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Optimal solution of the nonlinear time fractional diffusion-wave equation using generalized Laguerre polynomials

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Abstract

Determining the numerical solutions of a nonlinear fractional differential equation has been of interest for a long time. This study presents an optimization method with Lagrange coefficients based on operational matrices of derivatives to approximate the solution by selecting appropriate basis functions as a linear combination of generalized Laguerre polynomials (GLPs). Achieving an exact solution using fewer basis functions is one of the outstanding features of this method. This feature and high accuracy make the use of this method inevitable. Finally, the application of the mentioned method is investigated in determining the approximate solution of the nonlinear fractional time diffusion-wave equation for different values of α .

Keywords. Fractional calculus, Fractional diffusion-wave equation, Generalized Laguerre polynomials, Operational matrix, Caputo fractional derivative, Control parameters, Free coefficients.

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1. Introduction

Fractional calculus is a part of mathematical analysis that studies derivative and integral operations on real numbers or numbers with fractional order. In this way, differential equations with fractional derivatives can be defined and physical systems can be modeled with it. Due to the importance and many applications of these equations, Various numerical methods have been used to solve this category of equations, among which the finite difference method [22], sub-equation method [23], Generalized fuzzy functions method [21] and the trapezoidal method [20] can be mentioned.

An important class of fractional equations are nonlinear fractional partial differential equations such as Klein-Gordon's equation, Sine-Gordon's equation, the fractional convection-(or advection-) diffusion equation (FCDE; or FADE) and diffusion-wave equation. Often, due to the presence of nonlinear and complex sentences in these types of equations, it is difficult or impossible to find an analytical solution for them. The absence of an analytical solution for such complex and nonlinear equations has led to the creation and expansion of numerical methods. The most important parameters for evaluating numerical methods are speed and accuracy of solving the equation. Due to the importance of using nonlinear fractional equations in physics and applied mathematics and the significant improvement in the speed and capacity of information processing in processors and computers in the late 19th century, numerical methods have also become more widespread and this improvement and expansion continues. Among the methods of solving this category of equations are: method based on Fibonacci polynomials [7], Generalized shifted Chebyshev polynomials [8], B-Spline functions [19], a new hybrid method resulting from the combination of Legendre polynomials and piecewise Legendre polynomials [11] and the new finite difference method for variable-order nonlinear fractional diffusion equation [14]. For a comprehensive list of the many methods available for the numerical solution of the FCDE, see the compilation of Li and Wang [5]. Akbari-Ganji's method (AGM) is a method that Ganji et al. have used to find the analytical solution of Bagley equation and some other nonlinear fractional differential equations using

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polynomials [2]. Ordokhani et al. obtained the approximate solution to the fractional ray equations and fractional Duffing-van der Pol oscillator equations using the Chelyshkov wavelets method [15]. In [16, 17], a technique based on general Lagrange scaling functions for time-fractional diffusion-wave equations and variable order fractional partial differential equations is described. In a study conducted by Bonyadi and colleagues, spectral shifted Jacobi collocation method was used to approximate the solution of space-time fractional partial differential equations with variable coefficients [4]. In [18], the author investigated approximate solutions to the Two-dimensional temporal fractional advection-diffusion problem using the Sinc-Galerkin method. Recent studies by Lakstani et al. have been presented on analytical soliton solutions of the space-time fractional nonlinear Schrödinger equations [13]. A Pseudospectral approach based on Chebyshev functions has been used to solve fractional Sturm-Liouville problem by Afarideh and colleagues [1]. Jannelli et al. also conducted research on the solution of fractional convection diffusion equation using Lie symmetry [12]. In addition, Wang and other colleagues in 2023 investigated the solution of wavelet and propagation equations by the wavelet method [24]. In recent years, optimization using polynomials and generalized polynomials has become a common problem. Excellent algorithms with the above idea for fractional problems including nonlinear variable-order diffusion-wave equation are presented by Hassani, dahaghin et al in [6, 9, 10]. Other examples of these methods for solving fractional problems according to the most common types Polynomials, i.e. generalized Laguerre polynomials, see in [3].

