Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-27
DOI:10.22034/cmde.2025.64013.2879

Efficient numerical method for pricing option with underlying asset follows a fractal stochastic process

Santoshi Tarei, Ankur Kanaujiya, and Jugal Mohapatra*
Department of Mathematics, National Institute of Technology Rourkela, India.

Abstract

In this paper, three compact finite difference schemes on uniform mesh to solve the fractional Black-Scholes partial differential equation for European type option are presented. The time-fractional derivative is approximated by $L1$ formula, $L1-2$ formula and $L2-1_\sigma$ formula respectively, and three compact difference schemes with orders $O((\Delta t)^{2-\alpha}+(\Delta x)^4)$, $O((\Delta t)^{3-\alpha}+(\Delta x)^4)$ and $O((\Delta t)^2+(\Delta x)^4)$ are constructed. The stability and convergence analysis of the proposed method is also analyzed. Finally, a numerical example is carried out to verify the accuracy and effectiveness of the proposed methods, and the comparisons of these schemes are given. The paper also provides numerical studies including the effect of fractional orders and the effect of different parameters on option price in the time-fractional framework.

Keywords. Time fractional Black-Scholes equation, Caputo derivative, Convergence analysis.
2010 Mathematics Subject Classification. 26A33, 34K37, 65M06.

1. Introduction

Option pricing is an emerging research topic in the field of computational finance owing to its growing applications in stock and commodity these days. The pioneer of option pricing theory goes back to Fischer Black and Myron Scholes when they proposed an explicit formula to evaluate the value of an option in 1973. The study of the Black-Scholes (BS) equation for European option pricing earned the Nobel Prize in Economic Sciences in 1997. The BS model is an attempt to simplify the markets for both financial assets and derivatives into a set of mathematical rules. The model serves as the basis for a wide range of analyses of markets. In [4, 11, 20, 21] by replacing the standard Brownian motion with fractional Brownian motion in the BS equation, fractional BS models were derived. The emergence of fractal-based asset pricing models after the financial market's fractal nature was discovered has accelerated the search for precise and reliable numerical methods to solve these complex but helpful asset pricing models.

Although there are many numerical approaches for traditional asset pricing models, there are very few for fractional calculus-based models. Since fractional models are, in some ways, a generalization of classical models, a number of numerical algorithms for solving classical models already exist. The fractional Black-Scholes model and the classical Black-Scholes model differ significantly from one another in terms of the Black-Scholes model in that the derivatives involved in the former are globally defined and the latter can only capture localized information about a function in a point-wise manner. As a result, among other things, the non-locality of fractional derivatives-based models considerably adds to the complexity of the design, analysis, and implementation of the solution methods for fractional models.

In literature, Dura and Mosneagu [5] used the implicit, semi-implicit and explicit methods to solve time fractional Black-Scholes (TFBS) equation. Kumar et al. [9], presented a numerical algorithm to investigate the TFBS equation with boundary conditions for a European option problem by employing the homotopy perturbation method and homotopy analysis method. Cen et al. [23], converted the TFBS equation into an integral–differential equation with

Received: 15 October 2024 ; Accepted:02 September 2025.
* Corresponding author. Email: jugal@nitrkl.ac.in.

a weakly singular kernel, and for time discretization they used an integral discretization scheme on an adapted mesh. Roul [16], proposed a numerical approach based on quintic B-spline and backward Euler methods to study the TFBS equation. An et al. [3], proposed a spectral method based on Jacobi polynomials and Fourier-like basis functions to solve the TFBS equation. Singh and Kumar [18], developed a numerical scheme of TFBS using the exponential B-spline collocation to discretize in space and a finite difference method to discretize in time. Kumar et al. [10], produce an analytical solution for fractional BS equation using the Laplace homotopy perturbation method.

Song and Wang [19] derived an implicit difference scheme to solve the TFBS equation of European put option having convergence rate $O(\Delta t + (\Delta x)^2)$ (Δt denotes temporal step size, Δx denotes spatial step size). In [22], Zhang et al. proposed an implicit difference scheme for the TFBS equation, demonstrating that the method attains a convergence rate of $O((\Delta t)^{(2-\alpha)} + (\Delta x)^2)$. Roul and Goura [17], derived a numerical scheme using the compact finite difference method for space derivatives and the $L1$ formula for time fractional derivative. From above, we can see that many numerical schemes having lower order of convergence is there to solve the TFBS equation. In this paper, we will propose three compact difference schemes for the TFBS model to improve the numerical accuracy.

The outline of the paper is as follows: In section 2.1, we presents the basic concept and convert the original TFBS model to a suitable form. In Section 3, three different schemes for temporal discretization have been given. In section 4, a scheme for spatial discretization is presented. In section 5, three different numerical schemes are shown for solving the TFBS model. The stability and convergence of our suggested numerical technique are shown in section 6. Numerical examples are performed in section 7 to confirm the great accuracy and effectiveness of our suggested approach. Finally, a brief conclusion is provided in section 8.

2. Model

This section outlines the basic understanding of fractional differentiation while describing the relevant fractional BS model and its brief derivation history.

2.1. **Time fractional Black-Scholes model.**

Let us first suppose that the stock price S dynamics follow the following fractional stochastic process [13, 14]

$$dS = (R - D)Sd\tau + \delta S\omega(\tau)(d\tau)^{\alpha/2}, \qquad 0 < \alpha \leq 1, \tag{2.1}$$

where $\omega(\tau)$ is the typical Wiener process, R is the risk-free interest rate, and D and δ, respectively, are the continuous dividend and volatility. We should additionally take into account the following significant identities, which Jumarie [7] claims are in excellent agreement with the Jumarie fractional (generalized) Taylor series in [14]:

$$d^\alpha \tau = \frac{1}{(2-\alpha)} \tau^{1-\alpha}(d\tau)^\alpha, \qquad 0 < \alpha \leq 1, \tag{2.2}$$

$$d^\alpha S = \Gamma(1+\alpha)dS, \qquad 0 < \alpha \leq 1, \tag{2.3}$$

and

$$\frac{d^\alpha S}{(dS)^\alpha} = \frac{1}{\Gamma(2-\alpha)} S^{1-\alpha}, \qquad 0 < \alpha \leq 1. \tag{2.4}$$

Combining (2.4) and (2.5), we get a formula, which convert integer derivative to fractional derivative and vice-versa:

$$dS = \frac{S^{(1-\alpha)}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}(dS)^\alpha, \qquad 0 < \alpha \leq 1. \tag{2.5}$$

Let $\Phi(S,\tau)$ is the value of a European put option, and suppose that $\Phi(S,\tau)$ is smooth with respect to S and its α derivative with respect to time exists for some $\alpha(0 < \alpha \leq 1)$. Consider the risk-free investment interest rate dynamic equation

$$d\Phi = R\Phi d\tau. \tag{2.6}$$

Now multiplying $\Gamma(1-\alpha)$ in both side of Eq. (2.6), we get:

$$\Gamma(1-\alpha)d\Phi = \Gamma(1-\alpha)R\Phi d\tau. \tag{2.7}$$

Now from Eq. (2.7) and Eq. 2.3, we get variational fractional increment process:

$$d^\alpha \Phi = \Gamma(1+\alpha) R\Phi d\tau. \tag{2.8}$$

Using Eq. (2.8) and Eq. (2.5), we get fractional interest rate dynamic equation:

$$d^\alpha \Phi = \frac{R\Phi}{\Gamma(2-\alpha)} \tau^{1-\alpha} (d\tau)^\alpha. \tag{2.9}$$

Since $\Phi(S,\tau)$ is sufficiently smooth with respect to S and its α order derivative$(0 < \alpha \leq 1)$ with respect to τ exists, applying the fractional Taylor series [14] of order α on $\Phi(S,\tau)$ we get

$$d\Phi = \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha \Phi}{\partial \tau^\alpha} (d\tau)^\alpha + \frac{\partial \Phi}{\partial S} dS + \frac{1}{2} \frac{\partial^2 \Phi}{\partial S^2} (dS)^2. \tag{2.10}$$

Using the Eqs. (2.1)–(2.9) and Itô's lemma [14] we get,

$$\frac{\partial^\alpha \Phi}{\partial \tau^\alpha} = \left(R\Phi - (R-D)\, S \frac{\partial \Phi}{\partial S} \right) \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2} \delta^2 S^2 \frac{\partial^2 \Phi}{\partial S^2}, \qquad 0 < \alpha \leq 1. \tag{2.11}$$

The following TFBS model can therefore be obtained by setting $\tau = T - t$ and $S = e^x$ in Eq. (2.11) and then noting $\Psi(x,t) = \Phi(e^x, T-t)$:

$$\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} = \frac{\Gamma(1+\alpha)}{2} \delta^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2} + \left(\frac{(R-D)(T-t)^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)\delta^2}{2} \right) \frac{\partial \Psi(x,t)}{\partial x} - \frac{R(T-t)^{1-\alpha}}{\Gamma(2-\alpha)} \Psi(x,t), \tag{2.12}$$

with initial and boundary conditions

$$\Psi(x,0) = v(x),\ \Psi(-\infty,t) = u(t),\ \Psi(\infty,t) = w(t), \tag{2.13}$$

where the fractional derivative $\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha}$ is defined as

$$\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\Psi(x,\eta) - \Psi(x,0)}{(t-\eta)^\alpha} d\eta. \tag{2.14}$$

Since the problem (2.12)–(2.14) defined on an unbounded domain $\mathbb{R} \times (0,T)$, so we have to truncate it into a finite domain to solve it numerically. We consider the problem on a finite domain $(D_l, D_r) \times (0,T)$. Thus, we have

$$\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} = p \frac{\partial^2 \Psi(x,t)}{\partial x^2} + q \frac{\partial \Psi(x,t)}{\partial x} - r\Psi(x,t) + f(x,t), \qquad (x,t) \in (D_l, D_r) \times (0,T), \tag{2.15}$$

$$\Psi(x,0) = v(x), \quad \Psi(D_l,t) = u(t), \quad \Psi(D_r,t) = w(t), \tag{2.16}$$

where, $p = \frac{\Gamma(1+\alpha)}{2}\delta^2$, $q(t) = r - p$ and $r = \frac{R(T-t)^{1-\alpha}}{\Gamma(2-\alpha)}$ and $D = 0$. Note that here we have added an extra term $f(x,t)$. By putting $f(x,t) = 0$ in Eq. (2.15) we can get back to our original model.

3. Temporal Discretaization

In this section, we are giving three different approximations namely, $L1,\ L1-2,\ L2-1_\sigma$ to approximate the time fractional derivative.

3.1. **L1 Approximation [12].** Let us first discretize the Eqs. (2.15)–(2.16) over the temporal interval $[0,T]$. Let the partition on the time domain $[0,T]$ be $0 = t_1 < t_2 < \ldots < t_N < t_{N+1} = T$, where $t_j = (j-1)\Delta t,\ j = 1,2,3,\ldots,N+1$ and $\Delta t = \frac{T}{N}$. Using Eq. (2.14), we get

$$\begin{aligned}
\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\Psi(x,\eta) - \Psi(x,0)}{(t-\eta)^\alpha} d\eta \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\frac{d}{dt} \int_0^t \frac{\Psi(x,\eta)}{(t-\eta)^\alpha} d\eta - \frac{d}{dt} \int_0^t \frac{\Psi(x,0)}{(t-\eta)^\alpha} d\eta \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\Psi(x,0) t^{-\alpha} + \int_0^t (t-\eta)^{-\alpha} \frac{\partial \Psi(x,\eta)}{\partial \eta} d\eta - \Psi(x,0) t^{-\alpha} \right]
\end{aligned}$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial \Psi(x,\eta)}{\partial \eta} d\eta$$
$$= {}_0^C D_t^\alpha \Psi(x,t), \tag{3.1}$$

where ${}_0^C D_t^\alpha \Psi(x,t)$ denotes the Caputo fractional derivative. At the time $t = t_j$, from the definition of Caputo derivative, we have

$$\begin{aligned} {}_0^C D_t^\alpha \Psi(x,t_j) &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j-1} \int_{t_s}^{t_{s+1}} (t-\zeta)^{-\alpha} \frac{\partial \Psi(x,\zeta)}{\partial \zeta} d\zeta \\ &= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=1}^{j-1} \left(\Psi(x,t_{s+1}) - \Psi(x,t_s)\right) \left[(j-s)^{1-\alpha} - (j-s-1)^{1-\alpha}\right] + (R_1)_{\Delta t}^j \\ &= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=1}^{j-1} a_{j-s}^\alpha \left(\Psi(x,t_{s+1}) - \Psi(x,t_s)\right) + (R_1)_{\Delta t}^j, \end{aligned} \tag{3.2}$$

where $a_m^\alpha = m^{1-\alpha} - (m-1)^{1-\alpha}$ for $m = 1, 2, \ldots, N+1$ and $(R_1)_{\Delta t}^j$ is the truncation error which is given by:

$$(R_1)_{\Delta t}^j \leq c_\Psi \Delta t^{(2-\alpha)}, \tag{3.3}$$

where c_Ψ is a constant which only depends on Ψ.

3.2. **L1-2 Approximation** [6]**.** Let us consider same uniform mesh given in previous subsection $\bar{v}_{\Delta t} = \{t_j = (j-1)\Delta t,\ j = 1, 2, \ldots, N+1,\ \Delta t = \frac{T}{N}\}$. Then for $j \geq 1$, $t_{j+\frac{1}{2}} = \frac{t_j + t_{j+1}}{2}$. Suppose, $\Psi(t) \in C^1[0, t_j]$ $(j \geq 1)$ and Ψ_j denotes the value of $\Psi(t)$ at grid point t_j. Let us introduce difference quotient operators,

$$\delta_t \Psi_{j+\frac{1}{2}} = \frac{\Psi_{j+1} - \Psi_j}{\Delta t},\ \delta_t^2 \Psi_j = \frac{1}{\Delta t}(\delta_t \Psi_{j+\frac{1}{2}} - \delta_t \Psi_{j-\frac{1}{2}}). \tag{3.4}$$

Then the Caputo fractional derivative of order α, where $0 < \alpha < 1$, for the function $\Psi(t)$ evaluated at the discrete time point t_j $(j = 1, 2, \ldots, N+1)$ is given by,

$$\begin{aligned} {}_0^C D_{t_j}^\alpha \Psi(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} \frac{1}{(t_j-\eta)^\alpha} \frac{\partial \Psi(\eta)}{\partial \eta} d\eta \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j-1} \int_{t_s}^{t_{s+1}} \frac{\Psi'(\eta)}{(t_j-\eta)^\alpha} d\eta. \end{aligned} \tag{3.5}$$

As outlined in [6], a quadratic interpolation $P_{2,s}\Psi(t)$ is defined for $\Psi(t)$ over each interval $[t_s, t_{s+1}]$, where $1 \leq s \leq j-1$, using three adjacent points $(t_{s-1}, \Psi(t_{s-1}))$, $(t_s, \Psi(t_s))$ and $(t_{s+1}, \Psi(t_{s+1}))$, is given by,

$$P_{2,s}\Psi(t) = P_{1,s}\Psi(t) + \frac{1}{2}\delta_t^2 \Psi_s (t-t_s)(t-t_{s+1}), \tag{3.6}$$

$$(P_{2,s}\Psi(t))' = (P_{1,s}\Psi(t))' + \delta_t^2 \Psi_s (t - t_{s+\frac{1}{2}}) = \delta_t \Psi_{s+\frac{1}{2}} + \delta_t^2 \Psi_s (t - t_{s+\frac{1}{2}}),\ t \in [t_s, t_{s+1}], \tag{3.7}$$

and

$$\Psi(t) - P_{2,s}\Psi(t) = \frac{\Psi'''(v_s)}{6}(t-t_{s-1})(t-t_s)(t-t_{s+1}),\ v_s \in (t_{s-1}, t_{s+1}),\ t \in [t_s, t_{s+1}],\ 2 \leq s \leq j-1. \tag{3.8}$$

where $P_{1,s}\Psi(t)$ is the linear interpolation of $\Psi(t)$ using two points $(t_s, \Psi(t_s))$ and $(t_{s+1}, \Psi(t_{s+1}))$, i.e.

$$P_{1,s}\Psi(t) = \frac{t-t_s}{t_{s+1}-t_s}\Psi(t_{s+1}) + \frac{t-t_{s+1}}{t_s - t_{s+1}}\Psi(t_s), \tag{3.9}$$

and from the linear interpolation theory,

$$\Psi(t) - P_{1,s}\Psi(t) = \frac{\Psi''(\tilde{\varepsilon}_s)}{6}(t-t_s)(t-t_{s+1}),\ \tilde{\varepsilon}_s \in (t_s, t_{s+1}),\ t \in [t_s, t_{s+1}],\ 1 \leq s \leq j-1. \tag{3.10}$$

