Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-12
DOI:10.22034/cmde.2025.65612.3026

Bézier curve technique for solving p-fractional differential equations

Fateme Ghomanjani[1] **and Khursheed J. Ansari**[2,∗]

[1]Department of Mathematics, Kashmar Higher Education Institute, Kashmar, Iran.

[2]Department of Mathematics, College of Science, King Khalid University, Abha, 61413, Saudi Arabia.

Abstract

The Bézier curve technique is a numerical method often adopted for solving complex differential equations, including fractional differential equations. The quantum analogue of fractional differential equations extends classical fractional differential equations into the quantum domain, involving fractional calculus within quantum mechanic frameworks. In this sequel, the stated Liouville-Caputo type p-fractional differential equation (pFDE) is solved by utilizing the Bézier curve method. Firstly, the p-fractional differential equation is transformed into the equivalent systems of weakly singular p-integral equations by many results of fractional p-calculus. Secondly, the Bézier curve method is used to solve the latter systems of weakly singular p-integral equations. The stated method is an approximation method which has very small errors as it gives very good results. Numerical examples are also given to check the validity of the BCM technique.

Keywords. p-fractional differential equation, Bézier curve, p-fractional derivative, Simulation.

2010 Mathematics Subject Classification. 65K10, 26A33, 49K15.

1. Introductions and Motivations

Fractional modeling has become applicable in different sciences during the past three decades or more. In addition, many physical and engineering topics has attracted much attention for researchers (see [7, 8, 17, 23]). The fractional calculuses (FC) generalizes the operations of differentiation and integration to non-integer orders. FC emerged as an main tool for the study of dynamical systems, since fractional order operators are non-local and capture the history of dynamics. The p-fractional calculuses is the Liouville-Caputo type p-fractional calculus (pFC). In stochastic analysis fields and quantum mechanics, we can find the p-fractional calculus. The Liouville-Caputo type p-fractional calculus is one of the classical p-fractional calculuses. Some scientists have studied the existence of solutions for the Liouville-Caputo type p-fractional boundary value problems, (see [2, 4, 6, 9, 16, 18]).

Jarad et al.[19] studied the stability of Liouville-Caputo type p-fractional non-autonomous systems by using Lyapunov's direct technique.

On the numerical methods, some techniques were presented to achieve an approximate solution of the pFDE [1]. Then, the convergence of these techniques was found (see [22]). In [26], the difference method was used to solve the Liouville-Caputo type pFDE . The variational iteration technique and the Lagrange multipliers technique are used to study the Liouville-Caputo type pFDE (see [24]). For this problem, the utilization of the Bézier curve method (BCM) [10] is a new idea. Also this approach is simple to utilize. Therefore, we consider BCM for this problem. Additionally many researchers utilized the BCM: For example, the numerical solution for delay differential equation (DDE) is obtained by [13] and [14], and some examples for linear optimal control systems with pantograph delays is achieved by BCM [15]. Although the use of this method is very straightforward and simple (see the results). For the collocation technique, one can refer to [3, 5, 20].

Received: 25 January 2025 ; Accepted: 24 August 2025.

∗ Corresponding author. Email: ansari.jkhursheed@gmail.com.

In this paper, The approximate solution of IVPCpF via BCM is stated. The lemma is exhibited for truncated error. Also, some numerical examples are stated to verify our technique.

The organization of the paper is sorted as follows: Section 1 describes the introduction and motivation of the basic idea. Basic preliminaries are appearing in section 2. The Bézier curve method is introduced in section 4. The application of the current method is simulated in section 5. The last section 6 is dedicated to a concluding remark on the article.

2. Basic Preliminaries

Definition 2.1. (see [11]) Let $p \in (0,1]$ and

$$[t]_p = \frac{1-p^t}{1-p},\ t \in \mathbb{R},$$

where $\mathbb{R}$ is the set of real numbers. The p-analogue of the power function $(t-s)^n$ with $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is

$$(t-s)^0 = 1,\ (t-s)^n = \prod_{i=0}^{n-1}(t-p^i s), n \in \mathbb{N}, \quad t,s \in \mathbb{R},$$

where $\mathbb{N}$ is the set of natural numbers. For $0 \le s \le t$, we have

$$(t-s)^{(\beta)} = t^\beta \prod_{i=0}^{\infty} \frac{t-p^i s}{t-p^{\beta+i}s}, \tag{2.1}$$

where $\beta > 0$ and $0 < |p| < 1$. Note that, if $s = 0$ then $t^{(\beta)} = t^\beta$. For $t_0 \le t_1 \le t$, we have

$$(t-t_0)^{(\beta)} \ge (t-t_1)^{(\beta)},$$

where $0 < |p| < 1, \beta \in \mathbb{C} \backslash \{-n : n \in \mathbb{N} \cup \{0\}\}$, $\mathbb{C}$ is the set of complex number, and $\mathbb{N} = \{1,2,3,\dots\}$, then

$$\Gamma_p(\beta) = (1-p)^{1-\beta}(1-p)^{(\beta-1)},$$
$$\Gamma_p(\beta+1) = [\beta]\Gamma_p(\beta), \Gamma_p(1) = 1,$$

where $[\beta] = \left(1-p^\beta\right)/(1-p)$.