The wave-diffusion equation is one of the prominent mathematical equations that is often used to describe wave and diffusion phenomena in the field of physics and engineering. Now, in order to introduce the studied problem, we define the nonlinear time fractional diffusion-wave equations (N-TFDWE) along with the boundary and initial conditions as follows:

$$ {}_a^C D_\tau^\alpha \mathcal{V}(x,\tau) + \mu_1 \mathcal{V}_\tau(x,\tau) + \mu_2 \mathcal{V}^\eta(x,\tau) = \frac{\partial^2 \mathcal{V}(x,\tau)}{\partial x^2} + F(x,\tau), \ (x,\tau) \in [0,1]\times[0,1], \ \alpha \in (1,2], \tag{1.1} $$

$$ \begin{cases} \mathcal{V}(x,0) = f_1(x), \quad \mathcal{V}_\tau(x,0) = f_2(x), \\ \mathcal{V}(0,\tau) = g_1(\tau), \quad \mathcal{V}(1,\tau) = g_2(\tau). \end{cases} \tag{1.2} $$

Where $\mu_1, \mu_2 \in \mathbb{R}$ are constant coefficients, $\eta = 2 \ or \ 3$, respectively represent the quadratic or cubic nonlinear term. $f_1(x), f_2(x), g_1(\tau), g_2(\tau), F(x,\tau)$ are the known functions and ${}_a^C D_\tau^\alpha$ is the Caputo's fractional derivative operator.

The aim of this study is to investigate the numerical solution of the nonlinear time fractional diffusion-wave equation with the method based on the generalization of Laguerre polynomials. Based on this, the method is designed in such a way that it is possible to calculate operator matrices for ordinary and fractional derivatives. After that, by approximating the unknown function and its derivatives in the equation and using operator matrices, the remaining function is obtained in terms of free coefficients and control parameters. By defining the quadratic norm and optimizing it, in order to simplify the obtained nonlinear approximation and turn it into an algebraic equation system, the method of Lagrange coefficients is used, which by calculating coefficients and unknown parameters, an approximate solution for the studied problem is achieved. This paper is organized in several sections as follows:

The second part includes an overview of some definitions and features used related to key concepts. In the third part, we will introduce the basic functions as the basis for the construction of the proposed method and the approximation function of the solution. The next section is dedicated to the introduction of Operational matrices. In the fifth and sixth sections, the design and implementation of the approximation method based on the previous sections are examined, respectively. In the seventh part, we present the solution of some examples of the propagation wave Equations (1.1) according to the optimization algorithm to confirm the correctness and efficiency of the method. In the last part, we will express the general result.

2. Basic concepts

In this section, we will introduce the basic concepts about Caputo fractional derivative (CFD).

Definition 2.1. The CFD operator, ${}_a^C D_\tau^\alpha$ of order $n-1<\alpha\leq n$ corresponding to function $f(\tau)$ is [10]:

$$
{}_a^C D_\tau^\alpha f(\tau) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^\tau (\tau-\vartheta)^{n-\alpha-1} f^{(n)}(\vartheta) d\vartheta, & \alpha \in (n, n-1),\ \ \tau > 0, \\ f^{(n)}(\tau), & \alpha = n, \end{cases} \tag{2.1}
$$

in which $\Gamma(.)$ is the gamma function and for it we have $\Gamma(z)=\int_0^\infty \tau^{z-1}\mathbf{e}^{-\tau}d\tau,\ z\in\mathbb{C},\ \Re(z)>0$. It is noteworthy that for $n\in\mathbb{Z}^+$, we will have $\Gamma(n)=(n-1)!$.

