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# Treatment of Fractional Burgers-Fisher equation Using Taylor Wavelets

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**Abstract**

This study introduces an approach for finding an approximate solution to the time fractional generalized Burgers-Fisher equation. The core idea of the method is to transform the nonlinear partial differential equation into a linear one through the application of the Picard iteration technique. Subsequently, the Taylor wavelet collocation method is employed to address the linear equation derived in the prior step. The proposed approach effectively resolves the time fractional generalized Burgers-Fisher equation. The numerical results are evaluated against the exact solutions obtained from the Haar wavelet Picard and homotopy perturbation methods.

**Keywords.** Numerical methods for wavelets, Fractional partial differential equation, Fractional derivatives and integrals, Taylor wavelet, Collocation method.
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## 1. Introduction

Many physical phenomena in engineering and applied sciences are modeled using nonlinear equations. These equations play a crucial role in describing complex systems such as fluid dynamics, heat transfer, wave propagation, and other dynamic processes. Fractional partial differential equations (FPDEs) have gained increasing attention in the modeling of such phenomena, as they provide more accurate descriptions of systems with memory and hereditary properties. In particular, FPDEs are widely applied in various fields including signal processing [34], economics [6], control theory [9], and solid mechanics [38]. However, solving these equations often presents significant challenges due to their inherent nonlinearity and fractional order, which makes traditional methods less effective.

Several methods have been proposed to solve these equations, including the variational iteration method (VIM) [11, 42], Adomian decomposition method (ADM) [12]-[18], homotopy analysis method [15], power series method [32], shifted Jacobi collocation method [8], and Laplace transform method [36]. These methods, however, can be computationally expensive and time-consuming. To overcome these limitations, more efficient and reliable methods are being sought.

The generalized Burgers-Fisher equation is one of the most important nonlinear partial differential equations used in various applications, including heat conduction, fluid dynamics, shock wave formation, turbulence, and traffic flow [19–21, 25, 28, 33, 45]. The generalized time-fractional Burgers-Fisher equation with the Caputo fractional derivative is expressed as:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a u^\delta u_x - u_{xx} = bu(1-u^\delta), \quad x \in [0,1], \quad t \geq 0, \tag{1.1}$$

where $a$, $b$, and $\delta$ are constants, and $0 < \alpha \leq 1$ represents the order of the fractional derivative. For $\alpha = 1$, the equation reduces to the classical Burgers equation, and its exact solution is well known [18, 37].

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Numerous researchers have explored both analytical and numerical techniques for solving the generalized Burgers-Fisher equation. Kumar et al. [23] used the discontinuous Legendre wavelet Galerkin method, Singh et al. [41] applied the B-spline collocation method, and Saeed et al. [39] utilized the CAS wavelet quasilinearization technique. Other approaches include the Adomian decomposition method [18], homotopy perturbation method (HPM) [37], and Haar wavelet Picard method (HWPM) [40]. These methods, although effective, often face difficulties in handling fractional derivatives, especially for nonlinear cases.

Wavelets, particularly orthogonal wavelets, have been widely used to solve both ordinary and partial differential equations due to their ability to efficiently represent functions in terms of compactly supported bases. In the context of fractional partial differential equations, wavelets have been shown to provide highly accurate and computationally efficient solutions [1–4, 7, 13, 17, 24, 26, 27, 30, 31, 44]. Among the various types of wavelets, the Taylor wavelet has gained considerable attention due to its beneficial properties, including fast convergence and better handling of singularities compared to other methods [22, 29, 43].

In this paper, we propose a combined approach using the Taylor wavelet collocation method along with the Picard iteration technique to solve the generalized Burgers-Fisher equation with a time-fractional derivative. We derive an operational matrix for the fractional integral of Taylor wavelet in the Riemann-Liouville sense. The numerical results demonstrate that the proposed method is highly efficient and provides accurate solutions to the generalized Burgers-Fisher equation.

The structure of the paper is as follows. In section 2, we review some basic preliminaries about fractional derivatives and integrals, which are fundamental to understanding the generalized Burgers-Fisher equation with a fractional time derivative. In section 3, we provide a brief introduction to wavelet theory, followed by a detailed explanation of the Taylor wavelet and its properties in section 4. Section 5 is dedicated to presenting the proposed method, combining the Taylor wavelet collocation technique with the Picard iteration method for solving the fractional Burgers-Fisher equation. Finally, in section 6, we present several numerical examples to demonstrate the effectiveness and accuracy of our method. The results show that the proposed approach is both reliable and efficient for solving fractional partial differential equations.

## 2. Preliminaries on Fractional calculus, Wavelets, Taylor wavelets

2.1. **Fractional derivative and integral.** The Caputo fractional derivative of order $\alpha(\alpha > 0)$ is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}} \mathrm{d}\tau, \quad x > 0, \quad n = \lceil \alpha \rceil,$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to $\alpha$.

