Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-21
DOI:10.22034/cmde.2025.64157.2890

Multiple positive symmetric solutions for the fourth-order iterative differential equations with IHPH operator

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Abstract

The purpose of this paper is to investigate the existence of positive symmetric solutions for fourth order iterative systems with two point integral boundary conditions involving increasing homeomorphism and positive homomorphism operator. Firstly, we establish the existence of at least two positive symmetric solutions by using Avery-Henderson fixed point theorem. Later, we derive conditions for the existence of at least three positive symmetric solutions by using Avery-Peterson fixed point theorem.

Keywords. Cone, Fixed point theorems, Green's function, Iterative system, Positive symmetric solutions.
2010 Mathematics Subject Classification. 34B18, 34B27.

1. Introduction

Fourth-order ordinary differential equations have gained significant attention due to their frequent occurrence in numerous academic and engineering disciplines, including chemistry, physics, biotechnology, and control theory for dynamical systems. Researchers have focused extensively on studying these equations, particularly with regard to the existence and multiplicity of positive solutions under various boundary conditions. To address these problems, a variety of mathematical approaches have been employed, such as fixed point theory[2, 3, 7, 8, 10, 15], the monotone iterative method[16, 24], upper and lower solution techniques[1, 30], and shooting methods[9, 28], resulting in valuable insights and advancements in this area. To explore turbulent flow through porous media, researchers frequently rely on mathematical models to capture the complexity of these systems. These models are essential in diverse applications, including petroleum engineering, groundwater hydrology, chemical engineering, environmental science and engineering, heat transfer applications, biomechanical and biomedical engineering, for example see [12–14, 18, 21, 22]. One important mathematical framework for studying such phenomena is the p-Laplacian equation, introduced by Leibenson [17]. The equation is given by:

$$(\varphi_p(y'(s)))' = f(s, y(s), y'(s)),$$

where the operator $\varphi_p(y) = |y|^{p-2}y$ represents the p-Laplacian for some $p > 1$. This nonlinear operator generalizes the standard Laplace operator and plays a crucial role in modeling systems that involve nonlinear diffusion, such as turbulent flow in porous media. The inverse function of the p-Laplacian operator is denoted by $\varphi_q(y)$, where $\varphi_q(y) = |y|^{q-2}y$ and the exponents p and q are related by $\frac{1}{p} + \frac{1}{q} = 1$. The p-Laplacian operator appears in many applied fields, including turbulent filtration in porous media, blood flow dynamics, rheology, viscoplastic material modeling, and material science. The study of differential equations involving this operator is therefore of great interest in both theory and practical applications.

The generalization and improvement of the p-Laplacian operator is an operator φ and is called increasing homeomorphism and positive homomorphism operator (IHPHO). Some of the works on IHPHO are mentioned here.

Received: 23 October 2024 ; Accepted: 12 August 2025.
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In [19] Liu and Zhang studied the following boundary value problem with increasing homeomorphism and positive homomorphism

$$(\varphi(y'(s)))' + a(s)f(y(s)) = 0,\ 0 < s < 1,$$
$$y(0) - \beta y(0) = y(1) + \delta y'(1) = 0,$$

and established the existence and uniqueness of positive solutions by using fixed-point index theorem in cones.

In [20] Miao, Zhou and Song studied the following boundary value problem with increasing homeomorphism and positive homomorphism

$$(\varphi(y'(s)))' + a(s)f(y(s)) = 0,\ 0 < s < 1,$$
$$y(0) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i),\ y'(1) = 0,$$

and established the existence and uniqueness of positive solutions by using fixed-point theorem on partially ordered sets.

In [31] Zhang, Feng and Ge considered the p-Laplacian fourth order boundary value problem

$$(\varphi_p(y''(s)))'' = w(s)f(s, y(s)) = 0,\ 0 < s < 1,$$
$$y(0) = y(1) = \int_0^1 v(q)y(q)dq,$$
$$\varphi_p(y''(0)) = \varphi_p(y''(1)) = \int_0^1 w(q)\varphi_p(y''(q))dq,$$

they obtained the existence and multiplicity of symmetric positive solutions by utilizing fixed point theory.

Recently, Prasad and Bhushanam[25] considered the iterative boundary value problem

$$y_n''(s) + \Psi(s)f_n(y_{n+1}(s)) = 0,\ 1 \le n \le \ell,\ s \in [0, \kappa],$$
$$y_{\ell+1}(s) = y_1(s),\ s \in [0, \kappa],$$
$$y_n(0) = \int_0^\kappa w(q)y_n(q)dq, 1 \le n \le \ell,\ y_n(\kappa) = \int_0^\kappa w(q)y_n(q)dq, 1 \le n \le \ell,$$

and established the denumerably many positive symmetric solutions by using the Krasnoselskii's fixed point theorem. Following that, the researchers have explored the study of symmetric positive solutions, see [6, 11, 23, 26, 27, 29].

Inspired by the works mentioned above, we investigate the existence of multiple positive symmetric solutions for the fourth order iterative system involving IHPHO and satisfying integral boundary conditions.

$$\begin{cases} (\varphi(g(s)y_n''(s)))'' = h(s)f_n(s, y_{n+1}(s)),\ 1 \le n \le \ell,\ 0 \le s \le \kappa, \\ y_{\ell+1}(s) = y_1(s),\ 0 \le s \le \kappa, \end{cases} \tag{1.1}$$

satisfying boundary conditions

$$\begin{cases} \xi_1 y_n(0) - \xi_2 y_n'(0) = \displaystyle\int_0^\kappa v(q)y_n(q)dq, \\ \xi_1 y_n(\kappa) + \xi_2 y_n'(\kappa) = \displaystyle\int_0^\kappa v(q)y_n(q)dq,\ 1 \le n \le \ell, \end{cases} \tag{1.2}$$

$$\begin{cases} \varphi(g(0)y_n''(0)) = \displaystyle\int_0^\kappa w(q)\varphi(g(q)y_n''(q))dq, \\ \varphi(g(\kappa)y_n''(\kappa)) = \displaystyle\int_0^\kappa w(q)\varphi(g(q)y_n''(q))dq,\ 1 \le n \le \ell, \end{cases} \tag{1.3}$$

where $\ell \in \mathbb{N}$, ξ_1, ξ_2 are positive constants and $\varphi : \mathbb{R} \to \mathbb{R}$ is an IHPHO with $\varphi(0) = 0$. A projection $\varphi : \mathbb{R} \to \mathbb{R}$ is called a IHPHO, if the following conditions are satisfied:

(a) If $x \leq y$, then $\varphi(x) \leq \varphi(y)$ where x, y are real-valued functions.
(b) $\varphi(x)$ is a continuous bijection and its inverse mapping is also continuous.
(c) $\varphi(xy) = \varphi(x)\varphi(y)$, where x, y are real-valued functions.

The following conditions are presumed to be valid in the entire paper:

(I1) $f_n : [0, \kappa] \times [0, \infty) \to [0, \infty)$ is continuous, $f_n(\kappa - s, y) = f_n(s, y)$, $1 \leq n \leq \ell$ for all $(s, y) \in [0, \kappa] \times [0, \infty)$.
(I2) $g(s), h(s) \in \mathrm{L}^1[0, \kappa]$ are positive, symmetric on $[0, \kappa]$ (i.e., $g(\kappa - s) = g(s)$ for $s \in [0, \kappa]$).
(I3) $v(s), w(s) \in \mathrm{L}^1[0, \kappa]$ are non-negative, symmetric on $[0, \kappa]$, and $\mu_1, \mu_2 \in (0, 1)$, $\mu_1 = \int_0^\kappa v(q)dq$, $\mu_2 = \int_0^\kappa w(q)dq$, $\xi_1 - \mu_1 > 0$.

The organization of the remaining part of the paper is as follows. In Section 2, we construct Green's function and estimate the bounds for Green's function. In section 3, we establish the existence of at least two positive symmetric solutions by using Avery-Henderson fixed point theorem. Using Avery-Peterson fixed point theorem, we establish the existence of at least three positive symmetric solutions. In Section 4, we provide examples to check the validity of results.

2. Green's function and its characteristics

Here, we derive the solution of (1.1)-(1.3) as a solution of integral equation that includes Green's function. After that, we derive a few characteristics of the Green's function which are useful in establishing our main results.

Lemma 2.1. *Assume that* (I2) − (I3) *hold. Then for any* $u_1(s) \in \mathrm{C}([0, \kappa], \mathbb{R})$, *the BVP*

$$\varphi(g(s)y_1''(s)) = u_1(s), \; 0 \leq s \leq \kappa, \tag{2.1}$$

$$\xi_1 y_1(0) - \xi_2 y_1'(0) = \int_0^\kappa v(q)y_1(q)dq, \; \xi_1 y_1(\kappa) + \xi_2 y_1'(\kappa) = \int_0^\kappa v(q)y_1(q)dq, \tag{2.2}$$

has one and only one solution,

$$y_1(s) = -\int_0^\kappa \mathtt{H}_1(s, t)\frac{1}{g(t)}\varphi^{-1}(u_1(t))dt,$$

where $\mathtt{H}_1(s, t)$ *is the Green's function and is given by*

$$\mathtt{H}_1(s, t) = \mathrm{G}_1(s, t) + \frac{1}{\xi_1 - \mu_1}\int_0^\kappa \mathrm{G}_1(q, t)v(q)dq, \tag{2.3}$$

in which

$$\mathrm{G}_1(s, t) = \frac{1}{\xi_1(\xi_1\kappa + 2\xi_2)}\begin{cases}(\xi_1 t + \xi_2)(\xi_1(\kappa - s) + \xi_2), & 0 \leq t \leq s,\\ (\xi_1 s + \xi_2)(\xi_1(\kappa - t) + \xi_2), & s \leq t \leq \kappa.\end{cases} \tag{2.4}$$

Proof. An equivalent integral equation for (2.1) is

$$y_1(s) = \int_0^s (s - t)\frac{1}{g(t)}\varphi^{-1}(u_1(t))dt + \mathfrak{a}_1 s + \mathfrak{a}_2.$$