Definition 2.2. (see [18]) If g is a real or complex valued function, $t \in A$ (on p geometry set A, where $p \in R$ be an invariant point, a subset A of C is called p geometric if $pt \in A$ whenever $t \in A$), $|p| \ne 1$, then

$$D_p g(t) = \frac{\mathrm{d}_p}{\mathrm{d}_p t} g(t) = \frac{g(pt) - g(t)}{(p-1)t}, \quad t \in A \setminus 0,$$
$$D_p g(0) = \left.\frac{\mathrm{d}_p}{\mathrm{d}_p t} g(t)\right|_{t=0} = \lim_{n\to\infty} \frac{g(t^n) - g(0)}{tp^n}, \quad |p| < 1.$$

Definition 2.3. (see [25]) Let $t \in A$(p-geometry set A), $\beta \ne -1, -2, \dots$ and $\beta \ge 0$. The β-order p-fractional integral of the Riemann-Liouville type with the lower limit point a is can be defined as $I_{p,a} g(t) = g(t)$,

$$I_{p,a}^\beta g(t) = \frac{1}{\Gamma_p(\beta)} \int_a^t (t-ps)^{\beta-1} g(s) \mathrm{d}_p s.$$

Definition 2.4. (see [25]) Let $a \in T_p$, where $T_p = \{p^n : 0 < p < 1, n \in \mathbb{Z} \cup\ \{0\}\}, \mathbb{Z} = \{0, +1, -1, +2, -2, \dots\}$. The β-order Riemann-Liouville type p-fractional derivative (FD) of a function $g(t) : (a, \infty) \to R$ can be defined as

$$\left(D_{p,a}^\beta g\right)(t) = \begin{cases} \left(I_{p,a}^{-\beta} g\right)(t), & \beta \le 0, \\ \left(D_{p,a}^{\lceil\beta\rceil} I_{p,a}^{\lceil\beta 1 - \beta} g\right)(t), & \beta > 0, \end{cases}$$

where $\lceil\beta\rceil$ be the smallest integer that is greater or equal to β.

Definition 2.5. (see [25]) Let $a \in T_p$. The β-order Liouville-Caputo type p-FD of a function $g(t):(a,\infty)\to R$ can be defined as

$$\left({}^{c}D_{p,a}^{\beta} g\right)(t)=\begin{cases}\left(I_{p,a}^{-\beta} g\right)(t), & \beta \leq 0,\\ \left(I_{p,a}^{[\beta]-\beta} D_{p,a}^{[\beta]} g(t)\right), & \beta>0.\end{cases}$$

For facility, one use the notations $I_p^{\beta} g(t), D_p^{\beta} g(t)$, and ${}^{c}D_p^{\beta} g(t)$ instead of $I_{p,0}^{\beta} g(t)$, $D_{p,0}^{\beta} g(t)$, and ${}^{c}D_{p,0}^{\beta} g(t)$ respectively.

Definition 2.6. (see [26]) A function g defined on $[0,b]$ is called p-absolutely continuous if g is continuous and for $t\in(pb,b]$, there exists a constant $M>0$, where

$$\sum_{j=0}^{\infty}\left|g\left(tp^{j}\right)-g\left(tp^{j+1}\right)\right| \leq M.$$

Introduce the space $\mathcal{AC}_p^{(n)}[0,b]=\left\{D_p^{k-1} g \in \mathcal{AC}_p[0,b], k=1,2,\ldots,n\right\}$, where $\mathcal{AC}_p[0,b]$ is composed of all functions which are p-absolutely continuous on $[0,b]$.

Definition 2.7. Let $g(t)$ on (t_0,t_1) be a real valued function on a set A. The p-integral can be defined as

$$\int_{t_0}^{t_1} g(t) \mathrm{d}_p t=\int_0^{t_1} g(t) \mathrm{d}_p t-\int_0^{t_0} g(t) \mathrm{d}_p t,$$

where

$$\int_0^x g(t) \mathrm{d}_p t=(1-p) \sum_{n=0}^{\infty} x p^n g\left(x p^n\right), \quad x \in A.$$

Also

$$\int_0^1 t \mathrm{d}_p t=\frac{1}{1+p}, \qquad \text{and} \qquad \int_0^1 t^n \mathrm{d}_p t=\frac{p-1}{p^{n+1}-1}.$$

Definition 2.8. (see [9]) If $\beta>0, n=\lceil\beta\rceil, t\in(0,b], g\in\mathcal{AC}_p^{(n)}[0,b]$ and $D_p^n g \in C[0,b]$, then

$${}^{c}D_p^{\beta} g(t)=D_p^{\beta}\left(g(t)-\sum_{j=0}^{n-1} \frac{D_p^j g(0)}{\Gamma_p(j+1)} t^j\right).$$

3. The Approximation Method

The initial value problem of the β-order $(0<\beta<1)$ Liouville-Caputo type p-fractional (IVPCpF) differential equation $(0<p<1)$ can be introduced by

$$\begin{aligned}&{}^{c}D_p^{\beta} z(t)=g(t,z(t)), \quad 0\leq t\leq 1,\\ &z\left(t_0\right)=z_0, \quad t_0\in T_p, \quad t_0=0,\end{aligned} \tag{3.1}$$

where $g: T_p\times R^n\to R^n$ is continuous function, the time scale $T_p=\{p^n: n\in\mathbb{Z}\}\cup\{0\}$, $\mathbb{Z}$ is the set of integers, and t denotes time here.

Now, a difference formula is constructed to discretize the pCF,

$${}^{c}D_p^{\beta} z(t)=D_p^{\beta} z(t)-\frac{z(0)}{\Gamma_p(1-\beta)} t^{-\beta}.$$

Let $0=t_0<t_1<\ldots<t_N=1\in T_p$ be a nonuniform partition of $[0,1], t_k=p^{N-k}(1\leq k\leq N), \quad \triangle t_1=p^{N-1}, \quad \triangle t_k=p^{N-k}-p^{N-k+1}(2\leq k\leq N)$, where $N\geq 1, \quad N\in Z^{+}, \quad g(t,s)=(t-ps)^{(-\beta)} z(s), \quad g(pt,t)=0,$

therefore one may have

$$\begin{aligned}{}^{c}D_p^{\beta} z\left(t_n\right) &= D_p I_p^{1-\beta} z\left(t_n\right)-\frac{z(0)}{\Gamma_p(1-\beta)} t_n^{-\beta} \\ &= D_p \frac{1}{\Gamma_p(1-\beta)} \int_0^{t_n}\left(t_n-p s\right)^{(-\beta)} z(s) \mathrm{d}_p s-\frac{z(0)}{\Gamma_p(1-\beta)} t_n^{-\beta} \\ &= \frac{1}{\Gamma_p(1-\beta)} \int_0^{t_n} D_p\left(t_n-p s\right)^{(-\beta)} z(s) \mathrm{d}_p s-\frac{z(0)}{\Gamma_p(1-\beta)} t_n^{-\beta},\end{aligned}$$