Proposition 2.2. *Let $k\in\mathbb{N}\cup\{0\}$ and $n-1<\alpha\leq n$, then we have*

$$
{}_a^C D_\tau^\alpha \tau^k = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\tau^{k-\alpha}, & n\leq k\in\mathbb{N}, \\ 0, & n>k. \end{cases} \tag{2.2}
$$

3. Basis functions (Their definition and properties)

In this section, along with the definition of Laguerre polynomials, we will introduce an important class of generalized polynomials (GPs) called generalized Laguerre polynomials (GLPs). In the following, by using GLPs, we obtain the approximation of the solution function and the operational matrices related to their derivatives.

3.1. Laguerre polynomials.

The Laguerre polynomials are as follows:

$$
L_n(\tau) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(k!)^2(n-k)!}\tau^k, \tag{3.1}
$$

and according to this

$$
\begin{cases} L_0(\tau) = 1, \\ L_1(\tau) = -\tau+1, \\ L_2(\tau) = \frac{1}{2}(\tau^2-4\tau+2), \\ L_3(\tau) = \frac{1}{6}(-\tau^3+9\tau^2-18\tau+6). \end{cases} \tag{3.2}
$$

For any arbitrary function $u(\tau)\in L^2[0,1]$, using the first $n+1$ Laguerre polynomials, we can write:

$$
u(\tau) \simeq \sum_{i=0}^{n} a_i L_i(\tau) = \mathrm{A}^T \mathbb{L}_n(\tau) = \mathrm{A}^T B T_n(\tau), \tag{3.3}
$$

where

$$
A = [a_0\ \ a_1\ \ \ldots\ \ a_n]^T,
$$

$$
\mathbb{L}_n(\tau) = [L_0\ \ L_1\ \ \ldots\ \ L_n]^T = BT_n(\tau), \quad T_n(\tau) = [1\ \ \tau\ \ \ldots\ \ \tau^n],
$$

and

$$
B = [b_{00}\ \ b_{01}\ \ b_{02}\ \ \ldots\ \ b_{0n} | b_{10}\ \ b_{11}\ \ b_{12}\ \ \ldots\ \ b_{1n} | \ldots | b_{n0}\ \ b_{n1}\ \ b_{n2}\ \ \ldots\ \ b_{nn}], \quad b_{\imath\jmath} = \begin{cases} \dfrac{(-1)^\jmath \imath!}{(\jmath!)^2(\imath-\jmath)!}, & \imath\geq\jmath, \\ 0, & \text{Otherwise.} \end{cases}
$$

3.2. GLPs and function approximation.

Generalized Laguerre polynomials, $\ell_n(\tau)$, are easily obtained by substituting the invariant variable $\tau^{\imath+\beta_\imath}$ $(i+\beta_\imath>0)$ instead of $\tau^\imath$ in (1.1). Therefore, the relation (1.1) becomes as follows

$$
\ell_n(\tau) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(k!)^2(n-k)!}\tau^{k+\beta_k}, \tag{3.4}
$$

in which β_k are control parameters. Note that for $\beta_k=0$, GLPs are the same as Laguerre polynomials.

Now, according to the expansion of the $\mathcal{V}(\tau)$ function using GLPs, we have

$$\mathcal{V}(\tau) \simeq \sum_{i=0}^{n} \mathrm{v}_i \ell_i(\tau) = \mathrm{V}^T \widehat{\mathbb{L}}_n(\tau) = \mathrm{V}^T M \Theta(\tau),$$

where

$$\mathrm{V} = [\mathrm{v}_0 \ \ \mathrm{v}_1 \ \ \ldots \ \ \mathrm{v}_n]^T,$$

$$\widehat{\mathbb{L}}_n(\tau) = [\ell_0 \ \ \ell_1 \ \ \ldots \ \ \ell_n]^T = M\Theta(\tau), \quad \Theta(\tau) = [\theta_0(\tau) \ \ \theta_1(\tau) \ \ \ldots \ \ \theta_n(\tau)]^T, \ \ \theta_\jmath(\tau) = \tau^{\jmath+\beta_\jmath}, \ \ j = 0, 1, ...,$$

and

$$M = [m_{00} \ \ m_{01} \ \ m_{02} \ \ \ldots \ \ m_{0n} | m_{10} \ \ m_{11} \ \ m_{12} \ \ \ldots \ \ m_{1n} | \ldots | m_{n0} \ \ m_{n1} \ \ m_{n2} \ \ \ldots \ \ m_{nn}],$$

with

$$m_{\imath\jmath} = \begin{cases} \dfrac{(-1)^\jmath \imath!}{(\jmath!)^2 (\imath-\jmath)!}, & \imath \geq \jmath, \\ 0, & \imath < \jmath. \end{cases}$$