By using boundary conditions (2.2), we get

$$\mathfrak{a}_1 = \frac{-1}{(\xi_1\kappa + 2\xi_2)}\int_0^\kappa (\xi_1(\kappa - t) + \xi_2)\frac{1}{g(t)}\varphi^{-1}(u_1(t))dt,$$
$$\mathfrak{a}_2 = \frac{-\xi_2}{\xi_1(\xi_1\kappa + 2\xi_2)}\int_0^\kappa (\xi_1(\kappa - t) + \xi_2)\frac{1}{g(t)}\varphi^{-1}(u_1(t))dt + \frac{1}{\xi_1}\int_0^\kappa v(q)y_1(q)dq.$$

So, we have

$$y_1(s) = \int_0^s (s - t)\frac{1}{g(t)}\varphi^{-1}(u_1(t))dt + \frac{-s}{(\xi_1\kappa + 2\xi_2)}\int_0^\kappa (\xi_1(\kappa - t) + \xi_2)\frac{1}{g(t)}\varphi^{-1}(u_1(t))dt$$
$$+ \frac{-\xi_2}{\xi_1(\xi_1\kappa + 2\xi_2)}\int_0^\kappa (\xi_1(\kappa - t) + \xi_2)\frac{1}{g(t)}\varphi^{-1}(u_1(t))dt + \frac{1}{\xi_1}\int_0^\kappa v(q)y_1(q)dq$$

$$= \frac{-1}{\xi_1(\xi_1\kappa + 2\xi_2)} \Big[\int_0^s (\xi_1 t + \xi_2)(\xi_1(\kappa - s) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt$$
$$+ \int_s^\kappa (\xi_1 s + \xi_2)(\xi_1(\kappa - t) + \xi_2) \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt \Big] + \frac{1}{\xi_1} \int_0^\kappa v(q) y_1(q)\, dq$$
$$= -\int_0^\kappa \mathrm{G}_1(s,t) \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt + \frac{1}{\xi_1} \int_0^\kappa v(q) y_1(q)\, dq.$$

After certain computations, we obtain

$$\int_0^\kappa v(q) y_1(q)\, dq = \frac{-\xi_1}{\xi_1 - \mu_1} \int_0^\kappa \int_0^\kappa \mathrm{G}_1(q,t) v(q) \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt dq.$$

Therefore,

$$y_1(s) = -\int_0^\kappa \mathrm{G}_1(s,t) \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt + \frac{-1}{\xi_1 - \mu_1} \int_0^\kappa \int_0^\kappa \mathrm{G}_1(q,t) v(q) \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt dq$$
$$= -\int_0^\kappa \Big[\mathrm{G}_1(s,t) + \frac{1}{\xi_1 - \mu_1} \int_0^\kappa \mathrm{G}_1(q,t) v(q)\, dq \Big] \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt$$
$$= -\int_0^\kappa \mathtt{H}_1(s,t) \frac{1}{g(t)} \varphi^{-1}(u_1(t))\, dt.$$

□

In establishing the unique solution of the problem the following Lemma is required.

Lemma 2.2. *Assume that* (I2) – (I3) *hold. Then for any* $u_2(s) \in \mathrm{C}([0,\kappa], \mathbb{R})$*, the BVP*

$$x_1''(s) = u_2(s), \ 0 \le s \le \kappa, \tag{2.5}$$

$$x_1(0) = \int_0^\kappa w(q) x_1(q)\, dq, \ x_1(\kappa) = \int_0^\kappa w(q) x_1(q)\, dq, \tag{2.6}$$

has one and only one solution

$$x_1(s) = -\int_0^\kappa \mathtt{H}_2(s,t) u_2(t)\, dt,$$

where $\mathtt{H}_2(s,t)$ *is the Green's function and is given by*

$$\mathtt{H}_2(s,t) = \mathrm{G}_2(s,t) + \frac{1}{1-\mu_1} \int_0^\kappa \mathrm{G}_2(q,t) w(q)\, dq, \tag{2.7}$$

in which

$$\mathrm{G}_2(s,t) = \frac{1}{\kappa} \begin{cases} t(\kappa - s), & 0 \le t \le s, \\ s(\kappa - t), & s \le t \le \kappa. \end{cases} \tag{2.8}$$

Proof. An equivalent integral equation for (2.5) is

$$x_1(s) = \int_0^s (s-t) u_2(t)\, dt + \mathfrak{a}_3 s + \mathfrak{a}_4.$$

By using boundary conditions (2.6), we get

$$\mathfrak{a}_3 = \frac{-1}{\kappa} \int_0^\kappa (\kappa - t) u_2(t)\, dt, \ \text{and} \ \mathfrak{a}_4 = \int_0^\kappa w(q) x_1(q)\, dq.$$

So, we have

$$x_1(s) = \int_0^s (s-t)u_2(t)dt + \frac{-1}{\kappa}\int_0^\kappa (s)(\kappa - t)u_2(t)dt + \int_0^\kappa w(q)x_1(q)dq$$
$$= -\int_0^\kappa \mathrm{G}_2(s,t)u_2(t)dt + \int_0^\kappa w(q)x_1(q)dq.$$

After certain computations, we obtain

$$\int_0^\kappa w(q)x_1(q)dq = \frac{-1}{1-\mu_2}\int_0^\kappa\int_0^\kappa \mathrm{G}_2(q,t)w(q)u_2(t)dtdq.$$

Therefore,

$$x_1(s) = -\int_0^\kappa \mathrm{G}_2(s,t)u_2(t)dt + \frac{-1}{1-\mu_2}\int_0^\kappa\int_0^\kappa \mathrm{G}_2(q,t)w(q)u_2(t)dtdq$$
$$= -\int_0^\kappa \left[\mathrm{G}_2(s,t) + \frac{1}{1-\mu_2}\int_0^\kappa \mathrm{G}_2(q,t)w(q)dq\right]u_2(t)dt$$
$$= -\int_0^\kappa \mathtt{H}_2(s,t)u_2(t)dt.$$

□

Lemma 2.3. *Assume that* (I2) − (I3) *hold. Then for any* $u_3(s) \in \mathrm{C}([0,\kappa],\mathbb{R})$, *the BVP*

$$(\varphi(g(s)y_1''(s)))'' = u_3(s),\ 0 \le s \le \kappa, \tag{2.9}$$

satisfying boundary conditions

$$\begin{cases}\xi_1 y_1(0) - \xi_2 y_1'(0) = \int_0^\kappa v(q)y_1(q)dq,\ \xi_1 y_1(\kappa) + \xi_2 y_1'(\kappa) = \int_0^\kappa v(q)y_1(q)dq,\\ \varphi(g(0)y_1''(0)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq,\ \varphi(g(\kappa)y_1''(\kappa)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq,\end{cases} \tag{2.10}$$

has a unique solution

$$y_1(s) = \int_0^\kappa \mathtt{H}_1(s,t)\frac{1}{g(t)}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t,q)u_3(q)dq\right]dt,$$

where $\mathtt{H}_1(s,t)$, $\mathtt{H}_2(s,t)$ *are defined in Lemma 2.1, Lemma 2.2 respectively.*

Proof. Let $u_2(s) = \varphi(g(s)y_1''(s))$ for $0 \le s \le \kappa$. Then the BVP

$$(\varphi(g(s)y_1''(s)))'' = u_3(s),\ 0 \le s \le \kappa,$$
$$\varphi(g(0)y_1''(0)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq,\ \varphi(g(\kappa)y_1''(\kappa)) = \int_0^\kappa w(q)\varphi(g(q)y_1''(q))dq$$

is equivalent to the problem

$$u_2''(s) = u_3(s),\ 0 \le s \le \kappa, \tag{2.11}$$
$$u_2(0) = \int_0^\kappa w(q)u_2(q)dq,\ u_2(\kappa) = \int_0^\kappa w(q)u_2(q)dq. \tag{2.12}$$

By Lemma 2.2, the BVP (2.11)-(2.12) has unique solution $u_2(s) = -\int_0^\kappa \mathtt{H}_2(s,t)u_3(t)dt$. That is

$$\varphi(g(s)y_1''(s)) = -\int_0^\kappa \mathtt{H}_2(s,t)u_3(t)dt. \tag{2.13}$$

Again by Lemma 2.1, the differential Equation (2.13) with boundary conditions

$$\xi_1 y_1(0) - \xi_2 y_1'(0) = \int_0^\kappa v(q)y_1(q)dq,\ \xi_1 y_1(\kappa) + \xi_2 y_1'(\kappa) = \int_0^\kappa v(q)y_1(q)dq,$$

has a unique solution

$$y_1(s)=\int_0^{\kappa}\mathtt{H}_1(s,t)\frac{1}{g(t)}\varphi^{-1}\Big[\int_0^{\kappa}\mathtt{H}_2(t,q)u_3(q)dq\Big]dt.$$

This completes the proof. □

Lemma 2.4. *Suppose* (I3) *holds. For* $\lambda_1\in(0,\frac{\kappa}{2})$, *let* $\sigma_1(\lambda_1)=\frac{\xi_1\lambda_1+\xi_2}{\xi_1\kappa+\xi_2}$, $\sigma_2(\lambda_1)=\frac{\lambda_1}{\kappa}$, $\beta_1=\frac{\xi_1}{\xi_1-\mu_1}$, $\beta_2=\frac{1}{1-\mu_2}$. *Then* $\mathrm{G}_{\mathtt{i}}(s,t)$, $\mathtt{H}_{\mathtt{i}}(s,t)$, $\mathtt{i}=1,2$ *have the following properties:*

(A1) $0\le \mathrm{G}_{\mathtt{i}}(s,t)\le \mathrm{G}_{\mathtt{i}}(t,t),\ \forall\ s,t\in[0,\kappa]$,
(A2) $0\le \mathtt{H}_{\mathtt{i}}(s,t)\le \beta_{\mathtt{i}}\mathrm{G}_{\mathtt{i}}(t,t),\ \forall\ s,t\in[0,\kappa]$,
(A3) $\mathrm{G}_{\mathtt{i}}(s,t)\ge\sigma_{\mathtt{i}}(\lambda_1)\mathrm{G}_{\mathtt{i}}(t,t),\ \forall\ s\in[\lambda_1,\kappa-\lambda_1]$ *and* $t\in[0,\kappa]$,
(A4) $\mathtt{H}_{\mathtt{i}}(s,t)\ge\sigma_{\mathtt{i}}(\lambda_1)\beta_{\mathtt{i}}\mathrm{G}_{\mathtt{i}}(t,t),\ \forall\ s\in[\lambda_1,\kappa-\lambda_1]$ *and* $t\in[0,\kappa]$,
(A5) $\mathrm{G}_{\mathtt{i}}(\kappa-s,\kappa-t)=\mathrm{G}_{\mathtt{i}}(s,t),\ \mathtt{H}_{\mathtt{i}}(\kappa-s,\kappa-t)=\mathtt{H}_{\mathtt{i}}(s,t),\ \forall\ s,t\in[0,\kappa]$.

