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Analytical solutions and conservation laws of a generalized (3+1)-dimensional nonlinear evolution equation appearing in mathematical physics

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Abstract

In mathematical physics, the study of solutions to nonlinear evolution equations has always been important, especially in the fields of nonlinear optics, fluid dynamics, and condensed matter physics. We study a generalized (3+1)-dimensional nonlinear evolution equation as a key consequence. This underlying equation is discovered to admit an endless number of conservation laws and point symmetries. Traveling wave solutions of physical interest are demonstrated by combining the Lie symmetry method with ansatz techniques. Furthermore, we use the multiplier approach to obtain the underlying equation's infinitely many conservation laws. It is predicted that these findings may be utilized to better understand how nonlinear waves propagate in a range of nonlinear physical systems, such as fluid mechanics and nonlinear optics. The solution dynamics are presented graphically.

Keywords. Extended tan method, Extended tanh method, Point symmetries, Conservation laws.

2010 Mathematics Subject Classification. 35G20, 35C05, 35C07.

1. Introduction

Many scientific domains, including mechanics, oceanography, aeronautics, nonlinear optics, and plasma physics, have extremely clear concepts on the dynamical implications of nonlinear evolution equations. Their broad effect may be seen in every field of invention and research. Understanding the precise solutions of nonlinear evolution equations is of great interest to the scientific community, since it is necessary to understand their mathematical and practical applications. These equations are generated from several mathematical and physical models and have significant practical implications. In recent years, nonlinear events have revealed exciting qualities with a wide variety of applications, which has captivated researchers in mathematical physics and engineering. In the study of mathematical physics, nonlinear evolution equations [9, 17–19, 26, 33–36, 42] often arise in various contexts. These equations can describe complex physical phenomena such as fluid dynamics, heat transfer, and quantum mechanics. To further understand and analyze these phenomena, it is important to obtain exact solutions of these nonlinear evolution equations. Exact solutions of nonlinear evolution equations provide valuable information about the behavior and dynamics of physical systems. They allow us to make precise predictions and gain insights into the underlying physical processes. Overall, the search for exact solutions of nonlinear evolution equations in mathematical physics is a crucial endeavor that allows us to better understand and analyze complex physical phenomena [10–12, 27, 31, 32, 43–46]. Nonlinear evolution equations are an area of focus research that offers valuable insights into the complexities of several physical processes. In a number of disciplines, such as atmospheric science, fluid mechanics, plasma waves, optical fiber communications, and soliton theory, these formulae are very useful for clarifying complicated processes. Nonlinear evolution equations research opens openings for innovative solutions to urgent issues in various domains

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by paving the road for the understanding and management of many systems. Since there are no standard methods that can be used to analyze all of these nonlinear evolution equations, these equations are often challenging to solve analytically, necessitating a thorough examination of each equation separately. There are two main ways to solve nonlinear evolution equations: analytically or numerically. Recent years have witnessed tremendous advancements that have resulted in the creation of many dependable and effective mathematical techniques for obtaining precise solutions for nonlinear evolution equations [4, 14, 15, 20–25, 37–41].

A $(3+1)$-dimensional nonlinear evolution equation

$$3\, w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x\partial_x^{-1}w_y)_x = 0, \tag{1.1}$$

where ∂_x^{-1} stands for an inverse operator of $\partial_x = \partial/\partial x$ with the condition $\partial_x\partial_x^{-1} = \partial_x^{-1}\partial_x = 1$, which can be defined as $(\partial_x^{-1}f)(x) = \int_{-\infty}^{x} f(\zeta)\mathrm{d}\zeta$ under the decaying condition at infinity [13]. The dimensions interaction between a long wave along the x-axis and a Riemann wave moving along the y-axis was described by (1.1) who was the subject of several works [5–8, 16]. The method of inverse scattering can be employed to solve Eq. (1.1). It has been demonstrated that (1.1) has an unlimited number of symmetries and a Hamiltonian structure. Inspired by the works of [5–8, 13, 16], we aim to investigate a generalized version of Eq. (1.1) as

$$\delta\, w_{xz} - (\beta w_t + w_{xxx} - \beta ww_x)_y + \beta\left(w_x\partial_x^{-1}w_y\right)_x = 0. \tag{1.2}$$

On making use of $w = u_x$, where $w = w(t,x,y,z)$ then the generalized nonlinear evolution equation is now

$$\delta\, u_{xxz} - (\beta u_{tx} + u_{xxxx} - \beta u_x u_{xx})_y + \beta\left(u_{xx}u_y\right)_x = 0, \tag{1.3}$$

which after expanding, gives

$$\delta\, u_{xxz} - \beta\, u_{txy} - u_{xxxxy} + \lambda\, u_{xx}u_{xy} + \beta\, u_x u_{xxy} + \beta\, u_{xxx}u_y = 0, \tag{1.4}$$

where (δ, β, λ) are real non-zero constants while u is a function of the three scaled spatial variables (x, y, z) and t the temporal variable.

Due to the generalization in (1.4) we end up with novel symmetry reductions and associated group invariant solutions of physical interest. We like to emphasize due the arbitrary non-zero parameters in (1.4) leads to infinitely many conservation laws of (1.4).

The organization is structured as follows. Section 2 is concerned with the Lie point symmetry analysis of (1.4). Conservation laws of (1.4), with their physical ramifications are illustrated in section 3. Finally in section 4 an ansatz methodology is employed to seek further travelling wave solutions of physical interest.

2. Group invariant solutions

Lie symmetry analysis [1–3] is a powerful tool used to study differential equations and understand their underlying symmetries. By analyzing the Lie symmetries of a differential equation, we can identify transformations that leave the equation invariant. These symmetries can provide valuable insights into the behavior and properties of the equation, such as the existence of conservation laws or special solutions. Lie symmetry analysis involves finding transformations that can be applied to differential equations without changing their form. This technique, named after the Norwegian mathematician Sophus Lie, is based on the concept of a symmetry operator. A symmetry operator is a transformation that maps solutions of a differential equation to other solutions. By applying Lie symmetry analysis to a differential equation, we can determine the forms of the symmetry operators and use them to construct new solutions or simplify the equation.

