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Heat distribution analysis of nonlinear fin equation with variable thermal conductivity using group hidden symmetries techniques

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Abstract

The fin equation is a very important model describing the heat distribution along the surface of the fins. In this work, some cases with variable thermal conductivity are considered, underlining the influence of the fin parameter. The group theoretical method and hidden symmetry technique are applied for obtaining some new closed-form solutions that develop the understanding of heat distribution along the fin surface. It can be seen that the fin geometry causes the variation in the fin parameter. Thermal conductivity of constant, power-law, and exponential types is considered. New solutions involving Kummer and imaginary error functions are obtained. fin parameter and thermal conductivity are found to show considerable impact. With increasing value of fin parameter, there is an enhancement of heat distribution along the fin surface which decreases with time. equations.

Keywords. fin equation, thermal conductivity, symmetry methods.
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1. Introduction

In the context of differential equations, a hidden symmetry is an unexpected appearance of point symmetries due to changes in the order of ordinary or partial differential equations. For ODEs, such symmetries fall into two classes: Type I, which originates from the increase of the order of the equation, and Type II, which originates from the decrease in the order. Type II symmetries can be further classified as being contact or nonlocal symmetries. In the case of PDEs, it has been demonstrated that the hidden symmetries arise from the point symmetries of some associated PDE. Coggeshall et al. [12] studied hidden symmetries of PDEs by increasing the order of a given PDE by a symmetry vector and then reducing the new equation to a simpler form. Guo et al. [18] investigated the hidden symmetries in second-order ODEs of the type arising in the motion of a particle in a potential. These equations are usually of energy-conserving form. Shrauner et al.[1, 2, 6, 7] studied the hidden symmetries for ODE, with applications in Modified Painlevé-Ince, Pinney, and Semiconductor transport equations. The authors extended these equations to third order and, by invoking hidden symmetries, were able to reduce the equations to second-order forms [7]. Type I and Type II hidden symmetries were applied for general first and second-order ODEs, respectively, with applications to the Vlasov-Maxwell and Painlevé-Ince equations [6]. Other systems which have been treated by the same authors are a two-dimensional quadratic system, dynamical equations, and the Lotka-Volterra system [1]. Shrauner et al continued the research of nonlocal group generators for ODEs, including an exponential nonlocal group generator obtained from a higher-order ODE in canonical variables. The generators were shown to be invariant under a two-parameter Lie group [2]. Edwards et al. [13] have classified the hidden symmetries of two- and three-dimensional Burgers' equations by obtaining the optimal system of symmetry vectors. In their works connected with the PDEs, Shrauner et al. [4–6, 16, 17] investigated hidden symmetries by reducing variables with the help of Lie symmetries.

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Applying linear combinations of symmetry vectors, they reduced the number of variables and established type II hidden symmetries for non-inherited vectors in the process. This was also extended to Burgers' equation [5]. They also studied the latent symmetries of the second heavenly equation of gravitational physics that resulted in its reduction to the Monge-Ampère equation [3]. They also proposed model PDEs and reduced them to ODEs using Lie vectors with the aim of obtaining master PDEs which correspond to the same ODEs either by guessing or reverse procedures [4]. Other origins of hidden symmetries were discussed by Shrauner et al. pointing out that this could be related to either contact or nonlocal symmetries, rather than standard Lie point symmetries, using as illustrative example the shallow water wave equation [17]. The issues involved in finding the reason for hidden symmetries in PDEs were discussed in [16]. Type II hidden symmetries in wave equations and nonlinear PDEs were considered by Gandarias et al. [14, 15], as for example, two-dimensional Burgers' equation and Ferapontov's equation. In the present paper, the solution of the fin equation is subjected to a group similarity transformation. Variable heat transfer coefficients and thermal conductivities are allowed. This type of equation has been investigated in a number of earlier papers using different methods [11, 25, 37]. In the present paper, the solution of the fin equation is subjected to a group similarity transformation. Variable heat transfer coefficients and thermal conductivities are allowed. This type of equation has been investigated in a number of earlier papers using different methods. The key distinction of the present work is that certain cases now reveal hidden symmetries. A fin is an extended surface attached to the surface of a structure in order to augment heat transfer between the solid surface and the environment. Bokhari et al. [11] obtained exact solutions for the nonlinear fin equation by symmetry analysis. Popovych et al. [25] determined the exact solutions of a significant fin equation with Lie and nonclassical symmetries. Vaneeva et al. [37] performed a group classification of the fin equation. Khani et al. [21] proposed analytical solutions to the nonlinear fin problem by taking into account cases of temperature-dependent thermal conductivity and variable heat transfer coefficients. They also studied the performance of the fin by the homotopy analysis method, which is an analytical method in solving nonlinear PDEs based on series expansion. Moitsheki et al. [24] obtained the exact solutions for the fin equation where only power-law temperature-dependent thermal conductivity was considered using symmetry analysis. Khani et al. [20] utilized the homotopy analysis method (HAM) in order to derive analytical expressions for the thermal performance of a straight fin with a trapezoidal profile under the conditions where the thermal conductivity and heat transfer coefficient are temperature-dependent. Furthermore, this type of nonlinear differential equations may be described in fractional calculus using various types of fractional derivatives such as Caputo, Riemann–Liouville, Grunwald–Letnikov, Riesz, Hadamard derivatives, and others [8–10, 22, 23, 32, 33, 38] In the present paper, three kinds of thermal conductivities are considered. For each type, a group transformation method is applied and hidden symmetries are found for some cases.