Proof. From (2.3)-(2.4) and (2.7)-(2.8), it is clear that (A1) and (A2) hold.
For inequality (A3), let $s\in[\lambda_1,\kappa-\lambda_1]$ and $t\le s$, then

$$\frac{\mathrm{G}_1(s,t)}{\mathrm{G}_1(t,t)}=\frac{(\xi_1t+\xi_2)(\xi_1(\kappa-s)+\xi_2)}{(\xi_1t+\xi_2)(\xi_1(\kappa-t)+\xi_2)}\ge\sigma_1(\lambda_1),$$

and for $s\le t$,

$$\frac{\mathrm{G}_1(s,t)}{\mathrm{G}_1(t,t)}=\frac{(\xi_1s+\xi_2)(\xi_1(\kappa-t)+\xi_2)}{(\xi_1t+\xi_2)(\xi_1(\kappa-t)+\xi_2)}\ge\sigma_1(\lambda_1).$$

Similarly, $\mathrm{G}_2(s,t)\ge\sigma_2(\lambda_1)\mathrm{G}_2(t,t),\ \forall\ s\in[\lambda_1,\kappa-\lambda_1]$ and $t\in[0,\kappa]$. Hence, the inequality (A3). For the inequality (A4), consider

$$\begin{aligned}\mathtt{H}_1(s,t)=&\mathrm{G}_1(s,t)+\frac{1}{\xi_1-\mu_1}\int_0^{\kappa}\mathrm{G}_1(q,t)v(q)dq\\ \ge&\sigma_1(\lambda_1)\mathrm{G}_1(t,t)+\frac{1}{\xi_1-\mu_1}\int_0^{\kappa}\sigma_1(\lambda_1)\mathrm{G}_1(t,t)v(q)dq\\ \ge&\sigma_1(\lambda_1)\beta_1\mathrm{G}_1(t,t).\end{aligned}$$

Similarly, $\mathtt{H}_2(s,t)\ge\sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t,t)$. For inequality (A5), consider

$$\begin{aligned}\mathrm{G}_1(\kappa-s,\kappa-t)&=\frac{1}{\xi_1(\xi_1\kappa+2\xi_2)}\begin{cases}(\xi_1(\kappa-t)+\xi_2)(\xi_1(\kappa-(\kappa-s))+\xi_2),\ 0\le\kappa-t\le\kappa-s,\\(\xi_1(\kappa-s)+\xi_2)(\xi_1(\kappa-(\kappa-t))+\xi_2),\ \kappa-s\le\kappa-t\le\kappa,\end{cases}\\ &=\frac{1}{\xi_1(\xi_1\kappa+2\xi_2)}\begin{cases}(\xi_1s+\xi_2)(\xi_1(\kappa-t)+\xi_2),\ s\le t\le\kappa,\\(\xi_1t+\xi_2)(\xi_1(\kappa-s)+\xi_2),\ 0\le t\le s,\end{cases}\\ &=\mathrm{G}_1(s,t).\end{aligned}$$

Similarly, $\mathrm{G}_2(\kappa-s,\kappa-t)=\mathrm{G}_2(s,t)$.
Consider

$$\begin{aligned}\mathtt{H}_1(\kappa-s,\kappa-t)=&\mathrm{G}(\kappa-s,\kappa-t)+\frac{1}{\xi_1-\mu_1}\int_0^{\kappa}\mathrm{G}(q,\kappa-t)v(q)dq\\ =&\mathrm{G}(s,t)+\frac{1}{\xi_1-\mu_1}\int_{\kappa}^{0}\mathrm{G}(\kappa-q,\kappa-t)v(\kappa-q)d(\kappa-q)\\ =&\mathrm{G}(s,t)+\frac{1}{\xi_1-\mu_1}\int_0^{\kappa}\mathrm{G}(q,t)v(q)dq\\ =&\mathtt{H}_1(s,t).\end{aligned}$$

Similarly, $\mathtt{H}_2(\kappa-s,\kappa-t)=\mathtt{H}_2(s,t),\ \forall\ s,t\in[0,\kappa]$. □

Note that an ℓ-tuple $\big(y_1(s), y_2(s), \cdots, y_\ell(s)\big)$ is a solution of (1.1)-(1.3) if and only if

$$y_n(s) = \int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_n(t_2,y_{n+1}(t_2))dt_2\right]dt_1,\ \ n=1,2,\cdots,\ell,$$
$$y_{\ell+1}(s) = y_1(s),\ \ s\in[0,\kappa],\ \ 1\le n\le \ell.$$

Therefore,

$$y_1(s) = \int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\left(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\right.\right.\right.$$
$$\cdots f_{\ell-1}\left(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\right]\right.$$
$$\left.\left.\left.\left.dt_{2\ell-1}\right)\cdots dt_4\right]dt_3\right)dt_2\right]dt_1.$$

Now we state the following Avery-Henderson and Avery and Peterson fixed point theorems to establish the existence of at least two and at least three positive symmetric solutions respectively.

Theorem 2.5. *[4] Let* K *be a cone in a real Banach space* B*. Suppose* α_1 *and* γ_1 *are increasing, nonnegative continuous functionals on* K *and* θ_1 *is a nonnegative continuous functional on* K *with* $\theta_1(0)=0$ *such that, for some positive numbers* $\mathtt{c}_1$ *and* $\mathtt{k}, \gamma_1(y)\le\theta_1(y)\le\alpha_1(y)$ *and* $\|y\|\le \mathtt{k}\gamma_1(y)$*, for all* $y\in\overline{\mathrm{P}(\gamma_1,\mathtt{c}_1)}$*. Suppose that there exist positive numbers* $\mathtt{a}_1$ *and* $\mathtt{b}_1$ *with* $\mathtt{a}_1<\mathtt{b}_1<\mathtt{c}_1$ *such that* $\theta_1(\lambda y)\le\lambda\theta_1(y)$*, for all* $0\le\lambda\le1$ *and* $y\in\partial\mathrm{K}(\theta_1,\mathtt{b}_1)$*. Further, let* $\mathrm{T}: y\in\overline{\mathrm{P}(\gamma_1,\mathtt{c}_1)}\to\mathrm{K}$ *be a completely continuous operator such that*

(B1) $\gamma_1(\mathrm{T}y)>\mathtt{c}_1$*, for all* $y\in\partial\mathrm{K}(\gamma_1,\mathtt{c}_1)$*,*
(B2) $\theta_1(\mathrm{T}y)<\mathtt{b}_1$*, for all* $y\in\partial\mathrm{K}(\theta_1,\mathtt{b}_1)$*,*
(B3) $\mathrm{K}(\alpha_1,\mathtt{a}_1)\ne\emptyset$ *and* $\alpha_1(\mathrm{T}y)>\mathtt{a}_1$*, for all* $y\in\partial\mathrm{K}(\alpha_1,\mathtt{c}_1)$*.*

Then, T *has atleast two fixed points* $y^*, y^{**}\in\overline{\mathrm{P}(\gamma_1,\mathtt{c}_1)}$ *such that* $\mathtt{a}_1<\alpha_1(y^*)$ *with* $\theta_1(y^*)<\mathtt{b}_1$ *and* $\mathtt{b}_1<\theta_1(y^{**})$ *with* $\gamma_1(y^{**})<\mathtt{c}_1$*.*

Theorem 2.6. *[5] Let* K *be a cone in a real Banach space* B*. Let* γ_2 *and* θ_2 *be a nonnegative continuous convex functionals on* K*,* α_2 *be a nonnegative continuous concave functional on* K*, and* ψ_2 *be a nonnegative continuous functional on* K *satisfying* $\psi_2(\lambda u)\le\lambda\psi_2(u)$ *for* $0\le\lambda\le1$*, such that for some positive numbers* $\mathcal{M}$ *and* $\mathtt{d}_2$*,*

$$\alpha_2(u)\le\psi_2(u)\ \text{and}\ \|u\|\le\mathcal{M}\gamma_2(u),$$

for all $u\in\overline{\mathrm{K}(\gamma_2,\mathtt{d}_2)}$*. Suppose* $\mathrm{T}:\overline{\mathrm{K}(\gamma_2,\mathtt{d}_2)}\to\overline{\mathrm{K}(\gamma_2,\mathtt{d}_2)}$ *is completely continuous and there exist positive numbers* $\mathtt{a}_2,\mathtt{b}_2$ *and* $\mathtt{c}_2$ *with* $\mathtt{a}_2<\mathtt{b}_2$ *such that*

(B4) $\{u\in\mathrm{K}(\gamma_2,\theta_2,\alpha_2,\mathtt{b}_2,\mathtt{c}_2,\mathtt{d}_2):\alpha_2(u)>\mathtt{b}_2\}\ne\emptyset$ *and* $\alpha_2(\mathrm{T}u)>\mathtt{b}_2$ *for* $u\in\mathrm{K}(\gamma_2,\theta_2,\alpha_2,\mathtt{b}_2,\mathtt{c}_2,\mathtt{d}_2)$*;*

(B5) $\alpha_2(\mathrm{T}u)>\mathtt{b}_2$ *for* $u\in\mathrm{K}(\gamma_2,\alpha_2,\mathtt{b}_2,\mathtt{d}_2)$ *with* $\theta_2(\mathrm{T}u)>\mathtt{c}_2$*;*

(B6) $0\notin R(\gamma_2,\psi_2,\mathtt{a}_2,\mathtt{d}_2)$ *and* $\psi_2(\mathrm{T}u)<\mathtt{a}_2$ *for* $u\in R(\gamma_2,\psi_2,\mathtt{a}_2,\mathtt{d}_2)$ *with* $\psi_2(u)=\mathtt{a}_2$*.*