In Eq. (3.5), We use $P_{1,2}\Psi(t)$ to approximate $\Psi(t)$ on the small interval $[t_1, t_2]$ and $P_{2,s}\Psi(t)$ to approximate $\Psi(t)$ on the interval $[t_s, t_{s+1}]$ $(s \geq 2)$. Noticing

$$\int_{t_s}^{t_{s+1}} (\eta - t_{s+\frac{1}{2}})(t_j - \eta)^{-\alpha} d\eta = \frac{(\Delta t)^{2-\alpha}}{1-\alpha} b_{j-s}^{(\alpha)}, \ 2 \leq s \leq j-1 \tag{3.11}$$

with

$$b_m^{(\alpha)} = \frac{1}{2-\alpha}[m^{2-\alpha} - (m-1)^{2-\alpha}] - \frac{1}{2}[m^{1-\alpha} + (m-1)^{1-\alpha}], \ m \geq 1.$$

Using Eq. (3.7) and Eq. (3.9), we can find a new approximation of the Caputo fractional derivative of order α, where $0 < \alpha < 1$, for the function $\Psi(t)$ is given by,

$$\begin{aligned}
{}_0^C D_t^\alpha \Psi(t)|_{t=t_j} &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j-1} \int_{t_s}^{t_{s+1}} \frac{\Psi'(\eta)}{(t_j - \eta)^\alpha} d\eta \\
&\approx \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t_2} \frac{(P_{1,2}\Psi(\eta))'}{(t_j-\eta)^\alpha} d\eta + \frac{1}{\Gamma(1-\alpha)} \sum_{s=2}^{j-1} \int_{t_s}^{t_{s+1}} \frac{(P_{2,s}\Psi(\eta))'}{(t_j-\eta)^\alpha} d\eta \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\delta_t \Psi_{\frac{3}{2}} \int_{t_1}^{t_2} (t_j - \eta)^{-\alpha} d\eta + \sum_{s=2}^{j-1} \int_{t_s}^{t_{s+1}} (t_j - \eta)^{-\alpha} [\delta_t \Psi_{s+\frac{1}{2}} + (\delta_t^2 \Psi_s)(\eta - t_{s+\frac{1}{2}})] d\eta \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\sum_{s=1}^{j-1} \delta_t \Psi_{s+\frac{1}{2}} \int_{t_s}^{t_{s+1}} (t_j - \eta)^{-\alpha} d\eta + \sum_{s=2}^{j-1} (\delta_t^2 \Psi_s) \int_{t_s}^{t_{s+1}} (\eta - t_{s+\frac{1}{2}})(t_j - \eta)^{-\alpha} d\eta \right] \\
&= \tilde{D}_t^\alpha \Psi(t)|_{t=t_j} + \frac{(\Delta t)^{2-\alpha}}{\Gamma(2-\alpha)} \sum_{s=2}^{j-1} b_{j-s}^{(\alpha)} \delta_t^2 \Psi_s,
\end{aligned}$$

where $\tilde{D}_t^\alpha \Psi(t)$ will be found by the classical $L1$ operator which is derived from a piece-wise linear interpolation approximation of $\Psi(t)$ on each small interval $[t_s, t_{s+1}]$ $(1 \leq s \leq j-1)$, which is defined by

$$\begin{aligned}
\tilde{D}_t^\alpha \Psi(t)|_{t=t_j} =& \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=1}^{j-1} a_{j-s}^{(\alpha)} \delta_t \Psi_{s+\frac{1}{2}} \\
=& \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left[a_1 \Psi_j - \sum_{s=2}^{j-1} (a_{(j-s)} - a_{(j-s+1)}) \Psi_s - a_{j-1} \Psi_1 \right],
\end{aligned} \tag{3.12}$$

with

$$a_m^{(\alpha)} = m^{1-\alpha} - (m-1)^{1-\alpha}, \ 1 \leq m \leq j.$$

We define

$$\mathbb{D}_t^\alpha \Psi(t)|_{t=t_j} := \tilde{D}_t^\alpha \Psi(t)|_{t=t_j} + \frac{(\Delta t)^{2-\alpha}}{\Gamma(2-\alpha)} \sum_{s=2}^{j-1} b_{j-s}^{(\alpha)} \delta_t^2 \Psi_s, \tag{3.13}$$

where $\mathbb{D}_t^\alpha$ is the $L1-2$ operator. Now Eq. (3.13) can be rewritten as

$$\begin{aligned}
\mathbb{D}_t^\alpha \Psi(t)|_{t=t_j} =& \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{s=1}^{j-1} a_{j-s}^{(\alpha)}\delta_t\Psi_{s+\frac{1}{2}} + \sum_{s=2}^{j-1} b_{j-s}^{(\alpha)}(\delta_t\Psi_{s+\frac{1}{2}} - \delta_t\Psi_{s-\frac{1}{2}})\right] \\
=& \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{s=1}^{j-1} a_{j-s}^{(\alpha)}\delta_t\Psi_{s+\frac{1}{2}} + \sum_{s=2}^{j-1} b_{j-s}^{(\alpha)}\delta_t\Psi_{s+\frac{1}{2}} - \sum_{s=2}^{j-1} b_{j-s}^{(\alpha)}\delta_t\Psi_{s-\frac{1}{2}}\right] \\
=& \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{j-1} \hat{c}_{j-s}^{(\alpha)}\delta_t\Psi_{s+\frac{1}{2}} \\
=& \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\hat{c}_1\Psi(t_j) - \sum_{s=2}^{j-1}(\hat{c}_{j-s} - \hat{c}_{j-s+1})\Psi(t_s) - \hat{c}_{j-1}\Psi(t_1)\right],
\end{aligned} \tag{3.14}$$

where for $j=2, \hat{c}_1^{(\alpha)} = a_1^{(\alpha)} = 1$ and for $j \geq 3$,

$$\hat{c}_s^{(\alpha)} = \begin{cases} a_1^{(\alpha)} + b_1^{(\alpha)}, & s = 1, \\ a_s^{(\alpha)} + b_s^{(\alpha)} - b_{s-1}^{(\alpha)}, & 2 \leq s \leq j-2, \\ a_s^{(\alpha)} - b_{s-1}^{(\alpha)}, & s = j-1. \end{cases}$$

where, $a_m^{(\alpha)} = m^{1-\alpha} - (m-1)^{1-\alpha}$ and $b_m^{(\alpha)} = \frac{1}{2-\alpha}[m^{2-\alpha} - (m-1)^{2-\alpha}] - \frac{1}{2}[m^{1-\alpha} + (m-1)^{1-\alpha}],\ m \geq 1.$

3.3. $L2-1_\sigma$ **Approximation** [2]**.** Let us again consider the same uniform mesh given in the previous subsection. Let $\sigma = 1 - \frac{\alpha}{2}$, then the Caputo fractional derivative of order α, where $0 < \alpha < 1$ for the function $\Psi(t) \in C^3[0,T]$ at the discrete time point $t_{j+\sigma}$ $(j = 1, 2, \ldots, N+1)$ is given by,

$$\begin{aligned}
{}_0^C D_{t_{j+\sigma}}^\alpha \Psi(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{j+\sigma}} \frac{1}{(t_{j+\sigma}-\eta)^\alpha}\frac{\partial\Psi(\eta)}{\partial\eta}d\eta \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{j-1}\int_{t_s}^{t_{s+1}} \frac{\Psi'(\eta)}{(t_{j+\sigma}-\eta)^\alpha}d\eta + \frac{1}{\Gamma(1-\alpha)}\int_{t_j}^{t_{j+\sigma}} \frac{\Psi'(\eta)}{(t_{j+\sigma}-\eta)^\alpha}d\eta.
\end{aligned} \tag{3.15}$$

As outlined in [6], a quadratic interpolation $P_{2,s}\Psi(t)$ is defined for $\Psi(t)$ over each interval $[t_s, t_{s+1}]$, where $1 \leq s \leq j-1$, using three adjacent points $(t_{s-1}, \Psi(t_{s-1}))$, $(t_s, \Psi(t_s))$ and $(t_{s+1}, \Psi(t_{s+1}))$, is given by,

$$P_{2,s}\Psi(t) = P_{1,s}\Psi(t) + \frac{1}{2}\delta_t^2\Psi_s(t-t_s)(t-t_{s+1}), \tag{3.16}$$

where, $P_{1,s}\Psi(t)$ is the linear interpolation of $\Psi(t)$ using two points $(t_s, \Psi(t_s))$ and $(t_{s+1}, \Psi(t_{s+1}))$. Hence

$$P_{1,s}\Psi(t) = \frac{t-t_s}{t_{s+1}-t_s}\Psi(t_{s+1}) + \frac{t-t_{s+1}}{t_s-t_{s+1}}\Psi(t_s). \tag{3.17}$$

Now

$$(P_{2,s}\Psi(t))' = (P_{1,s}\Psi(t))' + \delta_t^2\Psi_s(t-t_{s+\frac{1}{2}}) = \Psi_{t,s} + \Psi_{tt,s}(t-t_{s+\frac{1}{2}}), \tag{3.18}$$

and $\Psi(t) - (P_{2,s}\Psi(t)) = \frac{\Psi'''(\tilde{\kappa})}{6}(t-t_{s-1})(t-t_s)(t-t_{s+1}),\ \tilde{\kappa} \in (t_{s-1}, t_{s+1}),\ t \in [t_s, t_{s+1}]$,
where $t_{s+\frac{1}{2}} = t_s + \frac{1}{2}\Delta t,\ \Psi_{t,s} = \frac{\Psi(t_{s+1})-\Psi(t_s)}{\Delta t}$.
In Eq. (3.15), we use $P_{2,s}\Psi(t)$ to approximate $\Psi(t)$ on the interval $[t_s, t_{s+1}]$ $(1 \leq s \leq j-1)$. Noticing

$$\int_{t_s}^{t_{s+1}} \frac{(\eta - t_{s+\frac{1}{2}})}{(t_{j+\sigma}-\eta)^\alpha}d\eta = \frac{\Delta t^{2-\alpha}}{1-\alpha}b_{j-s}^{(\alpha,\sigma)},\ 1 \leq s \leq j-1 \tag{3.19}$$

with $b_n = \frac{1}{2-\alpha}[(n+\sigma)^{2-\alpha} - (n+\sigma-1)^{2-\alpha}] - \frac{1}{2}[(n+\sigma)^{1-\alpha} + (n+\sigma-1)^{1-\alpha}],\ n \geq 1,$
from Eq. (3.15) and Eq. (3.18) we can obtain the numerical approximation of the Caputo fractional derivative of order α, where $0<\alpha<1$, for the function $\Psi(t)$ is given by

$$\begin{aligned}
{}_0^C D_{t_{j+\sigma}}^{\alpha}\Psi(t) &= \frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{j-1}\int_{t_s}^{t_{s+1}}\frac{\Psi'(\eta)}{(t_{j+\sigma}-\eta)^{\alpha}}d\eta + \frac{1}{\Gamma(1-\alpha)}\int_{t_j}^{t_{j+\sigma}}\frac{\Psi'(\eta)}{(t_{j+\sigma}-\eta)^{\alpha}}d\eta \\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{j-1}\int_{t_s}^{t_{s+1}}\frac{(P_{2,s}\Psi(\eta))'}{(t_{j+\sigma}-\eta)^{\alpha}}d\eta + \frac{1}{\Gamma(1-\alpha)}\int_{t_j}^{t_{j+\sigma}}\frac{\Psi'(\eta)}{(t_{j+\sigma}-\eta)^{\alpha}}d\eta \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{j-1}\int_{t_s}^{t_{s+1}}\frac{\Psi_{t,s}+\Psi_{tt,s}(t-t_{s+\frac{1}{2}})}{(t_{j+\sigma}-\eta)^{\alpha}}d\eta + \frac{1}{\Gamma(1-\alpha)}\int_{t_j}^{t_{j+\sigma}}\frac{\Psi'(\eta)}{(t_{j+\sigma}-\eta)^{\alpha}}d\eta \\
&= \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}[(\sum_{s=1}^{j-1}(a_{j-s}^{(\alpha,\sigma)}\Psi_{t,s} + b_{j-s}^{(\alpha,\sigma)}\Psi_{tt,s}\Delta t)) + a_0^{(\alpha,\sigma)}\Psi_{t,j}] \\
&= \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{j}\tilde{c}_{j-s}^{(j,\alpha)}\Psi_{t,s} \\
&= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{j}\tilde{c}_{j-s}^{(j,\alpha)}(\Psi(t_{s+1})-\Psi(t_s)),
\end{aligned} \tag{3.20}$$

where, for $j=1$, $\tilde{c}_0^{(j,\alpha)} = a_0^{(\alpha,\sigma)}$ and for $j \geq 2$,

$$\tilde{c}_s^{(j,\alpha)} = \begin{cases} a_0^{(\alpha,\sigma)} + b_1^{(\alpha,\sigma)}, & s=0, \\ a_s^{(\alpha,\sigma)} + b_{s+1}^{(\alpha,\sigma)} - b_s^{(\alpha,\sigma)}, & 1 \leq s \leq j-2, \\ a_s^{(\alpha,\sigma)} - b_s^{(\alpha,\sigma)}, & s=j-1. \end{cases}$$

where, $a_0^{(\alpha,\sigma)} = \sigma^{1-\alpha}, a_m^{(\alpha,\sigma)} = (m+\sigma)^{1-\alpha} - (m+\sigma-1)^{1-\alpha},\ m \geq 1,\ b_m^{(\alpha,\sigma)} = \frac{1}{2-\alpha}[(m+\sigma)^{2-\alpha} - (m+\sigma-1)^{2-\alpha}] - \frac{1}{2}[(m+\sigma)^{1-\alpha} + (m+\sigma-1)^{1-\alpha}],\ m \geq 1.$

4. Spatial Discretization

Let us discretize the space domain $[D_l, D_r]$ into uniform partition $D_l = x_1 < x_2 < x_3 < \ldots < x_{M+1} = D_r$, so $x_k = D_l + (k-1)\Delta x,\ k = 1,2,\ldots,M+1$ and $\Delta x = \frac{D_r - D_l}{M}$. The first derivative $\Psi'(x_k)$ can be expressed by the finite difference approximation as

$$\delta_x\Psi(x_k) = \frac{\Psi(x_{k+1}) - \Psi(x_{k-1})}{2(\Delta x)} + \hat{r}_1,\ k = 2,3,\ldots,M, \tag{4.1}$$

where

$$\hat{r}_1 = \frac{(\Delta x)^2}{3!}\Psi^{(3)}(x_k) + \frac{(\Delta x)^4}{5!}\Psi^{(5)}(x_k) + O((\Delta x)^6).$$

Similarly, for the second derivative $\Psi''(x_k)$, the finite difference approximation can be expressed as

$$\delta_x^2\Psi(x_k) = \frac{\Psi(x_{k-1}) - 2\Psi(x_k) + \Psi(x_{k+1})}{(\Delta x)^2} + \hat{r}_2,\ k = 2,3,\ldots,M, \tag{4.2}$$

where

$$\hat{r}_2 = \frac{2(\Delta x)^2}{4!}\Psi^{(4)}(x_k) + \frac{2(\Delta x)^4}{6!}\Psi^{(6)}(x_k) + O((\Delta x)^6).$$

Theorem 4.1. *Consider the differential equation as*

$$\gamma_1 \frac{d^2\Psi(x)}{dx^2} + \gamma_2 \frac{d\Psi(x)}{dx} - \gamma_3 \Psi(x) = h(x), \tag{4.3}$$

and the compact finite difference method of fourth order for Eq. (4.3) is given as

$$\begin{aligned} &\left[\gamma_1 - \frac{\Delta x^2}{12}\left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right)\right]\delta_x^2\Psi(x_k) + \left[\gamma_2 - \frac{\Delta x^2}{12}\frac{\gamma_2\gamma_3}{\gamma_1}\right]\delta_x\Psi(x_k) - \gamma_3\Psi(x_k) \\ &= h(x_k) + \frac{\gamma_2}{\gamma_1}\frac{\Delta x^2}{12}\delta_x h(x_k) + \frac{\Delta x^2}{12}\delta_x^2 h(x_k) + \tilde{r}, \;\; k = 2, 3, \ldots, M, \end{aligned} \tag{4.4}$$

where $\tilde{r}$ is truncation error of $O((\Delta x)^4)$,

$$\tilde{r} = -\frac{\Delta x^4}{144}\left[\left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right)\Psi^{(4)}(x_k) + \frac{2\gamma_2\gamma_3}{\gamma_1}\Psi^{(3)}(x_k) + \frac{2\gamma_2}{\gamma_1}h^{(3)}(x_k) + h^{(4)}(x_k) - \frac{2\gamma_1}{5}\Psi^{(6)}(x_k) - \frac{6\gamma_2}{5}\Psi^{(5)}(x_k)\right]. \tag{4.5}$$