2. Mathematical Formulation

Consider the dimensionless nonlinear fin equation [25],

$$\frac{\partial}{\partial x}\left(K(\theta)\frac{\partial \theta}{\partial x}\right) - N^2 f(x)\theta = \frac{\partial \theta}{\partial t}, \tag{2.1}$$

subjected to the initial condition,

$$\theta(x,0) = g(x), \tag{2.2}$$

where, $\theta(x,t)$ is the dimensionless temperature, x and t are the dimensionless spatial and time variables, respectively. The thermal conductivity, $K(\theta)$, is a function in terms of temperature and $f(x)$ is an arbitrary function of the heat transfer coefficient. N is the fin parameter. Three different cases of thermal conductivities are to be considered in this work.

3. Case 1: Constant thermal Conductivity, $K(\theta) = D$

In this case the Equation (2.1) is rewritten as:

$$D\frac{\partial^2 \theta}{\partial x^2} - N^2 f(x)\theta - \frac{\partial \theta}{\partial t} = 0. \tag{3.1}$$

The initial condition, Equation (2.2), is normalized by setting

$$\theta(x,t) = g(x)\varphi(x,t). \tag{3.2}$$

Thus, Equations (2.2) and (3.1) are rewritten as:

$$D\left(\varphi\frac{d^2g}{dx^2} + 2\frac{dg}{dx}\frac{\partial\varphi}{\partial x} + g\frac{\partial^2\varphi}{\partial x^2}\right) - N^2 fg\varphi - g\frac{\partial\varphi}{\partial t} = 0, \tag{3.3}$$

subjected to the initial condition; $\varphi(x,0) = 1$.

3.1. **Group formulation of the problem.** A one-parameter group,G, is used in order to reduce the partial differential equation into an ordinary one as described in [19, 26–31, 34–36]

$$G : \bar{S} = Q^s(a)S + T^s(a), \tag{3.4}$$

where, $\overline{S}$ and S stand for the system variables in case of pre- and post-transformation, Q^s, T^s are real valued coefficients at least differentiable in the group parameter (a). The first and second partial derivatives of the dependent variables with respect to dependent variables are defined as:

$$\left.\begin{array}{r} \bar{S}_{\bar{i}} = (\frac{Q^s}{Q^i})S_i, \\ \bar{S}_{\bar{i}\bar{j}} = (\frac{Q^s}{Q^iQ^j})S_{ij}, \end{array}\right\}, i,j = x,y, \tag{3.5}$$

where, S stands for the dependent variables(θ, f, g). A transformation of Equation (3.2) following Equation (3.4) and Equation (3.5) definitions leads to:

$$D\left(\frac{Q^gQ^\varphi}{(Q^x)^2}\right)\varphi\frac{d^2g}{dx^2} + 2D\left(\frac{Q^gQ^\varphi}{(Q^x)^2}\right)\frac{dg}{dx}\frac{\partial\varphi}{\partial x} + D\left(\frac{Q^gQ^\varphi}{(Q^x)^2}\right)g\frac{\partial^2\varphi}{\partial x^2}, \tag{3.6}$$

where

$$\begin{aligned} R = {} & D\left(\frac{T^\varphi Q^g}{(Q^x)^2}\right)\frac{d^2g}{dx^2} + D\left(\frac{T^gQ^\varphi}{(Q^x)^2}\right)\frac{\partial^2\varphi}{\partial x^2} - N^2\left(Q^f f + T^f\right)\left(Q^g g + T^g\right)T^\varphi \\ & - N^2\left(Q^f f + T^f\right)\left(Q^\varphi\varphi + T^\varphi\right)T^g - N^2\left(Q^\varphi\varphi + T^\varphi\right)\left(Q^g g + T^g\right)T^f - \left(\frac{T^gQ^\varphi}{Q^t}\right)\frac{\partial\varphi}{\partial t}. \end{aligned} \tag{3.7}$$

For invariant transformation, R is equated to zero. This leads to,

$$T^\varphi = T^g = T^f = 0. \tag{3.8}$$

The invariance of the initial condition leads to,

$$Q^\varphi = 1. \tag{3.9}$$

The invariance of Equation (3.7) leads to,

$$\frac{Q^gQ^\varphi}{(Q^x)^2} = Q^fQ^gQ^\varphi = \frac{Q^gQ^\varphi}{Q^t} = H(a). \tag{3.10}$$

Combining Equations (3.9) and (3.10) implies the following group structure,

$$G\left\{\begin{array}{l} G_1\left\{\begin{array}{l} \bar{x} = Q^x x + T^x, \\ \bar{t} = (Q^x)^2 t + T^t, \end{array}\right. \\ G_2\left\{\begin{array}{l} \bar{f} = \left(\frac{1}{Q^x}\right)^2 f(\eta), \\ \overline{g} = g(x), \\ \bar{\varphi} = \varphi(\eta). \end{array}\right. \end{array}\right. \tag{3.11}$$

where, G_1, G_2 stand for the subgroups of independent and dependent variables. The dash stands for the transformation of variables.