Then T *has at least three fixed points* $u_1,u_2,u_3\in\overline{\mathrm{K}(\gamma_2,\mathtt{d}_2)}$ *such that*

$$\gamma_2(u_i)\le\mathtt{d}_2,\ \text{for}\ i=1,2,3;$$
$$\mathtt{b}_2<\alpha_2(u_1);$$
$$\mathtt{a}_2<\psi_2(u_2)\ \text{with}\ \alpha_2(u_2)<\mathtt{b}_2;$$

and

$$\psi_2(u_3)<\mathtt{a}_2.$$

3. Existence of positive symmetric solutions

Let $\mathrm{B} = \{y : y \in \mathrm{C}^{(4)}([0,\kappa],\mathbb{R})\}$ be a Banach space with norm $\|y\| = \max_{s\in[0,\kappa]} |y(s)|$. For $\lambda_1 \in (0, \frac{\kappa}{2})$, we define the cone $\mathrm{K} \subset \mathrm{B}$ as

$$\mathrm{K} = \big\{y \in \mathrm{B} : y(s) \geq 0, y(s) \text{ is concave, symmetric on } [0,\kappa] \text{ and } \min_{s\in[\lambda_1,\kappa-\lambda_1]} y(s) \geq \sigma_1(\lambda_1)\|y\|\big\}.$$

Define operator $\mathrm{T} : \mathrm{K} \to \mathrm{B}$ by

$$\begin{aligned}\mathrm{T}y_1(s) = &\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Big(t_2, \int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\\ &\cdots f_{\ell-1}\Big(t_{2\ell-2}, \int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]\\ &dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1.\end{aligned}$$

Let,

$$\begin{aligned}m &= \max\Big\{\beta_1\int_0^\kappa \mathrm{G}_1(t_j,t_j)\frac{1}{g(t_j)}\varphi^{-1}\Big[\beta_2\int_0^\kappa \mathrm{G}_2(t_{j+1},t_{j+1})h(t_{j+1})dt_{j+1}\Big]dt_j,\ j=1,2,\cdots 2\ell-1\Big\},\\ M &= \min\Big\{\sigma_1(\lambda_1)\beta_1\int_\lambda^{\kappa-\lambda} \mathrm{G}_1(t_j,t_j)\frac{1}{g(t_j)}\varphi^{-1}\Big[\sigma_2(\lambda_1)\beta_2\int_\lambda^{\kappa-\lambda} \mathrm{G}_2(t_{j+1},t_{j+1})h(t_{j+1})dt_{j+1}\Big]dt_j\Big\},\\ &\quad j=1,2,\cdots 2\ell-1.\end{aligned}$$

Lemma 3.1. *For each $\lambda_1 \in (0,\frac{\kappa}{2})$, $\mathrm{T(K)} \subset \mathrm{K}$ and $\mathrm{T} : \mathrm{K} \to \mathrm{K}$ is completely continuous.*

Proof. Since $\mathtt{H}_1(s,t) \geq 0$, $\mathtt{H}_2(s,t) \geq 0$, $\forall\ s,t \in [0,\kappa]$, $(\mathrm{T}y_1)(s) \geq 0$. Let $y_1 \in \mathrm{K}$, then consider

$$\begin{aligned}(\mathrm{T}y_1)(\kappa - s) = &\int_0^\kappa \mathtt{H}_1(\kappa-s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Big(t_2, \int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\ &\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Big(t_{2\ell-2}, \int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\ &\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1\\ = &\int_\kappa^0 \mathtt{H}_1(\kappa-s,\kappa-t_1)\frac{1}{g(\kappa-t_1)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(\kappa-t_1,t_2)h(t_2)f_1\Big(t_2, \int_0^\kappa \mathtt{H}_1(t_2,t_3)\cdots\\ &\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]d(\kappa-t_1)\\ = &\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Big[\int_\kappa^0 \mathtt{H}_2(\kappa-t_1,\kappa-t_2)h(\kappa-t_2)f_1\Big(\kappa-t_2, \int_0^\kappa \mathtt{H}_1(\kappa-t_2,t_3)\cdots\\ &\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)d(\kappa-t_2)\Big]dt_1\\ &\vdots\\ = &\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Big(t_2, \int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\\ &\cdots f_{\ell-1}\Big(t_{2\ell-2}, \int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]\\ &dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1\end{aligned}$$

$$=(\mathrm{T}y_1)(s).$$

Hence $\mathrm{T}y_1$ is symmetric on $[0,\kappa]$. From Lemma 2.4, we get

$$(\mathrm{T}y_1)(s)=\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Bigg(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2 \cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg] dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1$$

$$\le\beta_1\int_0^\kappa \mathrm{G}(t_1,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Bigg(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4) f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell})) dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1.$$

So,

$$\|\mathrm{T}y_1\|\le\beta_1\int_0^\kappa \mathrm{G}(t_1,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Bigg(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2 \cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg] dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1.$$

Again from Lemma 2.4, we get

$$\min_{s\in[\lambda_1,\kappa-\lambda_1]}\{(\mathrm{T}y_1)(s)\}\ge\beta_1\ \sigma_1(\lambda_1)\int_0^\kappa \mathrm{G}(t_1,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Bigg(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)} \varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})} \varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1.$$

By using above two inequalities one can write

$$\min_{s\in[\lambda_1,\kappa-\lambda_1]}\{(\mathrm{T}y_1)(s)\}\ge\sigma_1(\lambda_1)\|\mathrm{T}y_1\|.$$

So, $\mathrm{T}y_1\in\mathrm{K}$ and thus $\mathrm{T}(\mathrm{K})\subset\mathrm{K}$. By using Arzela-Ascoli theorem and standard methods it can be proved T is completely continuous. □

3.1. **Existence of at least two positive solutions.** In obtaining existence of at least two positive symmetric solutions of the BVP (1.1)-(1.3), the Avery-Henderson functional fixed point theorem will be the fundamental tool. Let ψ_1 be a nonnegative continuous functional on a cone K of the real Banach space B. Then for a positive real number $\mathtt{c}_1$, we define the sets

$$\mathrm{K}(\psi_1,\mathtt{c}_1)=\{y\in\mathrm{K}:\psi_1(y)<\mathtt{c}_1\}$$

and

$$\mathrm{K}_{\mathtt{c}_1}=\{y\in\mathrm{K}:\|y\|<\mathtt{c}_1\}.$$

Also, we define the nonnegative, increasing, continuous functionals γ_1, θ_1 and α_1 on the cone K

$$\gamma_1(y) = \min_{s\in I_1} |y(s)|,$$
$$\theta_1(y) = \max_{s\in I_1} |y(s)|, \text{ and}$$
$$\alpha_1(y) = \max_{[0,\kappa]} |y(s)|.$$

For any $y_1 \in$ K, we have

$$\gamma_1(y_1) \le \theta_1(y_1) \le \alpha_1(y_1) \tag{3.1}$$

and

$$\|y_1\| \le \frac{1}{\sigma_1(\lambda_1)} \min_{s\in I_1} |y_1(s)| = \frac{1}{\sigma_1(\lambda_1)}\gamma_1(y_1) \le \frac{1}{\sigma_1(\lambda_1)}\theta_1(y_1) \le \frac{1}{\sigma_1(\lambda_1)}\alpha_1(y_1). \tag{3.2}$$

Theorem 3.2. *Suppose there exist* $0 < \mathtt{a}_1' < \mathtt{b}_1' < \mathtt{c}_1'$ *such that* f_n, *for* $n = 1, 2, \cdots, \ell$ *satisfies the following conditions:*

(I4) $f_n(s, y) > \varphi(\frac{\mathtt{c}_1'}{M})$, $s \in I_1$ *and* $y \in \left[\mathtt{c}_1', \frac{\mathtt{c}_1'}{\sigma_1(\lambda_1)}\right]$,

(I5) $f_n(s, y) < \varphi(\frac{\mathtt{b}_1'}{m})$, $s \in [0, \kappa]$ *and* $y \in \left[0, \frac{\mathtt{b}_1'}{\sigma_1(\lambda_1)}\right]$,

(I6) $f_n(s, y) > \varphi(\frac{\mathtt{a}_1'}{M})$, $s \in I_1$ *and* $y \in \left[\mathtt{c}_1', \frac{\mathtt{c}_1'}{\sigma_1(\lambda_1)}\right]$.

Then the iterative boundary value problem (1.1)-(1.3) has at least two positive solutions $(y_1^*, y_2^*, \cdots, y_\ell^*)$ *and* $(y_1^{**}, y_2^{**}, \cdots, y_\ell^{**})$ *such that*

$$\mathtt{a}_1' < \max_{s\in[0,\kappa]}\{|y_n^*(s)|\} \text{ with } \max_{s\in I_1}\{|y_n^*(s)|\} < \mathtt{b}_1', \ n = 1, 2, \cdots, \ell,$$
$$\mathtt{b}_1' < \max_{s\in I_1}\{|y_n^{**}(s)|\} \text{ with } \min_{s\in I_1}\{|y_n^{**}(s)|\} < \mathtt{c}_1', \ n = 1, 2, \cdots, \ell.$$

Proof. From Lemma 3.1, we know T : K $\to$ K is a completely continuous operator. From (3.1)-(3.2), for each $y_1 \in$ K, $\gamma_1(y_1) \le \theta_1(y_1) \le \alpha_1(y_1)$ and $\|y_1\| \le \frac{1}{\sigma_1(\lambda_1)}\gamma_1(y_1)$. Also for any $\lambda \in [0, 1]$ and $y_1 \in$ K, $\theta_1(\lambda y_1) = \max_{s\in I_1}\{\lambda(|y_1(s)|)\} = \lambda \max_{s\in I_1}\{|y_1(s)|\} = \lambda\theta_1(y_1)$. It is clear that $\theta_1(0) = 0$. Since $y_1 \in \partial$K$(\gamma_1, \mathtt{c}_1')$, from (3.2) we have that

$$\mathtt{c}_1' = \min_{s\in I_1} |y_1(s)| \le \|y_1\| \le \frac{\mathtt{c}_1'}{\sigma_1(\lambda_1)}.$$