then

$$\begin{aligned}{}^{c}D_p^{\beta} z\left(t_n\right) &= \frac{1}{\Gamma_p(-\beta)} \int_0^{t_n}\left(t_n-p s\right)^{(-\beta-1)} z(s) \mathrm{d}_p s-\frac{z(0)}{\Gamma_p(1-\beta)} t_n^{-\beta} \\ &= \frac{1}{\Gamma_p(-\beta)} \sum_{k=1}^n \int_{t_{k-1}}^{t_k}\left(t_n-p\right)^{(-\beta-1)} z(s) \mathrm{d}_p s-\frac{z(0)}{\Gamma_p(1-\beta)} t_n^{-\beta}.\end{aligned} \tag{3.2}$$

For discretizing the integral in (3.2), one may have

$$L_{1,k}(s)=\frac{s-t_{k-1}}{\triangle t_k} z\left(t_k\right)+\frac{t_k-s}{\triangle t_k} z\left(t_{k-1}\right), \qquad s \in\left[t_{k-1}, t_k\right], k=1,2, \ldots, N.$$

Theorem 3.1. *Assume that* $0<\beta, p<1, z(t)$ *be a continuous function on* $[0,1]$, $M_1=\sup_{0 \leq t \leq 1}\left|D_p^2 z(t)\right|$, *then for any fixed* $\delta\,(0<\delta<1, t_n(n \leq(1-\delta) N)$ *the truncation error* $r\left(t_n\right)$ *can be written by:*

$$\left|r\left(t_n\right)\right|<\frac{M_1(1-p)^2 p^{(2-\beta) \delta N}}{4\left(1-p^\beta\right)\left(1-p^{1-\beta}\right)\left|\Gamma_p(-\beta)\right|}.$$

Proof. See [26]. □

4. The Bézier Curve Method

Let k be a chosen positive integer and $\{t_0 \leq t_1 \leq \ldots \leq t_k=t_f\}$ be a partition of $[t_0, t_f]$. The idea is utilizing BCM to approximate the solutions $z(t)$. Define the Bézier polynomials of degree n over the interval $[t_{j-1}, t_j]$ as follows:

$$z_j(t)=\sum_{r=0}^n w_r^j B_{r, n}\left(\frac{t-t_{j-1}}{h}\right), \tag{4.1}$$

where $h=t_j-t_{j-1}$, and

$$B_{r, n}\left(\frac{t-t_{j-1}}{h}\right):=\binom{n}{r} \frac{1}{h^n}\left(t_j-t\right)^{n-r}\left(t-t_{j-1}\right)^r,$$

is the Bernstein polynomial of degree n over $[t_{j-1}, t_j]$, and w_r^j are unknown control points.
Also, one may have

$$C^T=<z_j(t), \phi(t)>Q_n^{-1},$$

$$C=\left[\begin{array}{llll}c_0 & c_1 & \ldots & c_n\end{array}\right].$$

For $\Omega=[t_{j-1}, t_j]$, and $T_n(t)=\left[\begin{array}{llll}1 & t & \ldots & t^n\end{array}\right]^T$, we have

$$\begin{aligned}\phi(t) &=<B_{0, n}(t), \ldots, B_{0, n}(t)>, \\ <z_j(t), \phi(t)> &=\int_{\Omega} z_j(t) \phi^T(t) \mathrm{d} t \\ &=\left[<z_j(t), B_{0, n}(t)><z_j(t), B_{1, n}(t)> \quad \ldots<z_j(t), B_{n, n}(t)>\right],\end{aligned}$$

where the $(n+1)\times(n+1)$ matrix Q_n are considered as

$$
\begin{aligned}
Q_n &=< \phi(t), \phi(t) >= \int_\Omega \phi(t)\phi^T(t)\mathrm{d}t \\
&= \int_\Omega \left(\psi_n^\Omega T_n(t)\right)\left(\psi_n^\Omega T_n(t)\right)^T \mathrm{d}t \\
&= \psi_n^\Omega \int_\Omega T_n(t)\left(T_n(t)\right)^T \mathrm{d}t \left(\psi_n^\Omega\right)^T = \psi_n^T G_{\Omega,n}\left(\psi_n^\Omega\right)^T,
\end{aligned}
$$

and

$$
G_{\Omega,n} = \begin{bmatrix}
(t_j - t_{j-1}) & \frac{(t_j^2 - t_{j-1}^2)}{2} & \dots & \frac{(t_j^{n+1} - t_{j-1}^{n+1})}{n+1} \\
\frac{(t_j^2 - t_{j-1}^2)}{2} & \frac{(t_j^3 - t_{j-1}^3)}{3} & \dots & \frac{(t_j^{n+2} - t_{j-1}^{n+2})}{n+2} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{(t_j^{n+1} - t_{j-1}^{n+1})}{n+1} & \frac{(t_j^{n+2} - t_{j-1}^{n+2})}{n+2} & \dots & \frac{(t_j^{2n+1} - t_{j-1}^{2n+1})}{2n+1}
\end{bmatrix}
$$

and ψ_n^Ω is the $(n+1)(n+1)$ matrix, where

$$
\psi_n^\Omega = \psi^{n+1}A^{n+1},
$$

$$
\psi^{n+1}(i+1, j+1) = \begin{cases} \frac{(-1)^{j-i}}{h^j}\binom{n}{i}\binom{n-i}{j-i}, & i \le j, \\ 0, & j > i, \end{cases}
$$

and for $i, j = 0, 1, \dots, n$, we have

$$
A^{n+1}(i+1, j+1) = \begin{cases} \binom{i}{j}(-a)^{i-j}, & j \le i, \\ 0, & j > i. \end{cases}
$$