In this section, we aim to compute group-invariant solutions of Eq. (1.4). This primarily attained by first obtaining the admitted generators of Eq. (1.4). The admitted generators are formulated by considering the vector field of the form

$$\mathbf{\Delta} = \xi^1(t,x,y,z,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,z,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,z,u)\frac{\partial}{\partial y} + \xi^4(t,x,y,z,u)\frac{\partial}{\partial z} + \eta(t,x,y,z,u)\frac{\partial}{\partial u}, \tag{2.1}$$

is a Lie point symmetry of (1.4) if

$$\mathbf{\Delta}^{[5]}\left\{\delta\, u_{xxz}-\beta\, u_{txy}-u_{xxxxy}+\lambda\, u_{xx}u_{xy}+\beta\, u_{x}u_{xxy}+\beta\, u_{xxx}u_{y}\right\}\Big|_{(1.4)}=0, \tag{2.2}$$

where $\mathbf{\Delta}^{[5]}$ is the fifth prolongation of (2.1). Applying the fifth extension of Eq. (2.1) to Eq. (1.4) and solving the resulting system of linear partial differential equations leads to the cases shown below.

Case 1. δ, λ, β arbitrary, but not in the form of case 2. In this case Eq. (1.4) admits the following point symmetries:

$$\mathbf{\Delta_1}=\frac{\partial}{\partial t},\ \text{ time translation;}$$

$$\mathbf{\Delta_2}=\frac{\partial}{\partial z},\ \text{ space translation;}$$

$$\mathbf{\Delta_3}=y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},\ \text{ scaling;}$$

$$\mathbf{\Delta_4}=-x\frac{\partial}{\partial x}+2\,y\frac{\partial}{\partial y}-3\,t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u},\ \text{ scaling;}$$

$$\mathbf{\Delta_5}=F\left(t,z\right)\frac{\partial}{\partial u},\ \text{ space \& time dependent shift}$$

$$\mathbf{\Delta_6}=\beta\, G\left(z\right)\frac{\partial}{\partial y}+\delta\, xG'\left(z\right)\frac{\partial}{\partial u},\ \text{ space dependent shift ;}$$

$$\mathbf{\Delta_7}=\left(\beta x\, H_t\left(t,z\right)-\delta y\, H_z\left(t,z\right)\right)\frac{\partial}{\partial u}-\beta\, H\left(t,z\right)\frac{\partial}{\partial x}\ \text{ space \& time dependent shift.}$$

Case 2. $\lambda=2\beta$. Here Eq. (1.4) permits the following point symmetries:

$$\mathbf{\Phi_1}=\frac{\partial}{\partial t},\ \text{ time translation;}$$

$$\mathbf{\Phi_2}=\frac{\partial}{\partial y},\ \text{ space translation;}$$

$$\mathbf{\Phi_3}=\frac{\partial}{\partial z},\ \text{ space translation;}$$

$$\mathbf{\Phi_4}=-x\frac{\partial}{\partial x}+2\,y\frac{\partial}{\partial y}-3\,t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u},\ \text{ scaling;}$$

$$\mathbf{\Phi_5}=F\left(z,t\right)\frac{\partial}{\partial u},\text{space \& time dependent shift;}$$

$$\mathbf{\Phi_6}=-x\frac{\partial}{\partial x}-2\,z\frac{\partial}{\partial z}-3\,t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u},\ \text{ scaling;}$$

$$\mathbf{\Phi_7}=\beta\, G\left(z\right)\frac{\partial}{\partial y}+\delta\, xG'\left(z\right)\frac{\partial}{\partial u},\ \text{ space dependent shift ;}$$

$$\mathbf{\Phi_8}=\left(\beta x\, H_t\left(z,t\right)-\delta y\, H_z\left(z,t\right)\right)\frac{\partial}{\partial u}-\beta\, H\left(z,t\right)\frac{\partial}{\partial x}\ \text{ space \& time dependent shift.}$$

2.1. Invariant solutions for case 1.

In order to obtain symmetry reductions, one has to solve the associated Lagrange equations

$$\frac{dt}{\xi^1(t,x,y,z,u)}=\frac{dx}{\xi^2(t,x,y,z,u)}=\frac{dy}{\xi^3(t,x,y,z,u)}=\frac{dz}{\xi^4(t,x,y,z,u)}=\frac{du}{\eta(t,x,y,z,u)}. \tag{2.3}$$

First we make use of the summation of symmetries Δ_1 and Δ_3. This summation leads to the invariants of the form shown below.

$$f=x,\quad g=\frac{z}{y},\quad h=t-\ln\left(y\right),\quad \theta=u. \tag{2.4}$$

Courtesy of the above invariants one naturally ends up with following nonlinear partial differential equation

$$\begin{aligned} &g\beta\,\theta_f\theta_{ffg} + g\beta\,\theta_{fff}\,\theta_g + g\lambda\,\theta_{ff}\theta_{fg} - g\beta\,\theta_{fgh} + \beta\,\theta_f\theta_{ffh} \\ &\quad + \beta\,\theta_{fff}\theta_h + \lambda\,\theta_{ff}\theta_{fh} - g\,\theta_{ffffg} - \beta\,\theta_{fhh} - \delta\,\theta_{ffg} - \theta_{ffffh} = 0. \end{aligned} \tag{2.5}$$

It can be shown that the point symmetries of (2.5) are as follows.

$$\begin{aligned} &\mathbf{\Psi_1} = \frac{\partial}{\partial h}, \quad \mathbf{\Psi_2} = g^2\,\beta\,\frac{\partial}{\partial g} + g\beta\,\frac{\partial}{\partial h} - f\delta\frac{\partial}{\partial \theta}, \quad \mathbf{\Psi_3} = F_1\,(g,h)\,\frac{\partial}{\partial \theta}, \\ &\mathbf{\Psi_4} = F_2\,(g,h)\,\frac{\partial}{\partial f} - f\left(\frac{\partial}{\partial h}F_2\,(g,h)\right)\frac{\partial}{\partial \theta}. \end{aligned}$$

The linear combination of $\mathbf{\Psi_1}$ and $\mathbf{\Psi_3}$ generates the invariant that are shown below.

$$X = f, \quad Y = g, \quad \psi = \theta - h. \tag{2.6}$$

The above invariant lead to the following nonlinear partial different equation that is illustrated below.

$$Y\,\psi_{XXXXY} - Y\beta\,\psi_{XXX}\,\psi_Y - Y\beta\,\psi_X\,\psi_{XXY} - Y\lambda\,\psi_{XX}\,\psi_{XY} - \beta\,\psi_{XXX} + \delta\,\psi_{XXY} = 0. \tag{2.7}$$

The point symmetries of (2.7) are as depicted below.