Similar to what was mentioned above, an arbitrary function $\mathcal{V}(x,\tau) \in L^2([0,1] \times [0,1])$ can be expanded in term of the GLPs as follows:

$$\mathcal{V}(x,\tau) \simeq \sum_{k=0}^{n_1} \sum_{s=0}^{n_2} \mu_{ks} \ell_k(x) \ell_s(\tau) = \mathbb{P}_{n_1}(x)^T \boldsymbol{\mu} \mathbb{Q}_{n_2}(\tau) = (\mathrm{CR}_{n_1}(x))^T \boldsymbol{\mu} (\mathrm{DR}'_{n_2}(\tau)), \tag{3.5}$$

where the vectors $\mathbb{P}_{n_1}(x)$ and $\mathbb{Q}_{n_2}(\tau)$ are as follows:

$$\mathbb{P}_{n_1}(x) = [1 \ \ x \ \ \ell_2(x) \ \ \ldots \ \ \ell_{n_1}(x)]^T = \mathrm{CR}_{n_1}(x), \tag{3.6}$$

$$\mathbb{Q}_{n_2}(\tau) = [1 \ \ \ell_1(\tau) \ \ \ell_2(\tau) \ \ \ldots \ \ \ell_{n_2}(\tau)]^T = \mathrm{DR}'_{n_2}(\tau), \tag{3.7}$$

and

$$\mathrm{R}_{n_1}(x) = [\mathrm{r}_0(x) \ \ \mathrm{r}_1(x) \ \ \ldots \ \ \mathrm{r}_{n_1}(x)]^T, \tag{3.8}$$

$$\mathrm{R}'_{n_2}(\tau) = \left[\mathrm{r}'_0(\tau) \ \ \mathrm{r}'_1(\tau) \ \ \ldots \ \ \mathrm{r}'_{n_2}(\tau)\right]^T, \tag{3.9}$$

with

$$\mathrm{r}_i(x) = \begin{cases} x^\imath, & \imath = 0, 1, \\ x^{\imath+k_\imath}, & \imath = 2, 3, ..., n_1, \end{cases} \tag{3.10}$$

and

$$\mathrm{r}'_j(\tau) = \begin{cases} \tau^\jmath, & \jmath = 0, \\ \tau^{\jmath+s_\jmath}, & \jmath = 1, 2, ..., n_2, \end{cases} \tag{3.11}$$

where $k_\imath$ and $s_\jmath$ are control parameters (CP).

Besides these, $\boldsymbol{\mu}$ is the matrix of free coefficients of the order $(n_1+1) \times (n_2+1)$ that is calculated and defined by:

$$\boldsymbol{\mu} = [\mu_{00} \ \ \mu_{01} \ \ \mu_{02} \ \ \ldots \ \ \mu_{0n_2} | \mu_{10} \ \ \mu_{11} \ \ \mu_{12} \ \ \ldots \ \ \mu_{1n_2} | \ldots | \mu_{n_10} \ \ \mu_{n_11} \ \ \mu_{n_12} \ \ \ldots \ \ \mu_{n_1n_2}],$$

and

$$\mathrm{C} = [1 \ \ 0 \ \ 0 \ \ \ldots \ \ 0 | 0 \ \ 1 \ \ 0 \ \ \ldots \ \ 0 | \mathrm{c}_{20} \ \ \mathrm{c}_{21} \ \ \mathrm{c}_{22} \ \ \ldots \ \ \mathrm{c}_{2n_1} | \ldots | \mathrm{c}_{n_10} \ \ \mathrm{c}_{n_11} \ \ \mathrm{c}_{n_12} \ \ \ldots \ \ \mathrm{c}_{n_1n_1}], \tag{3.12}$$