Lemma 4.1. *Let the function $z_{\text{exact}} : [t_{j-1}, t_j) \longrightarrow R$ be $n+1$ times continuously differentiable, $z_{\text{exact}} \in C^{m+1}[t_{j-1}, t_j]$ and $Y = \operatorname{Span}\{\psi_{i,m} \mid i = 0, 1, \dots, n, m = 0, 1, \dots, k-1\}$, if $C^T\phi$ is the best approximation z_{exact} out of Y, then we have*

$$
\begin{aligned}
&\|z_{\text{exact}} - C^T\phi\|_2 \le \frac{M}{(n+1)!k^{n+1}\sqrt{3n+3}}, \\
&M = \max_{t\in[t_{j-1}, t_j]} \left| z_{\text{exact}}^{(n+1)}(t) \right|.
\end{aligned}
$$

Proof. Suppose that the Taylor polynomial of order n for the function $z_{\text{exact}} = g_{\text{exact}}$ on $\left[\frac{m}{k}, \frac{m+1}{k}\right)$ as

$$
y_m(t) = g_{\text{exact}}\left(\frac{m}{k}\right) + g'\left(\frac{m}{k}\right)\left(t - \frac{m}{k}\right) + \dots + g^{(n)}\left(\frac{m}{k}\right)\frac{\left(t - \frac{m}{k}\right)^n}{n!},
$$

$m = 0, 1, \dots, k-1$, we can compute

$$
|g(t) - y_n(t)| \le \|g^{(n+1)}(\eta)\| \frac{\left(t - \frac{m}{k}\right)^{n+1}}{(n+1)!}, \quad \eta \in \left[\frac{m}{k}, \frac{m+1}{k}\right),
$$

then, we have

$$\begin{aligned}
\|g-C^T\phi\|_2^2 &= \int_{[t_{j-1},t_j]} \left|g(t)-C^T\phi(t)\right|^2 \mathrm{d}t \\
&= \sum_{m=1}^{k-1} \int_{[\frac{m}{k},\frac{m+1}{k}]} \left|g(t)-C^T\phi(t)\right|^2 \mathrm{d}t \\
&\le \sum_{m=1}^{k-1} \int_{\frac{m}{k}}^{\frac{m+1}{k}} \left|g(t)-y_m(t)\right|^2 \mathrm{d}t \\
&\le \sum_{m=1}^{k-1} \int_{[\frac{m}{k},\frac{m+1}{k}]} \left[g^{(n+1)}(\eta)\frac{\left(t-\frac{m}{k}\right)^{n+1}}{(n+1)!}\right]^2 \mathrm{d}t \\
&\le \frac{M^2}{(n+1)!^2} \sum_{m=0}^{k-1} \int_{[\frac{m}{k},\frac{m+1}{k}]} \left(t-\frac{m}{k}\right)^{2n+2} \mathrm{d}t \\
&= \frac{M^2}{[(n+1)!]^2 k^{2n+2}(2n+3)}.
\end{aligned}$$

Now, the proof is completed. □

Note: Also, we have

$$\frac{\mathrm{d}B_{r,n}(\tau)}{\mathrm{d}\tau} = n\left(B_{r-1,n-1}(\tau)-B_{r,n-1}(\tau)\right),$$

$$\begin{aligned}
\frac{\mathrm{d}z_j(\tau)}{\mathrm{d}\tau} &= \sum_{r=0}^{n-1} nw_r^j B_{r-1,n-1}(\tau) - \sum_{r=0}^{n-1} nw_r^j B_{r,n-1}(\tau) \\
&= \sum_{r=0}^{n-1} nw_{r+1}^j B_{r,n-1}(\tau) - \sum_{r=0}^{n-1} nw_r^j B_{r,n-1}(t) \\
&= \sum_{r=0}^{n-1} B_{r,n-1}(\tau) n\left(w_{r+1}^j - w_r^j\right).
\end{aligned}$$

By substituting $z_j(t)$, Eqs. (3.1) and (4.1), one may achieve a simplified problem, then, this problem can be solved by Maple 16. The convergence of this method is similar the method in [15]. In [12], for

$$\begin{aligned}
&{}^cD_p^\beta z_i(t) = f_i\left(t, z_1, z_2, \ldots, z_n\right), \quad t_0 \le t \le 1, i = 1, 2, \ldots, n, \\
&D_p^j z_i\left(t_0\right) = b_{ij}, \quad b_{ij} \in R, \quad i = 1, \ldots, n, \quad j = 0, 1, \ldots, \lceil\beta\rceil - 1,
\end{aligned}$$

we have

$$z_i(t) = \sum_{j=0}^{\lceil\beta\rceil-1} \frac{b_{ij}}{\Gamma_p(j+1)} t^j + \frac{(1-p)}{\Gamma_p(\beta)} \sum_{k=0}^{\infty} tp^k \left(t - tp^{(k+1)}\right)^{(\beta-1)} f_i\left(tp^k, z_1\left(tp^k\right), \ldots, z_n\left(tp^k\right)\right) \mathrm{d}_p s,$$

be done in the following step

- Degree raising case of the Bézier polynomial approximation.

Remark 4.2. Consider the following system:

$$\begin{aligned}
&{}^cD_p^\beta z(t) = g(t, z(t)) = A(t)z(t) + F(t), \quad 0 \le t \le 1, \qquad (4.2)\\
&z\left(t_0\right) = z_0.
\end{aligned}$$

Sufficient conditions for the existence and uniqueness of the solution of the pantograph delay differential Equation (4.2) are

- the $z(t)$ is continuous;
- the right-hand side of (4.2) satisfies Lipschitz condition;
- the right-hand side of (4.2) is bounded.