$$\begin{aligned} &\chi_{\mathbf{1}} = \frac{\partial}{\partial \psi}, \quad \chi_{\mathbf{2}} = \frac{\partial}{\partial X}, \quad \chi_{\mathbf{3}} = -X\,\frac{\partial}{\partial X} - 2Y\,\frac{\partial}{\partial Y} + (\psi + \ln{(Y)})\,\frac{\partial}{\partial \psi}, \\ &\chi_{\mathbf{4}} = -Y^2\,\beta\,\frac{\partial}{\partial Y} + (X\delta + Y\beta)\,\frac{\partial}{\partial \psi}. \end{aligned}$$

We employ symmetry $\chi_{\mathbf{4}}$ and it naturally produces two invariant that are demonstrated below,

$$\nu = X, \quad P = \psi - \frac{X\delta}{Y\beta} + \ln{(Y)}\,.$$

Consequently, using the above invariants one ends up with linear ordinary differential equation that is illustrated below

$$\nu\beta P'''\,(\nu) + \lambda P''\,(\nu) = 0. \tag{2.8}$$

The integration of the above linear ordinary differential equation and reverting back to our original variables x, y, z, t leads to the exotic invariant solution to Eq. (1.4) that takes the form as shown below.

$$u(x,y,z,t) = \frac{1}{\beta\,z}\left(\beta z c_1 + \beta x z\,c_2 + \beta\,x^{\frac{2\,\beta-\lambda}{\beta}}\,z c_3 - \beta z\,\ln{(y)} - \beta z\,\ln\left(\frac{z}{y}\right) + \beta t z + \delta x y\right). \tag{2.9}$$

In similar spirit of manipulating the underlying point symmetries of (1.4) one ends up with the following solutions:

$$u(x,y,z,t) = \frac{1}{\sqrt[3]{t}\,z}\left(\Theta\,\sqrt[3]{t}\,z\,\ln{(y)} + \Theta\,\sqrt[3]{t}\,z\,\ln\left(\frac{z}{y}\right) + \Omega\delta\,\sqrt[3]{t}\,y + \sqrt{\frac{\Pi}{t^{\frac{2}{3}}}}\,z c_1 - 2\Theta\,\sqrt[3]{t}\,z\right), \tag{2.10}$$

$$\begin{aligned} \Pi = {}&\Omega^2\beta^2\,\ln^2{(y)} + 2\Omega^2\beta^2\,\ln{(y)}\ln\left(\frac{z}{y}\right) + \Omega^2\beta^2\,\ln^2\left(\frac{z}{y}\right) + 4\Omega^2\beta^2\,\ln{(y)} + 4\Omega^2\beta^2\,\ln\left(\frac{z}{y}\right) \\ &- 2\Omega\beta x\,\ln{(y)} - 2\Omega\beta x\,\ln\left(\frac{z}{y}\right) + 4\Omega^2\beta^2 - 4\Omega\beta x + t^{\frac{2}{3}} + x^2, \end{aligned} \tag{2.11}$$

$$u(x,y,z,t) = \frac{1}{z\beta}\left(\beta z\,c_1 - (t-x)\,\beta z\,c_2 + \beta\,(x-t)^{\frac{2\,\beta-\lambda}{\beta}}\,z\,c_3 + \Omega\beta z\,\ln{(z)} - \delta y t + \delta y x\right). \tag{2.12}$$

Group invariant solution (2.10) is obtained from invoking the linear combination of $\mathbf{\Delta_3}$, $\mathbf{\Delta_5}$, $\mathbf{\Delta_7}$ with the arbitrary functions are confined to be non-zero constants $F(t,z) = \Theta$ and $H\,(t,z) = \Omega$. Similarity solution (2.12) is a consequence of the summation of $\mathbf{\Delta_3}$, $\mathbf{\Delta_5}$ with the restriction of $F\,(t,z) = \Omega$, where Ω is a non-zero constant.

2.2. **Invariant solutions for case 2.** We now turn our attention to case 2 and as a result of manipulation of the Lagrange Eqs. (2.3) leads to the observations depicted below.

Observation 1. The addition of symmetries $\mathbf{\Phi_5}, \mathbf{\Phi_7}$, and $\mathbf{\Phi_8}$ and letting $F_1(z,t) = T$, $G(z) = Y$, and $H(z,t) = \Omega$ then solving the resulting Lagrange equations, gives four invariants, viz.,

$$f = z, \quad g = t, \quad h = -\frac{\Omega\, y - Y\, x}{Y}, \quad \theta = -\frac{-Y\,\beta\, u + T\, y}{Y\,\beta}. \tag{2.13}$$

The substitution of the above invariants into Eq. (1.4) gives

$$2\Omega\beta\,\theta_{hhh}\,\theta_h + 2\Omega\beta\,\theta_{hh}^2 - \Omega\beta\,\theta_{ghh} - Y\delta\,\theta_{fhh} - T\theta_{hhh} - \Omega\,\theta_{hhhhh} = 0, \tag{2.14}$$

where θ is a function of (f, g, h). Solving Eq. (2.14) and make use of the invariants (2.13), we conclude that the group-invariant solution of Eq. (1.4) is

$$\begin{aligned} u(t,x,y,z) = {} & \frac{1}{2\Omega Y\beta\, c_4}\Bigg(8\Omega\left(\Omega\, y - Yx\right)c_4^3 + 12c_4^2\,\tanh\left(-zc_2 - tc_3 + \frac{(\Omega\, y - Y\, x)\, c_4}{Y} - c_1\right)\Omega Y \\ & -6c_4^2\ln\left(-\tanh\left(-zc_2 - tc_3 + \frac{(\Omega\, y - Y\, x)\, c_4}{Y} - c_1\right) - 1\right)\Omega Y \\ & +6c_4^2\,\ln\left(-\tanh\left(-zc_2 - tc_3 + \frac{(\Omega\, y - Y\, x)\, c_4}{Y} - c_1\right) + 1\right)\Omega Y + 2\Omega\, Y\beta\, F_1\left(z,t\right)\, c_4 \\ & -\Omega^2\beta y c_3 - \Omega Y\,\delta y c_2 + \Omega Y\,\beta x c_3 + Y^2\delta x c_2 + \Omega T\, y c_4 + TY\, x c_4\Bigg), \end{aligned} \tag{2.15}$$

where $c_1, c_2, c_3, c_4, \Omega, Y, \beta, \delta$ are arbitrary constants and $F_1\left(z,t\right)$ an arbitrary fuction of z, t.