Proof. Utilizing approximations Eq. (4.1) and Eq. (4.2), we can write the finite difference discretization for Eq. (4.3) at $x = x_k$ as

$$\gamma_1\delta_x^2\Psi(x_k) + \gamma_2\delta_x\Psi(x_k) - \gamma_3\Psi(x_k) - \tilde{r}_1 = h(x_k), \tag{4.6}$$

where

$$\tilde{r}_1 = \gamma_1\left(\frac{2(\Delta x)^2}{4!}\Psi^{(4)}(x_k) + \frac{2(\Delta x)^4}{6!}\Psi^{(6)}(x_k)\right) + \gamma_2\left(\frac{(\Delta x)^2}{3!}\Psi^{(3)}(x_k) + \frac{(\Delta x)^4}{5!}\Psi^{(5)}(x_k)\right). \tag{4.7}$$

We need to approximate $\Psi^{(3)}(x_k)$ and $\Psi^{(4)}(x_k)$ in Eq. (4.7) in order to find a fourth-order finite difference approach. Thus, differentiating Eq. (4.3) with respect to x. we get

$$\gamma_1\frac{d^3\Psi(x)}{dx^3} + \gamma_2\frac{d^2\Psi(x)}{dx^2} - \gamma_3\frac{d\Psi(x)}{dx} = \frac{dh(x)}{dx}. \tag{4.8}$$

From Eq. (4.8), we have

$$\Psi^{(3)}(x) = -\frac{\gamma_2}{\gamma_1}\Psi^{''}(x) + \frac{\gamma_3}{\gamma_1}\Psi^{'}(x) + \frac{1}{\gamma_1}h^{'}(x). \tag{4.9}$$

Putting $x = x_k$ in Eq. (4.9) we have

$$\Psi^{(3)}(x_k) = -\frac{\gamma_2}{\gamma_1}\Psi^{''}(x_k) + \frac{\gamma_3}{\gamma_1}\Psi^{'}(x_k) + \frac{1}{\gamma_1}h^{'}(x_k). \tag{4.10}$$

Differentiating Eq. (4.3) with respect to x twice, it gives

$$\gamma_1\frac{d^4\Psi(x)}{dx^4} + \gamma_2\frac{d^3\Psi(x)}{dx^3} - \gamma_3\frac{d^2\Psi(x)}{dx^2} = \frac{d^2h(x)}{dx^2}. \tag{4.11}$$

At $x = x_k$, Eq. (4.11) becomes

$$\gamma_1\Psi^{(4)}(x_k) = -\gamma_2\Psi^{(3)}(x_k) + \gamma_3\Psi^{''}(x_k) + h^{''}(x_k). \tag{4.12}$$

Substituting $\Psi^{(3)}(x)$ in Eq. (4.12), we get

$$\gamma_1\Psi^{(4)}(x_k) = \left(\gamma_3 + \frac{\gamma_2^2}{\gamma_1}\right)\Psi^{''}(x_k) - \frac{\gamma_2\gamma_3}{\gamma_1}\Psi^{'}(x_k) - \frac{\gamma_2}{\gamma_1}h^{'}(x_k) + h^{''}(x_k). \tag{4.13}$$

Using Eqs. (4.10) and (4.12) in Eq. (4.7), we get

$$\tilde{r}_1 = \frac{\Delta x^2}{12}\left[\left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right)\Psi^{''}(x_k) + \frac{\gamma_2\gamma_3}{\gamma_1}\Psi^{'}(x_k) + \frac{\gamma_2}{\gamma_1}h^{'}(x_k) + h^{''}(x_k)\right] + \frac{2\gamma_1\Delta x^4}{6!}\Psi^{(6)}(x_k) + \frac{\gamma_2\Delta x^4}{5!}\Psi^{(5)}(x_k). \tag{4.14}$$

Using Eqs. (4.1) and (4.3) in Eq. (4.14) we get

$$\tilde{r}_1 = \frac{\Delta x^2}{12}\left[\left(\gamma_3 - \frac{\gamma_2^2}{\gamma_1}\right)\delta_x^2\Psi(x_k) + \frac{\gamma_2\gamma_3}{\gamma_1}\delta_x\Psi(x_k) + \frac{\gamma_2}{\gamma_1}\delta_x h(x_k) + \delta_x^2 h(x_k)\right] + \tilde{r}, \tag{4.15}$$

where

$$\tilde{r}=-\frac{\Delta x^4}{144}\left[\left(\gamma_3-\frac{\gamma_2^2}{\gamma_1}\right)\Psi^{(4)}(x_k)+\frac{2\gamma_2\gamma_3}{\gamma_1}\Psi^{(3)}(x_k)+\frac{2\gamma_2}{\gamma_1}h^{(3)}(x_k)+h^{(4)}(x_k)-\frac{2\gamma_1}{5}\Psi^{(6)}(x_k)-\frac{6\gamma_2}{5}\Psi^{(5)}(x_k)\right]. \quad (4.16)$$

Using Eq. (4.15) in Eq. (4.6), we get

$$\begin{aligned}&\left[\gamma_1-\frac{\Delta x^2}{12}\left(\gamma_3-\frac{\gamma_2^2}{\gamma_1}\right)\right]\delta_x^2\Psi(x_k)+\left[\gamma_2-\frac{\Delta x^2}{12}\frac{\gamma_2\gamma_3}{\gamma_1}\right]\delta_x\Psi(x_k)-\gamma_3\Psi(x_k)\\&=h(x_k)+\frac{\gamma_2}{\gamma_1}\frac{\Delta x^2}{12}\delta_x h(x_k)+\frac{\Delta x^2}{12}\delta_x^2 h(x_k)+\tilde{r},\qquad k=2,3,\dots,M.\end{aligned} \quad (4.17)$$

□

5. Fully Discrete Schemes

In this section, we will combine the temporal and space discretization discussed in Section 3 and Section 4 to develop three different compact schemes namely, $L1$, $L1-2$, and $L2-1_\sigma$.

5.1. **Compact $L1$ Scheme [17].** In this section, a difference scheme of order $((\Delta t)^{2-\alpha}+(\Delta x)^4)$ is constructed for problem (2.15)–(2.16). Now consider, $m(x,t)={}_0^C D_t^\alpha\Psi(x,t)$. Then at point (x_k,t_j), we have $m(x_k,t_j)={}_0^C D_t^\alpha\Psi(x_k,t_j)$. Using Eq. (3.2), we get

$$\begin{aligned}m(x_k,t_j)&=\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{j-1}a_{j-s}(\Psi(x_k,t_{s+1})-\Psi(x_k,t_s))+(R_1)_{\Delta t}^j\\&=\sum_{s=1}^{j-1}da_{j-s}(\Psi(x_k,t_{s+1})-\Psi(x_k,t_s))+(R_1)_{\Delta t}^j,\end{aligned} \quad (5.1)$$

with $d=\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}$. We can rewrite Eq. (2.15) as

$$-p\frac{\partial^2\Psi(x,t)}{\partial x^2}-q\frac{\partial\Psi(x,t)}{\partial x}+r\Psi(x,t)=f(x,t)-m(x,t). \quad (5.2)$$

With $\gamma_1=-p$, $\gamma_2=-q$, $\gamma_3=-r$ and Theorem 4.1, Eq. (5.2) at the point (x_k,t_j) can now be expressed as

$$\begin{aligned}&\left[-p-\frac{(\Delta x)^2}{12}\left(-r+\frac{q^2}{p}\right)\right]\delta_x^2\Psi(x_k,t_j)+\left[-q+\frac{(\Delta x)^2}{12}\frac{qr}{p}\right]\delta_x\Psi(x_k,t_j)+r\Psi(x_k,t_j)=f(x_k,t_j)\\&-m(x_k,t_j)+\frac{q}{p}\frac{(\Delta x)^2}{12}\delta_x(f(x_k,t_j)-m(x_k,t_j))+\frac{(\Delta x)^2}{12}\delta_x^2(f(x_k,t_j)-m(x_k,t_j))+\tilde{r}.\end{aligned} \quad (5.3)$$

Now assuming $\Psi_k^j=\Psi(x_k,t_j)$ and $f_k^j=f(x_k,t_j)$ $(k=1,2,\dots,M+1;\ j=1,2,\dots,N+1)$ and using Eq. (5.3) and Eq. (5.1) and then using Eq. (4.1) and Eq. (4.2) we get

$$\begin{aligned}&\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}+\frac{q}{2\Delta x}-\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k-1}^j+\left(\frac{2p}{\Delta x^2}+\frac{5r}{6}+\frac{q^2}{6p}\right)\Psi_k^j\\&+\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}-\frac{q}{2\Delta x}+\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k+1}^j=\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)f_{k-1}^j+\frac{5}{6}f_k^j+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)f_{k+1}^j\\&+da_{j-1}\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^1+\frac{5}{6}\Psi_k^1+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\Psi_{k+1}^1\right)\right]\\&+\sum_{s=2}^{j-1}d(a_{j-s}-a_{j-s+1})\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^s+\frac{5}{6}\Psi_k^s+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^s\right]\\&-da_1\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^j+\frac{5}{6}\Psi_k^j+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^j\right]+\hat{R}_k^j,\qquad k=2,3,\dots,M,\ j=2,3,\dots,N+1,\end{aligned} \quad (5.4)$$

where $\hat{R}_k^j = O((\Delta t)^{(2-\alpha)} + (\Delta x)^4)$, and the boundary and initial conditions can be discretized as

$$\Psi_1^j = u(t_j) \;=\; u^j,\; \Psi_{M+1}^j \;=\; w(t_j) \;=\; w^j,\; j = 2, 3, \ldots N+1, \tag{5.5}$$

$$\Psi_k^1 = v(x_k) \;=\; v_k,\; k = 1, 2, \ldots, M+1. \tag{5.6}$$

Rearranging above Eq. (5.4) we get

$$\begin{aligned}
&\left(\frac{-p}{\Delta x^2} + \frac{r}{12} - \frac{q^2}{12p} + \frac{q}{2\Delta x} - \frac{\Delta x}{24}\frac{qr}{p} + \frac{da_1}{12} - \frac{da_1 q\Delta x}{24p}\right)\Psi_{k-1}^j + \left(\frac{2p}{\Delta x^2} + \frac{5r}{6} + \frac{q^2}{6p} + \frac{5da_1}{6}\right)\Psi_k^j \\
&+ \left(\frac{-p}{\Delta x^2} + \frac{r}{12} - \frac{q^2}{12p} - \frac{q}{2\Delta x} + \frac{\Delta x}{24}\frac{qr}{p} + \frac{da_1}{12} + \frac{da_1 q\Delta x}{24p}\right)\Psi_{k+1}^j \\
&= \left(\frac{1}{12} - \frac{qdx}{24p}\right)f_{k-1}^j + \frac{5}{6}f_k^j + \left(\frac{1}{12} + \frac{q\Delta x}{24p}\right)f_{k+1}^j + da_{j-1}\left[\left(\frac{1}{12} - \frac{q\Delta x}{24p}\right)\Psi_{k-1}^1 + \frac{5}{6}\Psi_k^1\right. \\
&\left. + \left(\frac{1}{12} + \frac{q\Delta x}{24p}\right)\Psi_{k+1}^1\right] + \sum_{s=2}^{j-1} d(a_{j-s} - a_{j-s+1})\left[\left(\frac{1}{12} - \frac{q\Delta x}{24p}\right)\Psi_{k-1}^s + \frac{5}{6}\Psi_k^s + \left(\frac{1}{12} + \frac{q\Delta x}{24p}\right)\Psi_{k+1}^s\right] + \hat{R}_k^j, \\
&\quad k = 2, 3, \ldots, M,\; j = 2, 3, \ldots, N+1.
\end{aligned} \tag{5.7}$$

Let's denote $\tilde{\Psi}_k^j$ be the approximate solution of Ψ_k^j and neglect the error term $\hat{R}_k^j$ in Eq. (5.7), we get

$$\begin{aligned}
&\left(\frac{-p}{\Delta x^2} + \frac{r}{12} - \frac{q^2}{12p} + \frac{q}{2\Delta x} - \frac{\Delta x}{24}\frac{qr}{p} + \frac{da_1}{12} - \frac{da_1 q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^j + \left(\frac{2p}{\Delta x^2} + \frac{5r}{6} + \frac{q^2}{6p} + \frac{5da_1}{6}\right)\tilde{\Psi}_k^j \\
&+ \left(\frac{-p}{\Delta x^2} + \frac{r}{12} - \frac{q^2}{12p} - \frac{q}{2\Delta x} + \frac{\Delta x}{24}\frac{qr}{p} + \frac{da_1}{12} + \frac{da_1 q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^j = \left(\frac{1}{12} - \frac{qdx}{24p}\right)f_{k-1}^j + \frac{5}{6}f_k^j \\
&+ \left(\frac{1}{12} + \frac{q\Delta x}{24p}\right)f_{k+1}^j + da_{j-1}\left[\left(\frac{1}{12} - \frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^1 + \frac{5}{6}\tilde{\Psi}_k^1 + \left(\frac{1}{12} + \frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^1\right] \\
&+ \sum_{s=2}^{j-1} d(a_{j-s} - a_{j-s+1})\left[\left(\frac{1}{12} - \frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^s + \frac{5}{6}\tilde{\Psi}_k^s + \left(\frac{1}{12} + \frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^s\right], \\
&k = 2, 3, \ldots, M,\; j = 2, 3, \ldots, N+1.
\end{aligned} \tag{5.8}$$

with boundaries conditions and initial condition discretized as

$$\tilde{\Psi}_1^j = u^j,\; \tilde{\Psi}_{M+1}^j \;=\; w^j,\; j = 2, 3, \ldots, N+1, \tag{5.9}$$

$$\tilde{\Psi}_k^1 = v_k,\; k = 1, 2, \ldots, M+1. \tag{5.10}$$

Now combining all the Eqs. (5.8) for $k = 2, 3, \ldots, M$ with initial and boundary conditions given in Eq. (5.9)) and Eq. (5.10) we will get a system of $M-1$ Eqns. with $M-1$ unknowns $\tilde{\Psi}_2^j,\ \tilde{\Psi}_3^j, \ldots,\ \tilde{\Psi}_M^j$ which can be represented in matrix form as

$$AW^j = D\left(d\left[a_{j-1}W^1 + \sum_{s=2}^{j-1}(a_{j-s} - a_{j-s+1})W^s\right]\right) + BF^j + E^j, \qquad 2 \le j \le N+1. \tag{5.11}$$

where $W^j = (\tilde{\Psi}_2^j, \tilde{\Psi}_3^j \ldots, \tilde{\Psi}_M^j)^T$ and $F^j = (f_1^j, f_2^j, \ldots, f_{M+1}^j)^T$ and

$$A = \begin{pmatrix} \beta & \Lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & \beta & \Lambda & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda & \beta & \Lambda \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda & \beta \end{pmatrix},\ E^j = \begin{pmatrix} \sum_{s=2}^{j-1} d(a_{j-s} - a_{j-s+1})\tilde{\lambda}\tilde{\Psi}_1^s + da_{j-1}\tilde{\lambda}\tilde{\Psi}_1^1 - \lambda\tilde{\Psi}_1^j \\ 0 \\ \vdots \\ 0 \\ \sum_{s=2}^{j-1} d(a_{j-s} - a_{j-s+1})\tilde{\Lambda}\tilde{\Psi}_{M+1}^s + da_{j-1}\tilde{\Lambda}\tilde{\Psi}_{M+1}^1 - \Lambda\tilde{\Psi}_{M+1}^j \end{pmatrix},$$

$$B=\begin{pmatrix}\tilde{\lambda} & \tilde{\beta} & \tilde{\Lambda} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \tilde{\lambda} & \tilde{\beta} & \tilde{\Lambda} & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \tilde{\lambda} & \tilde{\beta} & \tilde{\Lambda} & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{\lambda} & \tilde{\beta} & \tilde{\Lambda}\end{pmatrix},\ D=\begin{pmatrix}\tilde{\beta} & \tilde{\Lambda} & 0 & 0 & \cdots & 0 & 0 & 0\\ \tilde{\lambda} & \tilde{\beta} & \tilde{\Lambda} & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \tilde{\lambda} & \tilde{\beta} & \tilde{\Lambda}\\ 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{\lambda} & \tilde{\beta}\end{pmatrix},$$

where $\lambda=\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}+\frac{q}{2\Delta x}-\frac{\Delta x}{24}\frac{qr}{p}+\frac{da_1}{12}-\frac{da_1q\Delta x}{24p}\right)$, $\beta=\left(\frac{2p}{\Delta x^2}+\frac{5r}{6}+\frac{q^2}{6p}+\frac{5da_1}{6}\right)$, $\Lambda=\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}-\frac{q}{2\Delta x}+\frac{\Delta x}{24}\frac{qr}{p}+\frac{da_1}{12}+\frac{da_1q\Delta x}{24p}\right)$, $\tilde{\lambda}=\frac{1}{12}-\frac{q\Delta x}{24p}$, $\tilde{\beta}=\frac{5}{6}$, $\tilde{\Lambda}=\frac{1}{12}+\frac{q\Delta x}{24p}$.