TABLE 1. Similarity variables for $K = D$.

Case	Coefficients values	Similarity Variables
I	$\alpha_1 = \alpha_2 = 0, \beta_1 \neq 0, \beta_2 \neq 0$	$\eta(x,t) = x - ct, c = \frac{\beta_1}{\beta_2}$
II	$\alpha_1 = \alpha_2 = 0, \beta_1 = 0, \beta_2 \neq 0$	$\eta(x,t) = x$
III	$\alpha_1 = \alpha_2 = 0, \beta_1 \neq 0, \beta_2 = 0$	$\eta(x,t) = t$
IV	$\alpha_1 \neq \alpha_2 \neq 0, \beta_1 = 0, \beta_2 = 0$	$\eta(x,t) = tx^{\frac{1}{m}}, m = \frac{-\alpha_1}{\alpha_2}$

3.1.1. *Group transformation of the problem.*
Apply Morgan Theorem [19, 26–31, 34–36]

$$\sum_{i=1}^{5} (\alpha_i u_i + \beta_i) \frac{\partial \overline{u}_i}{\partial u_i} = 0, \tag{3.12}$$

where, u_i and $\overline{u}_i$ stand for the variables before and after transformation. Moreover,

$$\alpha_i = \frac{\partial Q^{s_i}(a)}{\partial a}, \tag{3.13}$$

$$\beta_i = \frac{\partial T^{s_i}(a)}{\partial a}. \tag{3.14}$$

3.1.2. *Transformation of the independent variables.*
The similarity variable $\eta(x,t)$ is obtained invoking Equation (3.12). The following equation is obtained.

$$(\alpha_1 x + \beta_1) \frac{\partial \eta}{\partial x} + (\alpha_2 t + \beta_2) \frac{\partial \eta}{\partial t} = 0. \tag{3.15}$$

Equation (3.15)leads to different similarity variables which are summarized in Table 1.

3.1.3. *Transformation of the dependent variables.*
The heat distribution function,φ, the heat transfer coefficient, $f(x)$, and the initial condition, $g(x)$, are transformed according to the group structure described by Equation (3.11).

$$\bar{\varphi}(\bar{x}, \bar{t}) = \varphi(\eta), \tag{3.16}$$
$$\bar{f}(\bar{x}) = f(x), \tag{3.17}$$
$$\bar{g}(\bar{x}) = g(x). \tag{3.18}$$

3.1.4. *Reduction of Fin equation to an ODE.*
- 3.1.4.1. Using $\eta(x,t) = x - ct$:

The Equation (3.3) can be written as:

$$D\left(\frac{d^2 g}{dx^2}\varphi + 2\frac{dg}{dx}\frac{d\varphi}{d\eta} + g\frac{d^2\varphi}{d\eta^2}\right) - N^2 f(x) g(x) \varphi(\eta) + c g(x) \frac{d\varphi}{d\eta} = 0. \tag{3.19}$$

Divide the previous equation by g to get:

$$D\left(\left(\frac{d^2 g(x)/dx^2}{g(x)}\right)\varphi(\eta) + 2\left(\frac{dg(x)/dx}{g(x)}\right)\frac{d\varphi(\eta)}{d\eta} + \frac{d^2\varphi(\eta)}{d\eta^2}\right) - N^2 f(x)\varphi(\eta) + c\frac{d\varphi(\eta)}{d\eta} = 0. \tag{3.20}$$

In order to ensure that Equation (3.20) is an ODE, the coefficients must be constants or functions of similarity variable,η. This implies: $f(x) = 1$, $g(x) = e^{-x}$. Hence, Equation (3.20)is reduced to:

$$\frac{d^2\varphi}{d\eta^2} + \left(\frac{c}{D} - 2\right)\frac{d\varphi}{d\eta} + \left(1 - \frac{N^2}{D}\right)\varphi = 0. \tag{3.21}$$

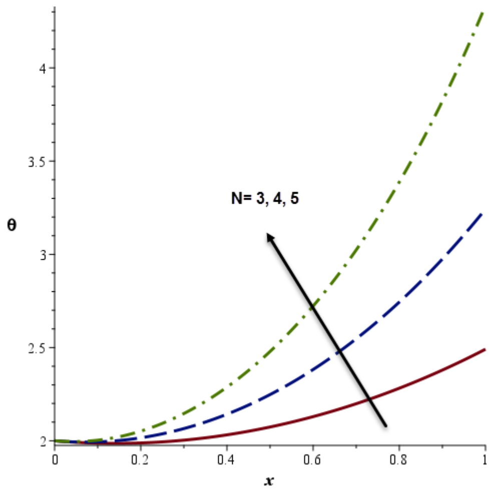

FIGURE 1. Effect of fin parameter, N, on heat distribution at t=0 , A=B=1, c=2, D=10.