The Riemann–Liouville fractional integral of order $\alpha(\alpha > 0)$ is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad \alpha > 0. \tag{2.1}$$

For $n-1 < \alpha \leq n$, where $n$ is a natural number, there are several beneficial properties associated with the Caputo operator and the Riemann–Liouville operator as follows:

$$D^{\alpha}(I^{\alpha} f(x)) = f(x), \tag{2.2}$$

$$I^{\alpha}(D^{\alpha} f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} x^k, \tag{2.3}$$

$$I^{\alpha}(I^{\beta} f(x)) = I^{\alpha+\beta} f(x), \quad \alpha, \beta > 0, \tag{2.4}$$

$$I^{\beta}(I^{\alpha} f(x)) = I^{\alpha} I^{\beta} f(x), \tag{2.5}$$

$$I^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{2.6}$$

For more details on the fractional derivative and integral see [35].

2.2. **Wavelets.** A wavelet is a function $\psi \in L^2(\mathbb{R})$ that meets the requirement

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \tag{2.7}$$

where $\hat{\psi}(\omega)$ denotes the Fourier transform of $\psi(t)$.

Condition (2.7) is called the *admissibility condition* (and $C_\psi$ is called the *admissibility constant*) which ensures that the inversion formula for the continuous wavelet transform is valid. The condition concludes that $\hat{\psi}(\omega) \to 0$ as $\omega$ goes to 0. In fact, if $\hat{\psi}(\omega)$ is continuous, then $\hat{\psi}(0) = 0$, i.e.,

$$\int_{-\infty}^{\infty} \psi(\omega) dt = 0.$$

This means that $\psi(\omega)$ must be an oscillatory function with zero mean.

Other characteristics, besides the admissibility requirement, might be helpful in specific situations. To demonstrate the regularity of the wavelet functions and the ability of a wavelet transform to extract localized information, for example, limitations on the support of $\psi$ and of $\hat{\psi}$ or $\psi$ may be necessary in order to have a specific number of vanishing moments. A wavelet $\psi(t)$ has $n$-vanishing moments if the following condition is satisfied:

$$m_k = \int_{-\infty}^{\infty} t^k \psi(t) dt = 0, \quad k = 0, 1, \ldots, n. \tag{2.8}$$

The regularity of the wavelet has a direct correlation with the number of vanishing moments. As a result, a wavelet with more regularity has more vanishing moments. The so-called localization property of wavelets is another advantageous characteristic that aids in capturing the localized effects of a signal in both the frequency and time domains. Regularity (vanishing moments) and localization have an inverse relationship. As a result, when the frequency $\omega$ is low, wavelets that possess a greater number of vanishing moments tend to be flatter.

Wavelets are a family of functions constructed from translation and dilation of a single function $\psi$, called the *mother wavelet*, we define wavelets by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0, \tag{2.9}$$

where $a$ is called a *scaling parameter* which measures the degree of compression or scale and $b$ is a *translation parameter* which determines the time location of the wavelet.

If we restrict the parameter $a$ discrete value as $a = a_0^m$, where $m \in \mathbb{Z}$ and the dilation step $a_0 \neq 1$ is fixed. Then, for $m = 0$, it becomes natural as well to discretize $b$ by taking only the integer multiples of one fixed $b_0$, where $b_0$ is approximately chosen so that the $\psi(t - nb_0)$ cover the whole line. For different values of $m$, the width $a_0^{-m/2}\psi(a_0^- mt)$ is $a_0^m$ times the width of $\psi(t)$, so that the choice $b = nb_0 a_0^m$, $m, n \in \mathbb{Z}$ will ensure that the discretized wavelets at level $m$ cover the whole line in the same way as the $\psi(t - nb_0)$ do. Thus, we choose $a = a_0^m, b = nb_0 a_0^m$, where the two positive constants $a_0$ and $b_0$ are fixed. With these choices of $a$ and $b$, the continuous family of wavelets $\psi_{a,b}$ as defined in (2.9) becomes

$$\psi_{m,n}(x) = a_0^{-m/2} \psi\left(\frac{t - nb_0 a_0^m}{a_0^m}\right) = a_0^{-m/2} \psi(a_0^{-m} x - nb_0), \tag{2.10}$$

where both $m$ and $n \in \mathbb{Z}$. Then, for $f \in L^2(\mathbb{R})$, we calculate the discrete wavelet coefficients $\langle f, \psi_{m,n} \rangle$.

2.3. **Taylor wavelets.** Taylor wavelets $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ have four arguments: $\hat{n} = n - 1$, $n = 1, 2, \ldots, 2^{k-1}$, $k$ can assume any positive integer, $m$ is the order for Taylor polynomials, and $x$ is the normalized time. We define them on the interval $[0, 1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{T}_m(2^{k-1}x - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \leq x < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{Otherwise}, \end{cases} \tag{2.11}$$

with

$$\tilde{T}_m(x) = \sqrt{2m+1}\, T_m(x),$$

where $m = 0, 1, 2, \dots, M-1$ and $n = 1, 2, \dots, 2^{k-1}$. The coefficient $\sqrt{2m+1}$ is for normality, the dilation parameter is $a = 2^{-(k-1)}$, and the translation parameter is $b = \hat{n}2^{-(k-1)}$. $T_m(x)$ are the Taylor polynomials of order $m$, which can be defined by $T_m(x) = x^m$. [29]

