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A Spectral Tau method based on Lucas polynomial approximation for solving the nonlinear fractional Riccati equation

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Abstract

The nonlinear fractional Riccati equation (NFRE) can be solved using a unique spectral tau approach in this study that uses Lucas polynomials as basis functions. The fractional Caputo derivative and nonlinear terms can be handled effectively by explicit operational formulations when the Lucas basis is used. A tau projection is used to convert the problem into a nonlinear algebraic system, which is then solved by Gaussian elimination. The correctness and quick convergence of the suggested method are shown by a number of numerical tests that are backed by error analysis.

Keywords. Lucas polynomials, Spectral methods, Nonlinear fractional Riccati equation, Convergence analysis.

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1. Introduction

In fields including viscoelasticity, biology, signal processing, and finance, fractional differential equations (FDEs) are essential tools for simulating anomalous diffusion, memory effects, and nonlocal dynamics [19]. Robust numerical methods are required because many fractional models, especially those of Riccati form, are inherently nonlocal and nonlinear, making analytical solutions unfeasible.

A number of semi-analytical approaches have been put out to approximate solutions to FDEs. While the variation of parameters technique [15] adapts standard ODE solvers to the fractional case, the modified homotopy perturbation method [18] and generalized homotopy analysis [23] use iterative series expansions. Wavelet-based collocation techniques also use localized basis functions to capture solution features [28]. When dealing with large nonlinearities, these approaches may show slow convergence or increased complexity, notwithstanding their effectiveness in some situations.

Spectral methods offer an attractive alternative by achieving spectral (exponential) convergence for smooth solutions and handling complex boundary conditions naturally. Different orthogonal bases have been incorporated into Galerkin and collocation frameworks in recent work. Specifically, collocation methods for time-fractional Newell–Whitehead–Segel models [9], refined Bernoulli polynomial Galerkin schemes [30], and Chebyshev–Galerkin operational matrix techniques [2] have shown high accuracy and efficiency. The adaptability of Chebyshev-based methods is further demonstrated by extensions to Ψ-contraction frameworks [8] and KdV–Burgers equations [35].

Beyond Chebyshev systems, alternative families have been explored. Fourth-kind Chebyshev Tau algorithms for Bagley–Torvik problems [29], Caputo-based corneal models [32], and Euler–Bernoulli beam formulations via second-kind Chebyshev polynomials [31] illustrate diverse applications. More recently, mildly singular integro-differential kernels have been tackled with Chebyshev Petrov–Galerkin methods [33], while modified shifted Chebyshev third-kind polynomials address hyperbolic telegraph equations [27]. Lucas polynomial–based Petrov–Galerkin schemes [1] and explicit Duffing collocation algorithms [34] showcase the expansion of spectral tools beyond classical bases.

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A further corpus of research extends these ideas to Gegenbauer, Jacobi, and other polynomial families. Representative studies include recursive expansion coefficient formulas [7], fractional Bratu problem algorithms [10], pseudo-spectral Tricomi solvers [36], and Robin boundary treatments via second-kind Chebyshev [26]. Boundary value diffusion and Bagley–Torvik formulations via tau methods [3, 24] co-exist with Jacobi-Galerkin surveys [13], Gegenbauer Fisher solvers [11], and seventh-kind shifted Gegenbauer techniques [25]. Additional advancements include fractional Fokker–Planck models [17], Romanovski–Jacobi spectral schemes [37], and fully Jacobi-Galerkin time-dependent PDE solvers [14].

Various numerical techniques have been developed to tackle differential equations of fractional and classical types. A fractional-order Lagrange polynomial approach was proposed in [20] to handle fractional differential equations efficiently. In [22], a forward Riccati formulation in the hybrid functions domain was applied to the control of time-varying systems. Spline-based strategies are also prominent, as demonstrated in [12], where cubic Hermit splines were used to approximate solutions of fractional problems. Scaling functions of fractional-order Lagrange type were constructed and analyzed in [21] for broader applications. A system of fractional Volterra integro-differential equations was solved numerically in [16] using cubic Hermit spline functions. Finally, the application of cubic B-splines to fractional Sturm–Liouville problems was investigated in [6], showing their accuracy and flexibility.

In this work, we introduce a spectral tau method based on Lucas polynomials for the nonlinear fractional Riccati equation. The main contributions are:

- The formulation of a Lucas polynomial basis within the Petrov–Galerkin tau framework for nonlinear fractional Riccati equations.
- Construction of operational matrices for the Caputo fractional derivative in the Lucas basis.
- Transformation of the FDE into a system of nonlinear algebraic equations solved by Gauss elimination.
- A comprehensive convergence study and numerical comparisons demonstrating the method's accuracy and efficiency.

The remainder of this paper is organized as follows. In section 2, we recall essential definitions and properties of Caputo fractional derivatives and introduce the Lucas polynomial basis along with its key operational relations. Section 3 details the construction of the spectral tau scheme, including the derivation of fractional derivative matrices and the formulation of the algebraic system. In section 4, a rigorous analysis of error bounds is presented. Section 5 provides numerical examples illustrating the accuracy and efficiency of the proposed method. Finally, concluding remarks and potential directions for future research are offered in section 6.

2. Fundamental Concepts

In this section, we present the essential definitions of the generalized Caputo fractional operators and summarize some important properties of the Lucas polynomials $\mathscr{L}_n(x)$, which will be utilized throughout the paper.

2.1. Caputo Fractional Derivative.

Definition 2.1. [19] The Caputo fractional derivative of a function $\psi(\sigma)$ of order $\alpha > 0$ is defined by

$$D^{\alpha}\psi(\sigma) = \frac{1}{\Gamma(m-\alpha)}\int_0^{\sigma}(\sigma-\tau)^{m-\alpha-1}\psi^{(m)}(\tau)\,d\tau, \tag{2.1}$$

where $m \in \mathbb{N}$ satisfies $m-1 < \alpha < m$, and $\sigma > 0$.

The operator D^{α} satisfies the following elementary properties for $m-1<\alpha<m$ and $m \in \mathbb{N}$:

$$D^{\alpha}\,k = 0, \quad \text{for any constant } k, \tag{2.2}$$

$$D^{\alpha}\,\sigma^{m} = \begin{cases} 0, & \text{if } m \in \mathbb{N}_0 \text{ and } m < \lceil\alpha\rceil, \\ \dfrac{m!}{\Gamma(m-\alpha+1)}\,\sigma^{m-\alpha}, & \text{if } m \in \mathbb{N}_0 \text{ and } m \geq \lceil\alpha\rceil, \end{cases} \tag{2.3}$$

where $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$, and $\lceil\alpha\rceil$ represents the ceiling of α.