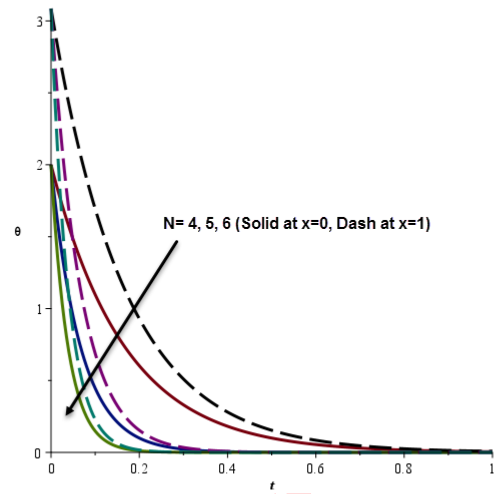

FIGURE 2. Effect of fin parameter, N, on heat distribution at x=0 and x=1,For A=B=1, D=10.

This is a second-order linear differential equation whose solution is given by:

$$\varphi(\eta) = Ae^{\left(\frac{-c+2D+n}{2D}\right)\eta} + Be^{\left(\frac{-c+2D-n}{2D}\right)\eta}, n = \sqrt{c^2 - 4cD + 4DN^2}. \tag{3.22}$$

As $\theta(x,t) = g(x)\varphi(\eta)$, the solution of fin equation can be written as:

$$\theta(x,t) = e^{-x}\left(Ae^{\left(\frac{-c+2D+n}{2D}\right)(x-ct)} + Be^{\left(\frac{-c+2D-n}{2D}\right)(x-ct)}\right). \tag{3.23}$$

The effect of fin parameter, N, is shown in Figure 1.

- 3.1.4.2. *Using* $\eta = t$.

Equation (3.2) is rearranged to be in the following form:

$$D\frac{d^2g}{dx^2}\varphi - N^2 fg\varphi - g\frac{d\varphi}{d\eta} = 0. \tag{3.24}$$

Dividing by (-g) leads to:

$$\frac{d\varphi}{d\eta} + N^2 f(x)\varphi - D\left(\frac{d^2g/dx^2}{g(x)}\right)\varphi = 0. \tag{3.25}$$

To ensure that Equation (3.25)is an ODE, the coefficients must be constants or functions of η. Thus, the following relations are obtained.

$$f(x) = 1, g(x) = A_1e^x + B_1e^{-x}. \tag{3.26}$$

Equation (3.25)is reduced to:

$$\frac{d\varphi}{d\eta} + (N^2 - D)\varphi = 0. \tag{3.27}$$

Hence,

$$\varphi(\eta) = A_2e^{-(N^2-D)t}. \tag{3.28}$$

Finally,

$$\theta(x,t) = \left(Ae^x + Be^{-x}\right)e^{-(N^2-D)t}. \tag{3.29}$$

This result is depicted in Figure 2 for different fin parameters.

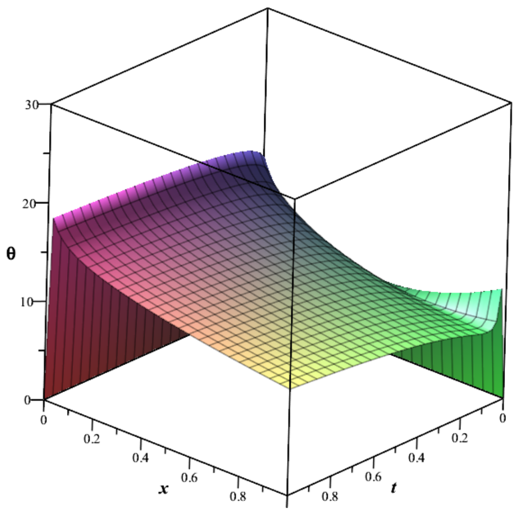

FIGURE 3. Temperature distribution of Equation (3.32) at $A = 1, B = 1, D = 10, N = 1$.

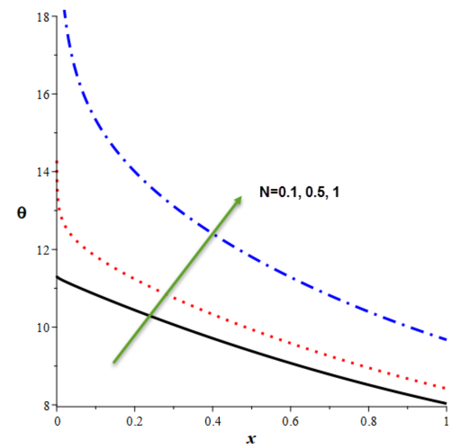

FIGURE 4. Effect of fin parameter on temperature distribution of Equation (3.32) at $A = 1, B = 1, D = 10, t = 0.5$.