The six Taylor wavelets corresponding to $k = 2$ with the order $m < 3$ are the following:

$$\psi_{1,0}(x) = \begin{cases} \sqrt{2}, & \text{if } 0 \le x < \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

$$\psi_{2,0}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ \sqrt{2}, & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

$$\psi_{1,1}(x) = \begin{cases} 2\sqrt{6}x, & \text{if } 0 \le x < \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

$$\psi_{2,1}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ \sqrt{6}(2x-1), & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

$$\psi_{1,2}(x) = \begin{cases} 4\sqrt{10}x^2, & \text{if } 0 \le x < \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

$$\psi_{2,2}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ \sqrt{10}(2x-1)^2, & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

2.4. **Function approximation.** It is possible to expand a square-integrable function $f(x)$ defined over $[0, 1]$, by the Taylor wavelets as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \psi_{n,m}(x), \tag{2.12}$$

where

$$c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle.$$

If we truncate the series (2.12), we obtain:

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = \mathbf{C}^T \Psi(x), \tag{2.13}$$

here, the coefficient vector $C$ and the Taylor wavelet function vector $\mathbf{\Psi}(x)$ are $m' = 2^{k-1}M$ column vectors given by

$$\mathbf{C} = \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1},M-1}\right]^T, \tag{2.14}$$

$$\mathbf{\Psi}(x) = \left[\psi_{10}(x), \psi_{11}(x), \dots, \psi_{1M-1}(x), \psi_{20}(x), \dots, \psi_{2M-1}(x), \dots, \psi_{2^{k-1}0}(x), \dots, \psi_{2^{k-1},M-1}(x)\right]^T. \tag{2.15}$$

For clarity, Eq. (2.13) may be rewrite as

$$f(x) \approx \sum_{i=1}^{m'} c_i \psi_i = C^T \Psi(x), \tag{2.16}$$

where $c_i = c_{n,m}, \psi_i(x) = \psi_{n,m}(x)$. The index $i$ can be derived by $i = M(n-1)+m+1$, thus

$$\mathbf{C} = [c_1, c_2, c_3, \ldots, c_{m'}]^T, \tag{2.17}$$

$$\mathbf{\Psi}(x) = [\psi_1, \psi_2, \psi_3, \ldots, \psi_{m'}]^T. \tag{2.18}$$

By applying the collocation points $x_i = \frac{2i-1}{2^k M}$, $i = 1, 2, 3, \ldots, 2^{k-1}M$, we consider the Taylor wavelet matrix $\Phi(x)_{m' \times m'}$ as

$$\Phi_{m' \times m'} = \Big[\mathbf{\Psi}\big(\frac{1}{2m'}\big), \mathbf{\Psi}\big(\frac{3}{2m'}\big), \ldots, \mathbf{\Psi}\big(\frac{2m'-1}{2m'}\big)\Big],$$

where $m' = 2^{k-1}M$. For example, for $M = 2$ and $k = 2$, the Taylor wavelet matrix is specified as

$$\Phi_{4\times 4} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 & 0 \\ \frac{\sqrt{\frac{3}{2}}}{2} & \frac{3\sqrt{\frac{3}{2}}}{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{3\sqrt{\frac{3}{2}}}{2} \end{bmatrix}.$$

Furthermore, a function $u(x,t) \in L_2([0,1]\times[0,1])$ may be also approximated as

$$u(x,t) = \mathbf{\Psi}^T(x) U \mathbf{\Psi}(t), \tag{2.19}$$

which $U$ is $m' \times m'$ matrix with $u_{ij} = \big\langle \psi_i(x), \langle u(x,t), \psi_j(t)\rangle\big\rangle$. To evaluate the coefficients $u_{i,j}$, we use the wavelet collocation method.

2.5. **The fractional integral of Taylor wavelet.** The formula for the fractional integral of the Taylor wavelet is obtained in the Riemann-Liouville sense by employing the Taylor polynomials $T_m$.

**Theorem 2.1.** *On the interval* $[0,1]$, *the fractional integral of Taylor wavelets which have the compact support* $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}])$, *is given by:*

$$I^\alpha \psi_{n,m}(t) = \begin{cases} 0, & t < \frac{2n-2}{2^k} \\ \frac{2^{\frac{k-1}{2}}\sqrt{2m+1}\, m! \left(2^{k-1}\right)^m \left(t - \frac{2n-2}{2^k}\right)^{\alpha+m}}{\Gamma(\alpha+m+1)}, & \frac{2n-2}{2^k} \le t \le \frac{2n}{2^k} \\ 2^{\frac{k-1}{2}}\sqrt{2m+1}\left(\frac{m!\left(2^{k-1}\right)^m \left(t-\frac{2n-2}{2^k}\right)^{\alpha+m}}{\Gamma(\alpha+m+1)} - \sum_{r=0}^{m} \frac{m!\left(2^{k-1}\right)^{m-r}\left(t-\frac{2n}{2^k}\right)^{\alpha+m-r}}{r!\,\Gamma(\alpha+m-r+1)}\right), & \frac{2n}{2^k} < t. \end{cases} \tag{2.20}$$