Observation 2. We now choose symmetries $\Xi\,\mathbf{\Phi_1}$, $R\,\mathbf{\Phi_3}$, $\mathbf{\Phi_5}, \mathbf{\Phi_7}$ and $\mathbf{\Phi_8}$. Again, letting $F(z,t) = T$, $G(z) = Y$ and $H(z,t) = \Omega$ and solve the characteristics equation to get the following four invariants

$$f = -\frac{R\, y - Y\beta z}{Y\beta}, \quad g = \frac{Y\,\beta\, t - \Xi\, y}{Y\,\beta}, \quad h = -\frac{\Omega\, y - Y\, x}{Y}, \quad \theta = -\frac{T\, y - Y\beta u}{Y\,\beta}.$$

The insertion of these invariants into Eq. (1.4) and solve the resulting partial deferential equation, yields

$$\begin{aligned} u(t,x,y,z) = {} & -\frac{1}{Y\beta}\Bigg(\tanh\Bigg(-\frac{T\,(R\, y - Y\beta z)\, c_2}{Y^2\delta\beta} + \frac{(\Omega Y\delta\beta - RT)\,(Y\beta t - \Xi\, y)\, c_2}{\Xi Y^2\delta\beta} + \frac{(\Omega\, y - Y\, x)\, c_2}{Y} \\ & -c_1\Bigg)\, Y\beta c_4 - Y\beta c_3 - T\, y\Bigg), \end{aligned} \tag{2.16}$$

as the group-invariant solution of Eq. (1.4), where $c_1, c_4, c_5, c_6, \Omega, \Xi, Y, T, R, \beta, \delta$ are arbitrary constants. **Observation 3**. Lastly, we take the linear combination of symmetries $\Xi\mathbf{\Phi_1}$, $Y\mathbf{\Phi_2}$, $R\mathbf{\Phi_3}$ together with symmetry $\Omega\mathbf{\Phi_8}$ where $\beta\, H(z,t)$ is taken as Ω, gives

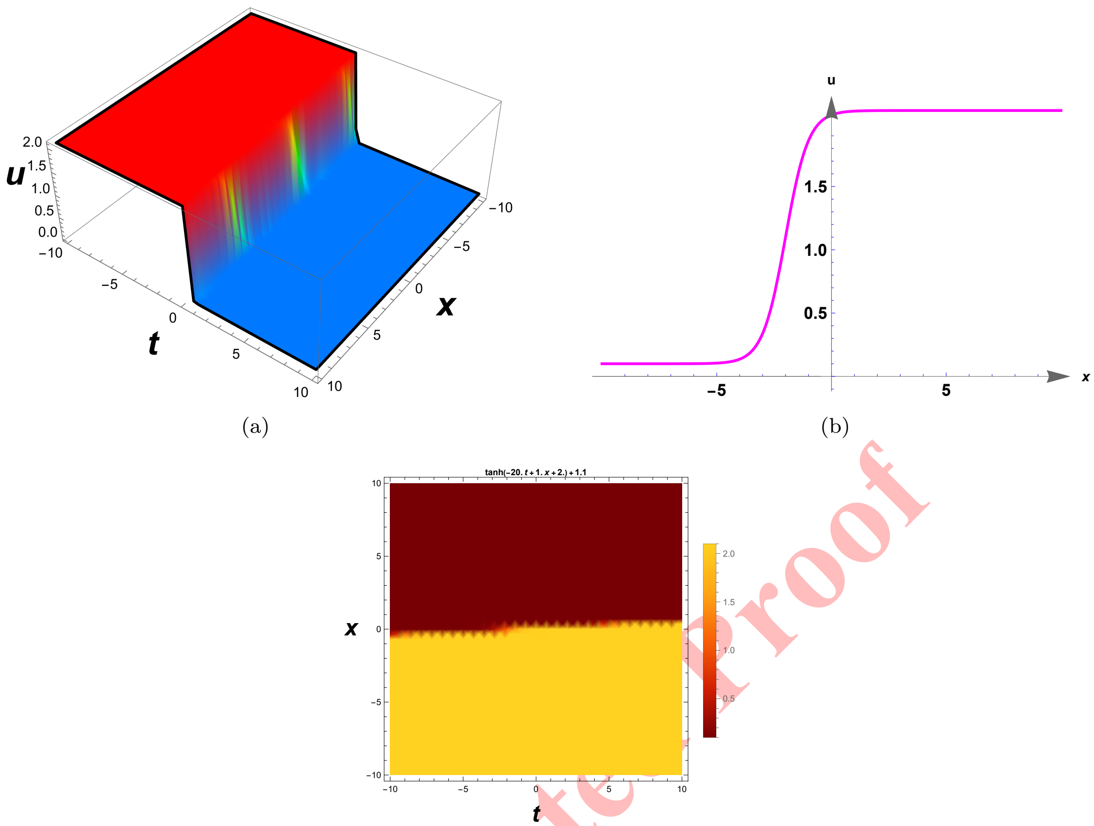
$$f = -\frac{R\, y - Y\, z}{Y}, \quad g = \frac{Y\, t - \Xi\, y}{Y}, \quad h = -\frac{\Omega\, y - Y\, x}{Y}, \quad \theta = u,$$

as the invariants.

Solving Eq. (1.4) and reverting back into the original variables, we conclude that the group-invariant solution of Eq. (1.4) is

$$\begin{aligned} u(t,x,y,z) = {} & -\frac{1}{\beta}\Bigg(6\,\tanh\Bigg(\frac{\left(4\,\Omega\, c_4^4 + 4\,\Xi\, c_3 c_4^3 + \Omega\,\beta c_3 c_4 + \Xi\,\beta\, c_3^2\right)(R\, y - Y\, z)}{\left(4\, R\, c_4^3 + R\,\beta c_3 + Y\,\delta\, c_4\right) Y} \\ & + \frac{(Y\, t - \Xi\, y)\, c_3}{Y} - \frac{(\Omega\, y - Y\, x)\, c_4}{Y} + c_1\Bigg)\, c_4 - \beta c_5\Bigg), \end{aligned} \tag{2.17}$$

where $c_1, c_3, c_4, c_5, \Omega, \Xi, Y, T, R, \beta, \delta$ are arbitrary constants.



(a) (b)

(c)

Figure 1. Profiles of solutions (2.16) with $T = 0.1, Y = 1, R = 1, \Xi = 0.1, \Omega = 1, \delta = -0.1$, $\beta = 1$, $y = 1$, $z = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$.

3. Local conservation laws

This section is dedicated to the construction of conservation laws [28–30] of Eq. (1.4) courtesy of multiplier method. Conservation laws play a fundamental role in understanding and analyzing systems governed by partial differential equations. They provide a mathematical framework for studying the behavior of physical quantities, such as mass, momentum, and energy, as they evolve in space and time. Conservation laws ensure that these quantities are preserved within a given domain, meaning that their total amount remains constant over time. They dictate the principles of mass, momentum, and energy balance, allowing us to quantify how these quantities are transferred, transformed, or stored within a system. By applying the laws of conservation to a system described by partial differential equations, we can gain insights into the underlying dynamics and predict how the system will evolve in response to external forces or internal interactions. These conservation laws are derived from the fundamental principles of physics and apply to a wide range of phenomena, including fluid dynamics, heat transfer, electromagnetism.