2.2. **An account on Lucas polynomials.** The power formula of Lucas polynomials $\mathcal{L}_i(x)$, $i > 0$, is defined as

$$\mathcal{L}_i(x) = \sum_{s=0}^{i} B_{s,i}\, x^s, \tag{2.4}$$

where

$$B_{s,i} = \frac{i\,\xi_{i-s}\Gamma\left(\frac{i+s}{2}\right)}{\Gamma\left(\frac{i-s}{2}+1\right)\, s!}. \tag{2.5}$$

Also, for $m \geq 0$, the inversion formula is

$$x^m = \frac{1}{2}\sum_{s=-m}^{m} (-1)^{\frac{m-s}{2}}\,\xi_{m-s}\binom{m}{\frac{m-s}{2}}\mathcal{L}_s(x). \tag{2.6}$$

Lemma 2.2. *The following linearization formula holds:*

$$\mathcal{L}_i(x)\mathcal{L}_j(x) = \mathcal{L}_{i+j}(x) + (-1)^i\mathcal{L}_{j-i}(x), \quad i, j \geq 0. \tag{2.7}$$

Lemma 2.3. *[4] The following formula holds:*

$$x^m\mathcal{L}_i(x) = \sum_{s=i-m}^{i+m} (-1)^{\frac{i+m-s}{2}}\,\xi_{i+m-s}\binom{m}{\frac{i+m-s}{2}}\mathcal{L}_s(x), \quad m, i \geq 0, \tag{2.8}$$

where

$$\xi_r = \begin{cases} 1, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases} \tag{2.9}$$

Theorem 2.4. *[4] The following formula holds for $\mathcal{L}_m(x)$, $m \geq 0$,*

$$\frac{d^s\mathcal{L}_m(x)}{dx^s} = \frac{1}{2}\sum_{r=s-m}^{m-s} \frac{(-1)^{\frac{m-r-s}{2}}\, m\,\xi_{m-r-s}\,\Gamma\left(\frac{1}{2}(m-r+s)\right)\,\Gamma\left(\frac{1}{2}(m+r+s)\right)}{\left(\frac{1}{2}(m-r-s)\right)!\,\left(\frac{1}{2}(m+r-s)\right)!\,\Gamma(s)}\mathcal{L}_r(x) \tag{2.10}$$

Lemma 2.5. *[4] The first derivative of $\mathcal{L}_m(x)$, $m \geq 0$, is*

$$\frac{d\mathcal{L}_m(x)}{dx} - \frac{1}{2}\sum_{r=1-m}^{m-1} (-1)^{\frac{m-r-1}{2}}\, m\,\xi_{m-r-1}\,\mathcal{L}_r(x). \tag{2.11}$$

Lemma 2.6. *[4] The following relations hold*

$$\int \mathcal{L}_i(x)\,dx = \sum_{r=0}^{i+1}\eta_{r,i}\mathcal{L}_r(x) + \gamma_i, \quad , i \geq 0, \tag{2.12}$$

where

$$\gamma_i = \begin{cases} 0, & \text{if } i < 4 \text{ or } i \quad \text{even,} \\ \frac{2i(i-3)}{(i-1)(i+1)}, & \text{Otherwise.} \end{cases}$$

$$\int_0^1 x^j\mathcal{L}_i(x)\,dx = \begin{cases} \frac{2}{1+j}, & \text{if } i = 0, \\ \displaystyle\sum_{s=0}^{i} \frac{i\,\xi_{i-s}\,\Gamma\left(\frac{i+s}{2}\right)}{\Gamma\left(\frac{i-s}{2}+1\right)\, s!\,(s+j+1)}, & \text{if } i \neq 0. \end{cases} \tag{2.13}$$

where

$$\eta_{i,j}=\begin{cases}2, & \text{if } i=1 \text{ and } j=0,\\ \frac{1}{i}, & \text{if } |i-j|=1 \text{ and } j>0,\\ -\frac{j}{j+1}, & \text{if } i=0 \text{ and } j \quad \text{odd},\\ 0, & \text{Otherwise}.\end{cases} \tag{2.14}$$

3. Tau approach for the NFRE

Consider the following NFRE [28]:

$$D^{\beta}\mathcal{K}(t)=a(t)+b(t)\,\mathcal{K}(t)+f(t)\,\mathcal{K}^2(t);\quad t\in[0,1], \tag{3.1}$$

where $0<\beta\leq 1$, subject to the initial condition:

$$\mathcal{K}(0)=\lambda, \tag{3.2}$$

where $a(t)$, $b(t)$ and $f(t)$ are continuous functions on [0,1].

Remark 3.1. Problem (3.1) is solved in two cases corresponding to

- $b(t)=b$ and $f(t)=f$ are constants.
- $b(t)=f(t)=t^m,\quad m\in\mathbb{N}$.

3.1. **Solution of NFRE for the case of $b(t)=b$ and $f(t)=f$ are constants.** Consider the NFRE (3.1), subject to the constant functions $b(t)=b$ and $f(t)=f$.
Now, the collection of $\mathcal{L}_i(t)$ enables us to write $\mathcal{K}(t)\in L_2[0,1]$ as a linearly combination as

$$\mathcal{K}(t)=\sum_{i=0}^{\infty}\hat{\mathcal{K}}_i\,\mathcal{L}_i(t), \tag{3.3}$$

which can be approximated as

$$\mathcal{K}(t)\approx\mathcal{K}_N(t)=\sum_{i=0}^{N}\hat{\mathcal{K}}_i\,\mathcal{L}_i(t). \tag{3.4}$$

The residual $\mathbf{R}(t)$ of NFRE (3.1) after putting $b(t)=b$ and $f(t)=f$ and using linearization formula (2.7) can be written as

$$\begin{aligned}\mathbf{R}(t)&=D^{\beta}\mathcal{K}_N(t)-a(t)-b\,\mathcal{K}_N(t)-f\,\mathcal{K}_N^2(t)\\ &=\sum_{i=0}^{N}\hat{\mathcal{K}}_i\,D^{\beta}\mathcal{L}_i(t)-a(t)-b\sum_{i=0}^{N}\hat{\mathcal{K}}_i\,\mathcal{L}_i(t)-f\sum_{i=0}^{N}\hat{\mathcal{K}}_i\,\mathcal{L}_i(t)\sum_{j=0}^{N}\hat{\mathcal{K}}_j\,\mathcal{L}_j(t)\\ &=\sum_{i=0}^{N}\hat{\mathcal{K}}_i\,D^{\beta}\mathcal{L}_i(t)-a(t)-b\sum_{i=0}^{N}\hat{\mathcal{K}}_i\,\mathcal{L}_i(t)-f\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{\mathcal{K}}_j\hat{\mathcal{K}}_i\,\left(\mathcal{L}_{i+j}(t)+(-1)^i\,\mathcal{L}_{j-i}(t)\right).\end{aligned} \tag{3.5}$$

