=0}^{i} \sum_{\mathfrak{r}=0}^{\mathfrak{n}} a_{\mathfrak{r},k} e^{\varkappa/2} \right)^2 - \left(\sum_{k=0}^{i-1} \sum_{\mathfrak{r}=0}^{\mathfrak{n}} a_{\mathfrak{r},k} e^{\varkappa/2} \right)^2 \left(\frac{1}{8} + \frac{3}{8} - \frac{1}{2} \right) = 0.$$

□

Remark 5.2. If $u_k = a_k + \sum_{\mathfrak{r}=0}^{\mathfrak{n}} a_{\mathfrak{r},k} \cosh(\varkappa/2) + \sum_{\mathfrak{r}=0}^{\mathfrak{n}} b_{\mathfrak{r},k} \sinh(\varkappa/2)$, then:

$$\begin{aligned} 3H_i + K_i - G_i = \frac{3}{8} &\left[\sum_{k=0}^{i-1} a_k \left\{ \sum_{k=0}^{i-1} \left(\sum_{\mathfrak{r}=0}^{\mathfrak{n}} a_{\mathfrak{r},k} \sinh(\varkappa/2) + \sum_{r=0}^{n} b_{\mathfrak{r},k} \cosh(\varkappa/2) \right) \right\} \right] \\ - \frac{3}{8} &\left[\sum_{k=0}^{i} a_k \left\{ \sum_{k=0}^{i} \left(\sum_{\mathfrak{r}=0}^{\mathfrak{n}} a_{\mathfrak{r},k} \sinh(\varkappa/2) + \sum_{\mathfrak{r}=0}^{\mathfrak{n}} b_{\mathfrak{r},k} \cosh(\varkappa/2) \right) \right\} \right]. \end{aligned}$$

6. Illustrative Examples

In this section, we will present two examples to illustrate the technique discussed above and its effectiveness, along with providing illustrative plots and tables for the absolute error.

Example 6.1. Consider the fractional FWE:

$$^{AB}D_t^{\vartheta} u - u_{\varkappa\varkappa t} + u_{\varkappa} + uu_{\varkappa} = 3u_{\varkappa}u_{\varkappa\varkappa} + uu_{\varkappa\varkappa\varkappa}, \quad 0 < \vartheta \leq 1, \tag{6.1}$$

with the initial condition:

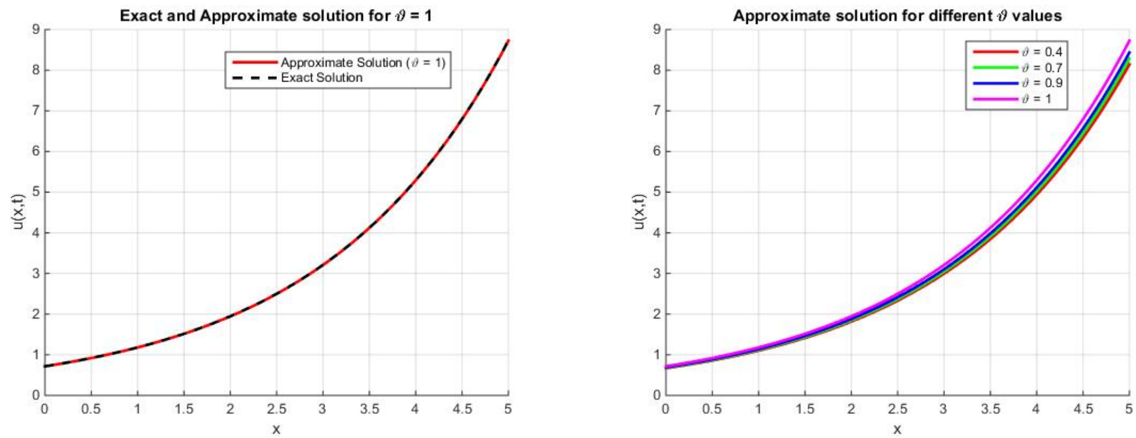
$$u(\varkappa, 0) = e^{\varkappa/2}. \tag{6.2}$$