In order to compute the multiplier for Eq. (1.4), one needs to solve over-determined system of linear partial differential equations that arise naturally from the expansion of

$$\frac{\delta(\Lambda E)}{\delta u} = 0, \tag{3.1}$$

where $\frac{\delta}{\delta u}$ is the Euler Lagrange operator, Λ denotes the multiplier function which in this context is assumed to be of order zero and E represent Eq. (1.4). The analysis of Eq. (3.1) prompts the following lemma.

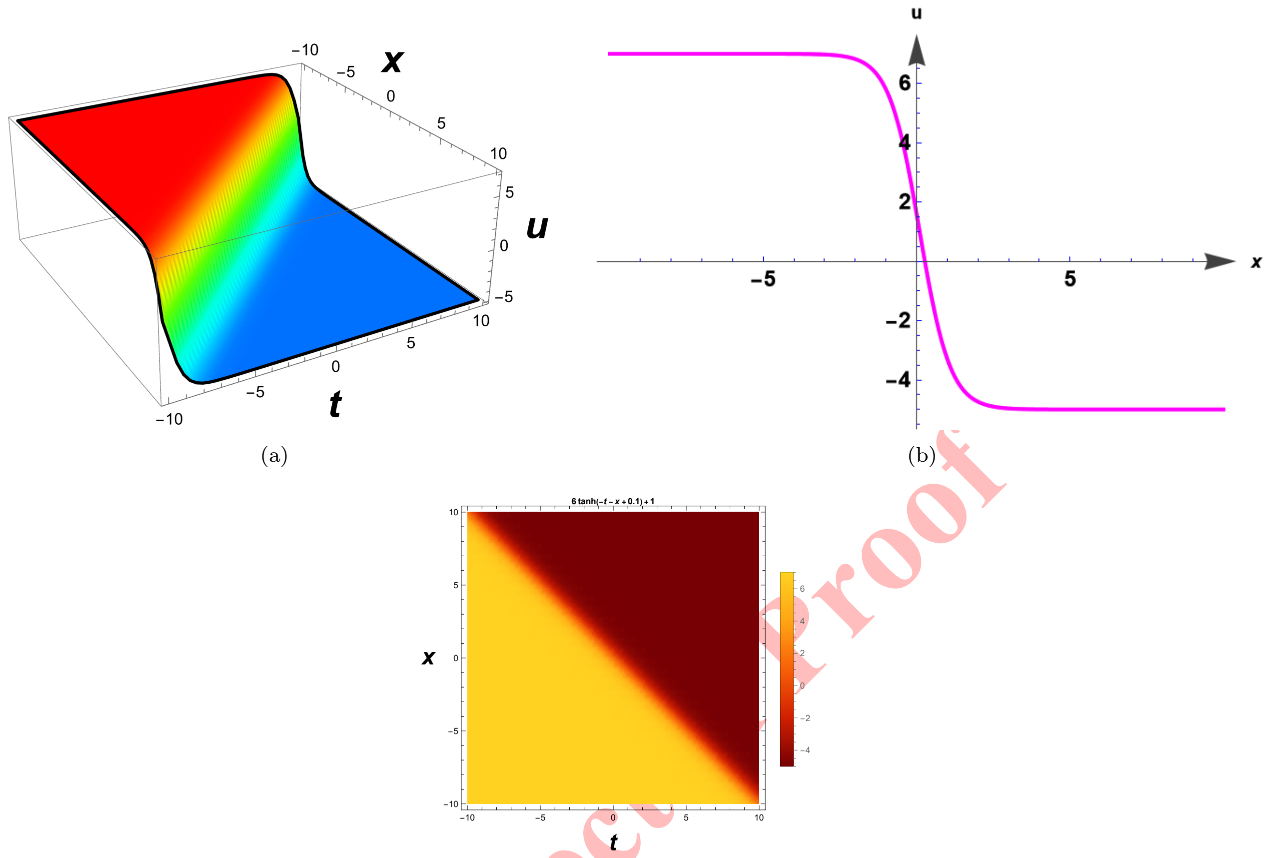


Figure 2. Profiles of solutions (2.17) with $\Upsilon = 1, R = 1, \Xi = 0.1, \Omega = 1, \delta = -0.1$, $\beta = 1$, $y = 1$, $z = 1$, $c_1 = 1$, $c_3 = 1$, $c_4 = 1$, $c_5 = 1$.

Lemma 3.1. *Let Λ be a zeroth order conservation law multiplier, then a generalized (3+1)-dimensional nonlinear evolution equation admits infinitely many zeroth order multipliers of the form*

$$\Lambda = uF_1(t) + \frac{1}{4}x^2 F_1'(t) + F_2(t, y, z) + (F_3(y, z) + F_4(t, z))\, x,$$

if and only if $\lambda = 2\beta$ and (F_1, F_2, F_3, F_4) being arbitrary functions with respect to their arguments.

Proof. A straightforward but lengthy computation from Eq. (3.1). □

The application of the above lemma reveals the following conservation laws:

$$T_1^t = -\frac{1}{12}\left(x^2 u_{xy} F'(t) + 4\, u u_{xy} F(t) - 2\, u_x u_y F(t) - x u_y F'(t)\right)\beta,$$

$$\begin{aligned} T_1^x = &-\frac{1}{6}\delta x u_z F'(t) - \frac{1}{12}\beta x^2 u_{ty} F'(t) + \frac{1}{6}\delta x^2 u_{xz} F'(t) + \frac{1}{6}\beta u u_y F'(t) + \frac{1}{6}\beta u_t u_y F(t) \\ &- \frac{1}{3}\delta u_x u_z F(t) + \frac{2}{3}\delta u u_{xz} F(t) - \frac{1}{3}\beta u u_{ty} F(t) - \frac{1}{9}\beta u^2 u_{xxy} F(t) - \frac{5}{9}\beta u_x^2 u_y F(t) \\ &- \frac{1}{5}u_{xy} F'(t) - \frac{2}{5}u_{xx} u_{xy} F(t) + \frac{1}{5}u_{xxx} u_y F(t) + \frac{3}{10}x u_{xxy} F'(t) + \frac{3}{5}u_x u_{xxy} F(t) \end{aligned}$$