The application of the Tau method leads to

$$\int_0^1\mathbf{R}(t)\,\mathcal{L}_r(t)\,dt=0,\quad r:0,1,...,N-1. \tag{3.6}$$

The previous equation can be written in another form as

$$\begin{aligned}\sum_{i=0}^{N}\hat{\mathcal{K}}_i\left(D^{\beta}\,\mathcal{L}_i(t),\mathcal{L}_r(t)\right)-b\sum_{i=0}^{N}\hat{\mathcal{K}}_i\left(\mathcal{L}_i(t)\,,\mathcal{L}_r(t)\right)-f\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{\mathcal{K}}_i\hat{\mathcal{K}}_j\left[\left(\mathcal{L}_{i+j}(t)\,,\mathcal{L}_r(t)\right)\right.\\ \left.+\left((-1)^i\,\mathcal{L}_{j-i}(t)\,,\mathcal{L}_r(t)\right)\right]=\left(a(t)\,,\mathcal{L}_r(t)\,\right),\end{aligned} \tag{3.7}$$

or

$$\sum_{i=0}^{N} \hat{\mathcal{K}}_i \mathcal{R}_{i,r} - b \sum_{i=0}^{N} \hat{\mathcal{K}}_i \mathcal{Z}_{i,r} - f \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{\mathcal{K}}_i \hat{\mathcal{K}}_j \left[\mathcal{Z}_{i+j,r} + (-1)^i \mathcal{Z}_{j-i,r} \right] = \mathcal{A}_r, \qquad r: 0, ..., N-1, \tag{3.8}$$

where

$$\begin{aligned} \mathcal{R}_{i,r} &= \left(D^{\beta} \mathcal{L}_i(t), \mathcal{L}_r(t) \right), \\ \mathcal{Z}_{i,r} &= \left(\mathcal{L}_i(t)\, , \mathcal{L}_r(t) \right), \\ \mathcal{A}_r &= \left(a(t)\, , \mathcal{L}_r(t)\, \right). \end{aligned} \tag{3.9}$$

Moreover, we get the following initial conditions

$$\sum_{i=0}^{N} \hat{\mathcal{K}}_i \, \mathcal{L}_i(0) = \lambda, \tag{3.10}$$

Finally, a system in (3.8)-(3.10) of dimension $(N+1)$ can be solved with the aid of the Gauss elimination method to get the unknown expansion coefficients $\hat{\mathcal{K}}_i$.

Remark 3.2. The inner product $(a(x)\, , b(x))$ is defined as

$$(a(x)\, , b(x)) = \int_0^1 a(x)\; b(x)\, d\, x. \tag{3.11}$$

Remark 3.3. Based on the relation (2.4) along with (2.3), we can write

$$D^{\beta} \mathcal{L}_i(t) = \sum_{s=0}^{i} \frac{B_{s,i}\; s!}{\Gamma(s-\beta+1)}\, t^{s-\beta}. \tag{3.12}$$

Theorem 3.4. *For all $i, r \geq 0$, the elements $\mathcal{R}_{i,r}$ and $\mathcal{Z}_{i,r}$ are given by*

$$\mathcal{R}_{i,r} = \left(D^{\beta} \mathcal{L}_i(t), \mathcal{L}_r(t) \right) = \sum_{k=1}^{i} \sum_{n=0}^{s} \frac{k!\, B_{k,i}\, B_{n,s}}{\Gamma(k-\beta+1)(-\beta+k+n+1)}, \tag{3.13}$$

$$\mathcal{Z}_{i,r} = \left(\mathcal{L}_i(t)\, , \mathcal{L}_r(t) \right) = \sum_{m=0}^{r+i+1} \left[\eta_{m,r+i} + (-1)^i \cdot \chi_{m,r,i} \cdot \eta_{m,r-i} \right] S_m, \tag{3.14}$$

where the indicator function $\chi_{m,r,i}$ is defined as:

$$\chi_{m,r,i} = \begin{cases} 1, & \textit{if } m \leq r-i+1, \\ 0, & \textit{Otherwise}, \end{cases}$$

and

$$S_m = \left(\mathcal{L}_m - (1 + (-1)^m) \right).$$

Proof. The proof of the first part can be easily obtained after using Remark 3.3 and imitating similar steps as in [5]. Now, to prove the second part

$$\begin{aligned}
\mathcal{Z}_{i,r} &= \int_0^1 \mathcal{L}_i(t)\,\mathcal{L}_r(t)\,dt \\
&= \int_0^1 \left(\mathcal{L}_{r+i}(t) + (-1)^i \mathcal{L}_{r-i}(t)\right)\,dt \\
&= \sum_{m=0}^{r+i+1} \eta_{m,r+i}\mathcal{L}_m(x) + (-1)^i \sum_{m=0}^{r-i+1} \eta_{m,r-i}\mathcal{L}_m(x)\Big|_0^1 \\
&= \sum_{m=0}^{r+i+1} \eta_{m,r+i}\mathcal{L}_m + (-1)^i \sum_{m=0}^{r-i+1} \eta_{m,r-i}\mathcal{L}_m - \sum_{m=0}^{r+i+1} \eta_{m,r+i}(1+(-1)^m) - (-1)^i \sum_{m=0}^{r-i+1} \eta_{m,r-i}(1+(-1)^m) \\
&= \sum_{m=0}^{r+i+1} \left[\eta_{m,r+i} S_m + (-1)^i \cdot \chi_{m,r,i} \cdot \eta_{m,r-i} S_m\right],
\end{aligned} \tag{3.15}$$

which completes the proof of the theorem. □

3.2. **Solution of NFRE for the case of** $b(t) = f(t) = t^m, \quad m \in \mathbb{N}$. Consider the NFRE (3.1), subject to the constant functions $b(t) = f(t) = t^m$.
The residual $\mathbf{R}(t)$ of NFRE (3.1) after putting $b(t) = f(t) = t^m$ and using formula (2.8) can be written as

