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Application of the interpolating element-free Galerkin method for the numerical solution of obstacle problem

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Abstract

The obstacle problem is a specific contact problem that can be formulated as a variational inequality or complementary problem in function spaces. Such problems often yield non-smooth solutions, making it challenging to find a suitable numerical approximation. In this paper, we present a meshfree method for numerically solving an obstacle problem. In proposed method the interpolating moving least square approximation is utilized in the element-free Galerkin approach. Implementing this method on a computer is straightforward and effective. To ensure the efficiency of the proposed method, we have investigated the convergent of the proposed method. Additionally, we have solved several examples of the obstacle problem using the proposed method. The numerical results obtained confirm the theoretical achievements and demonstrate the method's effectiveness and accuracy.

Keywords. Obstacle problem, Variational inequality, Active set method, Interpolating element free Galerkin method, Meshfree method, Interpolating moving least square.

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1. Introduction

Contact problems are a group of important problems in solid mechanics that study the deformation of solids that are in contact with each other at some point. The issues faced by this family can be used to explain a wide range of physical events, from modeling how a basketball hits the backboard to how a locomotive's brakes operate. One of the most important issues in this category is *the obstacle problem*. The obstacle problem deals with modeling the state change of an elastic membrane in contact with a smooth obstacle while being subjected to a known force. To put it more precisely, you can envision the edges of an elastic membrane fixed at the boundary of a region, with a force being applied to the surface of this membrane in the considered region. Along the path of deformation, there are one or more obstacles with smooth external surfaces within this region. In the obstacle problem, the objective is to determine the final shape of the membrane.

The obstacle problem can be formulated as an *elliptic variational inequality*. In mathematical literature, a variational inequality is an inequality involving a functional that must hold for all values within a convex subset of a space [26]. Such an issue is often encountered in optimization theory and related problems. For the obstacle problem explained above, if we assume the area where the elastic membrane is located is represented by $D \subset \mathbb{R}^n, n = 1, 2$, its boundary by ∂D, the position of the membrane with y and the force applied to it by $g \in C(D)$, and the surface of the obstacle situated in the path of the membrane by $z \in C(D)$, then the model of the obstacle problem will lead to the following inequality [13]:

Find

$$y \in \mathcal{O} := \left\{ f \in H^1(D) \mid \; f \le z \text{ a.e. in } D \;\&\; f\,|_{\partial D} = y_b \right\}, \tag{1.1}$$

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such that

$$(\nabla y, \nabla(y-u)) \leq (g, y-u) \quad \forall u \in \mathcal{O}, \tag{1.2}$$

where the standard notation $H^1(D)$ is used to refer to the Sobolov function space of order 1 from $L^2(D)$ with the inner product denoted by $(\cdot,\cdot)$. The important issue here is that the functional inequality (1.2) must hold for all values of $u \in \mathcal{O}$. Therefore, solving such a problem can be a significant challenge. Most variational inequality problems with similar characteristics do not have analytical solutions, thus highlighting the need for designing an efficient and accurate numerical method to address them. The set $\mathcal{O}$ represents the range of admissible changes for the desired elastic membrane. This set can also manifest in different forms; for instance, if obstacle are placed on both sides of the membrane, this set will appear as follows [25]:

$$\mathcal{O} := \left\{ f \in H^1(D) \mid \ t \leq f \leq z \text{ a.e. in } D \ \& \ f\mid_{\partial D} = y_b \right\}. \tag{1.3}$$

In this case, we refer to the problem as a *bilateral obstacle problem*.

Considering the physics of the problem, the computational domain D will be divided into two parts. In one part, the membrane will collide with the obstacle, while in the other part, at its maximum displacement under the assumed force, will not make contact with the obstacle. The main objective in this problem is to determine the boundary or boundaries between these two regions. In other words, this problem is one of *free boundary problems* [7]. This specific nature of the issue causes non-smoothness in the solution function along the priori unknown free boundary, which will affect the accuracy of numerical methods.

So far, numerous numerical methods have been proposed for solving obstacle problem. Finite difference methods [21] and finite element methods [2, 3] are among the most popular classical numerical approaches. Wavelet-baseds methods have also been utilized in some articles like as [14] for solving obstacle problem. Additionally, the discrete Galerkin method has been employed in [16, 30] to tackle this issue.

In recent decades, a new numerical method known as *meshless methods* has been increasingly used to solve mathematical problems. The main characteristic of this family of methods is their independence from a regular grid in the computational domain of the problem [6]. Notable examples include the element-free Galerkin method [17], generalized finite differences[4], and meshless local Petrov-Galerkin method [29], which have also been employed for the numerical solution of obstacle problem.

Most meshless methods for solving boundary value problems are based on obtaining the weak form of the problem, which can be applied in both local and global manners. In both cases, numerical integration can ultimately be used to derive a linear system, and solving this system will yield an approximation of the solution to the problem. In the global method, integration is performed over the entire computational domain. But in the case of local weak form methods, there is no need to define a mesh of points, and the integration operation is performed not over the entire region but just over subdomains that may have overlaps [9]. However, it should be noted that often in local methods, error analysis may not be possible. Additionally, these methods have a greater dependence on the selection of parameters such as shape parameters.