5.2. **Compact $L1-2$ Scheme.** In this section, a difference scheme of order $((\Delta t)^{(3-\alpha)}+(\Delta x)^4)$ is constructed for problem (2.15)–(2.16). Using Eq. 3.14, we have

$$\begin{aligned} m(x_k,t_j)&=\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{j-1}\hat{c}_{j-s}^{(j,\alpha)}\delta_t\Psi_{s+\frac{1}{2}}\\ &=\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\hat{c}_1\Psi(x_k,t_j)-\sum_{s=2}^{j-1}(\hat{c}_{j-s}-\hat{c}_{j-s+1})\Psi(x_k,t_s)-\hat{c}_{j-1}\Psi(x_k,t_1)\right]. \end{aligned}\tag{5.12}$$

Using Eq. (5.3) and Eq. (5.12) and then using Eq. (4.1) and Eq. (4.2) we get

$$\begin{aligned} &\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}+\frac{q}{2\Delta x}-\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k-1}^j+\left(\frac{2p}{\Delta x^2}+\frac{5r}{6}+\frac{q^2}{6p}\right)\Psi_k^j+\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}-\frac{q}{2\Delta x}+\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k+1}^j\\ &=\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)f_{k-1}^j+\frac{5}{6}f_k^j+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)f_{k+1}^j+d\hat{c}_{j-1}\left[\left(\frac{1}{12}\frac{q\Delta x}{24p}\right)\Psi_{k-1}^1+\frac{5}{6}\Psi_k^1+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^1\right]\\ &+\sum_{s=2}^{j-1}d(\hat{c}_{j-s}-\hat{c}_{j-s+1})\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^s+\frac{5}{6}\Psi_k^s+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^s\right]\\ &-d\hat{c}_1\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^j+\frac{5}{6}\Psi_k^j+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^j\right]+\mathbb{R}_k^j,\ k=2,3,\ldots,M,\ j=2,3,\ldots,N+1, \end{aligned}\tag{5.13}$$

where $\mathbb{R}_k^j=O((\Delta t)^{(3-\alpha)}+(\Delta x)^4)$, and the discretization of boundary and initial conditions is given in Eq. (5.5) and Eq. (5.6). Let's denote $\tilde{\Psi}_k^j$ be the approximate solution of Ψ_k^j and neglect the error term $\mathbb{R}_k^j$ in Eq. (5.13), we get

$$\begin{aligned} &\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}+\frac{q}{2\Delta x}-\frac{\Delta x}{24}\frac{qr}{p}+\frac{d\hat{c}_1}{12}-\frac{d\hat{c}_1q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^j+\left(\frac{2p}{\Delta x^2}+\frac{5r}{6}+\frac{q^2}{6p}+\frac{5d\hat{c}_1}{6}\right)\tilde{\Psi}_k^j\\ &+\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}-\frac{q}{2\Delta x}+\frac{\Delta x}{24}\frac{qr}{p}+\frac{d\hat{c}_1}{12}+\frac{d\hat{c}_1q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^j=\left(\frac{1}{12}-\frac{qdx}{24p}\right)f_{k-1}^j+\frac{5}{6}f_k^j+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)f_{k+1}^j\\ &+d\hat{c}_{j-1}\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^1+\frac{5}{6}\tilde{\Psi}_k^1+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^1\right]\\ &+\sum_{s=2}^{j-1}d(\hat{c}_{j-s}-\hat{c}_{j-s+1})\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^s+\frac{5}{6}\tilde{\Psi}_k^s+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^s\right],\ k=2,3,\ldots,M,\ j=2,3,\ldots,N+1. \end{aligned}\tag{5.14}$$

Combining all the Eqs. (5.14) for $k=2,3,\ldots,M$ with initial and boundary conditions given in Eq. (5.9) and Eq. (5.10) we will get a system of $M-1$ Eqs. with $M-1$ unknowns $\tilde{\Psi}_2^j,\ \tilde{\Psi}_3^j,\ldots,\ \tilde{\Psi}_M^j$ which can be represented in matrix form as

$$\mathbb{A}W^j=\mathbb{D}\left(d\left[\hat{c}_{j-1}W^1+\sum_{s=2}^{j-1}(\hat{c}_{j-s}-\hat{c}_{j-s+1})W^s\right]\right)+\mathbb{B}F^j+\mathbb{E}^j,\ 1\le j\le N.\tag{5.15}$$

where $W^j=(\tilde\Psi_2^j,\tilde\Psi_3^j\dots,\tilde\Psi_M^j)^T$, and $F^j=(f_1^j,f_2^j,\dots,f_{M+1}^j)^T$ and

$$\mathbb{A}=\begin{pmatrix}\Pi & \Theta & 0 & 0 & \cdots & 0 & 0 & 0\\ \theta & \Pi & \Theta & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \theta & \Pi & \Theta\\ 0 & 0 & 0 & \cdots & 0 & 0 & \theta & \Pi\end{pmatrix},\ \mathbb{E}^j=\begin{pmatrix}\sum_{s=2}^{j-1} d(\hat c_{j-s}-\hat c_{j-s+1})\tilde\theta\tilde\Psi_1^s+d\hat c_{j-1}\tilde\theta\tilde\Psi_1^1-\theta\tilde\Psi_1^j\\ 0\\ \vdots\\ 0\\ \sum_{s=2}^{j} d(\hat c_{j-s}-\hat c_{j-s+1})\tilde\Theta\tilde\Psi_{M+1}^s+d\hat c_{j-1}\tilde\Theta\tilde\Psi_{M+1}^1-\Theta\tilde\Psi_{M+1}^j\end{pmatrix},$$

$$\mathbb{B}=\begin{pmatrix}\tilde\theta & \tilde\Pi & \tilde\Theta & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \tilde\theta & \tilde\Pi & \tilde\Theta & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \tilde\theta & \tilde\Pi & \tilde\Theta & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \tilde\theta & \tilde\Pi & \tilde\Theta\end{pmatrix},\ \mathbb{D}=\begin{pmatrix}\tilde\Pi & \tilde\Theta & 0 & 0 & \cdots & 0 & 0 & 0\\ \tilde\theta & \tilde\Pi & \tilde\Theta & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \tilde\theta & \tilde\Pi & \tilde\Theta\\ 0 & 0 & 0 & \cdots & 0 & 0 & \tilde\theta & \tilde\Pi\end{pmatrix},$$

where $\theta=\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}+\frac{q}{2\Delta x}-\frac{\Delta x}{24}\frac{qr}{p}+\frac{d\hat c_1}{12}-\frac{d\hat c_1 q\Delta x}{24p}\right)$, $\Pi=\left(\frac{2p}{\Delta x^2}+\frac{5r}{6}+\frac{q^2}{6p}+\frac{5d\hat c_1}{6}\right)$, $\Theta=\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}-\frac{q}{2\Delta x}+\frac{\Delta x}{24}\frac{qr}{p}+\frac{d\hat c_1}{12}+\frac{d\hat c_1 q\Delta x}{24p}\right)$, $\tilde\theta=\frac{1}{12}-\frac{q\Delta x}{24p}$, $\tilde\Pi=\frac{5}{6}$, $\tilde\Theta=\frac{1}{12}+\frac{q\Delta x}{24p}$.

5.3. **Compact $L2-1_\sigma$ Scheme.** In this section, a difference scheme of order $((\Delta t)^2+(\Delta x)^4)$ is constructed for problem (2.15)–(2.16). Now consider, $m(x,t)={}_0^C D_t^\alpha\Psi(x,t)$. At the point $(x_k,t_{j+\sigma})$, we have

$$m(x_k,t_{j+\sigma})={}_0^C D_{t_{j+\sigma}}^\alpha\Psi(x_k,t_{j+\sigma}). \tag{5.16}$$

Using Eq. (3.20) and Eq. (5.16), we get

$$m(x_k,t_{j+\sigma})=\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{j}\tilde c_{j-s}^{(j,\alpha)}\Psi_{t,s} \tag{5.17}$$

$$=d\sum_{s=1}^{j}\tilde c_{j-s}^{(j,\alpha)}(\Psi(x_k,t_{s+1})-\Psi(x_k,t_s)), \tag{5.18}$$

with $d=\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}$. Now using Theorem 4.1 with $\gamma_1=-p$, $\gamma_2=-q$, $\gamma_3=-r$, Eq. (5.2) at the point $(x_k,t_{j+\sigma})$ can be written as

$$\begin{aligned}&\left[-p-\frac{\Delta x^2}{12}\left(-r+\frac{q^2}{p}\right)\right]\delta_x^2\Psi(x_k,t_{j+\sigma})+\left[-q+\frac{\Delta x^2}{12}\frac{qr}{p}\right]\delta_x\Psi(x_k,t_{j+\sigma})+r\Psi(x_k,t_{j+\sigma})=\\ &f(x_k,t_{j+\sigma})-m(x_k,t_{j+\sigma})+\frac{q}{p}\frac{\Delta x^2}{12}\delta_x(f(x_k,t_{j+\sigma})-m(x_k,t_{j+\sigma}))+\frac{\Delta x^2}{12}\delta_x^2(f(x_k,t_{j+\sigma})\\ &-m(x_k,t_{j+\sigma}))+\tilde r_1.\end{aligned} \tag{5.19}$$

Now let $\Psi_k^{j+\sigma}=\Psi(x_k,t_{j+\sigma})$ and $f_k^{j+\sigma}=f(x_k,t_{j+\sigma})$, $(k=1,2,\dots,M+1; j=1,2,\dots,N+1)$. Using Eq. (5.19) and (5.18) and then using Eq. (4.1) and Eq. (4.2) we get

$$\begin{aligned}&\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}+\frac{q}{2\Delta x}-\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k-1}^{j+\sigma}+\left(\frac{2p}{\Delta x^2}+\frac{5r}{6}+\frac{q^2}{6p}\right)\Psi_k^{j+\sigma}+\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}-\frac{q}{2\Delta x}+\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k+1}^{j+\sigma}\\ &=\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)f_{k-1}^{j+\sigma}+\frac{5}{6}f_k^{j+\sigma}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)f_{k+1}^{j+\sigma}-d\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\sum_{s=1}^{j}\tilde c_{j-s}(\Psi_{k-1}^{s+1}-\Psi_{k-1}^s)\\ &-\frac{5}{6}d\sum_{s=1}^{j}\tilde c_{j-s}(\Psi_k^{s+1}-\Psi_k^s)-\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)d\sum_{s=1}^{j}\tilde c_{j-s}(\Psi_{k+1}^{s+1}-\Psi_{k+1}^s)+R_k^{j+\sigma},\ k=2,3,\dots,M,\ j=1,2,\dots,N+1.\end{aligned} \tag{5.20}$$

i.e.

$$\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}+\frac{q}{2\Delta x}-\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k-1}^{j+\sigma}+\left(\frac{2p}{\Delta x^2}+\frac{5r}{6}+\frac{q^2}{6p}\right)\Psi_{k}^{j+\sigma}+\left(\frac{-p}{\Delta x^2}+\frac{r}{12}-\frac{q^2}{12p}-\frac{q}{2\Delta x}+\frac{\Delta x}{24}\frac{qr}{p}\right)\Psi_{k+1}^{j+\sigma}$$
$$=\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)f_{k-1}^{j+\sigma}+\frac{5}{6}f_{k}^{j+\sigma}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)f_{k+1}^{j+\sigma}+d\tilde{c}_{j-1}\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^{1}+\frac{5}{6}\Psi_{k}^{1}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^{1}\right]$$
$$+\sum_{s=2}^{j}d(\tilde{c}_{j-s}-\tilde{c}_{j-s+1})\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^{s}+\frac{5}{6}\Psi_{k}^{s}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^{s}\right]-d\tilde{c}_0\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^{j+1}+\frac{5}{6}\Psi_{k}^{j+1}\right.$$
$$\left.+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^{j+1}\right]+R_k^{j+\sigma},\ k=2,3,\dots,M,\ j=1,2,\dots,N+1, \tag{5.21}$$

where $R_k^{j+\sigma}=O((\Delta t)^{3-\alpha}+(\Delta x)^4)$, and the discretization of boundary and initial conditions is given in Eq. (5.5) and Eq. (5.6). Applying linear interpolation between t_{j+1} and t_j for term $\Psi(x_k,t_{j+\sigma})$, we have

$$\Psi(x_k,t_{j+\sigma})=\sigma\Psi(x_k,t_{j+1})-(\sigma-1)\Psi(x_k,t_j)+O((\Delta t)^2). \tag{5.22}$$

Using Eq. (5.22), Eq. (5.21) becomes

$$\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}+\frac{\sigma q}{2\Delta x}-\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\tilde{c}_0}{12}-\frac{d\tilde{c}_0 q\Delta x}{24p}\right)\Psi_{k-1}^{j+1}+\left(\frac{\sigma 2p}{\Delta x^2}+\frac{\sigma 5r}{6}+\frac{\sigma q^2}{6p}+\frac{d5\tilde{c}_0}{6}\right)\Psi_{k}^{j+1}$$
$$+\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}-\frac{\sigma q}{2\Delta x}+\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\tilde{c}_0}{12}+\frac{d\tilde{c}_0 q\Delta x}{24p}\right)\Psi_{k+1}^{j+1}=\left(\frac{1}{12}-\frac{qdx}{24p}\right)f_{k-1}^{j+\sigma}+\frac{5}{6}f_{k}^{j+\sigma}$$
$$+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)f_{k+1}^{j+\sigma}+\left(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}-\frac{(\sigma-1)q^2}{12p}+\frac{(\sigma-1)q}{2\Delta x}-\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)\Psi_{k-1}^{j}$$
$$+\left(\frac{(\sigma-1)2p}{\Delta x^2}+\frac{(\sigma-1)5r}{6}+\frac{(\sigma-1)q^2}{6p}\right)\Psi_{k}^{j}+\left(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}-\frac{(\sigma-1)q^2}{12p}-\frac{(\sigma-1)q}{2\Delta x}\right.$$
$$\left.+\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)\Psi_{k+1}^{j}+d\tilde{c}_{j-1}\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^{1}+\frac{5}{6}\Psi_{k}^{1}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^{1}\right]+\sum_{s=2}^{j}d(\tilde{c}_{j-s}-\tilde{c}_{j-s+1})$$
$$\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\Psi_{k-1}^{s}+\frac{5}{6}\Psi_{k}^{s}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\Psi_{k+1}^{s}\right]+R_k^{j+\sigma},\ k=2,3,\dots,M,\ j=1,2,\dots,N+1. \tag{5.23}$$

and the discretization of boundary and initial conditions is given in Eq. (5.5) and Eq. (5.6). Let's denote $\tilde{\Psi}_k^j$ be the approximate solution of Ψ_k^j and neglect the error term $R_k^{j+\sigma}$ in Eq. (5.23), we get

$$\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}+\frac{\sigma q}{2\Delta x}-\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\tilde{c}_0}{12}-\frac{d\tilde{c}_0 q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^{j+1}+\left(\frac{\sigma 2p}{\Delta x^2}+\frac{\sigma 5r}{6}+\frac{\sigma q^2}{6p}+\frac{d5\tilde{c}_0}{6}\right)\tilde{\Psi}_{k}^{j+1}$$
$$+\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}-\frac{\sigma q}{2\Delta x}+\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\tilde{c}_0}{12}+\frac{d\tilde{c}_0 q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^{j+1}=\left(\frac{1}{12}-\frac{qdx}{24p}\right)f_{k-1}^{j+\sigma}+\frac{5}{6}f_{k}^{j+\sigma}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)f_{k+1}^{j+\sigma}$$
$$+\left(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}-\frac{(\sigma-1)q^2}{12p}+\frac{(\sigma-1)q}{2\Delta x}-\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)\tilde{\Psi}_{k-1}^{j}+\left(\frac{(\sigma-1)2p}{\Delta x^2}+\frac{(\sigma-1)5r}{6}+\frac{(\sigma-1)q^2}{6p}\right)\tilde{\Psi}_{k}^{j}$$
$$+\left(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}-\frac{(\sigma-1)q^2}{12p}-\frac{(\sigma-1)q}{2\Delta x}+\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)\tilde{\Psi}_{k+1}^{j}+d\tilde{c}_{j-1}\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^{1}+\frac{5}{6}\tilde{\Psi}_{k}^{1}\right.$$
$$\left.+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^{1}\right]+\sum_{s=2}^{j}d(\tilde{c}_{j-s}-\tilde{c}_{j-s+1})\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k-1}^{s}+\frac{5}{6}\tilde{\Psi}_{k}^{s}+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\tilde{\Psi}_{k+1}^{s}\right],$$
$$k=2,3,\dots,M,\ j=1,2,\dots,N+1. \tag{5.24}$$