$$
\begin{aligned}
&-\frac{4}{5}uu_{xxxy}F(t)-\frac{1}{5}x^2u_{xxxy}F'(t)+\frac{1}{24}u_y\beta F''(t)+\frac{5}{24}\beta x^2u_{xx}u_yF'(t)-\frac{1}{24}\beta x^2uu_{xxy}F'(t)\\
&+\frac{1}{6}\beta x^2u_xu_{xy}F'(t)+\frac{8}{9}\beta uu_xu_{xy}F(t)+\frac{7}{9}\beta uu_{xx}u_yF(t)-\frac{1}{3}\beta xu_xu_yF'(t)+\frac{1}{6}\beta xuu_{xy}F'(t),\\
T_1^y=&\ \frac{1}{24}\beta x^2u_xF''(t)-\frac{1}{12}\beta x^2u_{tx}F'(t)-\frac{1}{6}\beta xuF''(t)+\frac{1}{12}\beta xu_tF'(t)+\frac{1}{9}\beta u^2u_{xxx}F(t)\\
&-\frac{1}{12}\beta xu_x^2F'(t)-\frac{1}{3}\beta uu_{tx}F(t)+\frac{1}{6}\beta u_tu_xF(t)-\frac{1}{10}u_{xx}^2F(t)-\frac{1}{10}u_{xx}F'(t)\\
&+\frac{1}{3}\beta uu_xu_{xx}F(t)+\frac{1}{24}\beta x^2uu_{xxx}F'(t)+\frac{1}{8}\beta x^2u_xu_{xx}F'(t)-\frac{1}{12}\beta xuu_{xx}F'(t)\\
&-\frac{1}{5}uu_{xxxx}F(t)-\frac{1}{9}\beta u_x^3F(t)+\frac{1}{5}u_xu_{xxx}F(t)-\frac{1}{20}x^2u_{xxxx}F'(t)+\frac{1}{10}u_{xxx}F'(t),\\
T_1^z=&\ \frac{1}{12}\delta x^2u_{xx}F'(t)+4uu_{xx}F(t)-2u_x^2F(t)-2xu_xF'(t)+2uF'(t);\\
T_2^t=&\ \frac{1}{6}\beta\left(u_xF_y(t,y,z)-2u_{xy}F(t,y,z)\right),\\
T_2^x=&-\frac{1}{6}\beta uu_{xx}F_y(t,y,z)-\frac{1}{6}\beta uu_{xxy}F(t,y,z)+\frac{2}{3}\beta u_xu_{xy}F(t,y,z)\\
&+\frac{5}{6}\beta u_{xx}u_yF(t,y,z)-\frac{1}{3}\beta uF_{ty}(t,y,z)-\frac{1}{6}\beta u_x^2F_y(t,y,z)+\frac{1}{6}\beta u_tF_y(t,y,z)-\frac{1}{3}\delta u_xF_z(t,y,z)\\
&-\frac{1}{3}\beta u_{ty}F(t,y,z)+\frac{2}{3}\delta u_{xz}F(t,y,z)+\frac{1}{6}u_y\beta F_t(t,y,z)+\frac{1}{5}u_{xxx}F_y(t,y,z)-\frac{4}{5}u_{xxxy}F(t,y,z),\\
T_2^y=&\ \frac{1}{6}\beta uu_{xxx}F(t,y,z)+\frac{1}{2}\beta u_xu_{xx}F(t,y,z)-\frac{1}{5}u_{xxxx}F(t,y,z)-\frac{1}{3}\beta u_{tx}F(t,y,z)+\frac{1}{6}\beta u_xF_t(t,y,z),\\
T_2^z=&\ \frac{1}{3}\delta u_{xx}F(t,y,z);\\
T_3^t=&\ \frac{1}{6}\beta xu_xF_y(y,z)+2xu_{xy}F(y,z)+2uF_y(y,z)-u_yF(y,z),\\
T_3^x=&-\frac{1}{3}\beta xu_{ty}F(y,z)+\frac{2}{3}\delta xu_{xz}F(y,z)-\frac{1}{3}\delta xu_xF_z(y,z)+\frac{1}{6}\beta xu_tF_y(y,z)\\
&+\frac{1}{3}\beta uu_xF_y(y,z)-\frac{1}{6}\beta xu_x^2F_y(y,z)+\frac{1}{3}\beta uu_{xy}F(y,z)-\frac{2}{3}\beta u_xu_yF(y,z)\\
&-\frac{2}{5}u_{xx}F_y(y,z)+\frac{3}{5}u_{xxy}F(y,z)-\frac{1}{6}\beta xuu_{xx}F_y(y,z)-\frac{1}{6}\beta xuu_{xxy}F(y,z)+\frac{2}{3}\beta xu_xu_{xy}F(y,z)\\
&+\frac{5}{6}\beta xu_{xx}u_yF(y,z)+\frac{1}{5}xu_{xxx}F_y(y,z)-\frac{4}{5}xu_{xxxy}F(y,z)-\frac{1}{3}\delta u_zF(y,z)+\frac{2}{3}\delta uF_z(y,z),\\
T_3^y=&-\frac{1}{3}\beta xu_{tx}F(y,z)-\frac{1}{6}\beta uu_{xx}F(y,z)-\frac{1}{6}\beta u_x^2F(y,z)+\frac{1}{6}\beta u_tF(y,z)\\
&-\frac{1}{5}xu_{xxxx}F(y,z)+\frac{1}{5}u_{xxx}F(y,z)+\frac{1}{6}\beta xuu_{xxx}F(y,z)+\frac{1}{2}\beta xu_xu_{xx}F(y,z),\\
T_3^z=&\ \frac{1}{3}\delta xu_{xx}F(y,z)-\frac{1}{3}\delta u_xF(y,z);\\
T_4^t=&-\frac{1}{3}\beta xu_{xy}F(t,z)+\frac{1}{6}\beta u_yF(t,z),\\
T_4^x=&-\frac{4}{5}xu_{xxxy}F(t,z)-\frac{1}{3}\delta u_zF(t,z)+\frac{2}{3}\delta uF_z(t,z)+\frac{1}{3}\beta uu_{xy}F(t,z)\\
&-\frac{2}{3}\beta u_xu_yF(t,z)-\frac{1}{3}\beta xu_{ty}F(t,z)+\frac{2}{3}\delta xu_{xz}F(t,z)-\frac{1}{3}\delta xu_xF_z(t,z)\\
&+\frac{1}{6}\beta xu_yF(t,z)+\frac{3}{5}u_{xxy}F(t,z)-\frac{1}{6}\beta xuu_{xxy}F(t,z)+\frac{2}{3}\beta xu_xu_{xy}F(t,z)+\frac{5}{6}\beta xu_{xx}u_yF(t,z),
\end{aligned}
$$