• 3.1.4.3. *Using* $\eta = tx^{\frac{1}{m}}$.
Following the same procedures as in the previous two similarity variables, Equation (3.2) is transformed to:

$$4\eta^2 \frac{d^2\varphi}{d\eta^2} + \left(6\eta - \frac{1}{D}\right) \frac{d\varphi}{d\eta} - \left(\frac{N^2}{D}\right) \varphi = 0, \tag{3.30}$$

where, $m = -\frac{1}{2}, f(x) = x^{-2}, g(x) = 1$. The solution of Equation (3.30) is given in the form:

$$\varphi(\eta) = A\eta^{\frac{1+\sqrt{1+4\frac{N^2}{D}}}{-4}} e^{\frac{-1}{4D\eta}} M\left(\frac{3+\sqrt{1+4\frac{N^2}{D}}}{4}, \frac{2+\sqrt{1+4\frac{N^2}{D}}}{2}, \frac{1}{4D\eta}\right). \tag{3.31}$$

Making back-substitution, one can get:

$$\theta(x,t) = A(tx^{-2})^{\frac{1+\sqrt{1+4\frac{N^2}{D}}}{-4}} e^{\frac{-x^2}{4Dt}} M\left(\frac{3+\sqrt{1+4\frac{N^2}{D}}}{4}, \frac{2+\sqrt{1+4\frac{N^2}{D}}}{2}, \frac{x^2}{4Dt}\right), \tag{3.32}$$

where, M(a,b,z) and U(a,b,z) are Kummer functions of first and second kinds, respectively. The temperature distribution is shown in Figure 3 at $A = 1, B = 1, D = 10, N = 1$ and the effect of Fin parameter is illustrated in Figure 4.

4. CASE 2: THERMAL CONDUCTIVITY OF POWER-LAW TYPE, $K(\theta) = \theta^p$

Equation (2.1) is rewritten as:

$$\theta^p \frac{\partial^2\theta}{\partial x^2} + p\left(\frac{\partial\theta}{\partial x}\right)^2 \theta^{p-1} - N^2 f(x)\theta - \frac{\partial\theta}{\partial t} = 0. \tag{4.1}$$

Using the normalization condition, $\theta(x,t) = g(x)\varphi(x,t)$, this equation is reduced to:

$$g^p\varphi^p\left(\varphi\frac{d^2g}{dx^2} + 2\frac{dg}{dx}\frac{\partial\varphi}{\partial x} + g\frac{\partial^2\varphi}{\partial x^2}\right) + pg^{p-1}\varphi^{p-1}\left(\frac{dg}{dx}\varphi + g\frac{\partial\varphi}{\partial x}\right)^2 - N^2fg\varphi - g\frac{\partial\varphi}{\partial t} = 0. \tag{4.2}$$

Table 2. Different similarity variables of Equation (4.5)

Case	Coefficients values	Similarity variable
I	$\alpha_1 = \alpha_2 = 0, \beta_1 \neq 0, \beta_2 \neq 0$	$\eta(x,t) = x - ct, c = \frac{\beta_1}{\beta_2}$
II	$\alpha_1 = \alpha_2 = 0, \beta_1 = 0, \beta_2 \neq 0$	$\eta(x,t) = x$
III	$\alpha_1 = \alpha_2 = 0, \beta_1 \neq 0, \beta_2 = 0$	$\eta(x,t) = t$
IV	$\alpha_1 \neq \alpha_2 \neq 0, \beta_1 = 0, \beta_2 = 0$	$\eta(x,t) = tx^{\frac{1}{m}}, m = \frac{-\alpha_1}{\alpha_2}$
V	$\alpha_1 = 0, \alpha_2 \neq 0, \beta_1 \neq 0, \beta_2 = 0$	$\eta(x,t) = te^{nx}, n = \frac{\alpha_2}{\beta_1}$

subjected to the initial condition:

$$\varphi(x,0) = 1. \tag{4.3}$$

4.1. **Group formulation of the problem.** Following the same steps as in case 1, the following group structure could be obtained.

$$G \begin{cases} G_1 \begin{cases} \bar{x} = Q^x x + T^x, \\ \bar{t} = \frac{(Q^x)^2}{(Q^g)^p} t + T^t, \end{cases} \\ \\ G_2 \begin{cases} \bar{f} = \frac{(Q^g)^p}{(Q^x)^2} f, \\ \bar{g} = Q^g g, \\ \\ \bar{\varphi} = \varphi. \end{cases} \end{cases} \tag{4.4}$$

4.1.1. *Group transformation of the problem.*
Apply Morgan Theorem as stated in Equations (3.12)-(3.14) to get the similarity variable, $\eta(x,t)$, and the invariant dependent variables.

4.1.2. *Transformation of the independent variables.*
The similarity variable, $\eta(x,t)$, is obtained invoking Equation (3.12) to obtain the following equation:

$$(\alpha_1 x + \beta_1) \frac{\partial \eta}{\partial x} + (\alpha_2 t + \beta_2) \frac{\partial \eta}{\partial t} = 0. \tag{4.5}$$

Equation (4.5) leads to different cases according to the choice of coefficients invoking the group structure Equation (4.4). The results are summarized in the following Table 2.