Now combining all the Eqs. (5.24) for $k=2,3,\dots,M$ with initial and boundary conditions given in Eq. (5.9) and Eq. (5.10) we will get a system of $M-1$ equations with $M-1$ unknowns $\tilde{\Psi}_2^j,\ \tilde{\Psi}_3^j,\dots,\ \tilde{\Psi}_M^j$ which can be represented in matrix form as

$$PY^{j+1} = Q\left(d\left[\tilde{c}_{j-1}Y^1 + \sum_{s=2}^{j}(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})Y^s\right]\right) + U\tilde{Y}^j + SF^{j+\sigma} + \mathcal{R}^j, \ 1 \le j \le N. \tag{5.25}$$

where $Y^j = (\tilde{\Psi}_2^j, \tilde{\Psi}_3^j \ldots, \tilde{\Psi}_M^j)^T, \tilde{Y}^j = (\tilde{\Psi}_1^j, \tilde{\Psi}_2^j \ldots, \tilde{\Psi}_{M+1}^j)^T$ and $F^{j+\sigma} = (f_1^{j+\sigma}, f_2^{j+\sigma}, \ldots, f_{M+1}^{j+\sigma})^T$ and

$$P = \begin{pmatrix} \omega & \kappa & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & \omega & \kappa & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mu & \omega & \kappa \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu & \omega \end{pmatrix}, \ \mathcal{R}^j = \begin{pmatrix} \sum_{s=2}^{j} d(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})\tilde{\mu}\tilde{\Psi}_1^s + d\tilde{c}_{j-1}\tilde{\mu}\tilde{\Psi}_1^1 - \mu\tilde{\Psi}_1^{j+1} \\ 0 \\ \vdots \\ 0 \\ \sum_{s=2}^{j} d(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})\tilde{\kappa}\tilde{\Psi}_{M+1}^s + d\tilde{c}_{j-1}\tilde{\kappa}\tilde{\Psi}_{M+1}^1 - \kappa\tilde{\Psi}_{M+1}^{j+1} \end{pmatrix},$$

$$U = \begin{pmatrix} \hat{\mu} & \hat{\omega} & \hat{\kappa} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \hat{\mu} & \hat{\omega} & \hat{\kappa} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \hat{\mu} & \hat{\omega} & \hat{\kappa} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \hat{\mu} & \hat{\omega} & \hat{\kappa} \end{pmatrix},$$

$$Q = \begin{pmatrix} \tilde{\omega} & \tilde{\kappa} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \tilde{\mu} & \tilde{\omega} & \tilde{\kappa} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{\mu} & \tilde{\omega} & \tilde{\kappa} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{\mu} & \tilde{\omega} \end{pmatrix}, \qquad S = \begin{pmatrix} \tilde{\mu} & \tilde{\omega} & \tilde{\kappa} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \tilde{\mu} & \tilde{\omega} & \tilde{\kappa} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{\mu} & \tilde{\omega} & \tilde{\kappa} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{\mu} & \tilde{\omega} & \tilde{\kappa} \end{pmatrix},$$

where

$$\mu = \left(\frac{-\sigma p}{\Delta x^2} + \frac{\sigma r}{12} - \frac{\sigma q^2}{12p} + \frac{\sigma q}{2\Delta x} - \frac{\sigma \Delta x}{24}\frac{qr}{p} + \frac{d\tilde{c}_0}{12} - \frac{d\tilde{c}_0 q\Delta x}{24p}\right), \qquad \omega = \left(\frac{\sigma 2p}{\Delta x^2} + \frac{\sigma 5r}{6} + \frac{\sigma q^2}{6p} + \frac{5d\tilde{c}_0}{6}\right),$$

$$\kappa = \left(\frac{-\sigma p}{\Delta x^2} + \frac{\sigma r}{12} - \frac{\sigma q^2}{12p} - \frac{\sigma q}{2\Delta x} + \frac{\sigma \Delta x}{24}\frac{qr}{p} + \frac{d\tilde{c}_0}{12} + \frac{d\tilde{c}_0 q\Delta x}{24p}\right), \qquad \tilde{\mu} = \frac{1}{12} - \frac{q\Delta x}{24p}, \ \tilde{\omega} = \frac{5}{6}, \ \tilde{\kappa} = \frac{1}{12} + \frac{q\Delta x}{24p},$$

$$\hat{\mu} = \left(\frac{-(\sigma-1)p}{\Delta x^2} + \frac{(\sigma-1)r}{12} - \frac{(\sigma-1)q^2}{12p} + \frac{(\sigma-1)q}{2\Delta x} - \frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right), \quad \hat{\omega} = \left(\frac{(\sigma-1)2p}{\Delta x^2} + \frac{(\sigma-1)5r}{6} + \frac{(\sigma-1)q^2}{6p}\right),$$

$$\hat{\kappa} = \left(\frac{-(\sigma-1)p}{\Delta x^2} + \frac{(\sigma-1)r}{12} - \frac{(\sigma-1)q^2}{12p} - \frac{(\sigma-1)q}{2\Delta x} + \frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right).$$

6. Stability and Convergence Analysis

6.1. **Stability Analysis.** In this subsection, we will use the Fourier analysis method to analyze the stability of the proposed scheme Eq. (5.24) for the problem (2.15)–(2.16). Let Ψ_k^j be approximate by $\tilde{\Psi}_k^j$. Now for $j = 1, 2, \ldots, N+1$. Define the error $\tilde{E}_k^j$ as

$$\tilde{E}_k^j = \tilde{\Psi}_k^j - \Psi_k^j. \tag{6.1}$$

Using Eq. (6.1), we get the error equation as

$$\left(\frac{-\sigma p}{\Delta x^2} + \frac{\sigma r}{12} - \frac{\sigma q^2}{12p} + \frac{\sigma q}{2\Delta x} - \frac{\sigma \Delta x}{24}\frac{qr}{p} + \frac{d\tilde{c}_0}{12} - \frac{d\tilde{c}_0 q\Delta x}{24p}\right)\tilde{E}_{k-1}^{j+1} + \left(\frac{\sigma 2p}{\Delta x^2} + \frac{\sigma 5r}{6} + \frac{\sigma q^2}{6p} + \frac{d5\tilde{c}_0}{6}\right)\tilde{E}_k^{j+1}$$
$$+ \left(\frac{-\sigma p}{\Delta x^2} + \frac{\sigma r}{12} - \frac{\sigma q^2}{12p} - \frac{\sigma q}{2\Delta x} + \frac{\sigma \Delta x}{24}\frac{qr}{p} + \frac{d\tilde{c}_0}{12} + \frac{d\tilde{c}_0 q\Delta x}{24p}\right)\tilde{E}_{k+1}^{j+1} = \left(\frac{-(\sigma-1)p}{\Delta x^2} + \frac{(\sigma-1)r}{12} - \frac{(\sigma-1)q^2}{12p} + \frac{(\sigma-1)q}{2\Delta x}\right.$$
$$\left. - \frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)\tilde{E}_{k-1}^j + \left(\frac{(\sigma-1)2p}{\Delta x^2} + \frac{(\sigma-1)5r}{6} + \frac{(\sigma-1)q^2}{6p}\right)\tilde{E}_k^j + \left(\frac{-(\sigma-1)p}{\Delta x^2} + \frac{(\sigma-1)r}{12} - \frac{(\sigma-1)q^2}{12p}\right.$$

$$- \frac{(\sigma-1)q}{2\Delta x} + \frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\Big)\tilde{E}^j_{k+1} + d\tilde{c}_{j-1}\Big[\Big(\frac{1}{12} - \frac{q\Delta x}{24p}\Big)\tilde{E}^1_{k-1} + \frac{5}{6}\tilde{E}^1_k + \Big(\frac{1}{12} + \frac{q\Delta x}{24p}\Big)\tilde{E}^1_{k+1}\Big] \tag{6.2}$$

$$+ \sum_{s=2}^{j} d(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})\Big[\Big(\frac{1}{12} - \frac{q\Delta x}{24p}\Big)\tilde{E}^s_{k-1} + \frac{5}{6}\tilde{E}^s_k + \Big(\frac{1}{12} + \frac{q\Delta x}{24p}\Big)\tilde{E}^s_{k+1}\Big], \; k = 2,3,\dots,M, \; j = 1,2,\dots,N+1.$$

and $\tilde{E}^j_1 = \tilde{E}^j_{M+1} = 0$. For $j = 1,2,\dots,N+1$, let us define a function

$$\tilde{E}^j(x) = \begin{cases} \tilde{E}^j_k, & \text{if} \quad x_{k-\frac{1}{2}} < x \le x_{k+\frac{1}{2}}, k = 2,3,\dots,M, \\ 0, & \text{if} \quad D_l < x \le D_l + \frac{\Delta x}{2} \text{ or } D_r - \frac{\Delta x}{2} < x \le D_r, \end{cases} \tag{6.3}$$

and let $\tilde{E}^j = [\tilde{E}^j_2, \tilde{E}^j_3, \dots, \tilde{E}^j_M]^T$ and let us consider norm

$$||\tilde{E}^j||_2 = \Big(\sum_{k=2}^{M} \Delta x |\tilde{E}^j_k|^2\Big)^{\frac{1}{2}} = \Big(\sum_{l=-\infty}^{\infty} |\chi_j(l)|^2\Big)^{\frac{1}{2}}, \tag{6.4}$$

where $\chi_j(l)$ denotes the Fourier coefficient of the series $E^j(x)$. Then, the error E^j_k can be expressed in terms of these coefficients as follows:

$$E^j_k = \chi_j e^{\iota\rho(D_l + k\Delta x)}, \; \iota = \sqrt{-1}. \tag{6.5}$$

Now substituting Eq. (6.5) into Eq. (6.2) we get

$$\chi_{j+1} e^{\iota\rho(D_l+k\Delta x)}\Big[\Big(\frac{-\sigma p}{\Delta x^2} + \frac{\sigma r}{12} - \frac{\sigma q^2}{12p} + \frac{\sigma q}{2\Delta x} - \frac{\sigma\Delta x}{24}\frac{qr}{p} + \frac{d\tilde{c}_0}{12} - \frac{d\tilde{c}_0 q\Delta x}{24p}\Big)e^{-\iota\rho\Delta x} + \Big(\frac{2\sigma p}{\Delta x^2} + \frac{5\sigma r}{6} + \frac{\sigma q^2}{6p} + \frac{5d\tilde{c}_0}{6}\Big)$$
$$+ \Big(\frac{-\sigma p}{\Delta x^2} + \frac{\sigma r}{12} - \frac{\sigma q^2}{12p} - \frac{\sigma q}{2\Delta x} + \frac{\sigma\Delta x}{24}\frac{qr}{p} + \frac{d\tilde{c}_0}{12} + \frac{d\tilde{c}_0 q\Delta x}{24p}\Big)e^{\iota\rho\Delta x}\Big] = \chi_j e^{\iota\rho(D_l+k\Delta x)}\Big[\Big(\frac{-(\sigma-1)p}{\Delta x^2} + \frac{(\sigma-1)r}{12}$$
$$- \frac{(\sigma-1)q^2}{12p} + \frac{(\sigma-1)q}{2\Delta x} - \frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\Big)e^{-\iota\rho\Delta x} + \Big(\frac{(\sigma-1)2p}{\Delta x^2} + \frac{(\sigma-1)5r}{6} + \frac{(\sigma-1)q^2}{6p}\Big) + \Big(\frac{-(\sigma-1)p}{\Delta x^2} + \frac{(\sigma-1)r}{12}$$
$$- \frac{(\sigma-1)q^2}{12p} - \frac{(\sigma-1)q}{2\Delta x} + \frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\Big)e^{\iota\rho\Delta x}\Big] + d\tilde{c}_{j-1}\chi_1 e^{\iota\rho(D_l+k\Delta x)}\Big[\Big(\frac{1}{12} - \frac{q\Delta x}{24p}\Big)e^{-\iota\rho\Delta x} + \frac{5}{6} + \Big(\frac{1}{12} + \frac{q\Delta x}{24p}\Big)e^{\iota\rho\Delta x}\Big]$$
$$+ \sum_{s=2}^{j} d(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})\chi_s e^{\iota\rho(D_l+k\Delta x)}\Big[\Big(\frac{1}{12} - \frac{q\Delta x}{24p}\Big)e^{-\iota\rho\Delta x} + \frac{5}{6} + \Big(\frac{1}{12} + \frac{q\Delta x}{24p}\Big)e^{\iota\rho\Delta x}\Big], \tag{6.6}$$

$k = 2,3,\dots M, \; j = 1,2,\dots,N+1.$

Which implies

$$\chi_{j+1}\Big[\Big\{\frac{-2\sigma p}{\Delta x^2} + \frac{\sigma r}{6} - \frac{\sigma q^2}{6p} + \frac{d\tilde{c}_0}{6}\Big\}\cos(\rho\Delta x) + \Big\{\frac{2\sigma p}{\Delta x^2} + \frac{5\sigma r}{6} + \frac{\sigma q^2}{6p} + \frac{5d\tilde{c}_0}{6}\Big\} + \Big\{\frac{-\sigma q}{\Delta x} + \frac{\sigma\Delta x qr}{12p} + \frac{q\Delta x d\tilde{c}_0}{12p}\Big\}$$
$$\iota\sin(\rho\Delta x)\Big] = \Big[\frac{d\tilde{c}_{j-1}\cos(\rho\Delta x)}{6} + \frac{5d\tilde{c}_{j-1}}{6} + \frac{q\Delta x d c_{j-1}}{12p}\iota\sin(\rho\Delta x)\Big]\chi_1 + \Big[\sum_{s=2}^{j}(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})\Big(\frac{d}{6}\cos(\rho\Delta x) + \frac{5d}{6}$$
$$+ \frac{qd\Delta x}{12p}\iota\sin(\rho\Delta x)\Big)\Big]\chi_s + \Big[\Big\{\frac{-2(\sigma-1)p}{\Delta x^2} + \frac{(\sigma-1)r}{6} - \frac{(\sigma-1)q^2}{6p}\Big\}\cos(\rho\Delta x) + \Big\{\frac{2(\sigma-1)p}{\Delta x^2}$$
$$+ \frac{5(\sigma-1)r}{6} + \frac{(\sigma-1)q^2}{6p}\Big\} + \Big\{\frac{-(\sigma-1)q}{\Delta x} + \frac{(\sigma-1)qr\Delta x}{12p}\Big\}\iota\sin(\rho\Delta x)\Big]\chi_j. \tag{6.7}$$

Define $\Im_1 = \frac{5}{6} + \frac{1}{6}\cos(\rho\Delta x)$, $\Im_2 = \frac{q\Delta x}{12p}\sin(\rho\Delta x)$, $\Im_3 = (\frac{-2p}{\Delta x^2} + \frac{r}{6} - \frac{q^2}{6p})\cos(\rho\Delta x) + (\frac{2p}{(\Delta x)^2} + \frac{5r}{6} + \frac{q^2}{6p})$, $\Im_4 = (\frac{q}{\Delta x} - \frac{\Delta x qr}{12p})\sin(\rho\Delta x)$. Thus we can write Eq. (6.7) as

$$\chi_{j+1} = \frac{1}{\sigma(\Im_3 - \iota\Im_4) + d\tilde{c}_0(\Im_1 + \iota\Im_2)}\Big[d\tilde{c}_{j-1}(\Im_1 + \iota\Im_2)\chi_1 + d\sum_{s=2}^{j}(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})(\Im_1 + \iota\Im_2)\chi_s + (\sigma-1)(\Im_3 - \iota\Im_4)\chi_j\Big]$$

$$= \frac{d(\Im_1 + \iota\Im_2)}{\sigma(\Im_3 - \iota\Im_4) + d\tilde{c}_0(\Im_1 + \iota\Im_2)}\left[\tilde{c}_{j-1}\chi_1 + \sum_{s=2}^{j}(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})\chi_s\right] + \frac{(\sigma-1)(\Im_3 - \iota\Im_4)}{\sigma(\Im_3 - \iota\Im_4) + d\tilde{c}_0(\Im_1 + \iota\Im_2)}\chi_j. \tag{6.8}$$

Taking modulus both sides in Eq. (6.8) we have

$$|\chi_{j+1}| \le \left|\frac{d(\Im_1 + \iota\Im_2)}{\sigma(\Im_3 - \iota\Im_4) + d\tilde{c}_0(\Im_1 + \iota\Im_2)}\left[\tilde{c}_{j-1}\chi_1 + \sum_{s=2}^{j}(\tilde{c}_{j-s} - \tilde{c}_{j-s+1})\chi_s\right]\right| + \left|\frac{(\sigma-1)(\Im_3 - \iota\Im_4)}{\sigma(\Im_3 - \iota\Im_4) + d\tilde{c}_0(\Im_1 + \iota\Im_2)}\chi_j\right|. \tag{6.9}$$

Now, define $\phi_1 = \left|\frac{d(\Im_1+\iota\Im_2)+(\sigma-1)(\Im_3-\iota\Im_4)}{+\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\right|$, $\phi_2 = \left|\frac{d(\Im_1+\iota\Im_2)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\right|$, and $\phi_3 = \left|\frac{(\sigma-1)(\Im_3-\iota\Im_4)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\right|$. Then we have the following stability result.

Theorem 6.1. *The numerical scheme* (5.24) *is stable when* $\phi_1 \le 1,\ \phi_2 \le 1,\ \phi_3 \le 1$.