$$T_4^y = -\frac{1}{6}\beta u_x^2 F(t,z) + \frac{1}{6}\beta u_t F(t,z) - \frac{1}{5}x u_{xxxx} F(t,z) - \frac{1}{3}\beta u F_t(t,z) - \frac{1}{3}\beta x u_{tx} F(t,z)$$
$$+ \frac{1}{6}\beta x u_x F_t(t,z) - \frac{1}{6}\beta u u_{xx} F(t,z) + \frac{1}{5}u_{xxx} F(t,z) + \frac{1}{6}\beta x u u_{xxx} F(t,z) + \frac{1}{2}\beta x u_x u_{xx} F(t,z),$$
$$T_4^z = \frac{1}{3}\delta x u_{xx} F(t,z) - \frac{1}{3}\delta u_x F(t,z).$$

It is noteworthy to mention that a generalized (3+1)-dimensional nonlinear evolution Eq. (1.4) admits an endless number of local conservation laws because of the arbitrary elements included in the conserved vectors. It is also worth pointing out that a commendable inspection demonstrated that the zeroth order multipliers is indeed identical to the first order, second order and third order multipliers. Hence one will end up with identical conservation laws. It remains to be systematically investigated elsewhere whether the n^{th} order multiplier is embedded in the zeroth order multiplier.

3.1. **Exact solutions using the extended tan method.** This section's goal is to introduce the extended tan method's approach for precisely solving the nonlinear evolution equations. We review the extended tan method's fundamental phases in brief. The extended tan function method's basic premise is to suppose that the solution to (1.4) may be expressed as follows

$$u(x,y,z,t) = F(p) \quad p = \kappa(x+y+z-\omega t). \tag{3.2}$$

From (3.2) we obtain the ordinary nonlinear differential equation

$$\kappa^5 F^{(5)}(p) - \beta\kappa^3\omega F^{'''}(p) - \delta\kappa^3 F^{'''}(p) - \kappa^4\lambda F''(p)^2 - 2\beta\kappa^4 F^{'''}(p)F'(p) = 0, \tag{3.3}$$

which has a solution of the form

$$F(p) = \sum_{i=-M}^{M} A_i H(p)^i, \tag{3.4}$$

where

$$H(p) = \tan(p),$$

satisfies the equation

$$H'(p) = 1 + H(p)^2. \tag{3.5}$$

The homogeneous balance approach between the highest order derivative and the least order nonlinear term that occur in (3.3) will be used to calculate the positive integer M. The parameters to be computed are A_i. Since the balancing process in this case yields $M = 1$, the solution to (3.3) have the following form:

$$F(p) = A_{-1}H^{-1} + A_0 + A_1 H. \tag{3.6}$$

With the aid of Mathematica, we obtain the following three cases :

Case 1

$$\beta = -\frac{6\kappa}{A_{-1}},$$
$$\delta = -\frac{2\left(2\kappa^2 A_{-1} - 3\kappa\omega\right)}{A_{-1}},$$
$$\lambda = -\frac{12\kappa}{A_{-1}},$$
$$A_1 = 0.$$

Case 2

$$\beta = \frac{6\kappa}{A_1},$$

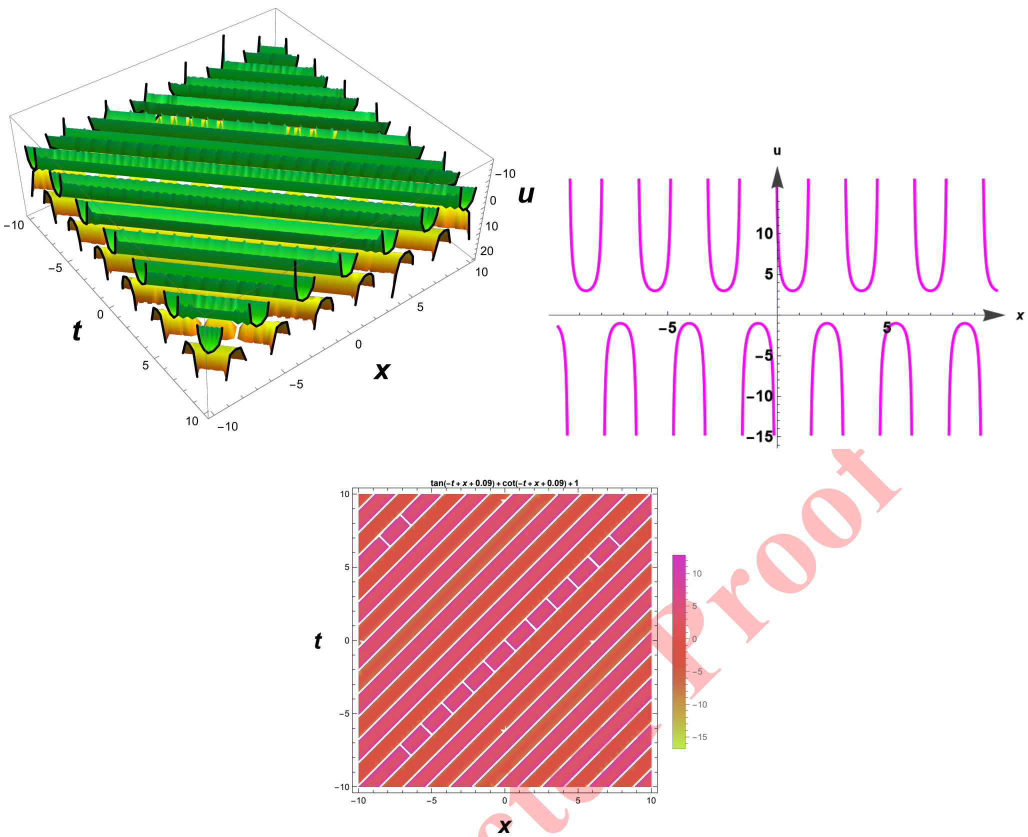


Figure 3. Profiles of solution (3.7) with $\delta = 1$, $\omega = 1$, $A_{-1} = 1$, $A_0 = 0$, $A_1 = 1$.

$$\delta = -\frac{2\left(2\kappa^2 A_1 + 3\kappa\omega\right)}{A_1},$$
$$\lambda = \frac{12\kappa}{A_1},$$
$$A_{-1} = 0.$$

Case 3

$$\delta = -\omega\beta - 16\kappa^2,$$
$$\lambda = 2\beta,$$
$$A_{-1} = -\frac{6\kappa}{\beta},$$
$$A_1 = \frac{6\kappa}{\beta}.$$

Thus, a solution of (1.4) is

$$u(x, y, z, t) = A_{-1}\cot(p) + A_0 + A_1 \tan(p), \tag{3.7}$$

where $p = \kappa(x + y + z - \omega t)$. The graphically representation of this solution is given in Figure 3.