4.1. **The convergence of the approximate solution.** One may consider the following problem:

$$I(z(t), {}^cD_p^\beta z(t)) = {}^cD_p^\beta z(t) - A(t)z(t) = F(t), \quad 0 \le t \le 1, \tag{4.3}$$
$$z(t_0) = z_0 = a \in \mathbb{R}.$$

Lemma 4.3. *For a polynomial in Bézier form*

$$z(t) = \sum_{i=0}^{n_1} c_{i,n_1} B_{i,n_1}(t),$$

then we have

$$\frac{\sum_{i=0}^{n_1} c_{i,n_1}^2}{n_1+1} \ge \frac{\sum_{i=0}^{n_1+1} c_{i,n_1+1}^2}{n_1+2} \ge \dots \ge \frac{\sum_{i=0}^{n_1+m_1} c_{i,n_1+m_1}^2}{n_1+m_1+1},$$

where c_{i,n_1+m_1} is the Bézier coefficients of $z(t)$ after it is degree-elevated to degree n_1+m_1.

Proof. See [27]. □

Theorem 4.4. *If the problem (4.3) has a unique C^1 continuous solutions $\bar{z}$, then the approximate solutions obtained by the control-point-based method converges to the exact solution $\bar{z}$ as the degree of the approximate solution tends to infinity.*

Proof. Given an arbitrary small positive number $\epsilon > 0$, by the Weierstrass theorem (see [21]) one can easily find polynomials $Q_{1,N_1}(t)$ of degree N_1 such that

$$||a - Q_{1,N_1}(0)||_\infty \le \frac{\epsilon}{16},$$

where $||.||_\infty$ stands for the L_∞-norm over $[0,1]$.
In general, $Q_{1,N_1}(t)$ does not satisfy the boundary conditions. After a small perturbation with linear and constant polynomials β, for $Q_{1,N_1}(t)$, we can obtain polynomial $P_{1,N_1}(t) = Q_{1,N_1}(t)+\beta$ such that $P_{1,N_1}(t)$ satisfies the boundary condition $P_{1,N_1}(0) = a$. Thus $Q_{1,N_1}(0) + \beta = a$. Then, we have

$$\|a - Q_{1,N_1}(0)\|_\infty = \|\beta\|_\infty \le \frac{\epsilon}{16},$$

$$\|{}^cD_p^\beta P_{1,N_1}(t) - {}^cD_p^\beta \bar{z}(t)\|_\infty = \left\| D_p^\beta \left((P_{1,N_1}(t) - \bar{z}(t)) - \sum_{j=0}^{n-1} \frac{D_p^j (P_{1,N_1}(0) - z(0))}{\Gamma_p(j+1)} t^j \right) \right\|_\infty$$
$$\le \left\| D_p^\beta (P_{1,N_1}(t) - \bar{z}(t)) \right\|_\infty < \frac{\epsilon}{5},$$

Now, let define

$$LP_N(t) = L(P_{1,N_1}(t), {}^cD_p^\beta P_{1,N_1}(t)) = {}^cD_p^\beta P_{1,N_1}(t) - A(t)P_{1,N_1}(t) = F(t),$$

for every $t \in [0,1]$. Thus for $N \ge N_1$, we have an upper bound for the following residual:

$$\|LP_N(t) - F(t)\|_\infty = \|L\left(P_{1,N_1}(t), {}^cD_p^\beta P_{1,N_1}(t)\right)\|_\infty \le \|{}^cD_p^\beta P_{1,N_1}(t) - {}^cD_p^\beta \bar{z}(t)\|_\infty + \|A(t)\|_\infty \|P_{1,N_1}(t) - \bar{z}(t)\|_\infty$$
$$\le C_1\left(\frac{\epsilon}{5} + \frac{\epsilon}{5}\right) < C_1 \epsilon,$$

where $C_1 = 1 + \|A(t)\|_\infty$ is a constant. Since the residual $R(P_N) := LP_N(t) - F(t)$ is a polynomial, we can represent it by a Bézier form. Therefore

$$R(P_N) := \sum_{i=0}^{m_1} d_{i,m_1} B_{i,m_1}(t),$$

then from Lemma 1 in [27], there exists an integer $M(\geq N)$ such that when $m_1 > M$, we have

$$\left|\frac{1}{m_1+1}\sum_{i=0}^{m_1} d_{i,m_1}^2 - \int_0^1 (R(P_N))^2 \mathrm{d}t\right| < \epsilon,$$

thus

$$\frac{1}{m_1+1}\sum_{i=0}^{m_1} d_{i,m_1}^2 < \epsilon + \int_0^1 (R(P_N))^2 \mathrm{d}t < \epsilon + C_1^2\epsilon^2.$$

Suppose $z(t)$ is approximated solution of (4.3) obtained by the control-point-based method of degree m_2 $(m_2 \geq m_1 \geq M)$. Let

$$R\left(z(t), {}^cD_p^\beta P_{1,N_1}(t)\right) = L\left(z(t), {}^cD_p^\beta P_{1,N_1}(t)\right) - F(t) = \sum_{i=0}^{m_2} c_{i,m_2} B_{i,m_2}(t), m_2 \geq m_1 \geq M, t \in [0,1].$$

Define the following norm for difference approximated solution $z(t)$ and exact solution $\bar{z}(t)$

$$\|z(t) - \bar{z}(t)\| := \int_0^1 \left|{}^cD_p^\beta z(t) - {}^cD_p^\beta \bar{z}(t)\right|^2 \mathrm{d}t. \tag{4.4}$$

By (4.4), Lemma 4.3, the boundary conditions $\bar{z}(0) = a = P_{1,N_1}(0) = z(0)$, we have

$$\begin{aligned}\|z(t) - \bar{z}(t)\| &\leq C|z(0) - \bar{z}(0)| + \left\|R\left((z(t), {}^cD_p^\beta z(t)) - (\bar{z}(t), {}^cD_p^\beta \bar{z}(t))\right)\right\|_2^2 \\ &= C\int_0^1 \sum_{i=0}^{m_2} (c_{i,m_2}B_{i,m_2}(t))^2 \mathrm{d}t \leq \frac{C}{m_2+1}\sum_{i=0}^{m_2} c_{i,m_2}^2.\end{aligned} \tag{4.5}$$

The last inequality in (4.5) is obtained by Lemma 4.3, where C is a constant positive number. Hence

$$\begin{aligned}\|z(t) - \bar{z}(t)\| &\leq \frac{C}{m_2+1}\sum_{i=0}^{m_2} c_{i,m_2}^2 \\ &\leq \frac{C}{m_2+1}\sum_{i=0}^{m_2} d_{i,m_2}^2 \leq \frac{C}{m_1+1}\sum_{i=0}^{m_1} d_{i,m_1}^2 \\ &\leq C(\epsilon + C_1^2\epsilon^2) = \epsilon_1, \quad m_1 \geq M.\end{aligned}$$

This completes the proof. □

5. Numerical examples

Now, a numerical example of IVPCpF is stated to illustrate the BCM (Bézier Curve Method). By utilizing Maple 16, all results are obtained.