Proof. We have to prove,

$$|\chi_{j+1}| \le \tilde{C}|\chi_1|,\ j \ge 1, \tag{6.10}$$

where, $\tilde{C}$ is any arbitrary constant.
If $\phi_1 \le 1$, for $j = 1$ Eq. (6.9) reduces to

$$|\chi_2| \le \tilde{C}|\chi_1|. \tag{6.11}$$

Thus, Eq. (6.10) holds for $j = 1$. Now assume that Eq. (6.10) holds for $j \le i - 1$. i.e.

$$|\chi_{j+1}| \le \tilde{C}|\chi_1|,\ j = 2, 3, \dots, i-1. \tag{6.12}$$

for $j = i$, Eq. (6.9) becomes

$$|\chi_{i+1}| \le \left|\left[\tilde{c}_{i-1}\chi_1 + \sum_{s=2}^{i-1}(\tilde{c}_{i-s} - \tilde{c}_{i-s+1})\chi_s + (\tilde{c}_0 - \tilde{c}_1)\chi_i\right]\right| + |(\sigma-1)||\chi_i|, \tag{6.13}$$

i.e.

$$|\chi_{i+1}| \le \left[\tilde{c}_{i-1}|\chi_1| + \sum_{s=2}^{i-1}(\tilde{c}_{i-s} - \tilde{c}_{i-s+1})|\chi_s| + (\tilde{c}_0 - \tilde{c}_1)|\chi_i|\right] + |(\sigma-1)||\chi_i|. \tag{6.14}$$

Using Eq. (6.11) in Eq. (6.12), we get

$$\begin{aligned} |\chi_{i+1}| &\le \left[\tilde{c}_{i-1}|\chi_1| + \sum_{s=2}^{i-1}(\tilde{c}_{i-s} - \tilde{c}_{i-s+1})|\chi_1| + (\tilde{c}_0 - \tilde{c}_1)|\chi_1|\right] + |(\sigma-1)||\chi_1| \\ &\le \tilde{c}_0|\chi_1| + |\sigma - 1||\chi_1| \\ &\le \tilde{C}\ |\chi_1|. \end{aligned} \tag{6.15}$$

Hence for $j = i$ the result holds. Therefore, by the principle of mathematical induction Eq. (6.10) is valid for every i. i.e.

$$|\chi_{j+1}| \le \tilde{C}|\chi_1|,\ j \ge 1. \tag{6.16}$$

For $j = 1$, Using Eq. (6.4) and Eq. (6.16), we have

$$||\tilde{E}^{j+1}||_2 = \left(\sum_{l=-\infty}^{\infty}|\chi_j(l)|^2\right)^{\frac{1}{2}} \le \left(\sum_{l=-\infty}^{\infty}|\tilde{C}\chi_1(l)|^2\right)^{\frac{1}{2}} \le |\tilde{C}|\ ||\tilde{E}^1||_2. \tag{6.17}$$

Therefore, it follows that

$$||\tilde{E}^{j+1}||_2 \le |\tilde{C}|\ ||\tilde{E}^1||_2. \tag{6.18}$$

Hence, proof of Theorem 6.1 is completed. □

6.2. **Convergence Analysis.** In this subsection, we will analyze the convergence rate of the numerical scheme defined by Eq. (5.24). Here we assume that $\tilde{\Psi}_k^j$ is the solution of Eq. (5.24) and Ψ_k^j is the approximate solution of Eq. (5.23). Now subtracting Eq. (5.24) from Eq. (5.23), we have the error equation as

$$\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}+\frac{\sigma q}{2\Delta x}-\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\tilde{c}_0}{12}-\frac{d\tilde{c}_0 q\Delta x}{24p}\right)\nu_{k-1}^{j+1}+\left(\frac{\sigma 2p}{\Delta x^2}+\frac{\sigma 5r}{6}+\frac{\sigma q^2}{6p}+\frac{d5\tilde{c}_0}{6}\right)\nu_k^{j+1}$$
$$+\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}-\frac{\sigma q}{2\Delta x}+\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\nu_0}{12}+\frac{d\tilde{c}_0 q\Delta x}{24p}\right)\nu_{k+1}^{j+1}=\left(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}-\frac{(\sigma-1)q^2}{12p}\right.$$
$$\left.+\frac{(\sigma-1)q}{2\Delta x}-\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)\nu_{k-1}^j+\left(\frac{(\sigma-1)2p}{\Delta x^2}+\frac{(\sigma-1)5r}{6}+\frac{(\sigma-1)q^2}{6p}\right)\nu_k^j+\left(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}\right.$$
$$\left.-\frac{(\sigma-1)q^2}{12p}-\frac{(\sigma-1)q}{2\Delta x}+\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)\nu_{k+1}^j+\sum_{s=2}^{j}d(\tilde{c}_{j-s}-\tilde{c}_{j-s+1})\left[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)\nu_{k-1}^s+\frac{5}{6}\nu_k^s\right.$$
$$\left.+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)\nu_{k+1}^s\right]+R_k^{j+1},\ k=2,3,\ldots,M,\ j=1,2,\ldots,N+1. \tag{6.19}$$

where $\nu_k^j=\Psi_k^j-\tilde{\Psi}_k^j$, with

$$\nu_1^j=0,\ \nu_{M+1}^j=0,\ \nu_k^1=0,\ k=1,2,\ldots,M+1,\ j=1,2,\ldots,N+1. \tag{6.20}$$

Now, we define the grid functions as:

$$R^j(x)=\begin{cases} R_k^j, & \text{if}\quad x_{k-\frac{1}{2}}<x\le x_{k+\frac{1}{2}},\ k=2,3,\ldots,M,\\ 0, & \text{if}\quad D_l\le x\le D_l+\frac{\Delta x}{2}\ \text{or}\ D_r-\frac{\Delta x}{2}<x\le D_r,\end{cases} \tag{6.21}$$

$$\nu^j(x)=\begin{cases} \nu_k^j, & \text{if}\quad x_{k-\frac{1}{2}}<x\le x_{k+\frac{1}{2}},\ k=2,3,\ldots,M,\\ 0, & \text{if}\quad D_l\le x\le D_l+\frac{\Delta x}{2}\ \text{or}\ D_r-\frac{\Delta x}{2}<x\le D_r,\end{cases} \tag{6.22}$$

and assuming $\nu^j=[\nu_2^j,\nu_3^j,\ldots,\nu_M^j]^T$, and $R^j=[R_2^j,R_3^j,\ldots,R_M^j]^T$. Let's define the following norms for $j=1,2,\ldots,N+1$

$$||R^j||_2=\left(\sum_{k=2}^{M}\Delta x|R_k^j|^2\right)^{\frac{1}{2}}=\left(\sum_{k=-\infty}^{\infty}|\zeta_j(k)|^2\right)^{\frac{1}{2}},\ j=1,2,\ldots,N+1, \tag{6.23}$$

and

$$||\nu^j||_2=\left(\sum_{k=2}^{M}\Delta x|\nu_k^j|^2\right)^{\frac{1}{2}}=\left(\sum_{k=-\infty}^{\infty}|\varsigma_j(k)|^2\right)^{\frac{1}{2}},\ j=1,2,\ldots,N+1, \tag{6.24}$$

where the functions $R^j(x)$ and $\nu^j(x)$ have Fourier coefficients denoted by $\zeta_j(k)$ and $\varsigma_j(k)$ respectively.
Now, choose R_k^j and ν_k^j as

$$R_k^j=\zeta_j e^{\iota\rho(D_l+k\Delta x)},\ \nu_k^j=\varsigma_j e^{\iota\rho(D_l+k\Delta x)}. \tag{6.25}$$

Substituting Eq. (6.25) into Eq. (6.19) we have

$$
\begin{aligned}
&\varsigma_{j+1} e^{\iota\rho(D_l+k\Delta x)}\Bigg[\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}+\frac{\sigma q}{2\Delta x}-\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\tilde{c}_0}{12}-\frac{d\tilde{c}_0 q\Delta x}{24p}\right)e^{-\iota\rho\Delta x}+\Bigg(\frac{2\sigma p}{\Delta x^2}\\
&+\frac{5\sigma r}{6}+\frac{\sigma q^2}{6p}+\frac{5d\tilde{c}_0}{6}\Bigg)+\left(\frac{-\sigma p}{\Delta x^2}+\frac{\sigma r}{12}-\frac{\sigma q^2}{12p}-\frac{\sigma q}{2\Delta x}+\frac{\sigma\Delta x}{24}\frac{qr}{p}+\frac{d\tilde{c}_0}{12}+\frac{d\tilde{c}_0 q\Delta x}{24p}\right)e^{\iota\rho\Delta x)}\Bigg]\\
&=\varsigma_j e^{\iota\rho(D_l+k\Delta x)}\Bigg[\left(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}-\frac{(\sigma-1)q^2}{12p}+\frac{(\sigma-1)q}{2\Delta x}-\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\right)e^{-\iota\rho\Delta x}\\
&+\left(\frac{(\sigma-1)2p}{\Delta x^2}+\frac{(\sigma-1)5r}{6}+\frac{(\sigma-1)q^2}{6p}\right)+\Bigg(\frac{-(\sigma-1)p}{\Delta x^2}+\frac{(\sigma-1)r}{12}-\frac{(\sigma-1)q^2}{12p}-\frac{(\sigma-1)q}{2\Delta x}\\
&+\frac{(\sigma-1)\Delta x}{24}\frac{qr}{p}\Bigg)e^{\iota\rho\Delta x}\Bigg]+\sum_{s=2}^{j} d(\tilde{c}_{j-s}-\tilde{c}_{j-s+1})\varsigma_s e^{\iota\rho(D_l+k\Delta x)}\Bigg[\left(\frac{1}{12}-\frac{q\Delta x}{24p}\right)e^{-\iota\rho\Delta x}+\frac{5}{6}\\
&+\left(\frac{1}{12}+\frac{q\Delta x}{24p}\right)e^{\iota\rho\Delta x}\Bigg]+\zeta_{j+1}e^{\iota\rho(D_l+k\Delta x)}.
\end{aligned}
$$

Using same $\Im_1, \Im_2, \Im_3, \Im_4$ as define in Subsection 6.1 the above equation can be written as

$$
\begin{aligned}
\varsigma_{j+1} &= \frac{1}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\Bigg[d\sum_{s=2}^{j}(\tilde{c}_{j-s}-\tilde{c}_{j-s+1})(\Im_1+\iota\Im_2)\varsigma_s\\
&\quad+(\sigma-1)(\Im_3-\iota\Im_4)\varsigma_j+\zeta_{j+1}\Bigg]\\
&= \frac{d(\Im_1+\iota\Im_2)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\Bigg[\sum_{s=2}^{j}(\tilde{c}_{j-s}-\tilde{c}_{j-s+1})\varsigma_s\Bigg]\\
&\quad+\frac{(\sigma-1)(\Im_3-\iota\Im_4)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\varsigma_j+\frac{\zeta_{j+1}}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}.
\end{aligned}
\tag{6.26}
$$

For $k=1,2,\dots,M+1,\ j=1,2,\dots,N+1$, $\exists$ a positive constant lets say B_1 such that

$$|R_k^j| \le B_1((\Delta t)^2+(\Delta x)^4). \tag{6.27}$$

Now using Eq. (6.23) we have

$$
\begin{aligned}
||R^j||_2 &= \left(\sum_{k=2}^{M}\Delta x|R_k^j|^2\right)^{\frac{1}{2}}\\
&\le \left(\sum_{k=2}^{M}\Delta x B_1((\Delta t)^2+(\Delta x)^4)^2\right)^{\frac{1}{2}}\\
&\le B_1\left((\Delta t)^2+(\Delta x)^4\right), B_1=\sqrt{M\Delta x B_1},\ j=2,3,\dots,N+1.
\end{aligned}
\tag{6.28}
$$

Since from the above $||R^j||_2$ is convergent, so using Eq. (6.23) we can easily say that $\left(\sum_{l=-\infty}^{\infty}|\zeta_j(l)|^2\right)^{\frac{1}{2}}$ converges. Therefore, $\exists$ a positive constant say B_2 such that

$$|\zeta_j| = |\zeta_j(l)| \le B_2\Delta t|\zeta_1| = B_2|\zeta_2(l)|,\ j=2,3,\dots,N+1. \tag{6.29}$$

Theorem 6.2. *If* $\left|\frac{d(\Im_1+\iota\Im_2)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\right| \le 1$, $\left|\frac{(\sigma-1)(\Im_3-\iota\Im_4)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\right| \le 1$, $\left|\frac{1}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\right| \le 1$, *then for* $j=1,2,\dots,N$ *the following holds true for Eq.* (5.24)

$$||\nu^{j+1}||_2 \le A_2(1+\Delta t)^{j+1}|\zeta_2|. \tag{6.30}$$

Proof. To prove the above theorem we will use mathematical induction. Now substituting $j = 1$ in Eq. (6.26)

$$\begin{aligned}|\varsigma_2| &= \left|\frac{(\sigma-1)(\Im_3-\iota\Im_4)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\varsigma_1+\frac{\zeta_2}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)}\right|\\ &\le |\sigma-1|\ |\varsigma_1|+|\zeta_2|\\ &\le A_2(1+\Delta t)^2|\zeta_2| \quad (\text{Assuming } |\sigma-1|\ |\varsigma_1|+|\zeta_2|\le A_2(1+\Delta t)^2|\zeta_2|).\end{aligned}$$

Thus, for $j = 1$ Eq. (6.30) is true. Now assume that for $j \le k-1$ the Eq. (6.30) holds.

$$|\varsigma_{j+1}| \le A_2(1+\Delta t)^{j+1}|\zeta_2|,\ \ j = 1, 2, \dots, k-1. \tag{6.31}$$

For $j = k-1$:

$$\begin{aligned}|\varsigma_k| &\le \sum_{s=2}^{k-1}(\tilde{c}_{k-s-1}-\tilde{c}_{k-s})|\varsigma_s|+|\sigma-1||\varsigma_{k-1}|+|\zeta_k|\\ &\le \sum_{s=2}^{k-1}(\tilde{c}_{k-s-1}-\tilde{c}_{k-s})A_2(1+\Delta t)^s|\zeta_2|+A_2(1+\Delta t)^k|\zeta_2| \quad (\text{assuming}|\sigma-1||\varsigma_{k-1}|+|\zeta_k|\le A_2(1+\Delta t)^k|\zeta_2|)\\ &\le (\tilde{c}_0-\tilde{c}_{k-2})(1+\Delta t)^{k-1}A_2|\zeta_2|+A_2(1+\Delta t)^k|\zeta_2|\\ &\le A_2|\zeta_2|(1+\Delta t)^k\left[\frac{\tilde{c}_0-\tilde{c}_{k-2}}{(1+\Delta t)}+1\right]\ (\tilde{c}_0-\tilde{c}_{k-2}\le 1)\\ &\le A_3|\zeta_2|(1+\Delta t)^k. \quad (A_3 = 2A_2)\end{aligned} \tag{6.32}$$

Hence for all $j \ge 1$, it is true i.e.

$$|\varsigma_j| \le A_2(1+\Delta t)^j|\zeta_2|.$$

So

$$|\varsigma_{j+1}| \le A_2(1+\Delta t)^{j+1}|\zeta_2|.$$

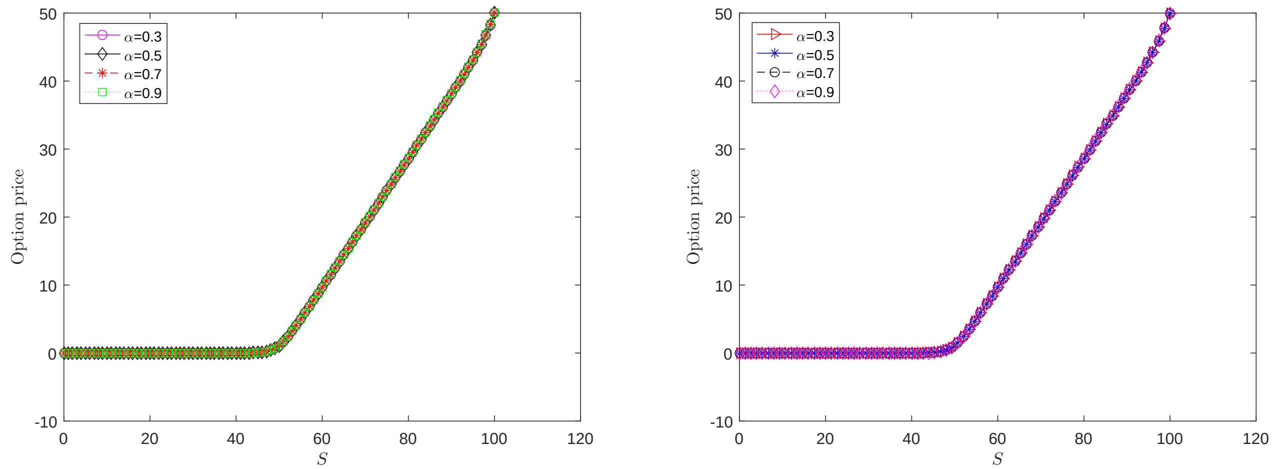
Hence the theorem proved. □

Theorem 6.3. *Let $\Psi(x,t)$ be the exact solution of the Eq. (2.15) and if $\frac{d(\Im_1+\iota\Im_2)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)} \le 1$, $\frac{(\sigma-1)(\Im_3-\iota\Im_4)}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)} \le 1$, $\frac{1}{\sigma(\Im_3-\iota\Im_4)+d\tilde{c}_0(\Im_1+\iota\Im_2)} \le 1$, the proposed numerical scheme defined in Eq. (5.24) is L_2-convergent and the resulting solution satisfies*

$$||\nu^j||_2 \le A((\Delta t)^2+(\Delta x)^4). \tag{6.33}$$

Proof. Consider the Eq. (6.24),

$$\begin{aligned}||\nu^j||_2 &= \left(\sum_{l=-\infty}^{\infty}|\varsigma_j(l)|^2\right)^{\frac{1}{2}}\\ &\le \left(\sum_{l=-\infty}^{\infty}(A_2(1+\Delta t)^j|\zeta_2(l)|)^2\right)^{\frac{1}{2}} (\text{using above theorem})\\ &= A_2(1+\Delta t)^j\left(\sum_{l=-\infty}^{\infty}|\zeta_2(l)|^2\right)\\ &= A_2(1+\Delta t)^j||R^2||_2(\text{using Eq. (6.23)})\\ &\le A_1A_2(1+\Delta t)^j((\Delta t)^2+(\Delta x)^4)\end{aligned}$$



(A) $L1-2$. (B) $L2-1_\sigma$.