3.2. **Exact solutions using the extended Tanh method.** We now apply the extended tanh function method in this part. This technique's fundamental premise is that the solution to (3.3) may be expressed as follows

$$F(p) = \sum_{i=-M}^{M} A_i H(p)^i, \tag{3.8}$$

where $H(p)$ satisfies the Riccati equation

$$H'(p) = 1 - H^2(p), \tag{3.9}$$

whose solution is

$$H(p) = \tanh(p).$$

The balancing process yields $M = 1$, thus we compute the positive integer M in the same manner as described in section 3.1. As a result, the solutions of (3.3) have the form

$$F(p) = A_{-1}H^{-1} + A_0 + A_1 H. \tag{3.10}$$

Again, we obtain the following three cases with the aid of Mathematica:

Case 1

$$\beta = -\frac{6\kappa}{A_{-1}},$$
$$\delta = \frac{2\left(2\kappa^2 A_{-1} + 3\kappa\omega\right)}{A_{-1}},$$
$$\lambda = -\frac{12\kappa}{A_{-1}},$$
$$A_1 = 0.$$

Case 2

$$\beta = -\frac{6\kappa}{A_1},$$
$$\delta = \frac{2\left(2\kappa^2 A_1 + 3\kappa\omega\right)}{A_1},$$
$$\lambda = -\frac{12\kappa}{A_1},$$
$$A_{-1} = 0.$$

Case 3

$$\delta = 16\kappa^2 - \beta\omega,$$
$$\lambda = 2\beta,$$
$$A_{-1} = -\frac{6\kappa}{\beta},$$
$$A_1 = -\frac{6\kappa}{\beta}.$$

Accordingly, a solution of (1.4) is

$$u(t, x, y, z) = A_{-1}\coth(p) + A_0 + A_1 \tanh(p), \tag{3.11}$$

where $p = \kappa(x + y + z - \omega t)$ and the corresponding profile representation of solution (3.11) is given in Figure 4.

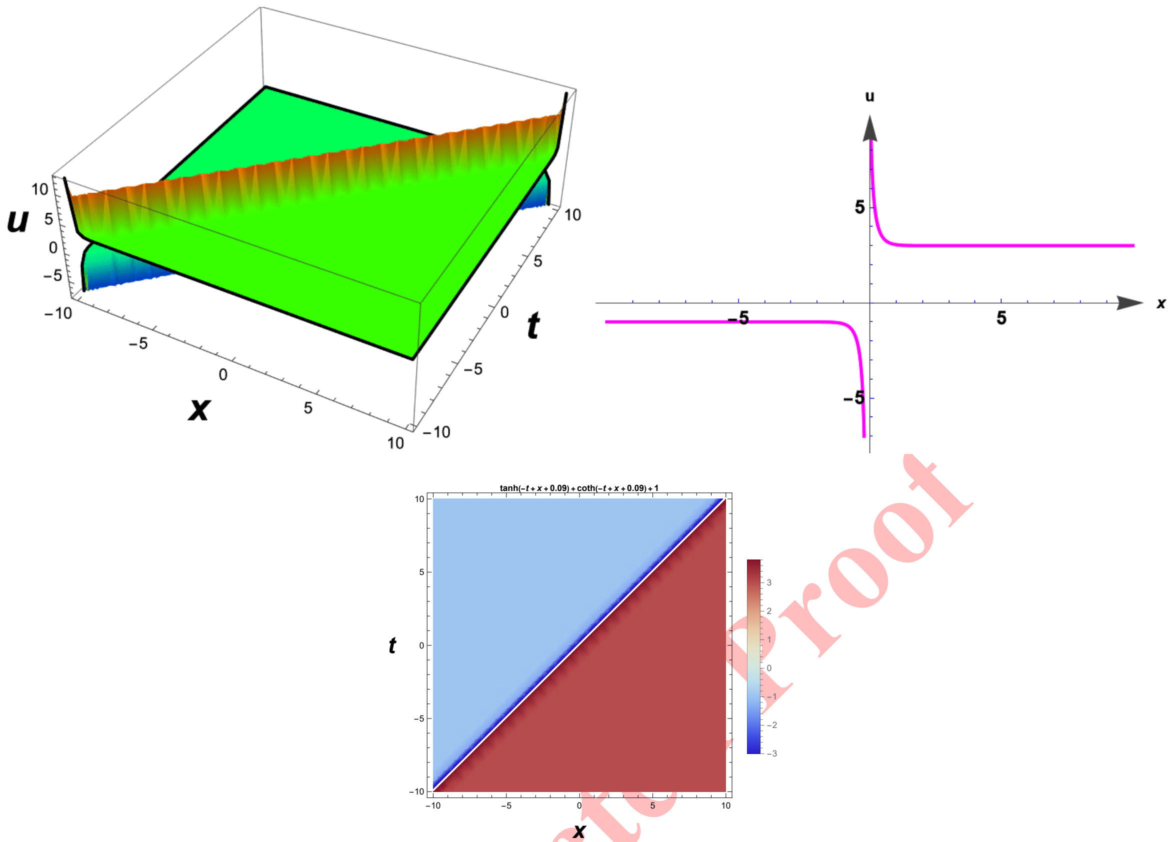


Figure 4. Profiles of solutions (3.11) with $\delta = 1$, $\omega = 1$, $A_{-1} = 1$, $A_0 = 0$, $A_1 = 1$.

4. Conclusions

Examining exact solutions for nonlinear evolution equations has consistently been crucial to the study of mathematical physics in areas including condensed matter physics, fluid dynamics and nonlinear optics. As a crucial consequence, we have investigated a generalized (3+1)-dimensional nonlinear evolution equation. It was found that this underlying equation admits infinitely many point symmetries and conservation laws. Lie symmetry method along with ansatz methods led to travelling wave solutions of physical interest. In addition, we also derived infinitely many conservation laws of the underlying equation via the multiplier method. The solution dynamics were shown graphically and it is anticipated that these findings may be used to better understand how nonlinear waves propagate in a variety of nonlinear physical systems, including fluid mechanics.

Declarations

Author Contributions This work was written with equal contributions from each author. The final manuscript was read and approved by all authors.

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