The element free Galerkin method (EFGM) uses a global weak form in which moving least squares (MLS) shape functions are employed as test and trial functions [24]. These basic functions do not possess the delta Kronecker property as an effective feature in approximation. For this reason, the essential boundary conditions cannot be implemented directly in this method. To overcome this weakness, the use of the interpolating moving least square (IMLS) shape function instead of MLS shape function is recommended in [15]. These functions take advantage of the Kronecker delta property. Subsequently, the obtained method is also referred to as the interpolating element free Galerkin method (IEFGM) [9].

In this work, we will present a meshless method based on the use of the interpolating element-free Galerkin method to solve the obstacle problem. To this end, a combination of this meshless method with an efficient algorithm for solving obstacle problems, known as the active set algorithm, has been employed. Additionally, the error analysis of the method has been investigated. In fact, the main objective of this paper is to demonstrate that the interpolation method is a globally weak form method that, just as it has been useful for solving PDE problems, can also be beneficial in solving variational inequalities such as obstacle problems.

The outline of the rest of the paper is as follows. In the next section, we will first address the active set algorithm for solving the obstacle problem. Then, in the third section, we will examine the IMLS shape functions and how to use them in IEFGM for solving obstacle problem. In the fourth section, we will present the convergence analysis of the method, and in the fifth section, we will provide several numerical examples to assess the accuracy and efficiency of the method.

2. Interpolating element free Galerkin method

First, we need to introduce the MLS and IMLS shape functions. Let $\Xi_I = \{\mathbf{x}_l\}_{l=1}^{N_I} \subset D$ and $\Xi_b = \{\mathbf{x}_l\}_{l=1}^{N_b} \subset \partial D$. By considering $N = N_I + N_b$, the set $\Xi = \{\mathbf{x}_l\}_{l=1}^{N} := \Xi_I \cup \Xi_b$ is all selected nodes in the computational domain. For y, the function which must be approximated, let $\mathbf{y} = \{y_l = y(\mathbf{x}_l)\}_{l=1}^{N}$. In MLS literature, we can approximate y by

$$y(\mathbf{x}) \simeq \bar{y}(\mathbf{x}, \bar{\mathbf{x}}) := \sum_{l=1}^{M} c_l(\mathbf{x}) P_l(\bar{\mathbf{x}}) = \mathbf{P}^t(\bar{\mathbf{x}}) \cdot \mathbf{c}(\mathbf{x}), \tag{2.1}$$

where $\bar{\mathbf{x}} \in \Xi$ is the point in local influenced domain of $\mathbf{x}_i$ determined by a radial weight function $w_i(\mathbf{x})$. The coefficients $c_l(\mathbf{x})$ are chosen in such a way as to minimize

$$\sum_{l=1}^{N} \left[\sum_{k=1}^{m} c_k(\mathbf{x}) P_k(\mathbf{x}_l) - y_l \right]^2 w_l(\mathbf{x}). \tag{2.2}$$

According to [22], we can prove that

$$\mathbf{c}(\mathbf{x}) = \left(\mathbf{P}^t \mathbf{Q}(\mathbf{x}) \mathbf{P}\right)^{-1} \mathbf{P}^t \mathbf{Q}(\mathbf{x}) \mathbf{y}, \tag{2.3}$$

where $\mathbf{P}$ is a $N \times M$ matrix with elements $P_m(\mathbf{x}_n)$ when $m = 1, \ldots, N$ and $m = 1, \ldots, M$ and $\mathbf{Q} = diag(w_1(\mathbf{x}), \ldots, w_N(\mathbf{x}))$. So, if we define the MLS shape function as

$$\mathbf{S}(\mathbf{x}) = (S_1(\mathbf{x}), \cdots, S_N(\mathbf{x})) = \mathbf{P}^t(\mathbf{x}) \left(\mathbf{P}^t \mathbf{Q}(\mathbf{x}) \mathbf{P}\right)^{-1} \mathbf{P}^t \mathbf{Q}(\mathbf{x}), \tag{2.4}$$

we have MLS approximation by

$$\bar{y}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \mathbf{y} = \sum_{l=1}^{N} S_l(\mathbf{x}) y_l. \tag{2.5}$$

An important point to consider is that this approximation cannot interpolate the function values at the desired points. To establish the interpolating conditions and from now on, we set the weight function as the following singular radial base function [5]

$$w_l(\mathbf{x}) = \begin{cases} \|\mathbf{x} - \mathbf{x}_l\|^{-\gamma}, & \|\mathbf{x} - \mathbf{x}_l\| \le r, \\ 0, & o.w. \end{cases} \tag{2.6}$$