$$\begin{aligned}
\mathbf{R}(t) &= D^\beta \mathcal{K}_N(t) - a(t) - t^m\,\mathcal{K}_N(t) - t^m\,\mathcal{K}_N^2(t) \\
&= \sum_{i=0}^{N} \hat{\mathcal{K}}_i\, D^\beta \mathcal{L}_i(t) - a(t) - \sum_{i=0}^{N} \hat{\mathcal{K}}_i\, t^m\,\mathcal{L}_i(t) - \sum_{i=0}^{N} \hat{\mathcal{K}}_i\, t^m\,\mathcal{L}_i(t) \sum_{j=0}^{N} \hat{\mathcal{K}}_j\,\mathcal{L}_j(t) \\
&= \sum_{i=0}^{N} \hat{\mathcal{K}}_i\, D^\beta \mathcal{L}_i(t) - a(t) - \sum_{i=0}^{N} \sum_{s=i-m}^{i+m} \hat{\mathcal{K}}_i\,(-1)^{\frac{i+m-s}{2}}\,\xi_{i+m-s} \binom{m}{\frac{i+m-s}{2}} \mathcal{L}_s(t) \\
&\quad - \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{\mathcal{K}}_j \hat{\mathcal{K}}_i \sum_{s=i-m}^{i+m} (-1)^{\frac{i+m-s}{2}}\,\xi_{i+m-s} \binom{m}{\frac{i+m-s}{2}} \left[\mathcal{L}_{s+j}(t) + (-1)^s\,\mathcal{L}_{j-s}(t)\right].
\end{aligned} \tag{3.16}$$

The application of the Tau method enables us to get The application of the Tau method leads to

$$\int_0^1 \mathbf{R}(t)\,\mathcal{L}_r(t)\,dt = 0, \quad r: 0, 1, ..., N-1, \tag{3.17}$$

which can be rewritten in another form as

$$\begin{aligned}
&\sum_{i=0}^{N} \hat{\mathcal{K}}_i\,\mathcal{R}_{i,r} - \sum_{i=0}^{N} \sum_{s=i-m}^{i+m} \hat{\mathcal{K}}_i\,(-1)^{\frac{i+m-s}{2}}\,\xi_{i+m-s} \binom{m}{\frac{i+m-s}{2}} \mathcal{Z}_{i,r} \\
&- \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{\mathcal{K}}_j \hat{\mathcal{K}}_i \sum_{s=i-m}^{i+m} (-1)^{\frac{i+m-s}{2}}\,\xi_{i+m-s} \binom{m}{\frac{i+m-s}{2}} \left[\mathcal{Z}_{s+j,r} + (-1)^s\,\mathcal{Z}_{j-s,r}\right] = \mathcal{A}_r, \qquad r: 0, ..., N-1.
\end{aligned} \tag{3.18}$$

Now, Eq. (3.18) along with the initial condition in (3.10) enables us to get a system of algebraic equations of dimension $(N+1)$ that can be solved with the aid of Gauss elimination method to get the unknown expansion coefficients $\hat{\mathcal{K}}_i$.

4. Error Analysis

In this section, we aim to establish that the residual term $\mathbf{R}(\tau)$ diminishes as M tends toward infinity.

Theorem 4.1. *Suppose that* $\frac{d^i\,\mathcal{K}_M(\tau)}{d\,\tau^i} \in \mathbf{C}([0,1])$ *for* $i = 0,1,2,\dots,M$*, where* $\mathcal{K}_M(\tau)$ *represents the approximate solution. Define*

$$\varrho_M = \sup_{\tau\in[0,1]} \left| \frac{d^{M+1}\,\mathcal{K}(\tau)}{d\,\tau^{M+1}} \right|.$$

Then, the following inequality holds:

$$\|\mathcal{K}(\tau) - \mathcal{K}_M(\tau)\|_2 \le \frac{\varrho_M}{(2M+3)^{1/2}(M+1)!}.$$

Proof. Consider the Taylor polynomial expansion of $\mathcal{K}(\tau)$ around $\tau = 0$:

$$\chi_M(\tau) = \sum_{i=0}^{M} \left(\frac{d^i\,\mathcal{K}(\tau)}{d\,\tau^i} \right)_{\tau=0} \frac{\tau^i}{i!}, \tag{4.1}$$

with the remainder given by

$$\mathcal{K}(\tau) - \chi_M(\tau) = \frac{\tau^{M+1}}{(M+1)!} \left(\frac{d^{M+1}\,\mathcal{K}(\tau)}{d\,\tau^{M+1}} \right)_{\tau=c}, \quad c \in [0,1]. \tag{4.2}$$

Since $\mathcal{K}_M(\tau)$ is the optimal approximation of $\mathcal{K}(\tau)$, it follows that

$$\begin{aligned} \|\mathcal{K}(\tau) - \mathcal{K}_M(\tau)\|_2^2 &\le \|\mathcal{K}(\tau) - \chi_M(\tau)\|_2^2 \\ &\le \int_0^1 \frac{\varrho_M^2\,\tau^{2(M+1)}}{((M+1)!)^2}\,d\tau \\ &= \frac{\varrho_M^2}{(2M+3)\,((M+1)!)^2}, \end{aligned} \tag{4.3}$$

which yields

$$\|\mathcal{K}(\tau) - \mathcal{K}_M(\tau)\|_2 \le \frac{\varrho_M}{(2M+3)^{1/2}(M+1)!}.$$

□

Theorem 4.2. *Let* $\mathcal{K}(\tau)$ *and* $\mathcal{K}_M(\tau)$ *satisfy the conditions of Theorem 4.1. Define*

$$\upsilon_{M,m} = \sup_{\tau\in[0,1]} \left| \left(\frac{d^{M+1}\,\mathcal{K}(\tau)}{d\,\tau^{M+1}} \right)^m \right|, \quad m = 2,3,4,\dots$$

Then, the following estimate holds:

$$\|\mathcal{K}^m(\tau) - \mathcal{K}_M^m(\tau)\|_2 \le \frac{\upsilon_{M,m}}{(2m(M+1)+1)^{1/2}\,((M+1)!)^m}.$$

Proof. Using the Taylor expansion (4.1), we have

$$\mathcal{K}^m(\tau) - \chi_M^m(\tau) = \frac{\tau^{m(M+1)}}{((M+1)!)^m} \left(\frac{d^{M+1}\,\mathcal{K}(\tau)}{d\,\tau^{M+1}} \right)_{\tau=c}^m, \quad c \in [0,1]. \tag{4.4}$$

Following a similar argument as in Theorem 4.1 based on best approximation, we obtain

$$\begin{aligned} \|\mathcal{K}^m(\tau) - \mathcal{K}_M^m(\tau)\|_2^2 &\le \|\mathcal{K}^m(\tau) - \chi_M^m(\tau)\|_2^2 \\ &\le \int_0^1 \frac{\upsilon_{M,m}^2\,\tau^{2m(M+1)}}{((M+1)!)^{2m}}\,d\tau \\ &= \frac{\upsilon_{M,m}^2}{(2m(M+1)+1)((M+1)!)^{2m}}, \end{aligned}$$

and consequently

$$\|\mathcal{K}^m(\tau) - \mathcal{K}_M^m(\tau)\|_2 \leq \frac{\upsilon_{M,m}}{(2m(M+1)+1)^{1/2}\,((M+1)!)^m}.$$