Based on the above method derivation, Equation (3.11) provides a basis for determining iterations, along with the insights from **Remark 5.1** The resulting solution is as follows:

$$\begin{aligned} u_0 &= e^{\varkappa/2}, \\ u_1 &= -\frac{1}{2} e^{\varkappa/2} \left[1 - \vartheta + \frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} \right], \\ u_2 &= -\left[\frac{(1-\vartheta)\vartheta t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)} \right] \frac{1}{8} e^{\varkappa/2} + \left[(1-\vartheta)^2 + 2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)} \right] \frac{1}{4} e^{\varkappa/2}, \\ u_3 &= -\left[\frac{(1-\vartheta)^2 \vartheta t^{\vartheta-2}}{\Gamma(\vartheta-1)} + \frac{2(1-\vartheta)\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)} + \frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)} \right] \frac{1}{32} e^{\varkappa/2} \\ &\quad + \left[\frac{3(1-\vartheta)^2 \vartheta t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{5(1-\vartheta)\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)} + \frac{2\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)} \right] \frac{1}{16} e^{\varkappa/2} \\ &\quad - \left[(1-\vartheta)^3 + \frac{3(1-\vartheta)^2 \vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{3(1-\vartheta)\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)} \right] \frac{1}{8} e^{\varkappa/2}. \end{aligned}$$

The approximate solution is:

$$u(\varkappa, t) = u_0 + u_1 + u_2 + u_3 + \cdots$$



(A) (B)

FIGURE 1. In **Example 6.1**, plots (A) and (B) illustrate that the curve increasingly converges toward the exact solution as ϑ approaches 1. Specifically, at $\vartheta = 0.9$, the curve nearly overlaps with that of $\vartheta = 1$.

$$
\begin{aligned}
&= e^{\varkappa/2} - \frac{1}{2}e^{\varkappa/2}\left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right] \\
&- \frac{1}{8}e^{\varkappa/2}\left[\frac{(1-\vartheta)\vartheta t^{\vartheta-1}}{\Gamma(\vartheta)}+\frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right] \\
&+ \frac{1}{4}e^{\varkappa/2}\left[(1-\vartheta)^2+2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right] \\
&- \frac{1}{32}e^{\varkappa/2}\left[\frac{(1-\vartheta)^2\vartheta t^{\vartheta-2}}{\Gamma(\vartheta-1)}+\frac{2(1-\vartheta)\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)}+\frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right] \\
&+ \frac{1}{16}e^{\varkappa/2}\left[\frac{3(1-\vartheta)^2\vartheta t^{\vartheta-1}}{\Gamma(\vartheta)}+\frac{5(1-\vartheta)\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}+\frac{2\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right] \\
&- \frac{1}{8}e^{\varkappa/2}\left[(1-\vartheta)^3+\frac{3(1-\vartheta)^2\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{3(1-\vartheta)\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)}+\frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]+\cdots
\end{aligned}
\tag{6.3}
$$

For $\vartheta = 1$, the exact solution is:

$$u(\varkappa, t) = e^{\varkappa/2-\frac{2t}{3}}. \tag{6.4}$$

Example 6.2. Consider the fractional Fornberg–Whitham equation:

$$^{AB}D_t^{\vartheta}u - u_{\varkappa\varkappa t} + u_{\varkappa} + uu_{\varkappa} = 3u_{\varkappa}u_{\varkappa\varkappa} + uu_{\varkappa\varkappa\varkappa}, \quad 0 < \vartheta \le 1, \tag{6.5}$$

with the initial condition:

$$u(\varkappa, 0) = \cosh^2(\varkappa/4). \tag{6.6}$$

Using Equation (3.11) and **Remark 5.2**, the iterations are obtained as follows:

$$
\begin{aligned}
u_0 &= \cosh^2(\varkappa/4) = \frac{1}{2}+\frac{1}{2}\cosh(\varkappa/2), \\
u_1 &= -\left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right]\frac{11}{32}\sinh(\varkappa/2),
\end{aligned}
$$