4.1.3. *Transformation of the dependent variables.*
The invariance transformation of the heat distribution function,φ, heat transfer coefficient function, $f(x)$, and the initial condition function, $g(x)$, is given by the following relations.

$$\bar{\varphi}(\bar{x},\bar{t}) = \varphi(\eta), \tag{4.6}$$
$$\bar{f}(\bar{x}) = f(x), \tag{4.7}$$
$$\bar{g}(\bar{x}) = g(x). \tag{4.8}$$

4.1.4. *Reduction to ordinary differential equation.*
Some similarity variables are used to reduce the equation and find some exact solutions.
• 4.1.4.1 *Using* $\eta = x - ct$.
Equation (4.2) becomes,

$$g^p \varphi^p \left(\varphi \frac{d^2 g}{dx^2} + 2 \frac{dg}{dx} \frac{\partial \varphi}{\partial \eta} + g \frac{\partial^2 \varphi}{\partial \eta^2} \right) + p g^{p-1} \varphi^{p-1} \left(\frac{dg}{dx} \varphi + g \frac{\partial \varphi}{\partial \eta} \right)^2 - N^2 f g \varphi(\eta) + g c \frac{\partial \varphi}{\partial \eta} = 0. \tag{4.9}$$

To ensure that Equation (4.9) is an ordinary differential equation, the arbitrary functions are set to be$f(x) = 1, g(x) = 1$. Hence, equation (4.9) is transformed to:

$$\varphi^p \frac{d^2\varphi}{d\eta^2} + p\varphi^{p-1}\left(\frac{d\varphi}{d\eta}\right)^2 - N^2\varphi + c\frac{d\varphi}{d\eta} = 0. \tag{4.10}$$

Equation (4.10) has a symmetry vector $V = \frac{\partial}{\partial\eta}$ which represents a hidden symmetry vector for the original P.D.E. This vector reduces the order of Equation (4.10) to a first order differential equation,

$$\frac{dF}{dy} = N^2\frac{y^{1-p}}{F} - cy^{-p} - p\frac{F}{y}, \tag{4.11}$$

where, $y = \varphi(\eta), F(y) = \frac{d\varphi}{d\eta}$. If c=0 (i.e. $\eta = x$), the equation is solved to give:

$$F(y) = \pm\sqrt{Ay^{-2p} + \frac{2N^2}{2+p}y^{2-p}}, \quad p \neq -2, A: \text{ integration constant.} \tag{4.12}$$

To make back substitution, Equation (4.12) may be solved at prescribed values of p. If, as an example, p is taken to be -1, Equation (4.12)is rewritten to be:

$$F(y) = \pm\sqrt{Ay^2 + 2N^2y^3}. \tag{4.13}$$

Making back-substitution, the following solution can be obtained.

$$\theta(x,t) = \frac{A}{2N^2}\left(\tan h^2\left(\frac{\sqrt{A}}{2}(x+B)\right) - 1\right), \tag{4.14}$$

where, A and B are the integration constants.
Another approach of solving Equation (4.10) is to increase the order of the equation, then try to find the symmetries of the new third-order ODE. Using one of its symmetries will result in a second-order O.D.E instead of Equation (4.10) which may give other solutions. If the order is increased by using the transformation$\eta = z, \varphi(\eta) = \frac{dG}{dz}$, Equation (4.10) is transformed to:

$$\left(\frac{dG}{dz}\right)^p\left(\frac{d^3G}{dz^3}\right) + p\left(\frac{dG}{dz}\right)^{p-1}\left(\frac{d^2G}{dz^2}\right)^2 - N^2\frac{dG}{dz} + c\left(\frac{d^2G}{dz^2}\right) = 0. \tag{4.15}$$

This third-order differential equation has two symmetry vectors, $V_1 = \frac{\partial}{\partial G}, V_2 = \frac{\partial}{\partial z}$ Using V_1 leads to the original Equation (4.10), while V_2 leads to the following second-order differential equation through the invariants, $\eta = G(z), \varphi(\eta) = \frac{dG}{dz}$

$$\varphi\frac{d^2\varphi}{d\eta^2} + (p+1)\left(\frac{d\varphi}{d\eta}\right)^2 + c\frac{d\varphi}{d\eta}\varphi^{-p} - N^2\varphi^{-p} = 0. \tag{4.16}$$

This is a second-order differential equation simpler than Equation (4.10). If c=0, i.e, $\eta = x$, Equation (4.16) has a solution on the form:

$$\varphi(\eta) = \left(\frac{N^2p}{2}\eta^2 + N^2\eta^2 - pA\eta - 2A\eta + pB + 2B\right)^{\frac{1}{p+2}}, \quad p \neq -2, \tag{4.17}$$

where, A and B are the integration constants to be evaluated according to the boundary conditions. By back substitution, the solution is given by:

$$\theta(x,t) = \left(\frac{N^2p}{2}x^2 + N^2x^2 - pAx - 2Ax + pB + 2B\right)^{\frac{1}{p+2}}, \quad p \neq -2. \tag{4.18}$$

The effect of the thermal conductivity power $p = -1.5, -1$, and -0.5 is shown in Figure 5.

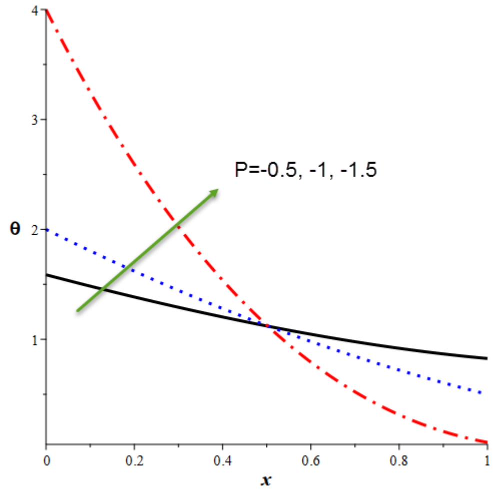

FIGURE 5. Effect of thermal conductivity power on the temperature distribution.