Example 5.1. We will examine our method by applying to the following IVPCpF discussed in ([25]) as given below:

$$\begin{aligned}&{}^cD_p^{\frac{1}{3}} z(t) = t^{-\frac{3}{2}} z^2(t), \quad 0 < t \leq 1, \quad z(0) = 10^{-4} \\ &z_{\text{exact}}(t) = \Gamma_p(13/6) t^{\frac{7}{6}} / \Gamma_p(11/6) + 10^{-4}.\end{aligned}$$

By BCM and the collocation points p^{N-i} $\left(N = 10, i = 0, 1, \ldots, N, \beta = \frac{1}{3}\right)$, we obtain Tables 1 and 2. The solution graphs z can be reveal in Figure 1 (with $n = 3$).

Example 5.2. The following IVPCpF is examined (see [26])

$$\begin{aligned}&{}^cD_p^\beta z(t) = t, \quad 0 < t \leq 1, \quad z(0) = z_0 = 0 \\ &z_{\text{exact}}(t) = \frac{\Gamma_p(2)}{\Gamma_p\left(\frac{5}{2}\right)} t^{\frac{3}{2}}.\end{aligned}$$

TABLE 1. Absolute error of $z(t)$ for this method and error in [25] for Example 5.1 ($N = 10, \beta = \frac{1}{3}, p = \frac{1}{2}$).

t	Absolute error in BCM	Absolute error in [25]
$\frac{1}{1024}$	$0.307597912584070 \times 10^{-3}$	–
$\frac{1}{512}$	$4.58331289587832 \times 10^{-14}$	7.48751×10^{-4}
$\frac{1}{256}$	0.000190894208127339	1.69380×10^{-3}
$\frac{1}{128}$	$5.70704941638134 \times 10^{-13}$	3.80772×10^{-3}
$\frac{1}{64}$	$4.88584867008868 \times 10^{-13}$	8.55030×10^{-3}
$\frac{1}{32}$	0.000103365853560924	1.19196×10^{-2}
$\frac{1}{16}$	$1.32436422939364 \times 10^{-11}$	4.30935×10^{-2}
$\frac{1}{8}$	0.00398404087914093	9.67418×10^{-2}
$\frac{1}{4}$	$3.90912302528079 \times 10^{-12}$	2.17178×10^{-1}
$\frac{1}{2}$	0.0388958273590442	4.87548×10^{-1}
1	0.0159066965555552	1.09451

TABLE 2. Absolute error of $z(t)$ for this method and error in [25] and [12] for Example 5.1 ($N = 10, \beta = \frac{1}{3}, p = \frac{1}{8}$).

t	Absolute error in BCM	Absolute error in [25]	Absolute error in [12]
$\frac{1}{8^{10}}$	$2.91031474126313 \times 10^{-11}$	–	–
$\frac{1}{8^{9}}$	$0.77659237026844 \times 10^{-14}$	7.10653×10^{-4}	1.35645×10^{-4}
$\frac{1}{8^{8}}$	$4.40869352346390 \times 10^{-10}$	1.59919×10^{-3}	1.35642×10^{-4}
$\frac{1}{8^{7}}$	$4.77766630895614 \times 10^{-14}$	3.59173×10^{-3}	1.35607×10^{-4}
$\frac{1}{8^{6}}$	$3.87464026063499 \times 10^{-14}$	8.06390×10^{-3}	1.35208×10^{-4}
$\frac{1}{8^{5}}$	0.00000193727690609905	1.81032×10^{-2}	1.30952×10^{-4}
$\frac{1}{8^{4}}$	$1.87868519569215 \times 10^{-14}$	4.06404×10^{-2}	1.05923×10^{-4}
$\frac{1}{8^{3}}$	0.000212000148251494	9.12346×10^{-2}	3.61824×10^{-3}
$\frac{1}{8^{2}}$	$3.88074988810772 \times 10^{-13}$	2.04815×10^{-1}	2.24448×10^{-3}
$\frac{1}{8}$	0.0144815408959622	4.59794×10^{-1}	7.76074×10^{-3}
1	$4.27017088355797 \times 10^{-10}$	1.03220	6.37084×10^{-3}

By BCM and the collocation points p^{N-i}, we achieve Tables 3 and 4 and Figure 2 (with $n = 3$).

$$z(t) = \begin{cases} 0.01636697759t - 0.1558044172 \times 10^{-4} - 0.5024058425t^2 + 81.81560373t^3, & 0 \le t \le \frac{1}{2^7}, \\ -0.2262052280 \times 10^{-4} + 0.01296279721t + .7150983986t^2 - 3.486727697t^3, & \frac{1}{2^7} \le t \le \frac{1}{2^4}, \\ -0.9705441908 \times 10^{-3} + 0.04755666544t + 0.1902266620t^2 - 0.06211482836t^3, & \frac{1}{2^4} \le t \le 1. \end{cases}$$

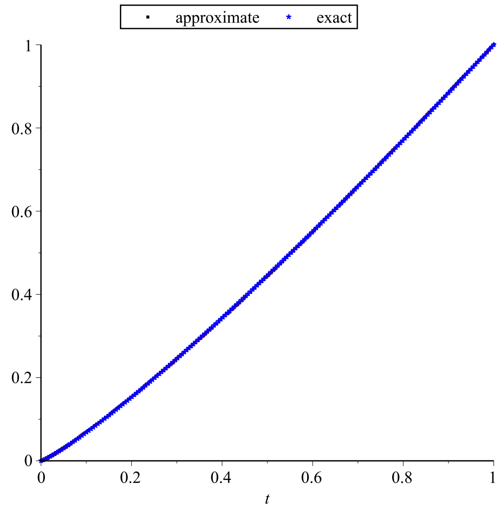

FIGURE 1. The graphs of approximated and exact solution $z(t)$ for Example 5.1 with $p = \frac{1}{2}$.