*Proof.* For Taylor polynomials

$$T_m(t) = t^m, \tag{2.21}$$

we obtain the operator $I^\alpha$ for $\Psi(t)$

$$I^\alpha \Psi(t) = P_t^\alpha.$$

To derive $I^\alpha \psi_{n,m}(t)$, we use the so-called Laplace transform. We derive the relation for the Taylor wavelets in the following manner:

$$\psi_{n,m}(t) = 2^{\frac{k-1}{2}}\sqrt{2m+1}\Bigg(\nu_{\frac{2n-2}{2^k}}(t) T_m(2^{k-1}t-(n-1)) - \nu_{\frac{2n}{2^k}}(t) T_m(2^{k-1}t-(n-1))\Bigg), \tag{2.22}$$

where $\nu_c(t)$ is the Heaviside function:

$$\nu_c(t) = \begin{cases} 1, & t \ge c, \\ 0, & t < c. \end{cases}$$

Using the Laplace transform from Eq. (2.22), we obtain

$$\begin{aligned}\mathcal{L}\{\psi_{n,m}(t)\} &= 2^{\frac{k-1}{2}}\sqrt{2m+1}\mathcal{L}\Big\{\nu_{\frac{2n-2}{2^k}}(t)T_m(2^{k-1}(t-\frac{2n-2}{2^k})) - \nu_{\frac{2n}{2^k}}(t)T_m(2^{k-1}(t-\frac{2n}{2^k})+1)\Big\} \\ &= 2^{\frac{k-1}{2}}\sqrt{2m+1}e^{-\frac{2n-2}{2^k}s}\mathcal{L}\Big\{T_m(2^{k-1}t)\Big\} - 2^{\frac{k-1}{2}}\sqrt{2m+1}e^{-\frac{2n}{2^k}s}\mathcal{L}\Big\{T_m(2^{k-1}t+1)\Big\}.\end{aligned} \tag{2.23}$$

From Eq. (2.21), we have

$$\begin{aligned}\mathcal{L}\{\psi_{n,m}(t)\} &= 2^{\frac{k-1}{2}}\sqrt{2m+1}\Big[e^{-\frac{2n-2}{2^k}s}\mathcal{L}\Big\{(2^{k-1}t)^m\Big\} - e^{-\frac{2n}{2^k}s}\mathcal{L}(2^{k-1}t+1)^m\Big] \\ &= 2^{\frac{k-1}{2}}\sqrt{2m+1}\Big[e^{-\frac{2n-2}{2^k}s}\mathcal{L}\Big\{(2^{k-1}t)^m\Big\} - e^{-\frac{2n}{2^k}s}\mathcal{L}\Big\{\sum_{r=0}^{m}\frac{m!}{r!(m-r)!}(2^{k-1})^{m-r}t^{m-r}\Big\}\Big] \\ &= 2^{\frac{k-1}{2}}\sqrt{2m+1}\Big[e^{-\frac{2n-2}{2^k}s}\frac{m!(2^{k-1})^m}{s^{m+1}} - e^{-\frac{2n}{2^k}s}\sum_{r=0}^{m}\frac{m!}{r!s^{m-r+1}}(2^{k-1})^{m-r}\Big] \\ &= 2^{\frac{k-1}{2}}\sqrt{2m+1}e^{-\frac{2n}{2^k}s}\Big[\frac{e^{\frac{2n}{2^k}s}m!(2^{k-1})^m}{s^{m+1}} - \sum_{r=0}^{m}\frac{m!}{r!s^{m-r+1}}(2^{k-1})^{m-r}\Big].\end{aligned}$$

By using the Riemann-Liouville fractional integral operator of order $\alpha$: $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t)$, where $*$ denotes the convolution product, we get

$$\begin{aligned}\mathcal{L}\{I^\alpha\psi_{n,m}(t)\} &= \mathcal{L}\Big\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big\}\mathcal{L}\{\psi_{n,m}(t)\} \\ &= 2^{\frac{k-1}{2}}\sqrt{2m+1}e^{-\frac{2n}{2^k}s}\Big\{\frac{e^{\frac{2n}{2^k}s}m!(2^{k-1})^m}{s^{m+1+\alpha}} - \sum_{r=0}^{m}\frac{m!}{r!s^{m-r+1+\alpha}}(2^{k-1})^{m-r}\Big\}.\end{aligned} \tag{2.24}$$

Applying the inverse Laplace transform of Eq. (2.24) yields

$$\begin{aligned}I^\alpha\psi_{n,m}(t) &= 2^{\frac{k-1}{2}}\sqrt{2m+1}\mathcal{L}^{-1}\Big\{\frac{e^{-\frac{2n-2}{2^k}s}}{s^{m+1+\alpha}} - \sum_{r=0}^{m}\frac{e^{-\frac{2n}{2^k}s}m!}{r!s^{m-r+1+\alpha}}(2^{k-1})^{m-r}\Big\} \\ &= 2^{\frac{k-1}{2}}\sqrt{2m+1}\Big(\frac{\nu_{\frac{2n-2}{2^k}}(2^{k-1})^m(t-\frac{2n-2}{2^k})^{m+\alpha}}{\Gamma(m+1+\alpha)} - \sum_{r=0}^{m}\frac{m!(2^{k-1})^{m-r}\nu_{\frac{2n}{2^k}}(t-\frac{2n}{2^k})^{m-r+\alpha}}{r!\Gamma(m-r+1+\alpha)}\Big)\end{aligned} \tag{2.25}$$