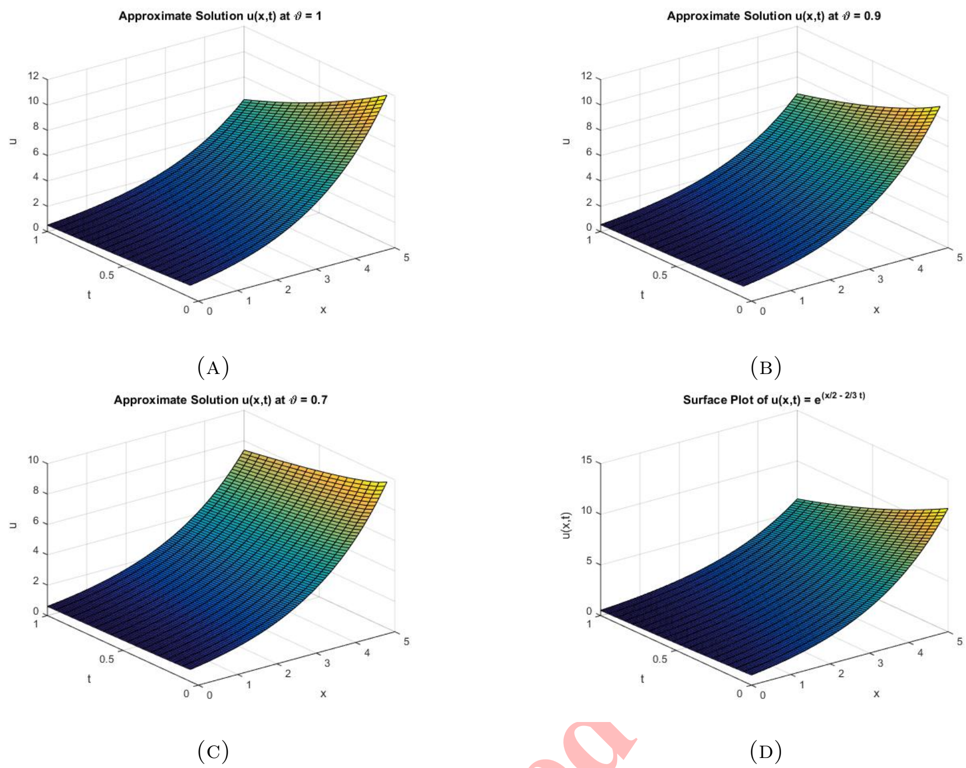
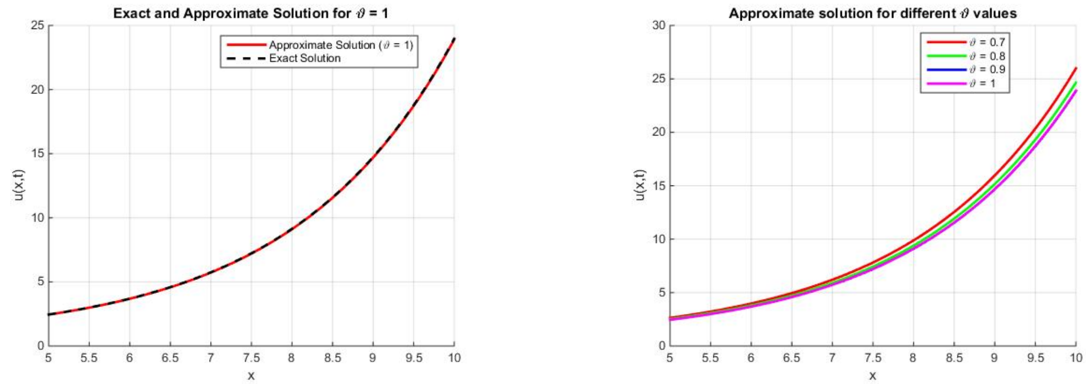


FIGURE 2. In **Example 6.1**, plots (A), (B), and (C) present surface visualizations that emphasize the strong agreement between the numerical and exact solutions, as depicted in plot (D).

$$
\begin{aligned}
u_2 &= \frac{121}{512}\cosh(\varkappa/2)\left[(1-\vartheta)^2 + 2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right] \\
&\quad - \frac{11}{128}\sinh(\varkappa/2)\left[\frac{\vartheta(1-\vartheta)t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right], \\
u_3 &= -\frac{1331}{8192}\sinh(\varkappa/2)\left[(1-\vartheta)^3 + \frac{3(1-\vartheta)^2\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{3(1-\vartheta)\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right] \\
&\quad + \frac{121}{2048}\cosh(\varkappa/2)\left[\frac{3(1-\vartheta)^2\vartheta t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{5(1-\vartheta)\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)} + \frac{2\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right] \\
&\quad - \frac{11}{512}\sinh(\varkappa/2)\left[\frac{(1-\vartheta)^2\vartheta t^{\vartheta-2}}{\Gamma(\vartheta-1)} + \frac{2(1-\vartheta)\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)} + \frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right].
\end{aligned}
$$

The approximate solution is:

$$
\begin{aligned}
u(\varkappa,t) &= u_0 + u_1 + u_2 + u_3 + \cdots \\
&= \frac{1}{2} + \frac{1}{2}\cosh\left(\frac{\varkappa}{2}\right) - \frac{11}{32}\sinh\left(\frac{\varkappa}{2}\right)\left[1-\vartheta + \frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right] \\
&\quad + \frac{121}{512}\cosh\left(\frac{\varkappa}{2}\right)\left[(1-\vartheta)^2 + 2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right] \\
&\quad - \frac{11}{128}\sinh\left(\frac{\varkappa}{2}\right)\left[\frac{\vartheta(1-\vartheta)t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right]
\end{aligned}
$$



(A) (B)

FIGURE 3. In **Example 6.2**, plots (A) and (B) demonstrate that the curve progressively converges toward the exact solution as ϑ approaches 1. At $\vartheta = 0.9$, the curve aligns almost perfectly with the one corresponding to $\vartheta = 1$.