$$\mathrm{D} = [1 \ \ 0 \ \ 0 \ \ \ldots \ \ 0 | \mathrm{d}_{10} \ \ \mathrm{d}_{11} \ \ \mathrm{d}_{12} \ \ \ldots \ \ \mathrm{d}_{1n_2} | \ldots | \mathrm{d}_{n_20} \ \ \mathrm{d}_{n_21} \ \ \mathrm{d}_{n_22} \ \ \ldots \ \ \mathrm{d}_{n_2n_2}], \tag{3.13}$$

with

$$\mathrm{c}_{\imath\jmath} = \begin{cases} \dfrac{(-1)^\jmath \imath!}{(\jmath!)^2 (\imath-\jmath)!}, & \imath \geq \jmath, \\ 0, & \imath < \jmath, \end{cases} \qquad \mathrm{d}_{\imath\jmath} = \begin{cases} \dfrac{(-1)^\jmath \imath!}{(\jmath!)^2 (\imath-\jmath)!}, & \imath \geq \jmath, \\ 0, & \imath < \jmath. \end{cases}$$

4. The operational matrix of fractional and ordinary derivatives

In this part, first, by using definition (2.2) and calculating the fractional derivative of the desired function, we calculate the related operational matrix. Then, by calculating ordinary derivatives with respect to x and τ, we obtain their operational matrices.

Consider the function $\mathrm{r}_j^{'}(\tau)$ defined in (3.11), using (2.2), we arrive at the following equation

$$ {}_a^C D_\tau^\alpha \mathrm{r}_j^{'}(\tau) = \begin{cases} 0, & \jmath = 0, \\ \dfrac{\Gamma(\jmath+1+s_\jmath)}{\Gamma(\jmath-\alpha+1+s_\jmath)}\tau^{\jmath-\alpha+s_\jmath}, & \jmath = 1,2,...,n_2, \end{cases} \tag{4.1}$$

Now, to calculate the operational matrix of the function $\mathrm{R}_{n_2}^{'}(\tau)$, we can write

$$ {}_a^C D_\tau^\alpha \mathrm{R}_{n_2}^{'}(\tau) = \mathfrak{D}_\tau^{(\alpha)} \mathrm{R}_{n_2}^{'}(\tau), \tag{4.2}$$

where $\mathfrak{D}_\tau^{(\alpha)} = \left[d_{\imath\jmath}^\alpha\right]_{(m_2+1)\times(m_2+1)}$, is called the operational matrix corresponding to the fractional derivative $\mathrm{R}_{m_2}^{'}(\tau)$ and its entries are:

$$ d_{\imath\jmath}^\alpha = \tau^{-\alpha} \begin{cases} \dfrac{\Gamma(\jmath+s_{\jmath-1})}{\Gamma(\jmath-\alpha+s_{\jmath-1})}, & \jmath = 1,2,...,n_2, \ \ \jmath=\imath, \\ 0, & \text{Otherwise.} \end{cases} \tag{4.3}$$

Considering the functions $\mathrm{r}_i(x)$ and $\mathrm{r}_j^{'}(\tau)$ defined in (3.10) and (3.11) and derivation with respect to x and τ respectively, we obtain

$$ \frac{d^2\mathrm{r}_i(x)}{dx^2} = \begin{cases} 0, & \imath = 0,1, \\ (\imath+k_\imath)(\imath-1+k_\imath)x^{\imath-2+k_\imath}, & \imath = 2,3,...,n_1, \end{cases} \tag{4.4}$$

and

$$ \frac{d\mathrm{r}_j^{'}(\tau)}{d\tau} = \begin{cases} 0, & \jmath = 0, \\ (\jmath+s_\jmath)\tau^{\jmath-1+s_\jmath}, & \jmath = 1,2,...,n_2, \end{cases} \tag{4.5}$$