By defining an inner product as

$$(y, z)_{\mathbf{w}} = \sum_{l=1}^{N} w_l(\mathbf{x}) y_l z_l, \tag{2.7}$$

its associated norm is as $\|y\|_{\mathbf{w}} = (y, y)_{\mathbf{w}}^{\frac{1}{2}}$.Now if we assume that $P_1(\mathbf{x}) = 1$, we can put

$$B_{1,\mathbf{x}}(\mathbf{x}) = \frac{P_1(\mathbf{x})}{\|P_1\|_{\mathbf{w}}} = \frac{1}{\left(\sum_{l=1}^{N} w_l(\mathbf{x})\right)^{\frac{1}{2}}}. \tag{2.8}$$

Performing orthogonalization algorithms on $\{P_2, \cdots, P_N\}$ respect to $B_{1,\mathbf{x}}$, we have

$$B_{k,\mathbf{x}}(\mathbf{x}) = P_k(\mathbf{x}) - (P_k, B_{1,\mathbf{x}})_{\mathbf{w}} B_1(\mathbf{x}) = P_k(\mathbf{x}) - \frac{\sum_{l=1}^{N} P_k(\mathbf{x}_l) w_l(\mathbf{x})}{\sum_{l=1}^{N} w_l(\mathbf{x})}, \; k = 2, \ldots, M. \tag{2.9}$$

Now, by orthogonality of $B_{1,\mathbf{x}}$ to $B_{k,\mathbf{x}}, k = 2, \ldots, M$, we can write [1]

$$y(\mathbf{x}) \simeq y_h(\mathbf{x}) = \mathbf{v}^t(\mathbf{x})\mathbf{y} + \mathbf{B}^t(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{L}(\mathbf{x})\mathbf{y}, \tag{2.10}$$

where

$$\begin{aligned}
\mathbf{v}^t(\mathbf{x}) &= \left[v_l := \frac{w_l(\mathbf{x})}{\sum_{l=1}^{N} w_l(\mathbf{x})} \right]_{l=1,\ldots,N}, \\
\mathbf{B}^t(\mathbf{x}) &= \left[B_{k,\mathbf{x}}(\mathbf{x}) \right]_{k=1,\ldots,M}, \\
\mathbf{L}(\mathbf{x}) &= \mathbf{D}^t(\mathbf{x})\mathbf{W}(\mathbf{x}), \\
\mathbf{R}(\mathbf{x}) &= \mathbf{L}(\mathbf{x})\mathbf{D}(\mathbf{x}), \\
\mathbf{D}(\mathbf{x}) &= \left[B_{k,\mathbf{x}}(\mathbf{x}_l) \right]_{l=1,\ldots,N,k=2,\ldots,M}.
\end{aligned}$$

By considering the approximation (2.10) the IMLS shape functions can be defined as

$$\mathbf{S}^*(\mathbf{x}) := [S_l^*(\mathbf{x})]_{l=1}^{N} = \mathbf{v}^t(\mathbf{x}) + \mathbf{B}^t(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{L}(\mathbf{x}), \tag{2.11}$$

and (2.10) can be summarized as

$$y_h(\mathbf{x}) = \mathbf{S}^*(\mathbf{x})\mathbf{y} = \sum_{l=1}^{N} S_l^*(\mathbf{x}) y_l. \tag{2.12}$$

It can be seen that the IMLS shape functions have the delta Kronecker property and the approximation (2.12) satisfies the interpolating conditions on Ξ.

Now, we are ready to use the obtained IMLS shape functions as the test and trail functions in the standard Galerkin scheme. The delta Kronecker property of these functions causes that the boundary conditions of the boundary value problems can be applied without the need for any other method. The obtained method is called *Interpolating element free Galerkin method* (IEFGM).

3. Obstacle Problem and method of solution

Consider the obstacle problem (1.1)-(1.2). This form of the obstacle problem can be considered the weak form of the problem. To obtain the strong form of the problem, take $u = y + \phi$ in Eq. (1.2) where $\phi \in C^\infty(D)$, $\phi|_{\partial D} = 0$ and $\phi \geq 0$:

$$\int_D \nabla y(x).\nabla \phi(x) - g(x)\phi(x)dx \leq 0. \tag{3.1}$$

It is weak form of the following differential inequality:

$$-\Delta y(x) - g(x) \leq 0, \ x \in D. \tag{3.2}$$

Moreover, if we have the point x_0 such that $y(x_0) < z(x_0)$, we can find a number $\epsilon > 0$ and a neighborhood like $N(x_0)$ where $y(x) < z(x) - \epsilon$ for all $x \in N(x_0)$. So, by taking $u = y \pm \epsilon\varphi$, where $\varphi \in C^\infty(N(x_0))$ and $\|\varphi\|_\infty \leq 1$, we have $-\Delta y(x_0) - g(x_0) = 0$. Therefore, we can write the strong form of obstacle problem as the following *complementarity problem (CP)* :

$$-\Delta y(x) - g(x) \leq 0, \ x \in D, \tag{3.3}$$

$$y(x) \leq z(x), \ x \in D, \tag{3.4}$$

$$\left(-\Delta y(x) - g(x)\right)\left(y(x) - z(x)\right) = 0, \ x \in D, \tag{3.5}$$

$$y(x) = y_b(x), \ x \in \partial D. \tag{3.6}$$

This complementary problem indicates that the computational domain is divided into two sections. The first section, where the elastic membrane will collide with the obstacle, is denoted by $\mathcal{A}$ and is referred to as the contact region [7].