FIGURE 1. Call option prices of Example 7.2 at different α for $M = N = 100$, $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$, and $D = 0$.

$$
\begin{aligned}
&\leq A_1 A_2 e^{j\Delta t}((\Delta t)^2 + (\Delta x)^4) \\
&\leq A((\Delta t)^2 + (\Delta x)^4). \qquad (6.34)
\end{aligned}
$$

where $A = A_1 A_2 e^{(j\Delta t)}$.
Hence, proof of Theorem 6.3 is completed. □

7. Numerical Experiments

In this section, some numerical experiments are carried out to establish the accuracy and effectiveness of the proposed schemes. Also, the results obtained by the proposed $L1-2$ and $L2-1_\sigma$ scheme are compared with those available in the existing literature.

Example 7.1. Consider the double barrier knock-out, European-type call option given by

$$\frac{\partial^\alpha \Phi(S,\tau)}{\partial \tau^\alpha} + S^2 \frac{\delta^2}{2} \frac{\partial^2 \Phi(S,\tau)}{\partial S^2} + S(r-D)\frac{\partial \Phi(S,\tau)}{\partial S} - Sr\Phi(S,\tau) = 0, \ (S,\tau) \in (0.1, 100) \times (0,1). \qquad (7.1)$$

This problem is subject to the terminal and boundary conditions:

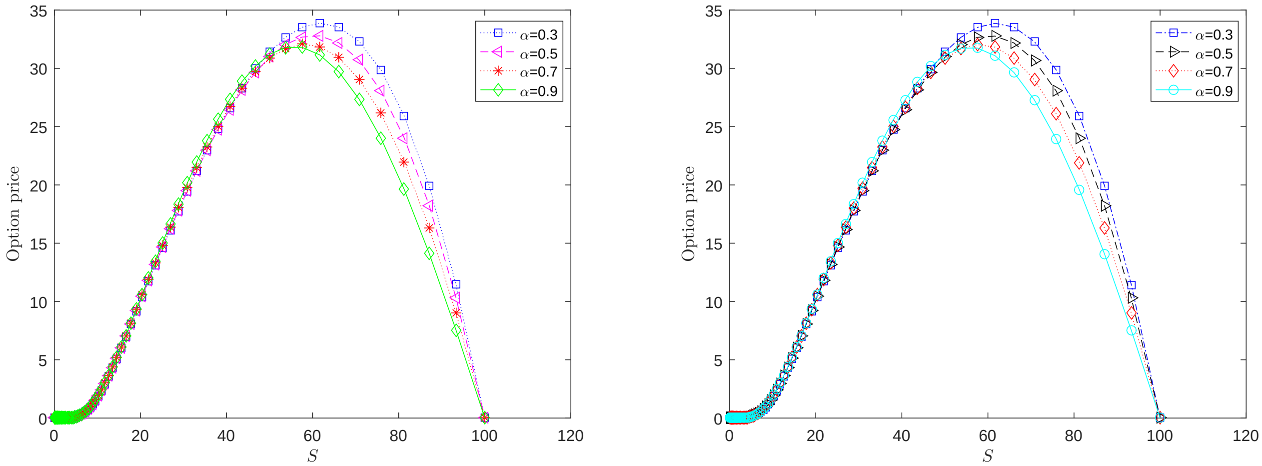
$$\Phi(S,T) = \max\{-K+S, 0\}, \ \Phi(0.1,\tau) = 0, \ \Phi(100,\tau) = 0. \qquad (7.2)$$

The parameters used in this model are volatility $\delta = 0.45$, risk-free interest rate $r = 0.03$, strike price $K = 10$, maturity time $T = 1$(year) and the dividend yield $D = 0.01$.

Here we explore the influence of α on the pricing of the double barrier option. The underlying problem is numerically solved by using both $L1-2$ and $L2-1_\sigma$ scheme. Figure 2 shows the numerical value of the price of the double barrier option for different values of α (0.3, 0.5, 0.7 and 0.9) with fixed $M = N = 100$. From Figure 2, It can be seen that, under the TFBS framework, the option price tends to be lower when the stock value is below K, and becomes higher once it surpasses K. From Figure 2, we can clearly say that fractional-order derivative has a significant impact on the behavior of the solution.

Example 7.2. Consider the European-type call option on the domain $(S,\tau) \in (0.01, 100) \times (0,1)$. The problem is subject to the following terminal and boundary conditions:

$$\Phi(S,T) = \max\{-K+S, 0\}, \ \Phi(0.01,\tau) = 0, \ \Phi(100,\tau) = 100 - K\exp(-r(1-\tau)). \qquad (7.3)$$



(A) $L1-2$.   (B) $L2-1_\sigma$.

FIGURE 2. Double barrier option prices of Example 7.1 at different α with fixed $M=N=100$.

TABLE 1. L_∞ error and OOC of Example 7.2 (non-smooth case) for $L1-2$ and $L2-1_\sigma$ scheme with $\delta=0.25$, $r=0.05$, $K=50$, $T=1$, and $D=0$.

	M	$L-12(N=100)$	OOC	$L2-1_\sigma$(N=100)	OOC
$\alpha=0.89$	32	2.71e-01	1.34	2.71e-01	1.35
	64	1.07e-01	2.13	1.06e-01	2.12
	128	2.44e-02	2.07	2.44e-02	2.07
	256	5.83e-03	-	5.82e-03	-
$\alpha=0.94$	32	2.78e-01	1.36	2.78e-01	1.36
	64	1.08e-01	2.20	1.08e-01	2.21
	128	2.35e-02	2.08	2.34e-02	2.08
	256	5.57e-03	-	5.55e-03	-
$\alpha=0.99$	32	2.85e-01	1.37	2.85e-01	1.37
	64	1.10e-01	2.29	1.09e-01	2.28
	128	2.25e-02	2.08	2.25e-02	2.09
	256	5.31e-03	-	5.29e-03	-

The exact solution of Example 7.2 is not available. In order to estimate the errors, we employ the double mesh principle. Let $\Phi_{k,j}^{M,N}$ denote the approximate solution at the grid point (S_k,τ_j), where M, N are the spatial and time discretization parameters, respectively. We find the L_∞ error as

$$\varepsilon^{M,N} = \max_{1\le k\le M+1} |\Phi_{k,N}^{M,N} - \Phi^{2M,N}(S_k,\tau_N)|,$$

and the order of convergence (OOC) is computed as

$$\text{OOC} = \log_2\left(\frac{\varepsilon^{M,N}}{\varepsilon^{2M,N}}\right).$$

Table 1 presents the L_∞ error and the OOC of $L1-2$ and $L2-1_\sigma$ schemes for a non-smooth payoff, using the parameters $\delta=0.25$, $r=0.05, K=50$, $T=1$(year) and the dividend yield $D=0$. Table 1 shows that $L1-2$ and $L2-1_\sigma$ schemes achieve only second-order convergence using the double mesh principle. To achieve the fourth

TABLE 2. L_∞ error and OOC of Example 7.2 (smooth case) with for $L1-2$ and $L2-1_\sigma$ scheme with $\epsilon = 10,\ \delta = 0.25,\ r = 0.05,\ K = 50,\ T = 1$ and $D = 0$.

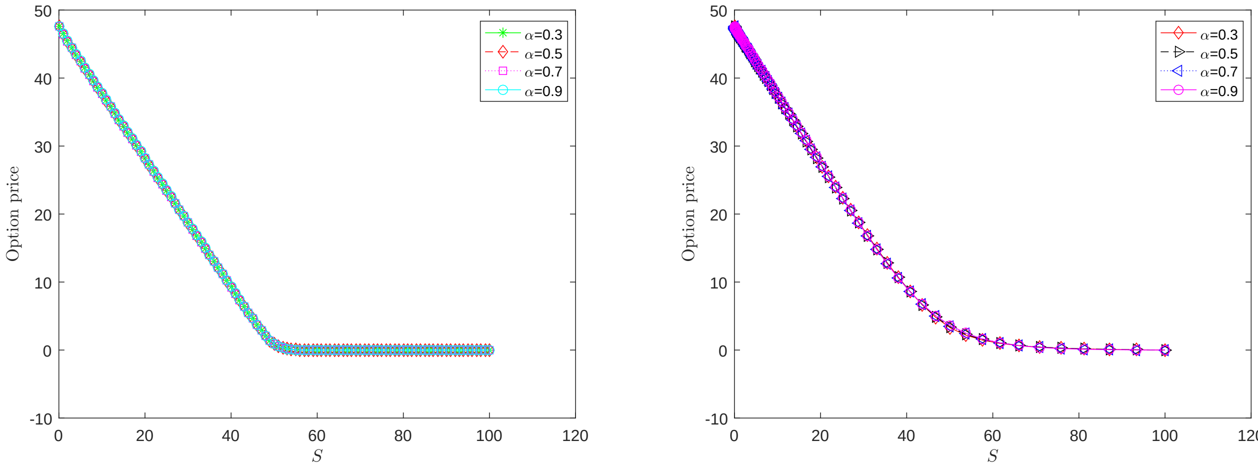
	M	$L-12(N=100)$	OOC	$L2-1_\sigma$ $(N=100)$	OOC
$\alpha = 0.89$	32	1.90e-02	0.47	1.91e-02	0.47
	64	1.37e-02	3.86	1.38e-02	3.87
	128	9.45e-04	4.02	9.45e-04	4.03
	256	5.81e-05	-	5.80e-05	-
$\alpha = 0.94$	32	1.99e-02	0.40	1.99e-02	0.40
	64	1.51e-02	3.92	1.51e-02	3.92
	128	1.00e-03	3.94	1.00e-03	3.94
	256	6.50e-05	-	6.50e-05	-
$\alpha = 0.99$	32	2.07e-02	0.32	2.07e-02	0.32
	64	1.66e-02	3.99	1.65e-02	3.99
	128	1.04e-03	3.76	1.04e-03	3.76
	256	7.65e-05	-	7.65e-05	-

TABLE 3. Comparison of L_∞ error and OOC of Example 7.2 (smooth case) using $L1-2$ and $L2-1_\sigma$ scheme with $A = 0.01,\ B = 40,\ \epsilon = 10,\ \delta = 0.3\ r = 0.04,\ K = 10,\ T = 1$ and $D = 0$.

	M	Cen[23] $(N=1024)$	OOC	Ahmad[1]	OOC	$L-12$ $(N=100)$	OOC	$L2-1_\sigma$ $(N=100)$	OOC
$\alpha = 0.1$	64	1.20e-03	2.03	3.78e-03	2.00	1.07e-04	3.97	1.07e-04	3.97
	128	2.94e-04	2.02	9.48e-04	2.09	6.78e-06	3.99	6.78e-06	3.99
	256	7.23e-05	2.01	2.22e-04	2.14	4.28e-07	4.00	4.28e-07	4.00
	512	1.79e-05	-	5.05e-05	-	2.67e-08	-	2.67e-08	-
$\alpha = 0.3$	64	1.15e-03	1.99	3.60e-03	2.00	7.48e-05	3.99	7.47e-05	3.99
	128	2.79e-04	2.04	9.03e-04	2.00	4.71e-06	3.99	4.70e-06	3.99
	256	6.88e-05	2.02	2.26e-04	2.09	2.97e-07	4.00	2.96e-07	4.00
	512	1.71e-05	-	5.32e-05	-	1.85e-08	-	1.85e-08	-
$\alpha = 0.5$	64	1.10e-03	2.04	3.41e-03	2.00	5.25e-05	3.96	5.23e-05	3.96
	128	2.68e-04	2.12	8.55e-04	2.00	3.36e-06	4.00	3.35e-06	4.00
	256	6.16e-05	1.91	2.14e-04	2.05	2.10e-07	4.00	2.09e-07	4.00
	512	1.64e-05	-	5.16e-05	-	1.31e-08	-	1.31e-08	-
$\alpha = 0.7$	64	1.08e-03	2.03	3.22e-03	2.00	4.67e-05	3.98	4.66e-05	3.98
	128	2.65e-04	2.02	8.05e-04	2.01	2.96e-06	4.00	2.95e-06	4.00
	256	6.55e-05	2.01	2.00e-04	2.17	1.85e-07	4.00	1.84e-07	4.00
	512	1.63e-05	-	4.45e-05	-	1.16e-08	-	1.15e-08	-
$\alpha = 0.9$	64	1.14e-03	2.03	3.05e-03	1.98	5.12e-05	3.98	5.10e-05	3.97
	128	2.79e-04	2.02	7.72e-04	2.15	3.25e-06	4.00	3.25e-06	4.00
	256	6.89e-05	2.01	1.74e-04	2.14	2.03e-07	4.00	2.03e-07	4.00
	512	1.71e-05	-	3.94e-05	-	1.27e-08	-	1.27e-08	-

order convergence of $L1-2$ and $L2-1_\sigma$ schemes, we transform the non-smooth payoff condition by a smooth payoff $\Phi(S,T) = \xi(S-K)$ using the function $\xi(x)$ [15] as

$$\xi(x) = \begin{cases} x, & x \geq -\epsilon, \\ c_0 + c_1(x) + c_2(x^2) + \cdots + c_9 x^9, & -\epsilon < x < \epsilon, \\ 0 & x \leq \epsilon. \end{cases} \tag{7.4}$$



(A) $L1-2$. (B) $L2-1_{\sigma}$.

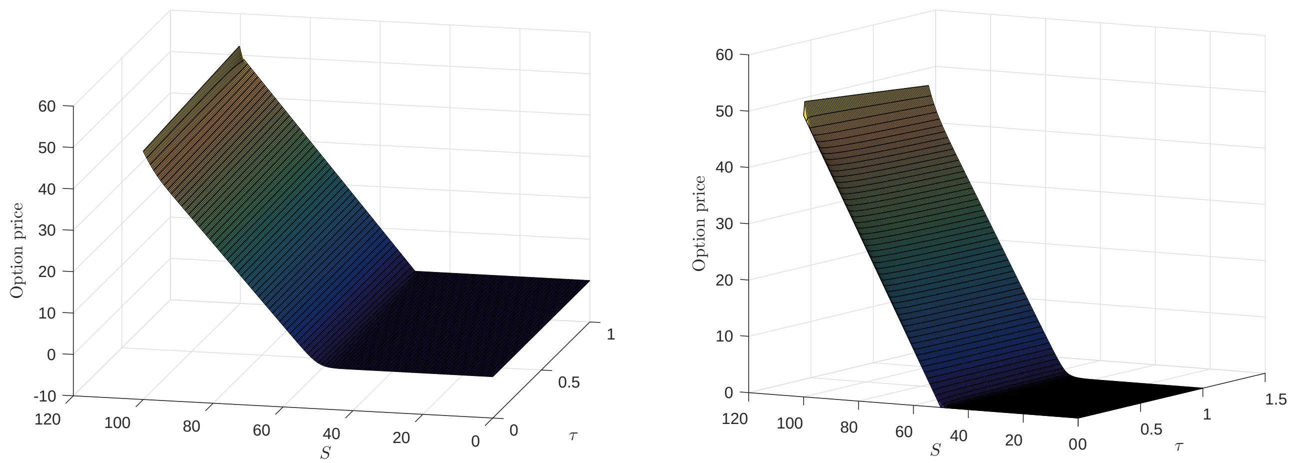
FIGURE 3. Put option prices of Example 7.3 at different α for $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$, and $D = 0$.