□

Theorem 4.3. *Suppose that the Caputo derivative $D^\beta \mathcal{K}(\tau) \in \mathbf{C}([0,1])$ and that the hypotheses of Theorem 4.1 are satisfied. Then,*

$$\|D^\beta[\mathcal{K}(\tau) - \mathcal{K}_M(\tau)]\|_2 \leq \frac{\varrho_M}{(2(M-\beta)+3)^{1/2}\,\Gamma(M-\beta+2)}.$$

Proof. Applying the operator D^β to the result of (4.2), we find

$$\left|D^\beta[\mathcal{K}(\tau) - \mathcal{K}_M(\tau)]\right| \leq \frac{\varrho_M\,\tau^{M-\beta+1}}{\Gamma(M-\beta+2)}. \tag{4.5}$$

Taking $\|.\|_2$ yields

$$\begin{aligned}\|D^\beta[\mathcal{K}(\tau) - \mathcal{K}_M(\tau)]\|_2^2 &\leq \int_0^1 \frac{\varrho_M^2\,\tau^{2(M-\beta+1)}}{(\Gamma(M-\beta+2))^2}\,d\tau \\ &= \frac{\varrho_M^2}{(\Gamma(M-\beta+2))^2}\int_0^1 \tau^{2(M-\beta+1)}\,d\tau \\ &= \frac{\varrho_M^2}{(2(M-\beta)+3)\,(\Gamma(M-\beta+2))^2},\end{aligned}$$

which proves the theorem. □

Theorem 4.4. *Let $\mathbf{R}(\tau)$ denote the residual corresponding to Eq. (3.1) as expressed in (3.16). Then, the following upper bound holds:*

$$\begin{aligned}\|\mathbf{R}(\tau)\|_2 \leq{} & \frac{\varrho_M}{(2(M-\beta)+3)^{1/2}\,\Gamma(M-\beta+2)} - b(\tau)\,\frac{\varrho_M}{(2M+3)^{1/2}(M+1)!} \\ & - f(\tau)\,\frac{\upsilon_{M,m}}{(2m(M+1)+1)^{1/2}((M+1)!)^m}.\end{aligned} \tag{4.6}$$

Proof. The residual $\mathbf{R}(\tau)$ of Eq. (3.1) can be written as

$$\mathbf{R}(\tau) = D^\beta \mathcal{K}_M(\tau) - a(\tau) - b(\tau)\mathcal{K}_M(\tau) - f(\tau)\mathcal{K}_M^2(\tau). \tag{4.7}$$

Based in Eq. (3.1), we can write

$$a(\tau) = D^\beta \mathcal{K}(\tau) - b(\tau)\mathcal{K}(\tau) - f(\tau)\mathcal{K}^2(\tau). \tag{4.8}$$

Now, inserting Eq. (4.8) into Eq. (4.7), we get

$$\mathbf{R}(\tau) = D^\beta\left[\mathcal{K}_M(\tau) - \mathcal{K}(\tau)\right] - b(\tau)\left[\mathcal{K}_M(\tau) - \mathcal{K}(\tau)\right] - f(\tau)\left[\mathcal{K}_M^2(\tau) - \mathcal{K}^2(\tau)\right]. \tag{4.9}$$

At the end, taking $|.|$ for both sides and using the bounds from Theorems 4.1–4.3, leads to the estimate (4.6). □

5. Illustrative examples

Example 5.1. [15, 18, 23, 28] Consider the following equation

$$D^\beta \mathcal{K}(t) + \mathcal{K}^2(t) = 1\,; \quad t \in [0,1], \tag{5.1}$$

subject to

$$\mathcal{K}(0) = 0, \tag{5.2}$$

where the exact solution of this problem is $\mathcal{K}(t) = \frac{e^{2\,t}-1}{e^{2\,t}+1}$.

TABLE 1. AE of Example 5.1 at $\beta = 1$.

t	$N=7$	$N=8$	$N=9$
0.1	3.47637×10^{-7}	1.1573×10^{-7}	9.76574×10^{-9}
0.2	6.34193×10^{-8}	7.86395×10^{-8}	7.58118×10^{-9}
0.3	4.16499×10^{-7}	7.1228×10^{-8}	3.91853×10^{-9}
0.4	1.60685×10^{-7}	1.22887×10^{-7}	9.23474×10^{-9}
0.5	3.98861×10^{-7}	3.1166×10^{-8}	3.03736×10^{-9}
0.6	1.0245×10^{-7}	1.16825×10^{-7}	1.05086×10^{-8}
0.7	3.46075×10^{-7}	1.03399×10^{-8}	5.28847×10^{-10}
0.8	1.13768×10^{-7}	8.00766×10^{-8}	1.08437×10^{-8}
0.9	1.66917×10^{-7}	7.12214×10^{-8}	1.04777×10^{-8}

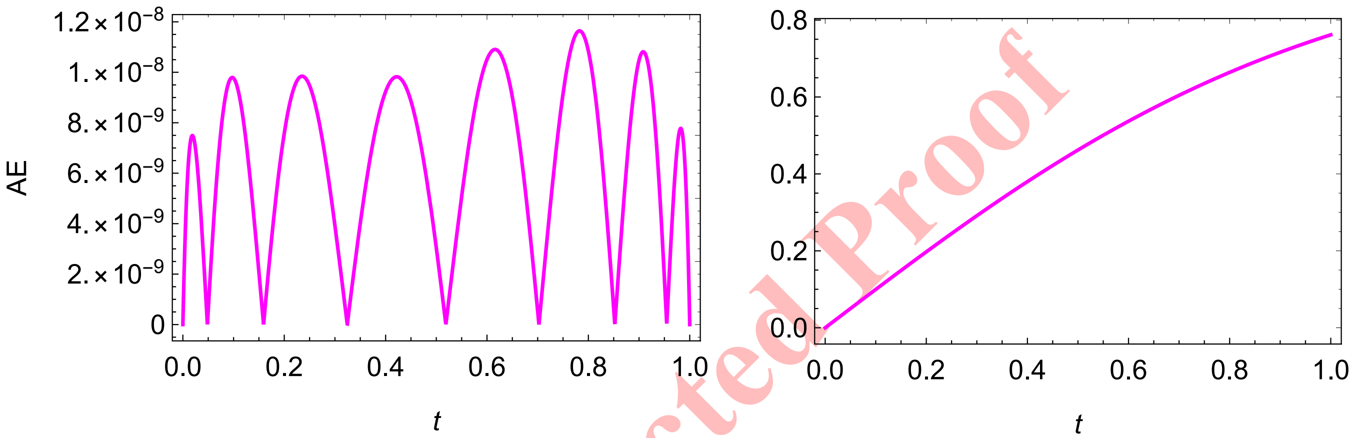

FIGURE 1. The AE (left) and approximate solution (right) of Example 5.1 at $\beta = 1$ and $N = 9$.