By (I4), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned}
&\int_0^\kappa \mathtt{H}_1(t_{2\ell-2}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1}, t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell}, y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}\\
&\ge \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1)\beta_1\mathrm{G}_1(t_{2\ell-1}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t_{2\ell}, t_{2\ell})h(t_{2\ell})\varphi(\frac{\mathtt{c}_1'}{M})dt_{2\ell}\right]dt_{2\ell-1}\\
&\ge \frac{\mathtt{c}_1'}{M}\sigma_1(\lambda_1)\beta_1\int_{\lambda_1}^{\kappa-\lambda_1} \mathrm{G}_1(t_{2\ell-1}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\sigma_2(\lambda_1)\beta_2\int_{\lambda_1}^{\kappa-\lambda_1} \mathrm{G}_2(t_{2\ell}, t_{2\ell})h(t_{2\ell})dt_{2\ell}\right]dt_{2\ell-1}\\
&\ge \mathtt{c}_1'.
\end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned}
&\int_0^\kappa \mathtt{H}_1(t_{2\ell-4}, t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3}, t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Bigg(t_{2\ell-2}, \int_0^\kappa \mathtt{H}_1(t_{2\ell-2}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
&\quad\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1}, t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell}, y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}\Bigg)dt_{2\ell-1}\Bigg]dt_{2\ell-3}
\end{aligned}$$

$$\geq \int_0^\kappa \mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Big(t_{2\ell-2},\mathtt{c}_1'\Big)dt_{2\ell-1}\right]dt_{2\ell-3}$$
$$\geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1)\beta_1\mathrm{G}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})\varphi(\frac{\mathtt{c}_1'}{M})dt_{2\ell-1}\right]dt_{2\ell-3}$$
$$\geq \frac{\mathtt{c}_1'}{M}\sigma_1(\lambda_1)\beta_1\int_{\lambda_1}^{\kappa-\lambda_1} \mathrm{G}_1(t_{2\ell-3},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t_{2\ell-2},t_{2\ell-2})h(t_{2\ell-2})dt_{2\ell-1}\right]dt_{2\ell-3}$$
$$\geq \mathtt{c}_1'.$$

Continuing in this fashion, we get

$$\gamma_1(\mathrm{T}y_1(s)) = \min_{s\in I_1}\left[\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Big(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\right.\right.$$
$$\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Big(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\right.$$
$$\left.\left.\left.\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}\Big)\cdots dt_4\right]dt_3\Big)dt_2\right]dt_1\right]$$
$$\geq \mathtt{c}_1'.$$

Since $y_1\in\partial\mathrm{K}(\theta_1,\mathtt{b}_1')$, from (3.2) we have that

$$0\leq y_1(s)\leq \|y_1\|\leq \frac{\mathtt{b}_1'}{\sigma_1(\lambda_1)},\ \text{ for } s\in[0,\kappa].$$

By (I5), and for $t_{2\ell-2}\in[0,\kappa]$, we have

$$\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}$$
$$\leq \int_0^\kappa \beta_1\mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})\varphi(\frac{\mathtt{b}_1'}{m})dt_{2\ell}\right]dt_{2\ell-1}$$
$$\leq \frac{\mathtt{b}_1'}{m}\beta_1\int_0^\kappa \mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\beta_2\int_0^\kappa \mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})dt_{2\ell}\right]dt_{2\ell-1}$$
$$\leq \mathtt{b}_1'.$$

Similarly for $t_{2\ell-4}\in[0,\kappa]$

$$\int_0^\kappa \mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Big(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\right.$$
$$\left.\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}\Big)dt_{2\ell-1}\right]dt_{2\ell-3}$$
$$\leq \int_0^\kappa \mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Big(t_{2\ell-2},\mathtt{b}_1'\Big)dt_{2\ell-1}\right]dt_{2\ell-3}$$
$$\leq \int_0^\kappa \beta_1\mathrm{G}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})\varphi(\frac{\mathtt{b}_1'}{m})dt_{2\ell-1}\right]dt_{2\ell-3}$$
$$\leq \frac{\mathtt{b}_1'}{m}\beta_1\int_0^\kappa \mathrm{G}_1(t_{2\ell-3},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell-2},t_{2\ell-2})h(t_{2\ell-2})dt_{2\ell-1}\right]dt_{2\ell-3}$$
$$\leq \mathtt{b}_1'.$$

Continuing in this fashion, we get

$$
\begin{aligned}
\theta_1(\mathrm{T}y_1(s)) = & \max_{s\in I_1}\Bigg[\int_0^{\kappa}\mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_1,t_2h(t_2)f_1\Bigg(t_2,\int_0^{\kappa}\mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\
& \varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^{\kappa}\mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
& \varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1\Bigg]\\
\le & \int_0^{\kappa}\mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_1,t_2h(t_2)f_1\Bigg(t_2,\int_0^{\kappa}\mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\
& \varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^{\kappa}\mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
& \varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1\\
\le & \mathtt{c}_1'.
\end{aligned}
$$

Since $0\in\mathrm{K}$ and $\mathtt{a}_1'>0$, we get $\mathrm{K}(\alpha_1,\mathtt{a}_1')\neq\emptyset$. Since $y_1\in\partial\mathrm{K}(\alpha_1,\mathtt{a}_1')$,

$$
\mathtt{a}_1' = \max_{s\in[0,\kappa]}|y_1(s)| \le \|y_1\| \le \frac{\mathtt{a}_1'}{\sigma_1(\lambda_1)}.
$$

By ($\mathtt{I6}$), and for $t_{2\ell-2}\in[0,\kappa]$, we have

$$
\begin{aligned}
& \int_0^{\kappa}\mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\\
& \ge \int_{\lambda_1}^{\kappa-\lambda_1}\sigma_1(\lambda_1)\beta_1\mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\int_{\lambda_1}^{\kappa-\lambda_1}\sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})\varphi(\frac{\mathtt{a}_1'}{M})dt_{2\ell}\Bigg]dt_{2\ell-1}\\
& \ge \frac{\mathtt{a}_1'}{M}\sigma_1(\lambda_1)\beta_1\int_{\lambda_1}^{\kappa-\lambda_1}\mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\sigma_2(\lambda_1)\beta_2\int_{\lambda_1}^{\kappa-\lambda_1}\mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})dt_{2\ell}\Bigg]dt_{2\ell-1}\\
& \ge \mathtt{a}_1'.
\end{aligned}
$$

Similarly for $t_{2\ell-4}\in[0,\kappa]$

$$
\begin{aligned}
& \int_0^{\kappa}\mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^{\kappa}\mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
& \varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\
& \ge \int_0^{\kappa}\mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_0^{\kappa}\mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Bigg(t_{2\ell-2},\mathtt{a}_1'\Bigg)dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\
& \ge \int_{\lambda_1}^{\kappa-\lambda_1}\sigma_1(\lambda_1)\beta_1\mathrm{G}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_{\lambda_1}^{\kappa-\lambda_1}\sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})\varphi(\frac{\mathtt{a}_1'}{M})dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\
& \ge \frac{\mathtt{a}_1'}{M}\sigma_1(\lambda_1)\beta_1\int_{\lambda_1}^{\kappa-\lambda_1}\mathrm{G}_1(t_{2\ell-3},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_{\lambda_1}^{\kappa-\lambda_1}\sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t_{2\ell-2},t_{2\ell-2})h(t_{2\ell-2})dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\
& \ge \mathtt{a}_1'.
\end{aligned}
$$

Continuing in this fashion, we get

$$\begin{aligned}\gamma_1(\mathrm{T}y_1(s)) = \min_{s\in I_1} & \Bigg[\int_0^{\kappa} \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^{\kappa} \mathtt{H}_2(t_1,t_2h(t_2)f_1\Bigg(t_2,\int_0^{\kappa} \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)} \\ & \varphi^{-1}\Bigg[\int_0^{\kappa} \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^{\kappa} \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})} \\ & \varphi^{-1}\Bigg[\int_0^{\kappa} \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1\Bigg] \\ & \geq \mathtt{a}_1'.\end{aligned}$$

So, proved all the conditions of Avery-Henderson fixed point Theorem 2.5. Hence T has at least two fixed points $y_1^*, y_1^{**} \in \overline{\mathrm{P}(\gamma_1,\mathtt{c}_1')}$ such that $\mathtt{a}_1' < \alpha_1(y^*)$ with $\theta_1(y^*) < \mathtt{b}_1'$ and $\mathtt{b}_1' < \theta_1(y^{**})$ with $\gamma_1(y^{**}) < \mathtt{c}_1'$. Setting $y_{\ell+1}(s) = y_1(s)$, we obtain a two positive symmetric solutions $(y_1^*, y_2^*, \cdots, y_\ell^*)$ and $(y_1^{**}, y_2^{**}, \cdots, y_\ell^{**})$ of the boundary value problem (1.1)-(1.3) given iteratively by

$$y_n(s) = \int_0^{\kappa} \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^{\kappa} \mathtt{H}_2(t_1,t_2)h(t_2)f_n(t_2,y_{n+1}(t_2))dt_2\Bigg]dt_1,\ n = 1,2,\cdots,\ell,$$

such that

$$\begin{aligned}&\mathtt{a}_1' < \max_{s\in[0,\kappa]}\{|y_n^*(s)|\} \text{ with } \max_{s\in I_1}\{|y_n^*(s)|\} < \mathtt{b}_1',\ n = 1,2,\cdots,\ell,\\ &\mathtt{b}_1' < \max_{s\in I_1}\{|y_n^{**}(s)|\} \text{ with } \min_{s\in I_1}\{|y_n^{**}(s)|\} < \mathtt{c}_1',\ n = 1,2,\cdots,\ell.\end{aligned}$$

□

3.2. **Existence of at least three positive solutions.** Let γ_2 and θ_2 be nonnegative continuous convex functionals on K, α_2 be a nonnegative continuous concave functional on a K, and ψ_2 be a nonnegative continuous functional K. Then for positive real numbers $\mathtt{a}_2$, $\mathtt{b}_2$, $\mathtt{c}_2$, and $\mathtt{d}_2$ we define the following convex sets:

$$\begin{aligned}\mathrm{K}(\gamma_2,\mathtt{d}_2) &= \{u\in\mathrm{K} : \gamma_2(u) < \mathtt{d}_2\},\\ \mathrm{K}(\gamma_2,\alpha_2,\mathtt{b}_2,\mathtt{d}_2) &= \{u\in\mathrm{K} : \mathtt{b}_2 \leq \alpha_2(u), \gamma_2(u)\leq \mathtt{d}_2\},\\ \mathrm{K}(\gamma_2,\theta_2,\alpha_2,\mathtt{b}_2,\mathtt{c}_2,\mathtt{d}_2) &= \{u\in\mathrm{K} : \mathtt{b}_2 \leq \alpha_2(u), \theta_2(u)\leq \mathtt{c}_2, \gamma_2(u)\leq \mathtt{d}_2\},\end{aligned}$$

and a closed set

$$R(\gamma_2,\psi_2,\mathtt{a}_2,\mathtt{d}_2) = \{u\in\mathrm{K} : \mathtt{a}_2\leq\psi_2(u), \gamma_2(u)\leq\mathtt{d}_2\}.$$