By using Eq. (2.25), we have

$$I^\alpha\psi_{n,m}(t) = \begin{cases} 0, & t < \frac{2n-2}{2^k} \\ \frac{2^{\frac{k-1}{2}}\sqrt{2m+1}m!\left(2^{k-1}\right)^m\left(t-\frac{2n-2}{2^k}\right)^{\alpha+m}}{\Gamma(\alpha+m+1)}, & \frac{2n-2}{2^k} \le t \le \frac{2n}{2^k} \\ 2^{\frac{k-1}{2}}\sqrt{2m+1}\left(\frac{m!\left(2^{k-1}\right)^m\left(t-\frac{2n-2}{2^k}\right)^{\alpha+m}}{\Gamma(\alpha+m+1)} - \sum_{r=0}^{m}\frac{m!\left(2^{k-1}\right)^{m-r}\left(t-\frac{2n}{2^k}\right)^{\alpha+m-r}}{r!\Gamma(\alpha+m-r+1)}\right), & \frac{2n}{2^k} < t, \end{cases} \tag{2.26}$$

and this complete the proof. □

Using the collocation points in Eq. (2.20), we obtain the fractional order integration matrix $P^\alpha_{m'\times m'} = I^\alpha\psi_{n,m}(x)$, then

$$P^\alpha_{2^{k-1}M\times 2^{k-1}M} = \begin{pmatrix} I^\alpha\psi_{1,0}(x(1)) & I^\alpha\psi_{1,0}(x(2)) & \dots & I^\alpha\psi_{1,0}(x(2^{k-1}M)) \\ I^\alpha\psi_{1,1}(x(1)) & I^\alpha\psi_{1,1}(x(2)) & \dots & I^\alpha\psi_{1,1}(x(2^{k-1}M)) \\ \vdots & \vdots & \ddots & \vdots \\ I^\alpha\psi_{2^{k-1},M}(x(1)) & I^\alpha\psi_{2^{k-1},M}(x(2)) & \dots & I^\alpha\psi_{2^{k-1},M}(x(2^{k-1}M)) \end{pmatrix}.$$

For instance, we fix $k = 2, M = 2$ and $\alpha = 0.8$, then

$$P^{\alpha}_{4\times 4} = \begin{bmatrix} 0.287683 & 0.692806 & 0.754851 & 0.671755 \\ 0.0692058 & 0.499989 & 0.686486 & 0.597872 \\ 0 & 0 & 0.287683 & 0.692806 \\ 0 & 0 & 0.0692058 & 0.499989 \end{bmatrix}.$$

Another fractional integration operational matrix can be obtain to solve the fractional boundary value problems. Assume $\eta > 0$ and $g : [0, \eta] \to \mathbb{R}$ be a continuous function, put

$$g(x) I^{\alpha} \psi_{n,m}(\eta) = v^{\alpha,\eta}. \tag{2.27}$$

We introduce a matrix $V$ by using the collocation points $x_i = \frac{2i-1}{2^k M}, i = 1, 2, \ldots, 2^{k-1}M$ in Eq. (2.27), we have

$$V^{\alpha,\eta,g(x)}_{2^{k-1}M\times 2^{k-1}M} = \begin{pmatrix} g(x_1)I^{\alpha}\psi_{1,0}(\eta) & g(x_2)I^{\alpha}\psi_{1,0}(\eta) & \ldots & g(x_{2^{k-1}M})I^{\alpha}\psi_{1,0}(\eta) \\ g(x_1)I^{\alpha}\psi_{1,1}(\eta) & g(x_2)I^{\alpha}\psi_{1,1}(\eta) & \ldots & g(x_{2^{k-1}M})I^{\alpha}\psi_{1,1}(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_1)I^{\alpha}\psi_{2^{k-1},M-1}(\eta) & g(x_2)I^{\alpha}\psi_{2^{k-1},M-1}(\eta) & \ldots & g(x_{2^{k-1}M})I^{\alpha}\psi_{2^{k-1},M-1}(\eta) \end{pmatrix}.$$

For instance, for $\eta = 1, g(x) = x, \alpha = 0.9, k = 2$ and $M = 3$

$$V^{\alpha,\eta,g(x)}_{2^{k-1}M\times 2^{k-1}M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2.6. **Convergence.** The goal of this section is to discuss the error analysis of the present method. Therefore, we present the following theorems.