TABLE 2. Comparison between approximate solutions of Example 5.1.

t	Our method	Method in[18]	Method in [23]	Method in [15]	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.1973753126	0.197375	0.1973753092	0.1973753160	0.1973753202
0.4	0.3799489714	0.379944	0.3799435784	0.3799469862	0.3799489622
0.6	0.5370495564	0.536857	0.5368572343	0.5369833784	0.5370495669
0.8	0.6640367811	0.661706	0.6617060368	0.6633009217	0.6640367702
1.0	0.7615941559	0.746032	0.746031746	0.7571662667	0.7615941559

Table 1 presents the absolute errors (AE) at $\beta = 1$ and different values of N. Figure 1 shows the AE (left) and approximate solution (right) at $\beta = 1$ and $N = 9$. Figure 2 shows the maximum absolute errors (MAE) and L_∞-errors at different values of N when $\beta = 1$. Figure 3 shows the approximate solutions at different values of β when $N = 7$. Table 2 presents a comparison between our method and methods in [15, 18, 23] of approximate solutions at $\beta = 1$.

Example 5.2. Consider the following equation

$$D^\beta \mathcal{K}(t) + \mathcal{K}^2(t) - \mathcal{K}(t) = a(t); \quad t \in [0, 1], \tag{5.3}$$

subject to

$$\mathcal{K}(0) = 1, \tag{5.4}$$

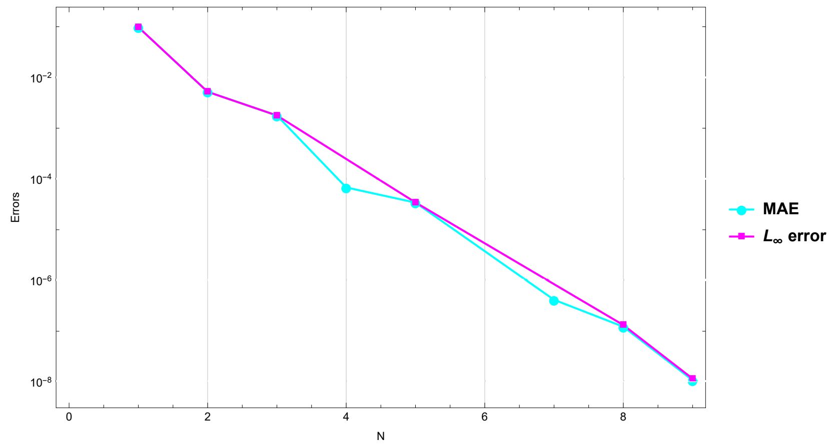

Figure 2. Errors of Example 5.1 at $\beta = 1$.

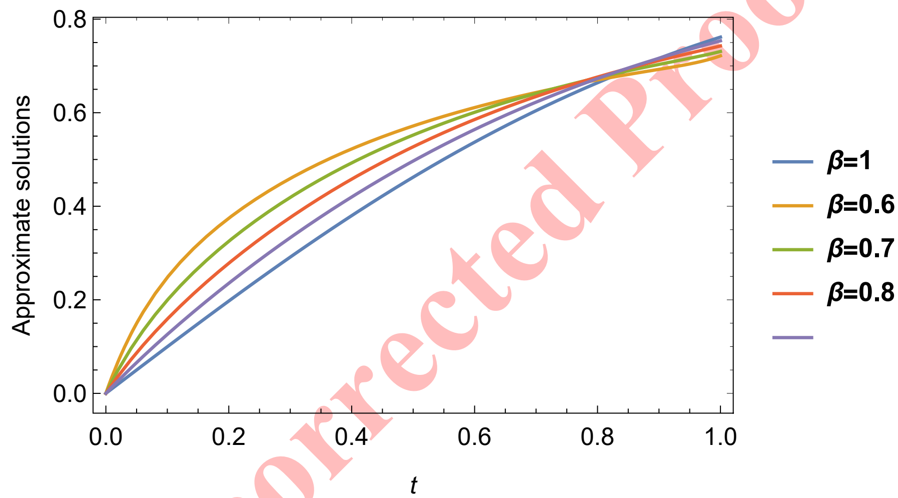

Figure 3. Approximate solutions of Example 5.1.

where $a(t)$ is chosen such that the exact solution of this problem is $\mathcal{K}(t) = e^{\beta\, t}$.

Figure 4 shows the AE at $\beta = 1$ and different values of N. Table 3 presents the AE of Example 5.2 at $\beta = 0.5$ and different values of N. Figure 5 shows the MAE and L_∞-errors at different values of N when $\beta = 0.9$. Table 4 presents the MAE and L_∞-errors at different values of N when $\beta = 0.3$.

Example 5.3. Consider the following equation

$$D^{\beta}\mathcal{K}(t) + t\,\mathcal{K}^2(t) - t^2\,\mathcal{K}(t) = a(t)\,; \quad t \in [0,1], \tag{5.5}$$

subject to

$$\mathcal{K}(0) = 0, \tag{5.6}$$

where $a(t)$ is chosen such that the exact solution of this problem is $\mathcal{K}(t) = \sin(\beta\, t)$.

Figure 6 shows the AE at $\beta = 1$ and different values of N. Table 5 presents the MAE and L_∞-errors at different values of N when $\beta = 0.7$. Figure 7 shows AE (left) and approximate solution (right) at $\beta = 1$ and $N = 12$.