TABLE 1. A table showing the absolute error for **Example 6.1**, where the Atangana-Baleanu operator is used.

$\varkappa$	$\vartheta = 1$	$\vartheta = 0.9$	$\vartheta = 0.7$	$\vartheta = 0.4$
0.5	0.00049496	0.031087	0.047460	0.061608
1.0	0.00063554	0.039917	0.060537	0.079106
1.5	0.00081606	0.051254	0.077731	0.101570
2.0	0.00104780	0.065811	0.099809	0.130420
2.5	0.00134540	0.084503	0.128160	0.167470
3.0	0.00172760	0.108500	0.164560	0.215030
3.5	0.00221830	0.139320	0.211300	0.276110
4.0	0.00284830	0.178890	0.271310	0.354530
4.5	0.00365730	0.229700	0.348370	0.455230
5.0	0.00469610	0.294950	0.447310	0.584520

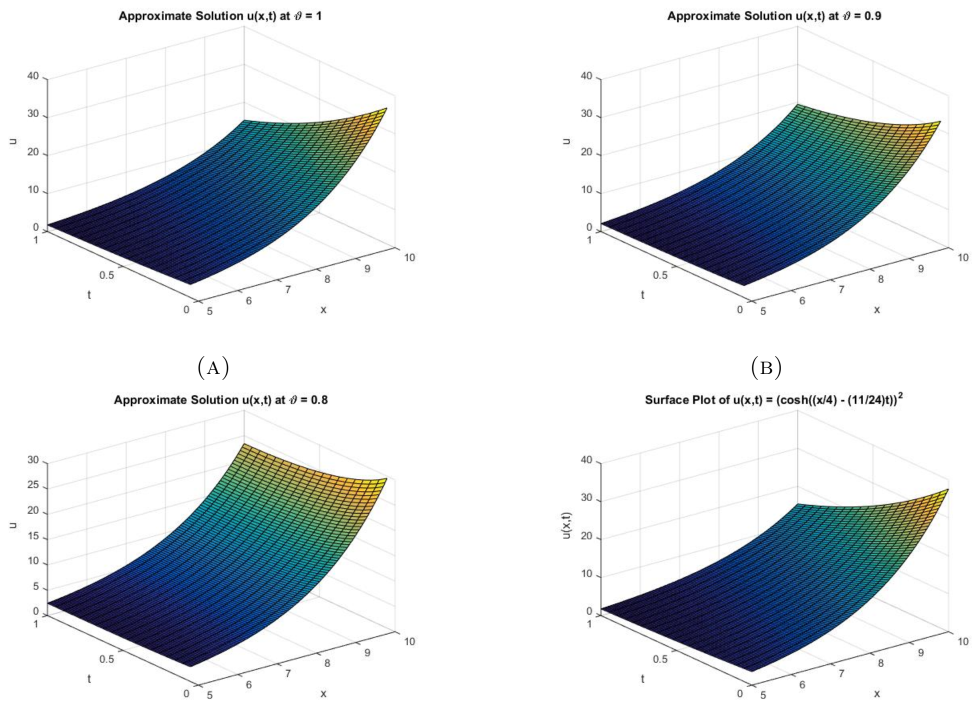
$$
\begin{aligned}
&-\frac{1331}{8192}\sinh\left(\frac{\varkappa}{2}\right)\left[(1-\vartheta)^3+\frac{3(1-\vartheta)^2\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{3(1-\vartheta)\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)}+\frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]\\
&+\frac{121}{2048}\cosh\left(\frac{\varkappa}{2}\right)\left[\frac{3(1-\vartheta)^2\vartheta t^{\vartheta-1}}{\Gamma(\vartheta)}+\frac{5(1-\vartheta)\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}+\frac{2\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right]\\
&-\frac{11}{512}\sinh\left(\frac{\varkappa}{2}\right)\left[\frac{(1-\vartheta)^2\vartheta t^{\vartheta-2}}{\Gamma(\vartheta-1)}+\frac{2(1-\vartheta)\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)}+\frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right]+\cdots
\end{aligned}
$$

For $\vartheta = 1$, the exact solution is:

$$u(\varkappa,t)=\cosh^2\left(\frac{\varkappa}{4}-\frac{11t}{24}\right). \tag{6.7}$$

7. CONCLUSION

In this study, we investigated the fractional Fornberg–Whitham equation using the Yang–Daftardar–Jafari Method (YDJM) in conjunction with the Atangana–Baleanu fractional derivative. The existence and uniqueness of the solution



(A) (B) (C) (D)

FIGURE 4. In **Example 6.2**, plots (A), (B), and (C) provide surface representations that highlight the close correspondence between the numerical solution and the exact solution, shown in plot (D).