Where $c_0 = \frac{35\epsilon}{256}$, $c_1 = \frac{1}{2}$, $c_2 = \frac{35}{64\epsilon}$, $c_3 = 0$, $c_4 = -\frac{35}{128\epsilon}$, $c_5 = 0$, $c_6 = \frac{7}{64\epsilon^3}$, $c_7 = 0$, $c_8 = \frac{-5}{256\epsilon^7}$, $c_9 = 0$ for $\epsilon > 0$. Due to the smoothness of $\xi(x)$, the error is significantly reduced, resulting in a higher convergence rate. With $\epsilon = 10$ in Eq. (7.4), fourth-order convergence in space is observed, as shown in Table 2. Table 3 compares the L_∞ error and the order of convergence (OOC) for the $L1$–2 and $L2$–1_σ schemes, along with the results reported in [1, 23], using the smooth payoff function $\xi(S-K)$. For comparison purpose in Table 3 we consider the spatial domain $(0.01, 40)$ and the parameters we used for simulations are $\delta = 0.3$, $r = 0.04$, $K = 10$, $T = 1$ year, and $D = 0$. Cen et al. [23] employed a fine temporal discretization with $N = 1024$, the proposed method attains higher accuracy using a significantly coarser temporal grid ($N = 100$) by varying the number of spatial steps M. This demonstrates the enhanced efficiency and accuracy of the developed schemes. We apply $L1-2$ and $L2-1_\sigma$ scheme to solve the Example 7.2 for different values of α with fixed $M = N = 100$. Figure 1 illustrates the call option prices computed using the parameters $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$ and $D = 0$. From Figure 1, we can notice that the call option price is affected by the fractional derivative order. Figure 4 shows the surface plot of the approximate solution obtained using the $L1$–2 and $L2$–1_σ schemes. The simulations were conducted with parameters $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$ (year), and $D = 0$.

Example 7.3. Consider the European type put option on the domain $(S, \tau) \in (0.1, 100) \times (0, 1)$. The problem is subject to the following terminal and boundary conditions:

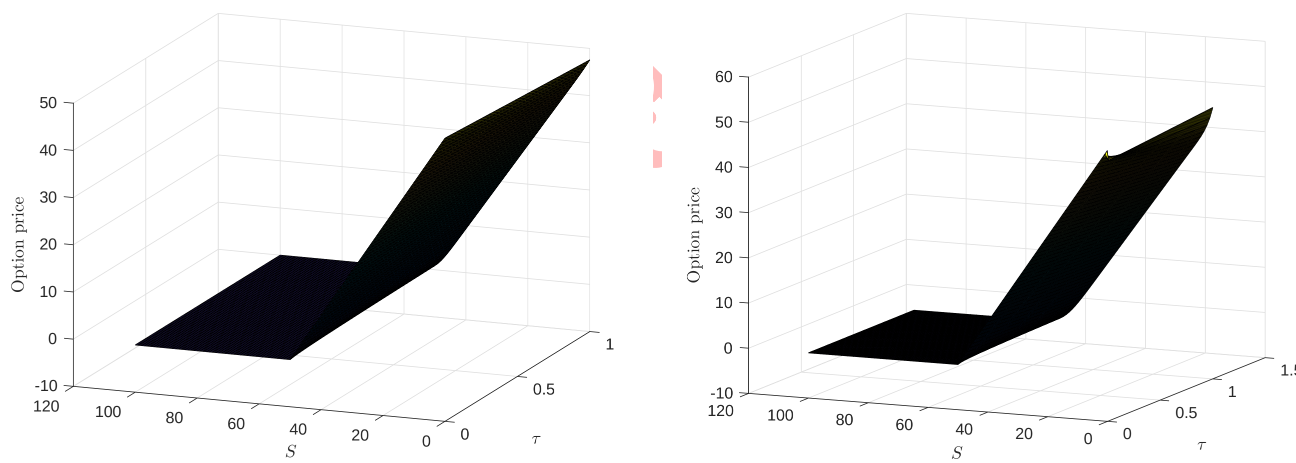
$$\Phi(S,T) = \max\{K-S, 0\},\ \Phi(0.1,\tau) = K\exp(-r(1-\tau)),\ \Phi(100,\tau) = 0. \tag{7.5}$$

The exact solution of Example 7.3 is not available. To estimate the numerical errors, we employ the double mesh principle as described in the preceding example. Table 4 shows that $L1-2$ and $L2-1_\sigma$ schemes achieve only second-order convergence using the double mesh principle with parameters $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$ and $D = 0$. To achieve the fourth order convergence of $L1-2$ and $L2-1_\sigma$ schemes, we transform the non-smooth payoff condition by a smooth payoff $\Phi(S,T) = \xi(K-S)$ using the function $\xi(x)$ as defined in Eq. (7.4). Because of its smoothness, this approach reduces error, resulting in a better convergence rate. By taking $\epsilon = 5$ in Eq. (7.4), we get fourth-order convergence in space given in Table 5. Table 6 compares the L_∞ error and the order of convergence (OOC) for the $L1$–2 and $L2$–1_σ schemes, along with the results reported in [8], using the smooth payoff function $\xi(K-S)$ with $\epsilon = 10$. The parameters used in the simulations are $\delta = 0.1$, $r = 0.01$, $K = 50$, $T = 1$ (year), and $D = 0$. Kazmi [8] employed a fine temporal discretization with $N = 2048$, the proposed method achieves superior accuracy even with a considerably



(A) $L1-2$. (B) $L2-1_\sigma$.

Figure 4. Approximate solution of Example 7.2 for $\alpha = 0.89$, $M = N = 100$, $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$, and $D = 0$.



(A) $L1-2$. (B) $L2-1_\sigma$.

Figure 5. Approximate solution of Example 7.3 for $\alpha = 0.89$, $M = N = 100$, $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$, and $D = 0$.

coarser temporal grid ($N = 100$), by varying the number of spatial steps M. This highlights the improved efficiency and accuracy of the developed numerical schemes. We apply $L1-2$ and $L2-1_\sigma$ scheme to solve the Example 7.5 for different values of α with fixed $M = N = 100$. Figure 3 shows the put option prices on the domain $S = [0.1, 100]$ with parameters $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$ and $D = 0$. From Figure 3, we can notice that the put option price is affected by the fractional derivative order. The surface plot of the approximate solution obtained by $L1-2$ and $L2-1_\sigma$ schemes is shown in Figure 5. The parameters used are $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$ (year), and $D = 0$.

Table 4. L_∞ error and OOC of Example 7.3 (nonsmooth case) for $L1-2$ and $L2-1_\sigma$ scheme with $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$ and $D = 0$.

	M	$L-12(N=100)$	OOC	$L2-1_\sigma(N=100)$	OOC
$\alpha = 0.89$	32	2.71e-01	1.34	2.72e-01	1.88
	64	1.07e-01	2.13	7.37e-02	2.21
	128	2.44e-02	2.07	1.59e-02	2.25
	256	5.83e-03	-	3.34e-03	-
$\alpha = 0.94$	32	2.78e-01	1.36	2.79e-01	1.94
	64	1.08e-01	2.20	7.26e-02	2.26
	128	2.35e-02	2.08	1.52e-02	2.26
	256	5.57e-03	-	3.18e-03	-
$\alpha = 0.99$	32	2.85e-01	1.37	2.88e-01	2.01
	64	1.10e-01	2.29	7.14e-02	2.30
	128	2.25e-02	2.08	1.45e-02	2.26
	256	5.31e-03	-	3.02e-03	-

Table 5. L_∞ error and OOC of Example 7.3 (smooth case) with $\epsilon = 5$ for $L1-2$ and $L2-1_\sigma$ scheme with $\delta = 0.25$, $r = 0.05$, $K = 50$, $T = 1$ and $D = 0$.

	M	$L-12(N=100)$	OOC	$L2-1_\sigma(N=100)$	OOC
$\alpha = 0.89$	32	8.41e-02	4.05	8.70e-02	3.60
	64	5.06e-03	4.18	7.16e-03	3.90
	128	2.79e-04	4.07	4.79e-04	3.95
	256	1.66e-05	-	3.09e-05	-
$\alpha = 0.94$	32	8.62e-02	4.05	9.35e-02	3.53
	64	5.21e-03	4.19	8.12e-03	3.94
	128	2.85e-04	4.07	5.28e-04	3.97
	256	1.70e-05	-	3.38e-05	-
$\alpha = 0.99$	32	8.83e-02	4.03	1.01e-01	3.43
	64	5.39e-03	4.21	9.34e-03	3.93
	128	2.92e-04	4.07	6.12e-04	4.01
	256	1.74e-05	-	3.80e-05	-

8. Conclusion

In this paper, the numerical approximation of the time fractional Black-Scholes on a uniform mesh has been considered. Firstly, we transfer the TFBS to a time-fractional diffusion equation by an exponential transformation. we approximate the time fractional derivative by $L1$ formula, $L1-2$ formula, $L2-1_\sigma$ formula respectively. Then, a fourth-order compact finite difference technique is used to approximate spatial derivatives. Further, we constructed three compact finite difference schemes with convergence rates $O((\Delta t)^{(2-\alpha)} + (\Delta x)^4)$, $O((\Delta t)^{(3-\alpha)} + (\Delta x)^4)$, $O((\Delta t)^2 + (\Delta x)^4)$. Truncation error and stability analysis are carried out. Error estimate of order is derived for the proposed scheme. Finally, numerical examples showed the accuracy and effectiveness of the proposed methods. To show the effectiveness of the new approach, two numerical experiments with exact answers are carried out. We also used the suggested approach to price three distinct instances of European options under the time-fractional framework. It has been noted that a change in the order of the time-fractional derivative has an impact on the option price. We conclude that the strategy is effective for solving various related time-fractional models governing European options and is in good agreement with the exact answer for the TFBS model.

TABLE 6. Comparison of L_∞ error and OOC of Example 7.3 (smooth case) using $L1-2$ and $L2-1_\sigma$ scheme for $\delta = 0.1$, $r = 0.01$, $K = 50$, $T = 1$, and $D = 0$.

	M	Kazmi[8] ($N = 2048$)	OOC	$L-12$ ($N = 100$)	OOC	$L2-1_\sigma$ ($N = 100$)	OOC
$\alpha = 0.1$	64	3.42e-02	1.98	1.37e-04	3.54	4.73e-03	3.38
	128	8.69e-03	2.00	1.18e-05	3.62	4.55e-04	3.96
	256	2.17e-03	2.00	9.63e-07	3.97	2.92e-05	3.99
	512	5.42e-04	2.00	6.17e-08	3.99	1.84e-06	4.00
	1024	1.36e-04	-	3.88e-09	-	1.15e-07	-
$\alpha = 0.3$	64	3.28e-02	1.99	3.76e-04	2.04	5.84e-03	3.38
	128	8.23e-03	2.00	9.16e-05	3.80	5.61e-04	3.96
	256	2.05e-03	2.00	6.59e-06	3.95	3.60e-05	3.99
	512	5.13e-04	2.00	4.28e-07	3.98	2.27e-06	4.00
	1024	1.28e-04	-	2.71e-08	-	1.42e-07	-
$\alpha = 0.5$	64	3.16e-02	2.01	6.69e-04	2.26	7.19e-03	3.51
	128	7.81e-03	2.00	1.40e-04	3.93	6.31e-04	3.97
	256	1.94e-03	2.00	9.15e-06	3.88	4.02e-05	3.99
	512	4.86e-04	2.00	6.19e-07	4.00	2.52e-06	3.99
	1024	1.21e-04	-	3.88e-08	-	1.59e-07	-
$\alpha = 0.7$	64	3.07e-02	2.03	6.69e-04	2.31	8.88e-03	3.85
	128	7.52e-03	2.01	1.35e-04	3.95	6.17e-04	3.87
	256	1.87e-03	2.00	8.73e-06	3.99	4.21e-05	4.00
	512	4.67e-04	2.00	5.49e-07	3.98	2.63e-06	4.00
	1024	1.17e-04	-	3.48e-08	-	1.65e-07	-
$\alpha = 0.9$	64	3.09e-02	2.04	3.15e-04	2.22	1.13e-02	3.77
	128	7.54e-03	2.01	6.76e-05	4.00	8.31e-04	4.02
	256	1.88e-03	2.00	4.21e-06	4.01	5.12e-05	4.00
	512	4.68e-04	2.00	2.62e-07	3.98	3.19e-06	3.99
	1024	1.17e-04	-	1.66e-08	-	2.01e-07	-

Acknowledgment

The First author would like to thank University Grants Commission, New Delhi for financial support in form of research grant(211610100146).

References

[1] H. Ahmad, M. N. Khan, I. Ahmad, M. Omri, and M. F. Alotaibi, *A meshless method for numerical solutions of linear and nonlinear time-fractional Black-Scholes models*, Aims Math *8*(8) (2023), 19677-19698.

[2] A. A. Alikhanov, *A new difference scheme for the time fractional diffusion equation*, J. Comput. Phys. *280* (2015), 424–438.

[3] X. An, F. Liu, M. Zheng, V. V. Anh and I. W. Turner, *A space-time spectral method for time-fractional Black-Scholes equation*, Appl. Numer. Math. *165* (2021), 152–166.

[4] T. Björk and H. Hult, *A note on Wick products and the fractional Black-Scholes model*, Finance Stoch. *9* (2005), 197–209.

[5] G. Dura and A. M. Mosneagu, *Numerical approximation of Black-Scholes equation*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. *56*(1) (2010), 39–64.

[6] G. H. Gao, Z. Z. Sun, and H. W. Zhang, *A new fractional numerical differentiation formula to approximate the caputo fractional derivative and its applications*, J. Comput. Phys. *259* (2014), 33–50.

[7] G. Jumarie, *Derivation and solutions of some fractional Black–Scholes equations in coarse-grained space and time. application to merton's optimal portfolio*, Comput. Math. Appl., *59*(3) (2010), 1142–1164.

[8] K. Kazmi, *A second order numerical method for the time-fractional Black–Scholes European option pricing model*, J. Comput. Appl. Math., *418* (2023), 114647.

[9] S. Kumar, D. Kumar, and J. Singh, *Numerical computation of fractional Black–Scholes equation arising in financial market*, Egypt. j. basic appl. sci., *1*(3-4) (2014), 177–183.

[10] S. Kumar, A. Yildirim, Y. Khan, H. Jafari, K. Sayevand, and L. Wei, *Analytical solution of fractional Black-Scholes european option pricing equation by using Laplace transform*, J. Fract. Calc. Appl., *2*(8) (2012), 1–9.

[11] J. R. Liang, J. Wang, W. J. Zhang, W. Y. Qiu, and F. Y. Ren, *Option pricing of a bi-fractional Black–Merton–Scholes model with the Hurst exponent H in $[\frac{1}{2}, 1]$*, Appl. Math. Lett., *23*(8) (2010), 859–863.

[12] Y. Lin and C. Xu, *Finite difference/spectral approximations for the time-fractional diffusion equation*, J. Comput. Phys., *225*(2) (2007), 1533–1552.

[13] J. Mohapatra, S. Santra, and A. Kanaujiya, *An Adaptive Mesh Based Computational Approach to the Option Price and Their Greeks in Time Fractional Black–Scholes Framework*, Differ. Equ. Dyn. Syst., (2025).

[14] S. Nuugulu, F. Gideon, and K. Patidar, *A robust numerical solution to a time-fractional Black–Scholes equation*, Adv. Differ. Equ., (2021).

[15] S. C. S. Rao and Manisha *Numerical solution of generalized Black–Scholes model*, Appl. Math Comput. *321* (2018), 401–421.

[16] P. Roul, *A high accuracy numerical method and its convergence for time-fractional Black-Scholes equation governing european options*, Appl. Numer. Math., *151* (2020), 472–493.

[17] P. Roul and V. P. Goura, *A compact finite difference scheme for fractional Black-Scholes option pricing model*, Appl. Numer. Math., *166* (2021), 40–60.

[18] A. Singh and S. Kumar, *An efficient numerical method based on exponential B-splines for time-fractional Black-Scholes equation governing european options*, Comput. Econom. (2023).

[19] L. Song and W. Wang, *Solution of the fractional Black-Scholes option pricing model by finite difference method*, Abstr. Appl. Anal., 2013, Article ID: 194286, Hindawi Limited, (2013), pp. 1–10.

[20] J. Wang, J. R. Liang, L. J. Lv, W. Y. Qiu, and F. Y. Ren, *Continuous time Black–Scholes equation with transaction costs in subdiffusive fractional Brownian motion regime*, Phys. A, *391*(3) (2012), 750–759.

[21] X. T. Wang, *Scaling and long-range dependence in option pricing I: Pricing European option with transaction costs under the fractional Black–Scholes model*, Phys. A, *389*(3) (2010), 438–444.

[22] H. Zhang, F. Liu, I. Turner, and Q. Yang, *Numerical solution of the time fractional Black–Scholes model governing european options*, Comput. Math. Appl., *71*(9) (2016), 1772–1783.

[23] C. Zhongdi, H. Jian, X. Aimin, and L. Anbo, *Numerical approximation of a time-fractional Black–Scholes equation*, Comput. Math. Appl., *75*(8) (2018), 2874–2887.