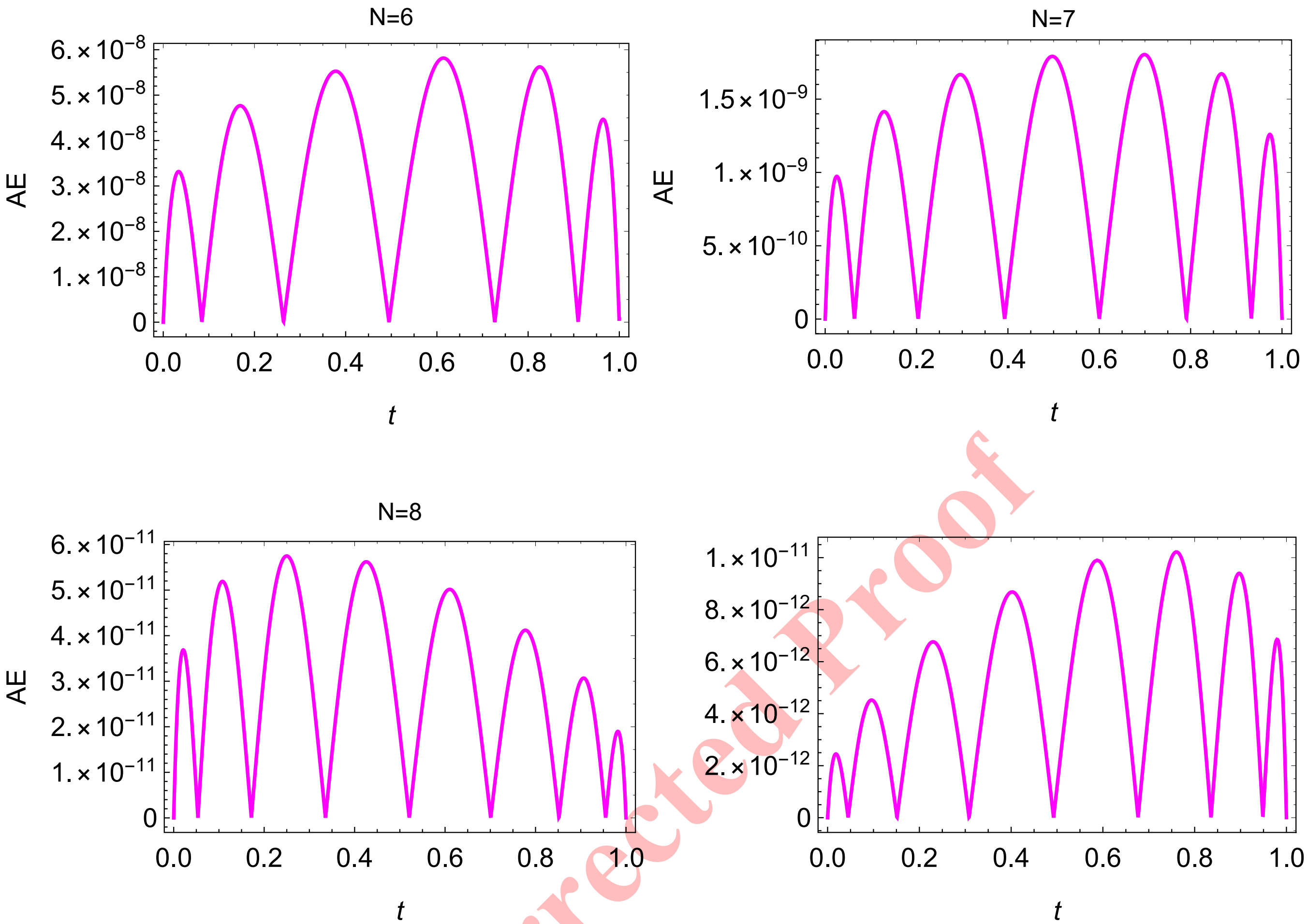

FIGURE 4. AE of Example 5.2 at $\beta = 1$.

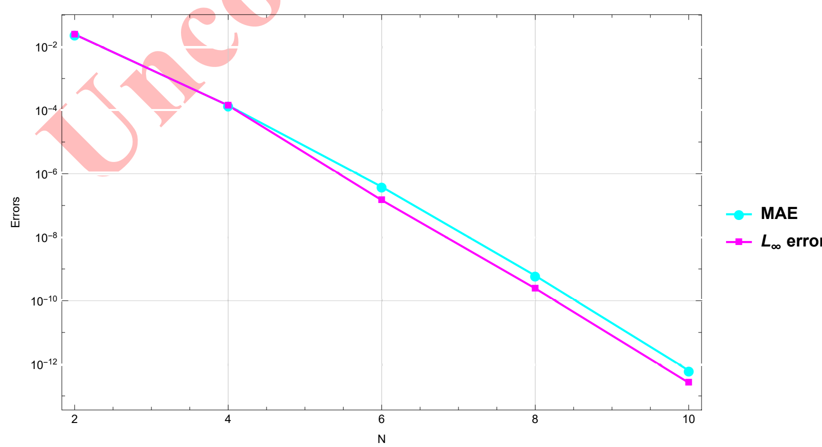

FIGURE 5. Errors of Example 5.2 at $\beta = 0.9$.