TABLE 2. A table displaying the absolute error for **Example 6.2**, where the Atangana-Baleanu operator is applied.

$\varkappa$	$\vartheta = 1$	$\vartheta = 0.9$	$\vartheta = 0.8$	$\vartheta = 0.7$
5.0	0.005757	0.0039514	0.065314	0.17893
5.5	0.0070306	0.0070339	0.079532	0.22314
6.0	0.008746	0.010558	0.098746	0.28137
6.5	0.011011	0.014746	0.124160	0.35728
7.0	0.013967	0.019860	0.157380	0.45564
7.5	0.017801	0.026222	0.200490	0.58262
8.0	0.022754	0.034232	0.256190	0.74621
8.5	0.029136	0.044392	0.327990	0.95667
9.0	0.037348	0.057341	0.420390	1.22720
9.5	0.047907	0.073892	0.539210	1.57490

were established, providing a solid theoretical foundation for the proposed approach. The approximate solutions obtained by the method were shown to converge accurately to the analytical solutions, demonstrating the method's reliability and efficiency in handling nonlinear fractional differential equations.

The integration of the YDJM with the Atangana–Baleanu operator proved to be a powerful analytical framework for exploring the complex dynamics of fractional systems characterized by nonlinearity and memory effects. In addition to reducing computational cost compared to purely numerical techniques, this method ensures fast convergence and

TABLE 3. Comparison of Absolute Errors for **Example 6.1**.

$\varkappa$	YDJM	mVIM	YDJM	mVIM	YDJM	mVIM
	$\vartheta = 1$		$\vartheta = 0.9$		$\vartheta = 0.7$	
0.5	0.00049496	0.00049496	0.031087	0.038857	0.04746	0.094969
1.0	0.00063554	0.00063554	0.039917	0.049893	0.060537	0.12194
1.5	0.00081606	0.00081606	0.051254	0.064064	0.077731	0.15658
2.0	0.0010478	0.0010478	0.065811	0.08226	0.099809	0.20105
2.5	0.0013454	0.0013454	0.084503	0.10562	0.12816	0.25815
3.0	0.0017276	0.0017276	0.1085	0.13562	0.16456	0.33147
3.5	0.0022183	0.0022183	0.13932	0.17415	0.2113	0.42562
4.0	0.0028483	0.0028483	0.17889	0.22361	0.27131	0.54651
4.5	0.0036573	0.0036573	0.2297	0.28712	0.34837	0.70173
5.0	0.0046961	0.0046961	0.29495	0.36867	0.44731	0.90104

TABLE 4. Comparison of absolute errors for **Example 6.2**.

$\varkappa$	YDJM	mVIM	YDJM	mVIM	YDJM	mVIM
	$\vartheta = 1$		$\vartheta = 0.9$		$\vartheta = 0.8$	
5	0.005757	0.005757	0.0039514	0.10332	0.065314	0.17664
5.5	0.0070306	0.0070306	0.0070339	0.13346	0.079532	0.22897
6	0.008746	0.008746	0.010558	0.17199	0.098746	0.29568
6.5	0.011011	0.011011	0.014746	0.22132	0.12416	0.38096
7	0.013967	0.013967	0.01986	0.28455	0.15738	0.49019
7.5	0.017801	0.017801	0.026222	0.36567	0.20049	0.63021
8	0.022754	0.022754	0.034232	0.46975	0.25619	0.80982
8.5	0.029136	0.029136	0.044392	0.60335	0.32799	1.0403
9	0.037348	0.037348	0.057341	0.77486	0.42039	1.3362
9.5	0.047907	0.047907	0.073892	0.99504	0.53921	1.716

acceptable accuracy. The analytical approximate solutions obtained provide explicit mathematical expressions that allow for easier interpretation and qualitative analysis of system behavior, which is often challenging with purely numerical results. Furthermore, the proposed approach can be extended to a wider class of nonlinear fractional partial differential equations, highlighting its potential in addressing real-world problems in physics, engineering, and other applied sciences. Thus, this work makes a valuable contribution to the advancement of fractional calculus and its applications.

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