In this case, we will have operational matrices $\mathfrak{D}_x^{(2)}$ and $\mathfrak{D}_\tau^{(1)}$ corresponding to the vectors $\mathrm{R}_{n_1}(x)$ and $\mathrm{R}_{n_2}^{'}(\tau)$ as follows:

$$ \frac{d^2\mathrm{R}_{n_1}(x)}{dx^2} = \mathfrak{D}_x^{(2)}\mathrm{R}_{n_1}(x), \qquad \frac{d\mathrm{R}_{n_2}^{'}(\tau)}{d\tau} = \mathfrak{D}_\tau^{(1)}\mathrm{R}_{n_2}^{'}(\tau), \tag{4.6}$$

where $\mathfrak{D}_x^{(2)} = \left[d_{\imath\jmath}^2\right]_{(n_1+1)\times(n_1+1)}$, and $\mathfrak{D}_\tau^{(1)} = \left[d_{\imath\jmath}^1\right]_{(n_2+1)\times(n_2+1)}$, are operational matrices and for each we have respectively:

$$ d_{\imath\jmath}^2 = x^{-2} \begin{cases} (\imath+k_\imath)(\imath-1+k_\imath), & \imath = 2,3,...,n_1, \ \ \imath=\jmath, \\ 0, & \text{Otherwise,} \end{cases} \tag{4.7}$$

and

$$ d_{\imath\jmath}^1 = \tau^{-1} \begin{cases} 1, & \jmath = 1, \ \ \jmath=\imath, \\ \jmath+s_\jmath, & \jmath = 2,3,...,n_2, \ \ \jmath=\imath, \\ 0, & \text{otherwise,} \end{cases} \tag{4.8}$$

5. Algorithm method

The purpose of this section is to present the technique of problem solving using basis functions and operational matrices introduced in the previous section. For this purpose, considering the approximation function (3.5) and inserting in Equation (1.1), as well as using the operational matrices obtained in (4.2) and (4.6), we calculate ${}^{C}_{a}D^{\alpha}_{\tau}\mathcal{V}(x,\tau)$,$\mathcal{V}_{xx}(x,\tau)$ and $\mathcal{V}_{\tau}(x,\tau)$ as follows:

$$ {}^{C}_{a}D^{\alpha}_{\tau}\mathcal{V}(x,\tau) \simeq (\mathrm{CR}_{n_1}(x))^T \boldsymbol{\mu}(\mathrm{D}\mathfrak{D}^{(\alpha)}_{\tau}\mathrm{R}'_{n_2}(\tau)), \tag{5.1} $$

$$ \mathcal{V}_{xx}(x,\tau) \simeq (\mathrm{C}\mathfrak{D}^{(2)}_{x}\mathrm{R}_{n_1}(x))^T \boldsymbol{\mu}(\mathrm{D}\mathrm{R}'_{n_2}(\tau)), \tag{5.2} $$

$$ \mathcal{V}_{\tau}(x,\tau) \simeq (\mathrm{CR}_{n_1}(x))^T \boldsymbol{\mu}(\mathrm{D}\mathfrak{D}^{(1)}_{\tau}\mathrm{R}'_{n_2}(\tau)). \tag{5.3} $$

Now by applying the initial and boundary conditions of equation (**??**) in (**??**), we can write:

$$ \begin{cases} \Lambda_1(x) = (\mathrm{CR}_{n_1}(x))^T \boldsymbol{\mu}(\mathrm{DR}'_{n_2}(0)) - f_0(x) \simeq 0, \\ \Lambda_2(x) = (\mathrm{CR}_{n_1}(x))^T \boldsymbol{\mu}(\mathrm{DR}'_{n_2}(0)) - f_1(x) \simeq 0, \\ \Lambda_3(\tau) = (\mathrm{CR}_{n_1}(0))^T \boldsymbol{\mu}(\mathrm{DR}'_{n_2}(\tau)) - g_0(\tau) \simeq 0, \\ \Lambda_4(\tau) = (\mathrm{CR}_{n_1}(1))^T \boldsymbol{\mu}(\mathrm{DR}'_{n_2}(\tau)) - g_1(\tau) \simeq 0. \end{cases} \tag{5.4} $$