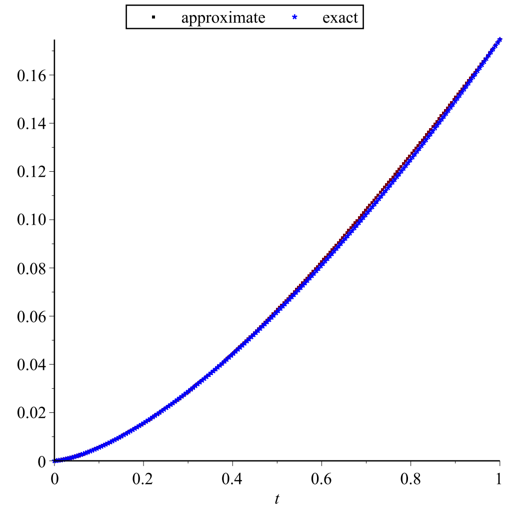

FIGURE 2. The graphs of approximated and exact solution $z(t)$ for Example 5.2 with $p = \frac{1}{2}, \beta = \frac{1}{3}$.

TABLE 3. Absolute error for $z(t)$ for Example 5.2 with $p = \frac{1}{2}, \beta = \frac{1}{3}$

t	Absolute error for this method
$\frac{1}{1024}$	0.0000305175781200000
$\frac{1}{512}$	$4.44582521838544 \times 10^{-15}$
$\frac{1}{256}$	0.000166726728261375
$\frac{1}{128}$	$1.10127076728694 \times 10^{-13}$
$\frac{1}{64}$	$1.24925243760732 \times 10^{-13}$
$\frac{1}{32}$	0.000533475738786126
$\frac{1}{16}$	$5.24244536670437 \times 10^{-12}$
$\frac{1}{8}$	0.000597492837855075
$\frac{1}{4}$	$3.08431058471115 \times 10^{-12}$
$\frac{1}{2}$	0.00477994278497773
1	$3.35516503469080 \times 10^{-10}$

Example 5.3. The following IVPCpF is examined (see [25])

$$
{}^{c}D_p^{\frac{1}{2}} z(t) = t + t^2, \quad 0 < t \leq 1, \quad z(0) = z_0 = 1,
$$

$$
z_{\text{exact}}(t) = 1 + \frac{\Gamma_p(2)}{\Gamma_p\left(\frac{5}{2}\right)} t^{\frac{3}{2}} + \frac{\Gamma_p(3)}{\Gamma_p\left(\frac{7}{2}\right)} t^{\frac{5}{2}}.
$$

By BCM and the collocation points p^{N-i}, we achieve Table 5 (with $n = 3$).

6. CONCLUSIONS

The approximate solution of IVPCpF via BCM is introduced in this paper. The lemma was proved for truncated error. This approach is computationally appealing, and decrease memory of the compuer, at the same time proceeds the precision. Finally, several examples are provided to verify our theoretical analysis by utilizing the technique

Table 4. Absolute error of $z(t)$ for this method and error in for Example 5.2 ($N = 20, p = \frac{3}{5}, \beta = \frac{1}{2}$).

t	Absolute error in BCM	Absolute error in [26]
$(3/5)^{19}$	$1.14677656602779 \times 10^{-16}$	4.44849×10^{-8}
$(3/5)^{18}$	$2.04019525271454 \times 10^{-8}$	–
$(3/5)^{17}$	$.36421444095043 \times 10^{-16}$	2.29471×10^{-8}
$(3/5)^{16}$	$9.75353885961235 \times 10^{-16}$	–
$(3/5)^{15}$	$6.80935073576306 \times 10^{-9}$	1.20204×10^{-7}
$(3/5)^{14}$	$4.83832736889063 \times 10^{-15}$	–
$(3/5)^{13}$	$4.54327696255453 \times 10^{-8}$	5.72027×10^{-8}
$(3/5)^{12}$	$2.20320829349898 \times 10^{-14}$	–
$(3/5)^{11}$	$2.10336874006543 \times 10^{-7}$	3.23198×10^{-8}
$(3/5)^{10}$	$1.34127287726958 \times 10^{-13}$	–
$(3/5)^{9}$	0.00556820984512183	1.90138×10^{-8}
$(3/5)^{8}$	$1.95646801439808 \times 10^{-12}$	–
$(3/5)^{7}$	$1.47896693264600 \times 10^{-12}$	1.13302×10^{-8}
$(3/5)^{6}$	$5.95253490595637 \times 10^{-12}$	–
$(3/5)^{5}$	$.205852355915861$	6.78151×10^{-9}
$(3/5)^{4}$	$8.38030166094850 \times 10^{-12}$	–
$(3/5)^{3}$	$2.01078628574158 \times 10^{-11}$	4.06534×10^{-9}
$(3/5)^{2}$	0.00283916294314055	–
$(3/5)^{1}$	$5.72063351550467 \times 10^{-11}$	1.22486×10^{-7}

Table 5. Absolute error for $z(t)$ for Example 5.3 with $p = \frac{1}{2}$.

t	Absolute error in BCM
$\frac{1}{1024}$	$0.305472210352359 \times 10^{-4}$
$\frac{1}{512}$	$6.07360770651151 \times 10^{-10}$
$\frac{1}{256}$	$0.167348306401367 \times 10^{-4}$
$\frac{1}{128}$	$3.96908105022170 \times 10^{-10}$
$\frac{1}{64}$	$4.81004684528357 \times 10^{-10}$
$\frac{1}{32}$	$0.514451038466408 \times 10^{-4}$
$\frac{1}{16}$	$1.58913326941956 \times 10^{-10}$
$\frac{1}{8}$	$0.656960275817142 \times 10^{-3}$
$\frac{1}{4}$	$2.44828601836389 \times 10^{-11}$
$\frac{1}{2}$	$8.98517371616947 \times 10^{-11}$
1	$1.00654995449645 \times 10^{-9}$

finding the approximate solutions of the IVPCpF and compared with the previous existing methods in the literature. Comparing with others, the outcomes state that the technique is more accurate than others.