- 4.1.4.2 *Using* $\eta = t$

Similarly, Equation (4.2) is transformed to:

$$\left(g^p \frac{d^2 g}{dx^2} + p\left(\frac{dg}{dx}\right)^2 g^{p-1}\right)\varphi^{p+1} - N^2 f g \varphi - g\frac{d\varphi}{d\eta} = 0. \quad (4.19)$$

Dividing Equation (4.19) by (g) leads to:

$$\left(g^{p-1} \frac{d^2 g}{dx^2} + p\left(\frac{dg}{dx}\right)^2 g^{p-2}\right)\varphi^{p+1} - N^2 f \varphi - \frac{d\varphi}{d\eta} = 0. \quad (4.20)$$

To ensure that Equation (4.20) is an ordinary differential equation, the coefficients must be constants or functions of symmetry variableη. This leads to $f(x) = 1$,while $g(x)$ has two possibilities, $g(x) = 1 or g(x) = \left(\frac{p}{\sqrt{4-2p}}x\right)^{\frac{2}{p}}$. if $g(x) = 1$, then Equation (4.20) is reduced to:

$$\frac{d\varphi}{d\eta} + N^2\varphi = 0. \quad (4.21)$$

Which has a solution in the form:

$$\varphi(\eta) = Ae^{-N^2\eta}. \quad (4.22)$$

And so,

$$\theta(x,t) = Ae^{-N^2 t}. \quad (4.23)$$

If $g(x) = \left(\frac{p}{\sqrt{4-2p}}x\right)^{\frac{2}{p}}$, Equation (4.20) is reduced to:

$$\frac{d\varphi}{d\eta} + N^2\varphi = \left(\frac{2+p}{2-p}\right)\varphi^{p+1}. \quad (4.24)$$

Which has a solution in the form:

$$\varphi(\eta) = \left(\frac{(p-2)N^2}{-(p+2) + AN^2(p-2)e^{pN^2\eta}}\right)^{\frac{1}{p}}. \quad (4.25)$$

And so,

$$\theta(x,t) = \left(\frac{p}{\sqrt{4-2p}}x\right)^{\frac{2}{p}} \left(\frac{(p-2)N^2}{-(p+2)+AN^2(p-2)e^{pN^2t}}\right)^{\frac{1}{p}}. \tag{4.26}$$

5. Case 3: Exponential thermal conductivity, $K(\theta) = e^{r\theta}$

The Equations (2.1) and (2.2) are reformulated to:

$$e^{r\theta}\frac{\partial^2\theta}{\partial x^2} + re^{r\theta}\left(\frac{\partial\theta}{\partial x}\right)^2 - N^2 f(x)\theta - \frac{\partial\theta}{\partial t} = 0. \tag{5.1}$$

Subjected to the same initial condition described in Equation (2.2). Using the normalization relation, $\theta(x,t) = g(x)\varphi(x,t)$, the Equation (5.1) is given by:

$$e^{rg\varphi}\left(\varphi\frac{d^2g}{dx^2} + 2\frac{dg}{dx}\frac{\partial\varphi}{\partial x} + g\frac{\partial^2\varphi}{\partial x^2}\right) + re^{rg\varphi}\left(\frac{dg}{dx}\varphi + g\frac{\partial\varphi}{\partial x}\right)^2 - N^2 fg\varphi - g\frac{\partial\varphi}{\partial t} = 0, \tag{5.2}$$

with

$$\varphi(x,0) = 1. \tag{5.3}$$

The same steps are executed to reduce the equation.

5.1. Reduction to ordinary differential equation.

5.1.1. *Using* $\eta(x,t) = x - ct$.
Equation (5.2) is transformed to the following ODE:

$$e^{\varphi}\left(\frac{d^2\varphi}{d\eta^2} + \left(\frac{d\varphi}{d\eta}\right)^2\right) - N^2\varphi + c\frac{d\varphi}{d\eta} = 0, \tag{5.4}$$

with

$$f(x) = 1, g(x) = \frac{1}{r}. \tag{5.5}$$

Equation (5.4) has a symmetry vector $V1 = \frac{\partial}{\partial\eta}$ which is not inherited from the original PDE. So, it represents a hidden symmetry vector. This vector transforms the Equation (5.4) through the invariants $y = \varphi(\eta), F(y) = \frac{d\varphi}{d\eta}$ to the following first order differential equation,

$$\frac{dF}{dy} = -F + \frac{N^2 y}{Fe^y} - \frac{c}{e^y}. \tag{5.6}$$

If c=0, (i.e. $\eta = x$), Equation (5.6) has a solution in the form:

$$F(y) = \pm e^{-y}\sqrt{A + 2N^2(y-1)e^y}, \text{A is the integration constant.} \tag{5.7}$$

Making back substitution, the exact solution could be obtained. The following implicit solution can be obtained if $A = 0$:

$$\frac{\sqrt{\pi e}}{N} erfi\left(\sqrt{\frac{\phi - 1}{2}}\right) = x - ct + B, \tag{5.8}$$

where, B is an integration constant and $erfi$ is the imaginary error function.