**Theorem 2.2.** [5] *Suppose a continuous function $f(x,t) \in L^2(\mathbb{R}^2)$ defined on $[0,1) \times [0,1)$ be bounded by $K$, then the approximation of $f(x,t)$ by the Taylor wavelets converges uniformly.*

**Theorem 2.3.** [5] *If a continuous function $f(x,t) \in L^2(\mathbb{R}^2)$ defined on $[0,1)^2$ be bounded, i.e. $\|f(x,t)\| \le K$, then*

$$||E_{u,kM,k'M'}||_{L^2([0,1]\times[0,1])} \le \Big[ \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \sum_{n'=2^{k'-1}+1}^{\infty} \sum_{m'=M}^{\infty} (\lambda^{k,k'}_{nm,n'm'} K)^2 \Big]^{\frac{1}{2}},$$

*where* $\lambda^{k,k'}_{nm,n'm'} = \frac{\sqrt{(2m+1)(2m'+1)}}{2^{\frac{k+k'-2}{2}}} \Big( \frac{1-(-1)^{m+1}}{m+1} \Big) \Big( \frac{1-(-1)^{m'+1}}{m'+1} \Big)$.

## 3. Description of the proposed method

Now, we propose the procedure of constructing a method for the generalized Burgers-Fisher equation with time-fractional derivative.

The generalized Burgers-Fisher equation with time-fractional derivative is the following equation:

$$u^{\alpha}_t - u_{xx} = bu(1-u^{\delta}) - au^{\delta}u_x, \qquad x \in [0,1], \quad t \ge 0, \quad 0 < \alpha \le 1 \tag{3.1}$$

with the conditions

$$\begin{aligned} & u(x,0) = g(x), \\ & u(0,t) = f_0(t), \qquad u(1,t) = f_1(t), \qquad 0 \le x, t \le 1, \end{aligned} \tag{3.2}$$

where $a, b, \delta$ are constants. Applying the proposed method to Eq. (3.1), we suppose

$$\frac{\partial^{2+\alpha} u(x,t)}{\partial x^2 \partial t^{\alpha}} \approx \Psi^T(x) U \Psi(t), \tag{3.3}$$

where $U = (u_{ij})_{2^{k-1}M\times 2^{k-1}M}$ is an undetermined matrix, and $\Psi(\cdot)$ is the Taylor wavelet matrix. By using the fractional operator $I^{\alpha}_t$ on Eq. (3.3), we have:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \Big( \frac{\partial^2 u(x,t)}{\partial x^2} \Big)_{t=0} + \Psi^T(x) U P^{\alpha}_t, \tag{3.4}$$

using initial condition, we have

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx g^{''}(x) + \Psi^T(x) U P_t^{\alpha}, \tag{3.5}$$

by integrating of Eq. (3.5), two times with respect to $x$, we have

$$u(x,t) \approx u(0,t) + x\Big(\frac{\partial u(x,t)}{\partial x}\Big)_{x=0} + g(x) - g(0) - xg'(0) + \int_0^x \int_0^s \Psi^T(s_1) ds_1 ds.UP_t^{\alpha}, \tag{3.6}$$

we put $x = 1$ in Eq. (3.6) and apply boundary condition, then

$$u(1,t) \approx u(0,t) + \Big(\frac{\partial u(x,t)}{\partial x}\Big)_{x=0} + g(1) - g(0) - g'(0) + \int_0^1 \int_0^s \Psi^T(s_1) ds_1 ds.UP_t^{\alpha}, \tag{3.7}$$

so

$$\Big(\frac{\partial u(x,t)}{\partial x}\Big)_{x=0} \approx f_1(t) - f_0(t) + g(0) + g'(0) - g(1) - \int_0^1 \int_0^s \Psi^T(s_1) ds_1 ds.UP_t^{\alpha}, \tag{3.8}$$

for simplicity,

$$H(t) = f_1(t) - f_0(t) + g(0) + g'(0) - g(1) - \int_0^1 \int_0^s \Psi^T(s_1) ds_1 ds UP_t^{\alpha}. \tag{3.9}$$

For $u(x,t)$ we get

$$u(x,t) \approx f_0(t) + xH(t) + g(x) - g(0) - xg'(0) + \int_0^x \int_0^s \Psi^T(s_1) ds_1 ds UP_t^{\alpha}, \tag{3.10}$$

by differentiating with respect to $x$ in Eq. (3.10), we have

$$\frac{\partial u(x,t)}{\partial x} \approx H(t) + g'(x) - g'(0) + \int_0^x \Psi^T(s) ds UP_t^{\alpha}, \tag{3.11}$$

by taking fractional differentiation of order $\alpha$ in Eq. (3.10) with respect to $t$:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} \approx D^{\alpha} f_0(t) + xD^{\alpha} H(t) + \int_0^x \int_0^s \Psi^T(s_1) ds_1 ds \, U\Psi(t), \tag{3.12}$$

where $D^{\alpha}H(t) = D^{\alpha}f_1(t) - D^{\alpha}f_0(t) - \int_0^1 \int_0^s \Psi^T(s_1) ds_1 ds \, U\Psi(t)$.