The second section, where the Poisson equation applies, will be denoted by $\mathcal{B}$ and is called the non-contact region. In other words

$$\mathcal{A} := \{x \in D | y(x) = z(x) \ \& \ -\Delta y(x) < g(x)\}, \tag{3.7}$$

$$\mathcal{B} := D \setminus \mathcal{A} = \{x \in D | -\Delta y(x) = g(x) \ \& \ y(x) < z(x)\}. \tag{3.8}$$

Then, it has been proven that the solution of CP (3.3)-(3.5) satisfies the following regularization conditions [12]

$$-\Delta y(x) + \lambda(x) = g(x), \tag{3.9}$$

$$\lambda(x) := \max\{\lambda(x) + c(y(x) - z(x)), 0\}, \tag{3.10}$$

where $\lambda(x)$ is Lagrange function and c is an arbitrary positive number.

3.1. **Active set strategy.** Due to non-differentiability of max operator in (3.10), the regularized system of Equations (3.9)-(3.10) cannot be solved by usual algorithms. To do that special algorithms are needed such as the semi-smooth Newton method [8, 28], interior point strategy [31] or active set algorithm [12]. In the following, the iteration of the active set strategy is presented to solve the obstacle problem.

Suppose that $y^{(l)}$ and $\lambda^{(l)}$ are the solution and Lagrange functions obtained in the l-th iteration of the active set algorithm. The algorithm is initialized with $y^{(0)} = z$ and $\lambda^{(0)} = g + \Delta y^{(0)} \geq 0$. In each iterate, the contact and non-contact set can be addressed as:

$$\mathcal{A}^{(l+1)} = \{x \in D| \ \lambda^{(l)}(x) + c(y^{(l)}(x) - z(x)) > 0\}, \tag{3.11}$$

$$\mathcal{B}^{(l+1)} = \{x \in D| \ \lambda^{(l)}(x) + c(y^{(l)}(x) - z(x)) \leq 0\}. \tag{3.12}$$

Then, the following updated in the contact and non-contact regions for unknown functions are considered:

$$y^{(l+1)}(x) = \mathbf{z}(x), \ \lambda^{(l+1)}(x) = g(x) + \Delta y^{(l+1)}(x), \ \text{for } x \in \mathcal{A}^{(l+1)}, \tag{3.13}$$

$$\lambda^{(l+1)}(x) = 0, \ -\Delta y^{(l+1)}(x) = g(x), \ \text{for } x \in \mathcal{B}^{(l+1)}. \tag{3.14}$$

This successive iteration continues until the stopping condition is met as $\mathcal{B}^{(l)} = \mathcal{B}^{(l+1)}$. In the modified active set method, the way of updating in the contact region is changed. In this case, instead of Equations (3.13), we use the following equations for updating in $\mathcal{A}^{(l+1)}$

$$\lambda^{(l+1)}(x) = \lambda^{(l)}(x) + c(y^{(l)}(x) - z(x)), \qquad \Delta y^{(l+1)}(x) = \lambda^{(l+1)}(x) - g(x). \tag{3.15}$$

In fact, the change that has been made in the modified active set method is that the Lagrange multiplier is updated first, and then the value of the unknown function y is obtained.

4. Implementation of IEFGM on obstacle problem

Although at first glance it seems that using Equations (1.1) and (1.2) are suitable for discretization and by considering V_h as a finite dimensional subspace of $H^1(D)$, one can define the weak form of the discretized obstacle problem as follows:

$$\text{Find } y_h \in \mathcal{O}_h \subset V_h \tag{4.1}$$

$$\langle Ay_h - g_h, y_h - u_h \rangle \leq 0, \quad \forall u_h \in \mathcal{O}_h, \tag{4.2}$$

but it should be noted that $V_h \subset H^1(D)$ and $\mathcal{O}_h \subset V_h$ cannot yield $\mathcal{O}_h \subset \mathcal{O}$ [7]. For this very reason, we use Equations (3.3)-(3.6) to obtain a discrete weak form for the obstacle problem.

Let $\Lambda = \{1, 2, \ldots, N\}$. Let Λ_I and Λ_b are subsets of Λ associated with the indices of Ξ_I and Ξ_b, respectively. Therefore, the IEFGM discretization of the obstacle problem can be defined as the following discreted complementarity problem (DCP):

$$\left.\begin{array}{l} \sum_{l\in\lambda_I} A(S_l^*, S_k^*) y_l - g_k \leq 0, \\ y_k - z_k \leq 0, \\ \left(\sum_{l\in\lambda_I} A(S_l^*, S_k^*) y_l - g_k\right)(y_l - z_l) = 0, \end{array}\right\} \text{where } x_k \in D, \tag{4.3}$$

$$y_k = y_b(\mathbf{x}_k), \ \text{where } x_k \in \partial D,$$

where

$$A(S_l^*, S_k^*) := \int_D \nabla S_l^*(\mathbf{x}) \nabla S_k^*(\mathbf{x}) d\mathbf{x}, \tag{4.4}$$