5.1.2. *Using* $\eta(x,t)=t$. Equation (5.2) is transformed to the following ODE:

$$\frac{d\varphi}{d\eta}+N^2\varphi=0. \tag{5.9}$$

Subjected to:

$$f(x)=1, g(x)=\frac{1}{r}. \tag{5.10}$$

From Equation (5.9), the following result can be obtained:

$$\varphi(\eta)=Ae^{-N^2\eta}. \tag{5.11}$$

Making a reverse substitution:

$$\theta(x,t)=Be^{-N^2t}. \tag{5.12}$$

The three presented cases of thermal conductivities play a vital role in the heat distribution and sensitivity to the fin parameter. Regarding the heat distribution, constant thermal conductivity provides a straightforward simpler equation and the temperature profile will be nice and predictable, but it does not account for material property changes with temperature; hence, it is less accurate in practical cases. In case of power-law type, thermal conductivity increases or decreases nonlinearly with temperature based on the power-law exponent. This model captures materials where thermal conductivity grows or diminishes sharply with temperature, adding realism but complexity. In case of exponential type, the variation of thermal conductivity is much stronger than in the power-law case, particularly at elevated temperatures. This case is suitable for materials exhibiting rapid changes in conductivity, such as metals at extremely high temperatures.

Regarding the sensitivity to Fin parameter, the constant conductivity case implies that the changes in N directly yield the heat transfer rate. In case of power-law case, for larger exponents, the system becomes highly sensitive for variations in N with large changes in the heat distribution and fin efficiency. In case of exponential type, the system is more sensitive to N when compared to the power-law case because of the rapid variation in conductivity with temperature.

6. Conclusions

In the current paper, the group method is used to obtain the exact solution of the nonlinear fin equation with variable heat transfer coefficient, which is to be determined by the condition of invariance of the whole equation under group transformation. Three types of thermal conductivity coefficients are considered, namely constant, power-law type, and exponential. In certain cases, the solutions possess hidden symmetries. The analysis provides the proper initial conditions that render the problem invariant under group transformation. Furthermore, two boundary conditions have been analyzed: a time-dependent one, and it follows from there that an increase in the fin parameter, N, increases the temperature distribution along the body of the fin. Whereas for a fin under steady-state boundary conditions, increase in the fin parameter, N, leads to a drop in the temperature distribution along the fin body. Temperature distribution for the positive range from the fin base at $x=0$ to the fin tip $x=1$, for different types of boundary conditions, also shows an increasing trend. These results are the same as the results reported by R.J. Moitsheki et al. [24].

Case1: Constant thermal conductivity,$K(\theta)=D$

Four types of symmetry variable were obtained from the group solution of the problem. Hidden symmetries are found for the symmetry variable,$\eta=x-ct$, and the solution is exactly the same as the ordinary solution showing the effectiveness of the hidden symmetries. The time dependent boundary conditions show that the increase of the fin parameter, N, leads to analogous increase in the temperature profile, while an inverse relation is achieved for steady state boundary conditions. For $\eta=tx^{-2}$, it yielded a novel solution to some two-point boundary value problems concerning Kummer functions of the first and second kinds. Temperature profile increases along the x-direction, whereas it increases initially for a small time before it decreases, as expected in such systems, with time.

Case 2: Power-law thermal conductivity, $K(\theta)=\theta^p$

Five types of symmetry variables were created through the group reduction of the problem. Two of them led to highly nonlinear ordinary differential equations which require a numerical solution which is out of scope the recent work. For the case,$\eta = x - ct$, hidden symmetries were obtained through two different ways. The first way is to decrease the order of the resulting ODE to a 1^{st} order differential equation and then solve it. The other is to increase the order of the equation to a 3^{rd} order equation and then reduce the order and solve the new equation. The two ways led to two different solutions in case of $c = 0$ which means that symmetry variable is restricted to be $\eta = x$. Steady state boundary condition is then applied, and fin parameter has also a reverse relation with temperature distribution. The effect of conductivity coefficients power, p, for negative values is studied for $\eta = x$and at unity fin parameter showing that the increase in p leads to a decrease in temperature distribution. For $\eta = t$, the solution gives two cases of initial conditions. If $g(x) = 1$, a solution is obtained as a function of time only. This makes the temperature constant along x-direction. This can be understood as the initial condition is constant and the symmetry variable is only a function of time. Three cases of boundary conditions are applied. All applied boundary conditions show an increasing profile with respect to x except with the first as the initial condition, g(x), is set to unity as previously discussed.

Case 3: Exponential thermal conductivity, $K(\theta) = e^{r\theta}$

Four types of symmetry variables were achieved. For $\eta = tx^{-2}$, a highly nonlinear ODE is obtained requiring a numerical solution which is out of scope of this study. For $\eta = x - ct$, hidden symmetry is obtained. A symmetry solution is obtained at the steady state case, c=0, which means that symmetry variable is reduced to be x. For $\eta = t$, a simple first order ODE is obtained while the initial condition is constant. So, the graph shows no change in temperature along the fin. The temperature decays only with time.

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