In obtaining existence of at least three positive symmetric solutions of the BVP (1.1)-(1.3), the Avery and Peterson fixed point theorem will be the fundamental tool. For that we define the nonnegative, continuous concave functional α_2, the nonnegative continuous convex functionals θ_2, γ_2 and the nonnegative continuous functional ψ_2 on the cone K by

$$\begin{aligned}&\gamma_2(y) = \max_{s\in\mathrm{I}}|y(s)|, \quad \theta_2(y) = \max_{s\in\mathrm{I}_1}|y(s)|,\\ &\psi_2(y) = \min_{s\in\mathrm{I}_2}|y(s)|, \quad \alpha_2(y) = \min_{s\in\mathrm{I}_1}|y(s)|,\end{aligned}$$

where $\mathrm{I}_1 = [\lambda_1, \kappa-\lambda_1]$, $\mathrm{I}_2 = [\lambda_2,\kappa-\lambda_2]$, $\lambda_1 < \lambda_2$, $\lambda_1,\lambda_2 \in (0,\frac{\kappa}{2})$. For any $y_1\in\mathrm{K}$, we have

$$\alpha_2(y_1)\leq\psi_2(y_1)\leq\theta_2(y_1)\leq\gamma_2(y_1), \tag{3.3}$$

and

$$\|y_1\| \leq \frac{1}{\sigma_1(\lambda_1)}\min_{s\in\mathrm{I}_1}|y_1(s)| = \frac{1}{\sigma_1(\lambda_1)}\alpha_2(y_1) \leq \frac{1}{\sigma_1(\lambda_1)}\psi_2(y_1) \leq \frac{1}{\sigma_1(\lambda_1)}\theta_2(y_1) \leq \frac{1}{\sigma_1(\lambda_1)}\gamma_2(y_1). \tag{3.4}$$

Theorem 3.3. *Suppose there exist* $0 < \mathtt{a}_2' < \mathtt{b}_2' < \mathtt{d}_2'$, $\mathtt{c}_2{}' = \frac{\mathtt{b}_2'}{\sigma_1(\lambda_1)}$ *such that* f_n, *for* $n = 1,2,\cdots,\ell$ *satisfies the following conditions:*

(I7) $f_n(s,y) \le \varphi(\frac{\mathtt{d}_2'}{m}),\ s \in \mathrm{I}$ *and* $y_n \in \Big[0, \sigma_1(\lambda_1)\mathtt{d}_2'\Big]$,

(I8) $f_n(s,y) > \varphi(\frac{\mathtt{b}_2'}{M}),\ s \in \mathrm{I}_1$ *and* $y_n \in \Big[\mathtt{b}_2', \mathtt{c}_2{}'\Big]$,

(I9) $f_n(s,y) < \varphi(\frac{\mathtt{a}_2'}{m}),\ s \in \mathrm{I}$ *and* $y_n \in \Big[0, \mathtt{a}_2'\Big]$.

Then the iterative boundary value problem (1.1)-(1.3) has at least three positive solutions $(y_1^*, y_2^*, \cdots, y_\ell^*)$, $(y_1^{**}, y_2^{**}, \cdots, y_\ell^{**})$ *and* $(y_1^{***}, y_2^{***}, \cdots, y_\ell^{***})$*such that*

$$\gamma_2(y_n^i) \le \mathtt{d}_2',\ \textit{for } i = *, **, ***,\ n = 1, 2, \cdots, \ell;$$
$$\mathtt{b}_2' < \alpha_2(y_n^*),\ n = 1, 2, \cdots, \ell;$$
$$\mathtt{a}_2' < \psi_2(y_n^{**})\ \textit{with } \alpha_2(y_n^{**}) < \mathtt{b}_2',\ n = 1, 2, \cdots, \ell;$$

and

$$\psi_2(y_n^{***}) < \mathtt{a}_2',\ n = 1, 2, \cdots, \ell.$$

Proof. From Lemma 3.1, $\mathrm{T} : \mathrm{K} \to \mathrm{K}$ is a completely continuous operator. From (3.3)-(3.4), for each $y_1 \in \mathrm{K}$, $\alpha_2(y_1) \le \psi_2(y_1) \le \theta_2(y_1) \le \gamma_2(y_1)$, and $\|y_1\| \le \frac{1}{\sigma_1(\lambda)}\gamma_2(y_1)$. Also for any $\lambda \in [0, 1]$ and $y_1 \in \mathrm{K}$, $\psi_2(\lambda y_1) = \min_{s \in \mathrm{I}_2}\{\lambda(|y_1(s)|)\} = \lambda \min_{s \in \mathrm{I}_2}\{|y_1(s)|\} = \lambda\psi_2(y_1)$. It is clear that $\psi_2(0) = 0 < \mathtt{a}_2'$, and thus $0 \notin R(\gamma_2, \psi_2, \mathtt{a}_2', \mathtt{d}_2')$. If $y_1 \in \overline{\mathrm{K}(\gamma_2, \mathtt{d}_2')}$, then $\|y_1\| \le \frac{1}{\sigma_1(\lambda)}\gamma_2(y_1) \le \frac{1}{\sigma_1(\lambda)}\mathtt{d}_2'$.
By (I7), and for $t_{2\ell-2} \in [0, \kappa]$, we have

$$\begin{aligned}
&\int_0^\kappa \mathtt{H}_1(t_{2\ell-2}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1}, t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell}, y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}\\
&\quad \le \int_0^\kappa \beta_1\mathrm{G}_1(t_{2\ell-1}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell}, t_{2\ell})h(t_{2\ell})\phi_{\mathrm{p}}(\frac{\mathtt{d}_2'}{m})dt_{2\ell}\right]dt_{2\ell-1}\\
&\quad \le \frac{\mathtt{d}_2'}{m}\beta_1\int_0^\kappa \mathrm{G}_1(t_{2\ell-1}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\left[\beta_2\int_0^\kappa \mathrm{G}_2(t_{2\ell}, t_{2\ell})h(t_{2\ell})dt_{2\ell}\right]dt_{2\ell-1}\\
&\quad \le \mathtt{d}_2'.
\end{aligned}$$

Similarly for $t_{2\ell-4} \in [0, \kappa]$

$$\begin{aligned}
&\int_0^\kappa \mathtt{H}_1(t_{2\ell-4}, t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3}, t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\left(t_{2\ell-2}, \int_0^\kappa \mathtt{H}_1(t_{2\ell-2}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\right.\right.\\
&\quad \left.\left.\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1}, t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell}, y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}\right)dt_{2\ell-1}\right]dt_{2\ell-3}\\
&\quad \le \int_0^\kappa \mathtt{H}_1(t_{2\ell-4}, t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3}, t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\left(t_{2\ell-2}, \mathtt{d}_2'\right)dt_{2\ell-1}\right]dt_{2\ell-3}\\
&\quad \le \int_0^\kappa \beta_1\mathrm{G}_1(t_{2\ell-4}, t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell-3}, t_{2\ell-2})h(t_{2\ell-2})\phi_{\mathrm{p}}(\frac{\mathtt{d}_2'}{m})dt_{2\ell-1}\right]dt_{2\ell-3}\\
&\quad \le \frac{\mathtt{d}_2'}{m}\beta_1\int_0^\kappa \mathrm{G}_1(t_{2\ell-3}, t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\left[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell-2}, t_{2\ell-2})h(t_{2\ell-2})dt_{2\ell-1}\right]dt_{2\ell-3}\\
&\quad \le \mathtt{d}_2'.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}\gamma_2(\mathrm{T}y_1) = \max_{s\in I} & \Bigg[\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_1,t_2h(t_2)f_1\Bigg(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\ &\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\ &\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1\Bigg]\\ &\leq \mathtt{d}_2'.\end{aligned}$$

Therefore, $\mathrm{T}:\overline{\mathrm{K}(\gamma_2,\mathtt{d}_2')}\to\overline{\mathrm{K}(\gamma_2,\mathtt{d}_2')}$, since we have $\mathrm{T}:\mathrm{K}\to\mathrm{K}$ by Lemma 3.1. It is immediate that $\{y_1\in \mathrm{K}(\gamma_2,\theta_2,\alpha_2,\mathtt{b}_2',\mathtt{c}_2{}',\mathtt{d}_2'):\alpha_2(y_1)>\mathtt{b}_2'\}\neq\emptyset$
Now, we prove that, if $y_1\in R(\gamma_2,\psi_2,\mathtt{a}_2',\mathtt{d}_2')$ with $\psi_2(y_1)=\mathtt{a}_2'$ then $\psi_2(\mathrm{T}y_1)<\mathtt{a}_2{}'$.
By (I9), and for $t_{2\ell-2}\in[0,\kappa]$, we have

$$\begin{aligned}&\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\\ &\quad\leq \int_0^\kappa \beta_1\mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})\phi_{\mathrm{p}}(\frac{\mathtt{a}_2'}{m})dt_{2\ell}\Bigg]dt_{2\ell-1}\\ &\quad\leq \frac{\mathtt{a}_2'}{m}\beta_1\int_0^\kappa \mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Bigg[\beta_2\int_0^\kappa \mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})dt_{2\ell}\Bigg]dt_{2\ell-1}\\ &\quad\leq \mathtt{a}_2'.\end{aligned}$$

Similarly for $t_{2\ell-4}\in[0,\kappa]$

$$\begin{aligned}&\int_0^\kappa \mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\ &\quad\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\ &\quad\leq \int_0^\kappa \mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Bigg(t_{2\ell-2},\mathtt{a}_2'\Bigg)dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\ &\quad\leq \int_0^\kappa \beta_1\mathrm{G}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})\phi_{\mathrm{p}}(\frac{\mathtt{a}_2'}{m})dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\ &\quad\leq \frac{\mathtt{a}_2'}{m}\beta_1\int_0^\kappa \mathrm{G}_1(t_{2\ell-3},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Bigg[\int_0^\kappa \beta_2\mathrm{G}_2(t_{2\ell-2},t_{2\ell-2})h(t_{2\ell-2})dt_{2\ell-1}\Bigg]dt_{2\ell-3}\\ &\quad\leq \mathtt{a}_2'.\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}\psi_2(\mathrm{T}y_1) = \min_{s\in \mathrm{I}_2} & \Bigg[\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_1,t_2h(t_2)f_1\Bigg(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\ &\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\ &\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Bigg]dt_{2\ell-1}\Bigg)\cdots dt_4\Bigg]dt_3\Bigg)dt_2\Bigg]dt_1\Bigg]\\ \leq \max_{s\in I} & \Bigg[\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_1,t_2h(t_2)f_1\Bigg(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\ &\varphi^{-1}\Bigg[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Bigg(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\end{aligned}$$

$$\varphi^{-1}\left[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\right]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1\Big]$$
$$\leq \mathtt{a}_2'.$$

Now, we prove that, if $y_1 \in \mathrm{K}(\gamma_2, \alpha_2, \mathtt{b}_2', \mathtt{d}_2')$ with $\theta_2(\mathrm{T}y_1) > \mathtt{c}_2{}'$, then $\alpha_2(\mathrm{T}y_1) > \mathtt{b}_2'$.