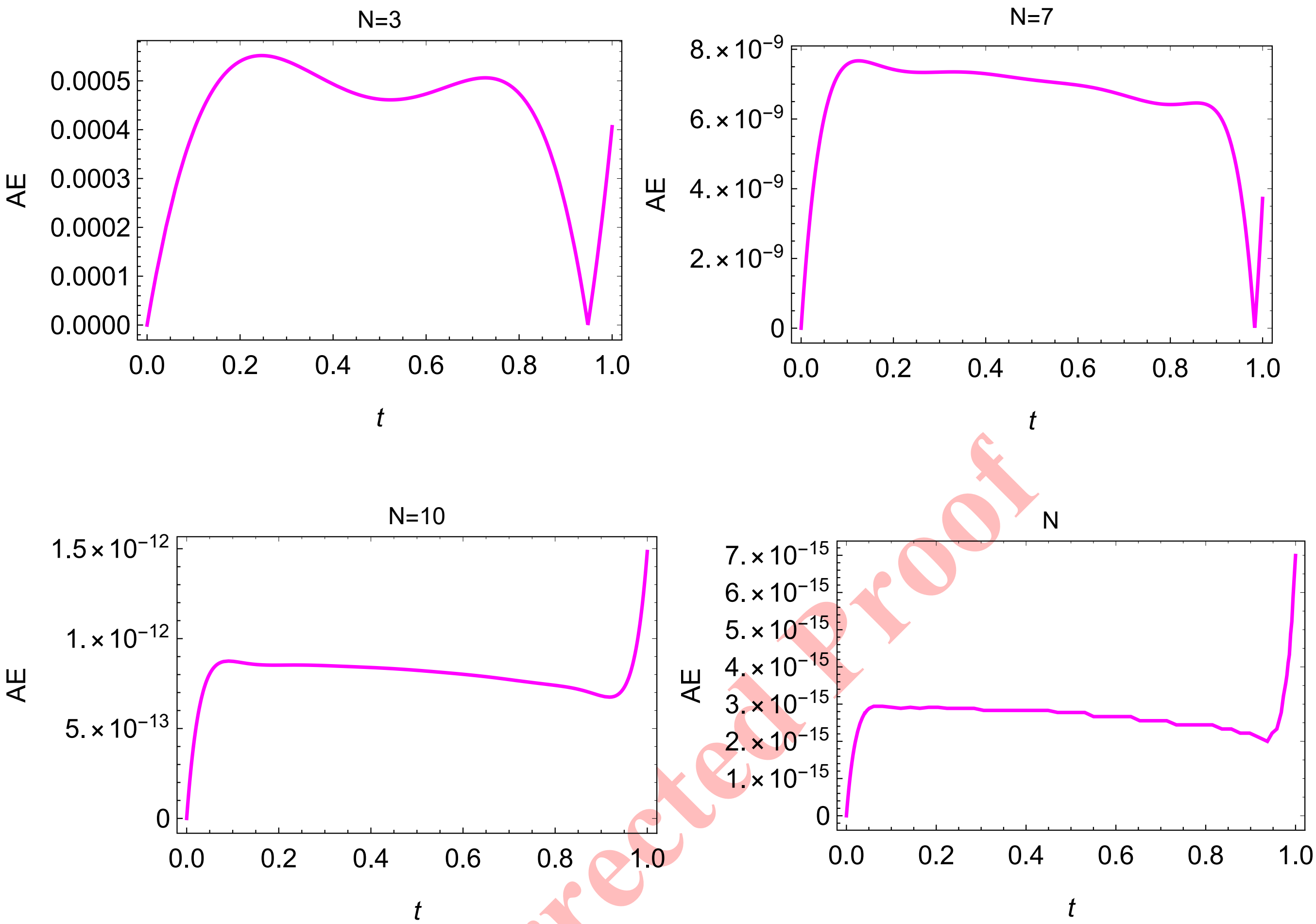

FIGURE 6. AE of Example 5.3 at $\beta = 1$.

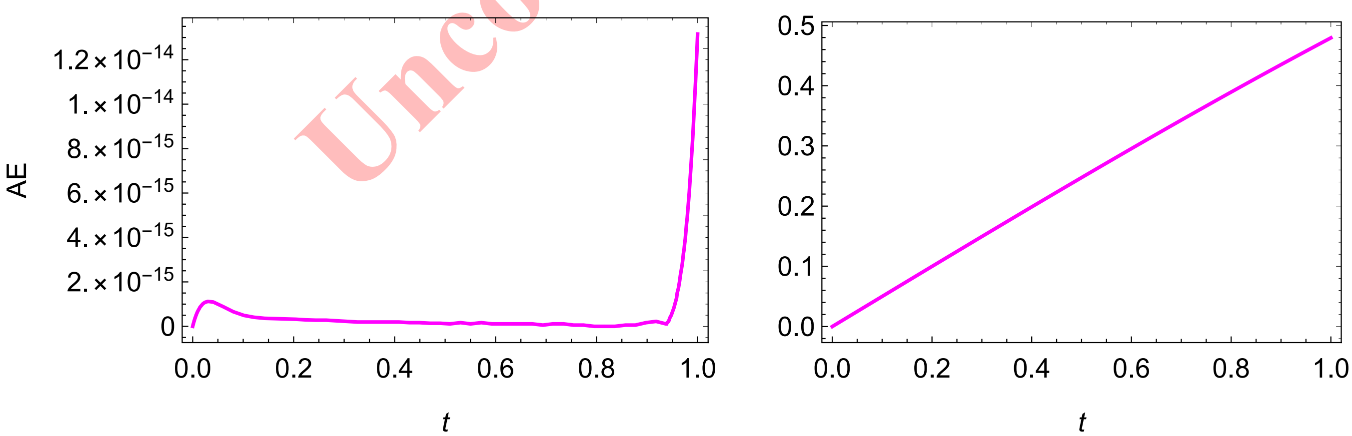

FIGURE 7. The AE (left) and approximate solution (right) of Example 5.3 at $\beta = 1$ and $N = 12$.

Table 3. AE of Example 5.2 at $\beta = 0.5$.

t	$N=7$	$N=8$	$N=9$	$N=10$
0.1	8.63354×10^{-12}	1.43885×10^{-13}	1.33227×10^{-15}	6.66134×10^{-16}
0.2	1.99685×10^{-12}	4.04121×10^{-14}	4.44089×10^{-16}	2.22045×10^{-16}
0.3	2.01483×10^{-12}	3.77476×10^{-14}	4.44089×10^{-16}	2.22045×10^{-16}
0.4	1.2963×10^{-12}	2.10942×10^{-14}	2.22045×10^{-16}	4.44089×10^{-16}
0.5	8.20233×10^{-13}	2.02061×10^{-14}	0	2.22045×10^{-16}
0.6	1.08846×10^{-12}	1.15463×10^{-14}	2.22045×10^{-16}	0
0.7	1.03917×10^{-13}	1.42109×10^{-14}	2.22045×10^{-16}	2.22045×10^{-16}
0.8	2.10409×10^{-12}	3.55271×10^{-15}	0	2.22045×10^{-16}
0.9	3.37796×10^{-12}	4.21885×10^{-15}	1.11022×10^{-15}	6.66134×10^{-16}

Table 4. Errors of example 5.2 at $\beta = 0.3$.

	$N=3$	$N=5$	$N=7$	$N=9$
MAE	3.16015×10^{-5}	1.5042×10^{-8}	3.6886×10^{-12}	3.01981×10^{-14}
L_∞ errors	3.16015×10^{5}	1.5042×10^{-8}	3.68842×10^{-12}	3.0263×10^{-14}

Table 5. Errors of example 5.3 at $\beta = 0.7$.

	$N=3$	$N=5$	$N=7$	$N=9$	$N=11$	$N=13$
MAE	1.72398×10^{-4}	4.16301×10^{-7}	5.30611×10^{-10}	4.08784×10^{-13}	8.21565×10^{-15}	4.44089×10^{-16}
L_∞ errors	1.72398×10^{4}	4.16301×10^{-7}	5.30577×10^{-10}	4.09006×10^{-13}	8.43769×10^{-15}	5.27356×10^{-16}

6. Concluding remarks

For the nonlinear fractional Riccati equation, a novel spectral tau scheme based on Lucas polynomials has been created. The approach offers a straightforward but incredibly precise framework for solving nonlinear FDEs by utilizing the orthogonal structure and derivative features of Lucas polynomials. The method's better accuracy, convergence, and flexibility in addressing scenarios with constant and variable coefficients are demonstrated by numerical examples. Extensions to space-time fractional models and systems of fractional equations might be investigated in future research. All codes were written and debugged by *Mathematica* 11 on an HP Z420 Workstation, Processor: Intel(R) Xeon(R) CPU E5-1620 v2 - 3.70GHz, 16 GB Ram DDR3, and 512 GB storage.

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