$$g_k := \int_D g(\mathbf{x}) S_k^*(\mathbf{x}) d\mathbf{x} - \sum_{l \in \lambda_b} A(S_l^*, S_k^*) y_b(\mathbf{x}_k), \tag{4.5}$$

and $z_k := z(\mathbf{x}_k)$. Now, let $\mathbf{A} := [A(S_l^*, S_k^*)]_{l,k \in \Lambda_I}$ and $\mathbf{g} := [g_k]_{k \in \Lambda_I}$ and $\mathbf{z} := [z_k]_{k \in \Lambda_I}$. Then, the matrix-vector form of DCP (4.3) is as follows:

$$\begin{aligned} &\mathbf{A}\mathbf{y} - \mathbf{g} \le 0, \\ &\mathbf{y} - \mathbf{z} \le 0, \\ &(\mathbf{A}\mathbf{y} - \mathbf{g})(\mathbf{y} - \mathbf{z}) = 0. \end{aligned} \tag{4.6}$$

According to the Theorem 2.1 in [12], the solution of (4.6) satisfies the following conditions

$$\mathbf{A}\mathbf{y} + \boldsymbol{\lambda} = \mathbf{g}, \tag{4.7}$$

$$\boldsymbol{\lambda} = \max\{\boldsymbol{\lambda} + c(\mathbf{y} - \mathbf{z}), 0\}, \tag{4.8}$$

where $\boldsymbol{\lambda}$ is the Lagrange variable and c is any positive number.

The Equations (4.7)-(4.8) are IMLS discritized form of (3.9)-(3.10). Therefore, active set strategy can be implemented in this case. Suppose that $\mathbf{y}^{(l)}$ and $\boldsymbol{\lambda}^{(l)}$ are the solution and Lagrange variable in l-th iterate. The algorithm is initialized with $\mathbf{y}^{(0)} = \mathbf{z}$ and $\boldsymbol{\lambda}^{(0)} \ge 0$. In each iterate, the computational indices can be divided into two disjoint part:

$$\Lambda_c^{(l)} = \{j \in \Lambda_I | \ \boldsymbol{\lambda}^{(l)} + c(\mathbf{y}^{(l)} - \mathbf{z}) > 0\}, \tag{4.9}$$

$$\Lambda_n^{(l)} = \{j \in \Lambda_I | \ \boldsymbol{\lambda}^{(l)} + c(\mathbf{y}^{(l)} - \mathbf{z}) = 0\}, \tag{4.10}$$

which are associated with contact and non-contact regions in (3.7) and (3.8), respectively. Then, the following new updated variables are considered:

$$\boldsymbol{\lambda}_i^{(l+1)} = 0, \ \text{for } i \in \Lambda_c^{(l)}, \tag{4.11}$$

$$\mathbf{y}_i^{(l+1)} = \mathbf{z}_i, \ \text{for } i \in \Lambda_n^{(l)}, \tag{4.12}$$

and by the remaining variables updated by solving the following linear system

$$\mathbf{A_{nn}}\mathbf{y}_{\mathbf{n}}^{(l+1)} = \mathbf{g_n} + \mathbf{A_{nc}}\mathbf{z_c}, \tag{4.13}$$

$$\boldsymbol{\lambda}_{\mathbf{c}}^{(l+1)} = \mathbf{g_c} - \mathbf{A_{cc}}\mathbf{z_c} - \mathbf{A_{cn}}\mathbf{y}_{\mathbf{n}}^{(l+1)}. \tag{4.14}$$

where the notation $\mathbf{A_{nc}} := [\mathbf{A}_{i,j}]_{i \in \Lambda_n^{(l)}, j \in \Lambda_c^{(l)}}$ denoted to the sum-matrix of $\mathbf{A}$ and all the other sub-matrix or sub-vector notations can be defined in a same manner. The algorithm continues until the stopping condition is met as $\Lambda_n^{(l)} = \Lambda_n^{(l+1)}$. In [12] , it has been proven that the active set algorithm converges uniformly under the given assumptions, and if it stops under the considered stopping condition, the obtained solution holds true in (4.7) and (4.8).

5. Error analysis

Suppose that y is the analytical solution of an obstacle problem. $y^{(l)}$ and $\lambda_{(l)}$ are the solutions obtained by active set strategy, analytically, and $y_h^{(l)}$ and $\lambda_h^{(l)}$ are the numerical solutions of IEFGM implementation, i.e.

$$y_h^{(l)}(\mathbf{x}) = \mathbf{S}^*(\mathbf{x})\mathbf{y}^{(l)} = \sum_{k=1}^{N} S_k^*(\mathbf{x}) y_k, \tag{5.1}$$

$$\lambda_h^{(l)} = \mathbf{S}^*(\mathbf{x})\boldsymbol{\lambda}^{(l)} = \sum_{k=1}^{N} S_k^*(\mathbf{x})\lambda_k, \tag{5.2}$$

where h is fill distance of D, i.e. $h = \sup_{\mathbf{x}\in D} \min_{\mathbf{x}_l\in\Xi} \|\mathbf{x}-\mathbf{x}_l\|_2$. We also define semi-norm for $f \in H^1(D)$ as follows

$$|f|_{1,D} = \left(\sum_{i=1}^{d} \|\frac{\partial f}{\partial x_i}\|^2_{L^2(D)}\right)^{\frac{1}{2}}. \tag{5.3}$$

Then we can prove the following theorem.