Statements and Declarations

Data Availability. Not applicable.

Ethical approval. Not applicable.

Conflict of interest. The authors have no competing interests.

Author Contributions. The authors contributed equally to the present work.

References

[1] T. Abdeljawad and D. Baleanu, *Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function*, Commun. Nonlinear Sci. Numer. Simul., *16*(12) (2011), 4682–4688.
[2] R. P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Camb. Phil. Soc., *66* (1969), 365–370.
[3] S. Ahmed, S. Jahan, K. J. Ansari, K. Shah, and T. Abdeljawad, *Wavelets collocation method for singularly perturbed differential–difference equations arising in control system*, Results Appl. Math., *21* (2024), 100415.
[4] H. Aktucglu and M. A. Özarslan, *On the solvability of Caputo q-fractional boundary value problem involving p-Laplacian operator*, Abstr. Appl. Anal., *2013*(1) (2013), 658617.
[5] H. Alrabaiah, I. Ahmad, R. Amin, and K. Shah, *A numerical method for fractional variable order pantograph differential equations based on Haar wavelet*, Eng. Comput., *38*(3) (2022), 2655–2668.
[6] W. A. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edinb. Math. Soc., *15*(2) (1966), 135–140.
[7] R. Amin, K. Shah, M. Asif, and I. Khan, *A computational algorithm for the numerical solution of fractional order delay differential equations*, Appl. Math. Comput., *402* (2021), 125863.
[8] R. Amin, H. Ahmad, K. Shah, M. B. Hafeez, and W. Sumelja, *Theoretical and computational analysis of nonlinear fractional integro-differential equations via collocation method*, Chaos Solitons Fractals, *151* (2021), 111252.
[9] M. H. Annaby and Z. S. Mansour, *q-Fractional Calculus and Equations*, Springer, Heidelberg, New York, 2012.
[10] G. E. Farin, *Curve and Surfaces for Computer Aided Geometric Design*, Academic Press, New York, 2014.
[11] R. A. C. Ferreira, *Nontrivial solutions for fractional q-difference boundary value problems*, Electron. J. Qual. Theory Differ. Equ., *70*(1) (2010), 1–10.
[12] F. A. Ghassabzadeh, E. Tohidi, H. Singh, and S. Shateyi, *RBF collocation approach to calculate numerically the solution of the nonlinear system of qFDEs*, J. King Saud Univ. Sci., *33*(2) (2020), 101288.
[13] F. Ghomanjani and M. H. Farahi, *The Bézier control points method for solving delay differential equation*, Intell. Control Autom., *3*(2) (2012), 188–196.
[14] F. Ghomanjani and M. H. Farahi, *Optimal control of switched systems based on Bézier control points*, Int. J. Intell. Syst. Appl., *4*(7) (2012), 16–22.
[15] F. Ghomanjani, M. H. Farahi, and A. V. Kamyad, *Numerical solution of some linear optimal control systems with pantograph delays*, IMA J. Math. Control Inf., *32*(2) (2015), 225–243.
[16] W. Hahn, *Beiträge zur theorie der Heineschen Reihen*, Math. Nachr., *2* (1949), 340–379.
[17] M. Izadi, S. Yüzbasi, and K. J. Ansari, *Application of Vieta-Lucas Series to Solve a Class of Multi-Pantograph Delay Differential Equations with Singularity*, Symmetry, *13*(12) (2021), 2370.
[18] F. H. Jackson, *On q-functions and a certain difference operator*, Trans. R. Soc. Edinb., *46* (1908), 64–72.
[19] F. Jarad, T. Abdeljawad, and D. Baleanu, *Stability of q-fractional non-autonomous systems*, Nonlinear Anal. RWA, *14*(1) (2013), 780–784.
[20] I. Khan, K. J. Ansari, R. Amin, Sundas, and H. Farheen, *Application of linear Legendre multi-wavelets collocation method for solution of fourth order boundary value problems*, Eng. Comput., (2024), 1–11.
[21] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1986.
[22] S. Salahshour, A. Ahmadian, and C. S. Chan, *Successive approximation method for Caputo q-fractional IVPs*, Commun. Nonlinear Sci. Numer. Simul., *24*(1-3) (2015), 153–158.
[23] H. M. Srivastava, *Operators of Basic (or q-) Calculus and Fractional q-Calculus and Their Applications in Geometric Function Theory of Complex Analysis*, Iran. J. Sci. Technol. Trans. A Sci., *44*(1) (2020), 327–344.
[24] G. C. Wu and D. Baleanu, *New applications of the variational iteration method-from differential equations to q-fractional difference equations*, Adv. Differ. Equ., *2013*(1) (2013), 21–37.
[25] Y. Tang and T. Zhang, *A remark on the q-fractional order differential equations*, Appl. Math. Comput., *350* (2019), 198–208.
[26] T. Zhang and Y. Tang, *A difference method for solving the q-fractional differential equations*, Appl. Math. Lett., *98* (2019), 292–299.
[27] J. Zheng, T. W. Sederberg, and R. W. Johnson, *Least squares methods for solving differential equations using Bézier control points*, Appl. Numer. Math., *48*(2) (2004), 237–252.