Consider

$$\begin{aligned}
\alpha_2(\mathrm{T}y_1(s)) =& \min_{s\in I_1}\Big[\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_1,t_2h(t_2)f_1\Big(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\
&\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Big(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
&\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1\Big]\\
&\geq \beta_1\ \sigma_1(\lambda_1)\int_0^\kappa \mathrm{G}(t_1,t_1)\frac{1}{g(t_1)}\phi_{\mathrm{q}}\Big[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Big(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\
&\phi_{\mathrm{q}}\Big[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Big(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
&\phi_{\mathrm{q}}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1\\
&\geq \sigma_1(\lambda_1)\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\phi_{\mathrm{q}}\Big[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_1\Big(t_2,\int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}\\
&\phi_{\mathrm{q}}\Big[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Big(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
&\phi_{\mathrm{q}}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1\\
&=\sigma_1(\lambda_1)\theta_2(\mathrm{T}y_1)\\
&\geq \sigma_1(\lambda_1)\mathtt{c}_2{}' = \mathtt{b}_2'.
\end{aligned}$$

Now, we prove that, if $y_1 \in \mathrm{K}(\gamma_2, \theta_2, \alpha_2, \mathtt{b}_2', \mathtt{c}_2{}', \mathtt{d}_2')$, then $\alpha_2(\mathrm{T}y_1) > \mathtt{b}_2'$.

By (I8), and for $t_{2\ell-2} \in [0,\kappa]$, we have

$$\begin{aligned}
&\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\\
&\geq \int_{\lambda_1}^{\kappa-\lambda_1}\sigma_1(\lambda_1)\beta_1\mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Big[\int_{\lambda_1}^{\kappa-\lambda_1}\sigma_2(\lambda_1)\beta_2\mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})\phi_{\mathrm{p}}(\frac{\mathtt{b}_2'}{M})dt_{2\ell}\Big]dt_{2\ell-1}\\
&\geq \frac{\mathtt{b}_2'}{M}\sigma_1(\lambda_1)\beta_1\int_{\lambda_1}^{\kappa-\lambda_1}\mathrm{G}_1(t_{2\ell-1},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Big[\sigma_2(\lambda_1)\beta_2\int_{\lambda_1}^{\kappa-\lambda_1}\mathrm{G}_2(t_{2\ell},t_{2\ell})h(t_{2\ell})dt_{2\ell}\Big]dt_{2\ell-1}\\
&\geq \mathtt{b}_2'.
\end{aligned}$$

Similarly for $t_{2\ell-4} \in [0,\kappa]$

$$\begin{aligned}
&\int_0^\kappa \mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Big(t_{2\ell-2},\int_0^\kappa \mathtt{H}_1(t_{2\ell-2},t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\\
&\quad\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1},t_{2\ell})h(t_{2\ell})f_\ell(t_{2\ell},y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)dt_{2\ell-1}\Big]dt_{2\ell-3}\\
&\quad\geq \int_0^\kappa \mathtt{H}_1(t_{2\ell-4},t_{2\ell-3})\frac{1}{g(t_{2\ell-3})}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-3},t_{2\ell-2})h(t_{2\ell-2})f_{\ell-1}\Big(t_{2\ell-2},\mathtt{b}_2'\Big)dt_{2\ell-1}\Big]dt_{2\ell-3}
\end{aligned}$$

$$\geq \int_{\lambda_1}^{\kappa-\lambda_1} \sigma_1(\lambda_1)\beta_1 \mathrm{G}_1(t_{2\ell-4}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1}\Big[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1)\beta_2 \mathrm{G}_2(t_{2\ell-3}, t_{2\ell-2}) h(t_{2\ell-2}) \phi_{\mathrm{p}}(\frac{\mathtt{b}_2'}{M}) dt_{2\ell-1}\Big] dt_{2\ell-3}$$
$$\geq \frac{\mathtt{b}_2'}{M}\sigma_1(\lambda_1)\beta_1 \int_{\lambda_1}^{\kappa-\lambda_1} \mathrm{G}_1(t_{2\ell-3}, t_{2\ell-3}) \frac{1}{g(t_{2\ell-3})} \varphi^{-1}\Big[\int_{\lambda_1}^{\kappa-\lambda_1} \sigma_2(\lambda_1)\beta_2 \mathrm{G}_2(t_{2\ell-2}, t_{2\ell-2}) h(t_{2\ell-2}) dt_{2\ell-1}\Big] dt_{2\ell-3}$$
$$\geq \mathtt{b}_2'.$$

Continuing in this fashion, we get

$$\alpha_2(\mathrm{T}y_1(s)) = \min_{s\in I_1} \Bigg[\int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_1, t_2 h(t_2) f_1\Big(t_2, \int_0^\kappa \mathtt{H}_1(t_2,t_3)\frac{1}{g(t_3)}$$
$$\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_3,t_4)h(t_4)f_2\cdots f_{\ell-1}\Big(t_{2\ell-2}, \int_0^\kappa \mathtt{H}_1(t_{2\ell-2}, t_{2\ell-1})\frac{1}{g(t_{2\ell-1})}\varphi^{-1}\Big[\int_0^\kappa \mathtt{H}_2(t_{2\ell-1}, t_{2\ell})$$
$$h(t_{2\ell})f_\ell(t_{2\ell}, y_1(t_{2\ell}))dt_{2\ell}\Big]dt_{2\ell-1}\Big)\cdots dt_4\Big]dt_3\Big)dt_2\Big]dt_1\Bigg]$$
$$\geq \mathtt{b}_2'.$$

So, proved all the conditions of Avery and Peterson fixed point theorem 2.6. Hence, there exist three positive fixed points $y_1^*, y_1^{**}, y_1^{***} \in \overline{\mathrm{K}(\gamma_2, \mathtt{d}_2)}$ such that

$\gamma_2(y_1^*) \leq \mathtt{d}_2'$;
$\mathtt{b}_2' < \alpha_2(y_1^*)$;
$\mathtt{a}_2' < \psi_2(y_1^{**})$ with $\alpha_2(y_1^{**}) < \mathtt{b}_2'$;

and

$\psi_2(y_1^{***}) < \mathtt{a}_2'$.

Setting $y_{\ell+1}(s) = y_1(s)$, we obtain a three positive symmetric solutions $(y_1^*, y_2^*, \cdots, y_\ell^*)$, $(y_1^{**}, y_2^{**}, \cdots, y_\ell^{**})$ and $(y_1^{***}, y_2^{***}, \cdots, y_\ell^{***})$ of the boundary value problem (1.1)-(1.3) given iteratively by

$$y_n(s) = \int_0^\kappa \mathtt{H}_1(s,t_1)\frac{1}{g(t_1)}\phi_{\mathrm{q}}\Big[\int_0^\kappa \mathtt{H}_2(t_1,t_2)h(t_2)f_n(t_2, y_{n+1}(t_2))dt_2\Big]dt_1, \ n = 1, 2, \cdots, \ell,$$

such that

$\gamma_2(y_n^i) \leq \mathtt{d}_2'$, for $i = *, **, ***$, $n = 1, 2, \cdots, \ell$;
$\mathtt{b}_2' < \alpha_2(y_n^*)$, $n = 1, 2, \cdots, \ell$;
$\mathtt{a}_2' < \psi_2(y_n^{**})$ with $\alpha_2(y_n^{**}) < \mathtt{b}_2'$, $n = 1, 2, \cdots, \ell$;

and

$\psi_2(y_n^{***}) < \mathtt{a}_2'$, $n = 1, 2, \cdots, \ell$.

□

4. Examples

Example 4.1. Consider the following problem

$$\begin{cases} (\varphi(g(s)y_n''(s)))'' = h(s)f_n(s, y_{n+1}(s)), & 1 \leq n \leq 2, 0 \leq s \leq 1, \\ y_3(s) = y_1(s), & 0 \leq s \leq 1, \end{cases} \tag{4.1}$$

satisfying boundary conditions

$$\begin{cases} 3y_n(0) - y_n'(0) = \int_0^1 v(q)y_n(q)dq, \ 3y_n(1) + y_n'(1) = \int_0^1 v(q)y_n(q)dq, & 1 \leq n \leq 2, \\ \varphi(g(0)y_n''(0)) = \int_0^1 w(q)\varphi(g(q)y_n''(q))dq, \ \varphi(g(1)y_n''(1)) = \int_0^1 w(q)\varphi(g(q)y_n''(q))dq, & 1 \leq n \leq 2, \end{cases} \tag{4.2}$$

where $g(s) = -5s^2 + 5s + 4,\ h(s) = 100,\ v(s) = \frac{1}{5},\ w(q) = \frac{1+s-s^2}{13}$,

$$f_1(s, y) = f_2(s, y) = \begin{cases} y^3s^4 - 2y^3s^3 + y^3s^2 + 15, & (s, y) \in [0, 1] \times (0, 6], \\ 6y^2s^4 - 12y^2s^3 + 6y^2s^2 + 3y^3e^y - 18y^2e^y + 15, & (s, y) \in [0, 1] \times [6, \infty). \end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{1}{5},\ \mu_2 = \frac{7}{78},\ \beta_1 = \frac{15}{14},\ \beta_2 = \frac{78}{71}$,

$$\begin{aligned} \mathtt{H}_1(s, t) =& \mathrm{G}_1(s, t) + \frac{1}{\xi_1 - \mu_1} \int_0^{\kappa} \mathrm{G}_1(q, t) v(q) dq, \\ \mathtt{H}_2(s, t) =& \mathrm{G}_2(s, t) + \frac{1}{1 - \mu_1} \int_0^{\kappa} \mathrm{G}_2(q, t) w(q) dq, \end{aligned}$$

in which

$$\mathrm{G}_1(s, t) = \frac{1}{15} \begin{cases} (3t + 1)(4 - 3s),\ 0 \le t \le s, \\ (3s + 1)(4 - 3t),\ s \le t \le 1, \end{cases}$$

$$\mathrm{G}_2(s, t) = \begin{cases} t(1 - s),\ 0 \le t \le s, \\ s(1 - t),\ s \le t \le 1. \end{cases}$$