Theorem 5.1. *Let $y \in H^1(D)$ is analytical solution of problem (1.1)-(1.2). Suppose r in (2.6) is chosen such that the number of points in the influenced domain of each point is a finite constant. If the active set algorithm stops at a finite number of steps then the solution of active set strategy $y^{(l)}$ converge to exact solution y as $\lim_{l\to\infty} |y-y^{(l)}|_{1,D} = 0$. Moreover, for IEFGM solution $y_h^{(l)}$, we have $\lim_{h\to 0, l\to\infty} |y-y_h^{(l)}|_{1,D} = 0$.*

Proof. We have

$$\begin{aligned}\int_D \left[\Delta y^{(l+1)} - \Delta y\right]^2 dx &= \int_D \left[\Delta y^{(l+1)} - \Delta y^{(l)} - c(y^{(l)}-z) + \Delta y^{(l)} + c(y^{(l)}-z) - \Delta y\right]^2 dx \\ &= \int_D \left[\Delta y^{(l)} + c(y^{(l)}-z) - \Delta y\right]^2 - \left[\Delta y^{(l+1)} - \Delta y^{(l)} - c(y^{(l)}-z)\right]^2 \\ &\quad + 2\left[\Delta y^{(l+1)} - \Delta y^{(l)} - c(y^{(l)}-z)\right]\cdot\left[\Delta y^{(l+1)} - \Delta y\right] dx. \end{aligned} \tag{5.4}$$

Now, for any $x \in \mathcal{A}^{(l+1)}$, according to (3.15) we have $\lambda^{(l+1)}(x) - \lambda^{(l)}(x) - c(y^{(l)}(x) - z(x)) = \Delta y^{(l+1)}(x) - \Delta y^{(l)}(x) - c(y^{(l)}(x)-z(x)) = 0$. On the other hand, for any $x \in \mathcal{B}^{(l+1)}$, we have $0 \geq \lambda^{(l)}(x) + c(y^{(l)}(x)-z(x)) = \Delta y^{(l)}(x) - g(x) + c(y^{(l)}(x)-z(x))$. Substituting $g(x) = -\Delta y^{(l+1)}(x)$ from (3.14) leads to $\Delta y^{(l+1)}(x) - \Delta y^{(l)}(x) - c(y^{(l)}(x)-z(x)) \geq 0$. Moreover, for exact solution y of obstacle problem, using $-\Delta y(x) \leq g(x)$ and Substituting $g(x) = -\Delta y^{(l+1)}(x)$ yields $\Delta y^{(l+1)}(x) - \Delta y(x) \leq 0$. Therefore,

$$\int_D \left[\Delta y^{(l+1)} - \Delta y^{(l)} - c(y^{(l)}-z)\right]\left[\Delta y^{(l+1)} - \Delta y\right] dx \leq 0. \tag{5.5}$$

By using (5.4) and (5.5)

$$\begin{aligned}\|\Delta y^{(l+1)} - \Delta y\|^2 &\leq \int_D \left[\Delta y^{(l)} - \Delta y\right]^2 - \left[\Delta y^{(l+1)} - \Delta y^{(l)}\right]^2 + 2c(y^{(l)}-z)\cdot\left[\Delta y^{(l+1)} - \Delta y\right] dx \\ &= \|\Delta y^{(l)} - \Delta y\|^2 - \|\Delta y^{(l+1)} - \Delta y^{(l)}\|^2 + 2c\int_D (y^{(l+1)}-y)\cdot\left[\Delta y^{(l+1)} - \Delta y\right] dx \\ &+ 2c\int_D (y^{(l+1)} - y^{(l)})\cdot\left[\Delta y - \Delta y^{(l+1)}\right] + (y-z)\cdot\left[\Delta y^{(l+1)} + g\right] dx \\ &- 2c\int_D (y-z)\cdot[\Delta y + g]\, dx. \end{aligned} \tag{5.6}$$

Using Gauss theorem for the first integral in right hand sight of (5.6) leads to

$$\int_D (y^{(l+1)}-y)\cdot\left[\Delta y^{(l+1)} - \Delta y\right] dx = -\int_D \left[\nabla y^{(l+1)} - \nabla y\right]^2 dx. \tag{5.7}$$

In addition, the value of the second integral is negative and the third integral will vanish since complementarity condition (3.6) holds. Therefore, we can conclude that

$$\|\Delta y^{(l+1)} - \Delta y\|^2 \leq \|\Delta y^{(l)} - \Delta y\|^2 - \|\Delta y^{(l+1)} - \Delta y^{(l)}\|^2 \leq \|\Delta y^{(l)} - \Delta y\|^2. \tag{5.8}$$