For simplicity, we put the right-hand-side of (3.1) equal to $S(x,t)$:

$$S(x,t) = bu_r(1 - u_r^{\delta}) - au_r^{\delta}(u_r)_x = \sum_{i=1}^{m'} \sum_{j=1}^{m'} m_{i,j} \psi_i(x) \psi_j(t) = \Psi^T(x) M \Psi(t), \tag{3.13}$$

where $m_{i,j} = \langle \psi_i(x), \langle S(x,t), \psi_j(t) \rangle \rangle$. Substituting Eqs. (3.10)–(3.13) and Eq. (3.5) into Eq. (3.1), changing $\approx$ by $=$, and applying the collocation points $x_i = \frac{2i-1}{2^k M}, t_j = \frac{2j-1}{2^k M}, i,j = 1,2,3,\dots,2^{k-1}M$ , we get a linear system of algebraic equations as follows:

$$D^{\alpha} f_0(t_j) + x_i D^{\alpha} H(t_j) + (P_{x_i}^2)^T U \Psi(t_j) = g^{''}(x_i) + \Psi^T(x_i) U P_{t_j}^{\alpha} + \Psi^T(x_i) M \Psi(t_j),, \tag{3.14}$$

where $\int_0^1 \int_0^s \Psi^T(s_1) ds_1 ds$ is a $2^{k-1}M \times 2^{k-1}M$ matrix of fractional order integration for boundary value problems by the Taylor wavelet (2.27), $(P_{x_i}^2)^T$ is $\int_0^x \int_0^s \Psi^T(s_1) ds_1 ds$, and $(P_{x_i}^1)^T$ is $\int_0^x \Psi^T(s_1) ds_1$.

Upon solving the system and determining $U$, we can derive an approximate solution by plugging $U$ into Eq. (3.10).

3.1. **Numerical Examples.** To demonstrate the efficiency of the Taylor wavelet collocation method (TWCM) for the generalized Burger and Burgers-Fisher equations, we solve some numerical examples for different values of $a, b, \delta$, and $\alpha$. The proposed method is compared with the Haar wavelet Picard method (HWPM) [40] to demonstrate its capability.

**Example 3.1.** We consider Eq. (3.1) with the following initial and boundary conditions:

$$u(x,0) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}x\right)\right)^{\frac{1}{\delta}},$$

$$u(0,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}\left[-\left(\frac{a^2+b(1+\delta)^2)}{a(1+\delta)}\right)t\right]\right)\right)^{\frac{1}{\delta}},$$

$$u(1,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}\left[1-\left(\frac{a^2+b(1+\delta)^2)}{a(1+\delta)}\right)t\right]\right)\right)^{\frac{1}{\delta}},$$

which the exact solution for $\alpha = 1$ is ([18, 37])

$$u(x,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}\left[x-\left(\frac{a^2+b(1+\delta)^2}{a(1+\delta)}\right)t\right]\right)\right)^{\frac{1}{\delta}}.$$

We suppose $u_0(x,t)$ as an initial approximation and apply the TWCM for different values of $a, b, \alpha$, and $\delta = 1$. Table 1 shows the absolute error for the approximate solutions obtained with different values of $\alpha, a$, and $b$ which the solutions at the different values of $\alpha$ converge to the exact solution at $\alpha = 1$ when $\alpha$ approaches 1. The results which are obtained by using the present method were compared with the Haar wavelet Picard method (HWPM) [37], and are shown in Table 2.

Table 1. Absolute error of approximate solutions obtained by using TWCM with $\delta = 1, M = 8, K = 2$ for different values $\alpha$ for Example 3.1

| | $\delta = 1 \quad J = 4 \quad a = 0.01, \; b = 0.01$ | | | $\delta = 1 \quad J = 4 \quad a = 0.5, \; b = 0.5$ | | |
|---|---|---|---|---|---|---|
| $(x,t)$ | $\alpha = 0.6$ | $\alpha = 0.9$ | $\alpha = 1$ | $\alpha = 0.6$ | $\alpha = 0.9$ | $\alpha = 1$ |
| $(\frac{1}{32}, \frac{1}{32})$ | 4.5205e-08 | 3.3245e-08 | 1.0019e-08 | 2.7754e-08 | 1.0041e-08 | 2.1744e-10 |
| $(\frac{7}{32}, \frac{7}{32})$ | 3.6311e-07 | 2.9908e-07 | 1.1157e-07 | 2.4961e-07 | 1.0982e-07 | 2.5308e-09 |
| $(\frac{13}{32}, \frac{13}{32})$ | 1.2902e-06 | 9.7632e-07 | 2.7286e-07 | 1.9755e-06 | 1.0041e-06 | 1.4702e-08 |
| $(\frac{19}{32}, \frac{19}{32})$ | 9.2368e-06 | 4.3876e-06 | 9.8771e-07 | 7.7541e-06 | 1.0011e-06 | 8.0278e-08 |
| $(\frac{25}{32}, \frac{25}{32})$ | 4.6208e-06 | 1.2366e-06 | 5.4407e-07 | 6.2517e-06 | 2.2346e-06 | 1.9275e-08 |
| $(\frac{31}{32}, \frac{31}{32})$ | 4.7193e-07 | 1.0199e-07 | 7.5547e-08 | 1.2533e-07 | 1.0045e-07 | 8.8047e-09 |