Let $\lambda_1 = \frac{1}{7}$, then $\sigma_1(\lambda_1) = \frac{5}{14},\ \sigma_2(\lambda_1) = \frac{1}{7},\ m = 0.3465210722,\ M = 0.03234219173$. Choose $\mathtt{a}_1' = 0.1,\ \mathtt{b}_1' = 2,\ \mathtt{c}_1' =$ 6.5, then

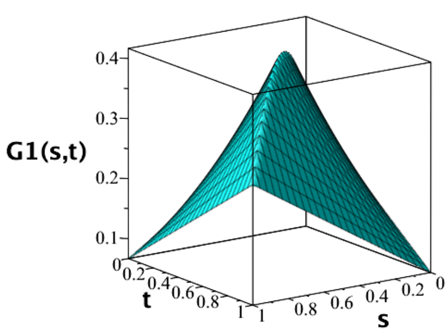

FIGURE 1. Pictorial representation of $\mathrm{G}_1(s, t)$.

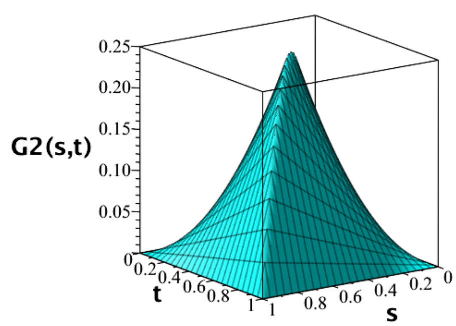

FIGURE 2. Pictorial representation of $\mathrm{G}_2(s, t)$.

$$\begin{aligned} f_n(s, y) >& \varphi(\frac{\mathtt{c}_1'}{M}) = 40391.29882,\ \text{for}\ \ s \in [0.14, 0.86],\ y \in [6.50, 18.20], \\ f_n(s, y) <& \varphi(\frac{\mathtt{b}_1'}{m}) = 33.31199881,\ \text{for}\ \ s \in [0, 1],\ y \in [0, 5.60], \\ f_n(s, y) >& \varphi(\frac{\mathtt{a}_1'}{M}) = 9.560070727,\ \text{for}\ \ s \in [0.14, 0.86],\ y \in [0.1, 0.28]. \end{aligned}$$

Hence by Theorem 3.2, the BVP (4.1)-(4.2) has at least two positive symmetric solutions (y_1^*, y_2^*) and (y_1^{**}, y_2^{**}) such that

$$\begin{aligned} &0.1 < \max_{s\in[0,1]} \{|y_n^*(s)|\} \text{ with } \max_{s\in[0.14,0.86]} \{|y_n^*(s)|\} < 2,\ n = 1, 2, \\ &2 < \max_{s\in[0.14,0.86]} \{|y_n^{**}(s)|\} \text{ with } \min_{s\in[0.14,0.86]} \{|y_n^{**}(s)|\} < 6.5,\ n = 1, 2. \end{aligned}$$

Example 4.2. Consider the following problem

$$\begin{cases} (\varphi(g(s)y_n''(s)))'' = h(s)f_n(s, y_{n+1}(s)), \ 1 \le n \le 2 & 0 \le s \le 4, \\ y_3(s) = y_1(s), & 0 \le s \le 4, \end{cases} \tag{4.3}$$

satisfying boundary conditions

$$\begin{cases} 5y_n(0) - 2y_n'(0) = \int_0^4 v(q)y_n(q)dq, \ 5y_n(1) + 2y_n'(1) = \int_0^4 v(q)y_n(q)dq, & 1 \le n \le 2, \\ \varphi(g(0)y_n''(0)) = \int_0^4 w(q)\varphi(g(q)y_n''(q))dq, \ \varphi(g(1)y_n''(1)) = \int_0^4 w(q)\varphi(g(q)y_n''(q))dq, & 1 \le n \le 2, \end{cases} \tag{4.4}$$

where $g(s) = \frac{137+300s-75s^2}{19}$, $h(s) = 78$, $v(s) = \frac{1+8s-2s^2}{17}$, $w(s) = \frac{4}{19}$,

$$f_1(s,y) = f_2(s,y) = \begin{cases} \frac{5ys^4-10ys^3+5ys^2+1}{24450}, & (s,y) \in [0,4] \times (0,2], \\ \frac{5ys^4-10ys^3+5ys^2+733500y-1466999}{24450}, & (s,y) \in [0,4] \times (2,3], \\ \frac{5ys^4-10ys^3+5ys^2+24450y+660151}{24450}, & (s,y) \in [0,4] \times (3,\infty). \end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{76}{51}$, $\mu_2 = \frac{16}{19}$, $\beta_1 = \frac{255}{179}$, $\beta_2 = \frac{19}{3}$,

$$\begin{aligned} \mathtt{H}_1(s,t) =& \mathrm{G}_1(s,t) + \frac{1}{\xi_1 - \mu_1}\int_0^\kappa \mathrm{G}_1(q,t)v(q)dq, \\ \mathtt{H}_2(s,t) =& \mathrm{G}_2(s,t) + \frac{1}{1 - \mu_1}\int_0^\kappa \mathrm{G}_2(q,t)w(q)dq, \end{aligned}$$

in which

$$\mathrm{G}_1(s,t) = \frac{1}{120}\begin{cases} (5t+2)(22-5s), \ 0 \le t \le s, \\ (5s+2)(22-5t), \ s \le t \le 4, \end{cases}$$

$$\mathrm{G}_2(s,t) = \begin{cases} \frac{t}{4}(4-s), \ 0 \le t \le s, \\ \frac{s}{4}(4-t), \ s \le t \le 4. \end{cases}$$

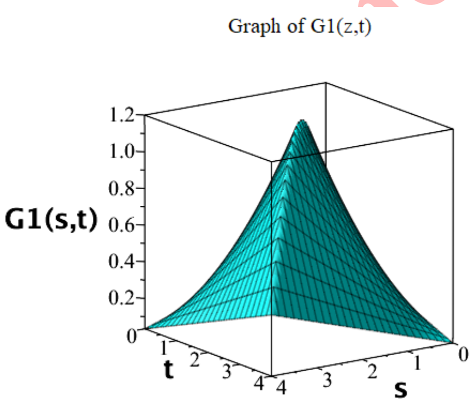

FIGURE 3. Pictorial representation of $\mathrm{G}_1(s,t)$.

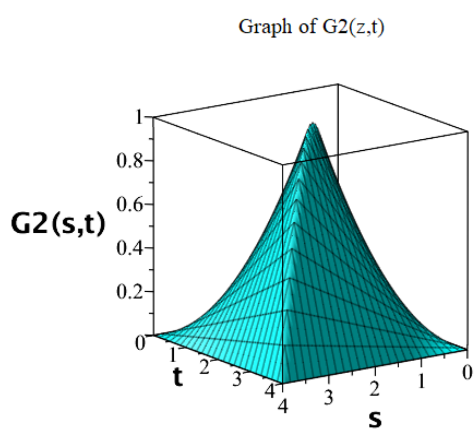

FIGURE 4. Pictorial representation of $\mathrm{G}_2(s,t)$.

Let $\lambda_1 = 1,\ \lambda_2 = 1.5$ then $\sigma_1(\lambda_1) = \frac{7}{22},\ \sigma_2(\lambda_1) = \frac{1}{4},\ m = 10.73716627,\ M = 0.7111785106$. Choose $\mathtt{a}_2' = 1.5,\ \mathtt{b}_2' = 2.5,\ \mathtt{c}_2' = 7.857142857,\ \mathtt{d}_2' = 100$ then

$$f_n(s,y) < \varphi(\frac{\mathtt{d}_2'}{m}) = 18.44215292, \text{ for } s \in [1,3],\ y \in [0, 31.82],$$
$$f_n(s,y) > \varphi(\frac{\mathtt{b}_2'}{M}) = 3.178111994, \text{ for } s \in [0,4],\ y \in [2.5, 7.857142857],$$
$$f_n(s,y) < \varphi(\frac{\mathtt{a}_2'}{m}) = 1.758390288, \text{ for } s \in [1,3],\ y \in [0, 1.5].$$

Hence by Theorem 3.3, the BVP (4.1)-(4.2) has at least three positive symmetric solutions (y_1^*, y_2^*) , (y_1^{**}, y_2^{**}) and (y_1^{***}, y_2^{***}) such that

$$\max_{s\in[0,4]} |y_n^i(s)| \le 100, \text{ for } i = *, **, ***,\ n = 1,2;$$
$$2.5 < \min_{s\in[1,3]} |y_n^*(s)|,\ n = 1,2;$$
$$1.5 < \min_{s\in[1.5,2.5]} |y_n^{**}(s)| \text{ with } \min_{s\in[1,3]} |y_n^{**}(s)| < 2.5,\ n = 1,2;$$

and

$$\min_{s\in[1.5,2.5]} |y_n^{***}(s)| < 1.5,\ n = 1,2.$$

5. Conclusion

This paper presents the existence of at least two positive symmetric solutions and three positive symmetric solutions for the fourth order iterative systems involving IHPH operator satisfying two point integral boundary conditions. The results are established by applying Avery-Henderson and Avery-Peterson fixed point theorems. These results may extended to the higher order multi point boundary value problems and fractional order boundary value problems with suitable fractional derivatives.

Acknowledgment

The authors thank to the reviewers for their appropriate comments and suggestions which help us to improve the manuscript. One of the authors (Kosuri Bhushanam) is thankful to UGC, Government of India, for awarding SRF; NTA Ref. No.:201610065189.

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