Consequently, (5.8) yields that $\lim_{l\to\infty} \Delta y^{(l)} = \Delta y$. So, by considering (5.7), we can obtain $\lim_{l\to\infty} |y^{(l)} - y|_{1,D} = 0$. Moreover, regarding to error estimate of IMLS approximation, it has been proven in Theorem 3.2 of reference [27], that if $y^{(l)} \in H^m(D)$ for $m \geq 1$, and if r in (2.6) is chosen such that the number of points in the influenced domain of each point is a finite constant, then

$$\|D^\alpha y^{(l)} - D^\alpha y_h^{(l)}\|_\infty \leq C_\alpha r^{m+1-|\alpha|} |y^{(l)}|_{1,D}, \ |\eta| \leq 2. \tag{5.9}$$

By considering the definition (5.3) and the inequality (5.9) for $\alpha = 1$ we have

$$|y^{(l)} - y_h^{(l)}|_{1,D} = \left(\sum_{i=1}^{n} \| \frac{\partial (y^{(l)} - y_h^{(l)})}{\partial x_i} \|^2_{L^2(D)} \right)^{\frac{1}{2}} \leq |D| \left(\sum_{i=1}^{n} \| \frac{\partial (y^{(l)} - y_h^{(l)})}{\partial x_i} \|^2_{\infty} \right)^{\frac{1}{2}} \leq n|D| C_1 r^m |y^{(l)}|_{1,D}. \tag{5.10}$$

Now, by considering $r = \beta h$, it can be concluded that $\lim_{h\to 0} |y^{(l)} - y_h^{(l)}|_{1,D} = 0$. Finally, by combining of this result with $\lim_{l\to\infty} |y^{(l)} - y|_{1,D} = 0$, the the desired result will be obtained. □

6. Numerical illustrations

In this section we solve two numerical examples of obstacle problems with the presented IEFGM method. The algorithm was implemented in MATLAB software and on a PC with a 3.4 Giga-Hertz Core i5 Processor and 8 Gigabytes of RAM. Here, we choose the radius of the influence as a multiplier of fill distance, i.e. $r = \beta h$. Choosing an β between 2 and 3 will yield more favorable results [9]. About regularization parameter c, it is important to stress that for numerical purposes at a solution y to the variational inequality, $\lambda(x) := \max\{\lambda(x) + c(y(x) - z(x)), 0\}$ holds for every $c > 0$ and there is no need to $c \to \infty$ [11]. So, in our examples the regularization parameter is chosen as $c = 1$.

Here, we have used composite Gaussian quadrature to calculate all the integrals. In fact, first we divided the integration domain into several separated subdomains and then we used a Gaussian integration formula in each subdomain. In the first example, where the integration domain is 1D, the number of divisions is 4 and the 8-point Gauss-Lobatto-Legendre formula is used in each subdomain. In the second example, where the integration region is 2D, the number of subdomains is 16 and the 8-point Gauss-Lobatto-Legendre formula is used in each subdomain.

In general, using Gaussian formulas with a higher number of points cannot help increase accuracy. It may even cause non-convergence and instability in the method. However, newer methods for calculating the integral in EFG and IEFGM have been proposed in references [18] and [19].

Example 6.1. Our first example is a one-dimensional obstacle problem in $D = [-1, 1]$. Consider the obstacle problem (1.1)-(1.2) with forcing function $g(x) = 0$ and homogeneous boundary conditions $y(-1) = y(1) = y_b = 0$. The obstacle is considered as $z(x) = (\frac{1}{2} - x^2)(1 - 4x^2) - 1$. The contact region in this example is the union of two disjoint intervals. The IMLS shape functions are constructed with a weight function (2.6) when $\gamma = 2$ and different values of r, and used as a basis functions in IEFGM. The distribution of nodes in the domain is assumed to be uniform. The obtained solution by applying IEFGM with $h = 0.01$ and the obstacle function are plotted in Figure 1. The ability of the method to obtain an acceptable solution is evident in this figure.

Example 6.2. The second example is devoted to a 2D obstacle problem. Let $D = [-1.5, 1.5]^2$. The obstacle and forcing function are constant, $z(\mathbf{x}) = 0$ and $g(\mathbf{x}) = 2$. The exact solution of this example is [20]

$$y^*(\mathbf{x}) = \begin{cases} \ln\sqrt{x_1^2 + x_2^2} - \frac{x_1^2 + x_2^2 - 1}{2}, & \text{for } \sqrt{x_1^2 + x_2^2} \geq 1, \\ 0, & \text{for } \sqrt{x_1^2 + x_2^2} < 1. \end{cases}$$

So, the contact region is the unit disc $x_1^2 + x_2^2 \leq 1$.

In this example, the distribution of nodes in the rectangular domain is assumed to be uniform. The IMLS shape functions are constructed with a weight function (2.6) when $\gamma = 2$ and different values of r, and used as a basis functions in IEFGM. The obtained solution by applying IEFGM with $h = 0.05$ is plotted in Figure 2. Figure 3 also shows the gridpoints that the presented algorithm has identified as contact and non-contact points.