Table 2. Comparison of approximate solutions by obtained using TWCM with Haar wavelet Picard method (HWPM) for $M = 8, K = 2$ in Example 3.1

| | $\delta = 1 \quad J = 4 \quad a = b = 0.5, \; \alpha = 1$ | | $\delta = 1 \quad J = 4 \quad a = b = 0.01, \; \alpha = 1$ | |
|---|---|---|---|---|
| (x,t) | TWCM | HWP[40] | TWCM | HWPM[40] |
| $(\frac{1}{32}, \frac{1}{32})$ | 2.1744e-10 | 1.035e-07 | 1.2973e-16 | 7.136e-13 |
| $(\frac{7}{32}, \frac{7}{32})$ | 2.5308e-09 | 3.454e-07 | 1.9987e-15 | 1.800e-12 |
| $(\frac{13}{32}, \frac{13}{32})$ | 1.4702e-08 | 8.077e-07 | 3.2177e-15 | 4.531e-12 |
| $(\frac{19}{32}, \frac{19}{32})$ | 8.0278e-08 | 8.634e-07 | 3.7963e-15 | 4.908e-12 |
| $(\frac{25}{32}, \frac{25}{32})$ | 1.9275e-08 | 6.102e-07 | 5.7517e-15 | 3.553e-12 |
| $(\frac{31}{32}, \frac{31}{32})$ | 8.8047e-09 | 1.356e-07 | 9.1127e-16 | 8.322e-13 |

**Example 3.2.** For $b = 0$, Eq. (3.1) is reduced to the generalized Burger equation. We have taken different values of $a, \delta$. Table 3 shows the absolute error for the approximate solutions obtained with different values of $\alpha$, where the solutions at different values of $\alpha$ converge to the exact solution at $\alpha = 1$, as $\alpha$ approaches 1.

The results of the present method were compared with the Haar wavelet Picard method (HWPM) [40] in Table 4.

TABLE 3. Absolute error of approximate solutions obtained by using TWCM with $b = 0, M = 8, K = 2$ for different value $\alpha$ and $\delta$ in Example 3.2

| | $\delta = 1 \quad J = 4 \quad a = 0.5, \quad b = 0$ | | | $\delta = 2 \quad J = 4 \quad a = 0.1, \quad b = 0$ | | |
|---|---|---|---|---|---|---|
| $(x, t)$ | $\alpha = 0.6$ | $\alpha = 0.9$ | $\alpha = 1$ | $\alpha = 0.6$ | $\alpha = 0.9$ | $\alpha = 1$ |
| $(\frac{1}{32}, \frac{1}{32})$ | 8.1755e-16 | 3.3245e-16 | 1.8641e-16 | 7.4865e-16 | 5.789e-16 | 1.4365e-16 |
| $(\frac{7}{32}, \frac{7}{32})$ | 8.5569e-15 | 4.4676e-15 | 2.6851e-15 | 6.8451e-16 | 2.2063e-16 | 2.0481e-16 |
| $(\frac{13}{32}, \frac{13}{32})$ | 9.5319e-14 | 6.9781e-15 | 3.6453e-15 | 7.5579e-16 | 2.0017e-16 | 1.8351e-16 |
| $(\frac{19}{32}, \frac{19}{32})$ | 7.4628e-14 | 5.9784e-15 | 2.9746e-15 | 6.4394e-16 | 2.0193e-16 | 1.8794e-16 |
| $(\frac{25}{32}, \frac{25}{32})$ | 3.1596e-15 | 2.5464e-15 | 1.9064e-15 | 3.2279e-16 | 2.0008e-16 | 1.0114e-16 |
| $(\frac{31}{32}, \frac{31}{32})$ | 6.9173e-16 | 4.7654e-16 | 1.8884e-16 | 8.4379e-17 | 6.7984e-17 | 2.2379e-17 |

TABLE 4. Comparison of the approximate solution obtained by using the TWCM with the Haar wavelet Picard method (HWPM) for $M = 8, K = 2$ in Example 3.2

| | $\delta = 1 \quad J = 4 \quad a = 0.1, \quad b = 0, \quad \alpha = 1$ | | $\delta = 0.5 \quad J = 4 \quad a = 0.2, \quad b = 0, \quad \alpha = 1$ | |
|---|---|---|---|---|
| (x,t) | TWCM | HWPM[40] | TWCM | HWPM[40] |
| $(\frac{1}{32}, \frac{1}{32})$ | 1.4365e-16 | 5.768e-14 | 7.0654e-14 | 4.153e-07 |
| $(\frac{7}{32}, \frac{7}{32})$ | 2.0481e-16 | 8.424e-12 | 6.831e-13 | 6.253e-05 |
| $(\frac{13}{32}, \frac{13}{32})$ | 1.8351e-16 | 1.407e-11 | 3.6372e-13 | 9.722e-05 |
| $(\frac{19}{32}, \frac{19}{32})$ | 1.8794e-16 | 1.440e-11 | 1.8324e-13 | 9.881e-05 |
| $(\frac{25}{32}, \frac{25}{32})$ | 1.0114e-16 | 1.022e-11 | 1.0011e-13 | 7.278e-05 |
| $(\frac{31}{32}, \frac{31}{32})$ | 2.2379e-17 | 1.937e-12 | 9.8046e-14 | 1.248e-05 |



## 4. CONCLUSION

In this study, we have successfully applied a combination of the operational matrix of the Taylor wavelet based on the collocation method to solve the time-fractional Burgers-Fisher equation.

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