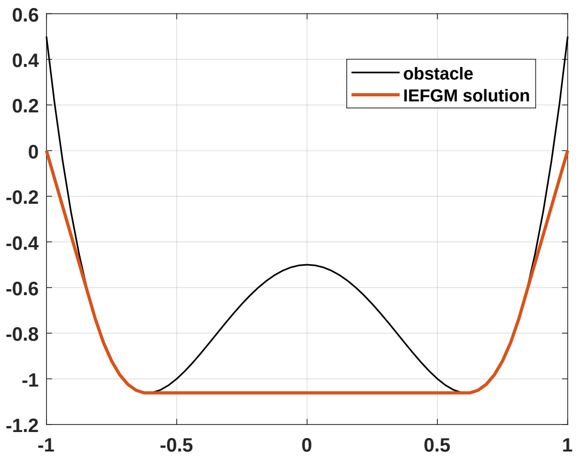

Figure 1. The solution function by IEFGM and the obstacle function for Example 6.1.

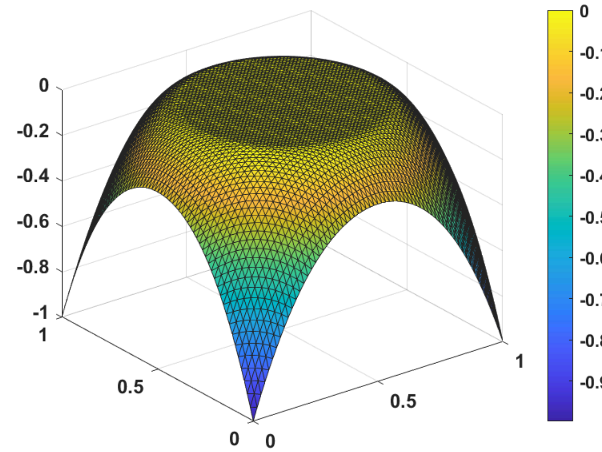

Figure 2. The solution function by IEFGM for Example 6.2.

Moreover, Table 1 contains L_2-error of the method for different choices of h and r. In this table the computational order of error is calculated by the following formula

$$order = \frac{\log(E_1) - \log(E_2)}{\log(h_1) - \log(h_2)},$$

where E_1 and E_2 are the errors associated with h_1 and h_2, respectively. In Table 2, the results of the current technique are compared to some other methods. The results of the element free Galerkin method (EFG) [17] and radial points interpolation method (RPIM) [23] and direct local boundary integral equation method (DLBIE) [10] are reported in this table. The discretization parameters in the methods are considered in such a way that fair conditions are established in the comparison. In all methods, the active set method is used to solve the discrete problem. The results indicate the appropriate accuracy of the proposed method for solving the obstacle problem.

Table 1. The computed errors for Example 6.2.

	$r = 2.01h$		$r = 2.20h$		$r = 2.50h$	
h	$\|y^* - y_h\|_2$	order	$\|y^* - y_h\|_2$	order	$\|y^* - y_h\|_2$	order
0.2	$2.8415e-02$	–	3.0891e-02	–	3.1940e-02	–
0.1	$7.6506e-03$	1.8930	8.3236e-03	1.8919	8.6350e-03	1.8871
0.05	$2.0255e-03$	1.9173	2.1971e-03	1.9216	2.2700e-03	1.9275
0.02	$3.4416e-04$	1.9344	3.6694e-04	1.9532	3.8705e-04	1.9306

Table 2. Comparison of absolute errors of different methods for Example 6.2.

h	DLBIE	RPIM	EFG	IEFGM
0.2	$1.5209e-01$	4.351e-02	2.638e-02	3.1940e-02
0.1	$5.3746e-02$	1.1028e-02	5.772e-03	8.6350e-03
0.05	$1.4279e-02$	6.3751e-03	1.801e-03	2.2700e-03
0.02	$4.9211e-03$	8.4162e-04	2.1914e-04	3.8705e-04

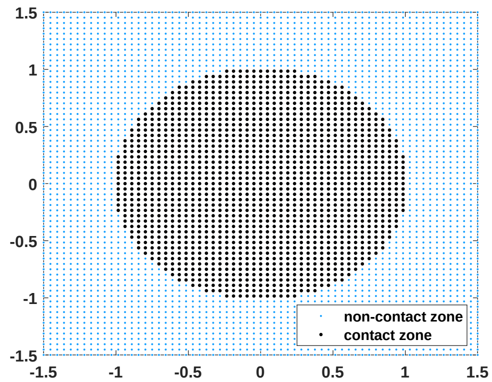

FIGURE 3. Grid points in contact and non-contact regions for Example 6.2.

7. Conclusion

In this paper, interpolating moving least square method is employed in the element free Galerkin approach, to solve the obstacle problem. The IEFGM is combined with an active set strategy to present an efficient algorithm for obstacle problem. The convergence theorem of the presented method has been proven and finally, two examples of obstacle problems have been solved by the presented method. The calculated results show that the IEFGM is efficient and it can provide accurate solutions for the variational inequalities like as obstacle problem.

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