At this stage, in order to find the coefficients and parameters of the problem that are unknown by putting Equations (3.5), (5.1), (5.2), and (5.3) in Equation (1.1), we first define the residual function $\mathbf{Res}(x,\tau)$ as follows:

$$ \begin{aligned} \mathbf{Res}(x,\tau) = (\mathrm{CR}_{n_1}(x))^T \left[\boldsymbol{\mu}\mathfrak{D}^{(\alpha)}_{\tau} + \mu_1\boldsymbol{\mu}\mathfrak{D}^{(1)}_{\tau} - \left(\mathfrak{D}^{(2)}_{x}\right)\boldsymbol{\mu}\right]\left(\mathrm{DR}'_{n_2}(\tau)\right) \\ + \mu_2\left[\left((\mathrm{CR}_{n_1}(x))^T\boldsymbol{\mu}(\mathrm{DR}'_{n_2}(\tau))\right)^d\right] - F(x,\tau). \end{aligned} \tag{5.5} $$

Now, using $\|\mathbf{Res}(x,\tau)\|_2$ and optimizing the problem under the following conditions, we have:

$$ \mathcal{N}\left(\boldsymbol{\mu}; k_2, k_3, ..., k_{n_1}; s_2, s_3, ..., s_{n_2}\right) = \int_0^1\int_0^1 \mathbf{Res}^2(x,\tau)dxd\tau, \tag{5.6} $$

$$ min\ \mathcal{N}\left[\boldsymbol{\mu}; k_2, k_3, ..., k_{n_1}; s_2, s_3, ..., s_{n_2}\right], \tag{5.7} $$

$$ \begin{cases} \Lambda_1(\frac{\imath-1}{n_1}) = 0, \ \ \imath = 2,3,...,n_1, \\ \Lambda_2(\frac{\imath-1}{n_1}) = 0, \ \ \imath = 2,3,...,n_1, \\ \Lambda_3(\frac{\jmath-1}{n_2}) = 0, \ \ \jmath = 1,2,...,n_2+1, \\ \Lambda_4(\frac{\jmath-1}{n_2}) = 0, \ \ \jmath = 1,2,...,n_2+1. \end{cases} \tag{5.8} $$

The above optimization problem is minimization with respect to coefficients and control parameters using $\lambda = \left[\lambda_1, \lambda_2, ..., \lambda_{(n_1+n_2)}\right]$ Lagrange coefficients. Thus, we have:

$$ \mathcal{L}^* = [\boldsymbol{\mu}; k_2, k_3, ..., k_{n_1}; s_2, s_3, ..., s_{n_2;\lambda}] = \mathcal{N}\left[\boldsymbol{\mu}; k_2, k_3, ..., k_{n_1}; s_2, s_3, ..., s_{n_2}\right] + \lambda\Lambda, \tag{5.9} $$

$$ \begin{cases} \frac{\partial\mathcal{L}^*}{\partial\boldsymbol{\mu}} = 0, \ \ \frac{\partial\mathcal{L}^*}{\partial\lambda} = 0, \\ \frac{\partial\mathcal{L}^*}{\partial k_\imath} = 0, \ \ \imath = 2,3,...,n_1, \\ \frac{\partial\mathcal{L}^*}{\partial s_\jmath} = 0, \ \ \jmath = 2,3,...,n_2, \end{cases} \tag{5.10} $$

where, $\Lambda = [\Lambda_1 \;\; \Lambda_2 \;\; \Lambda_3 \;\; \Lambda_4]^T$ includes the restrictions defined in (5.8). By solving the nonlinear equations of the above system according to the proposed algorithm, the unknowns can be obtained and from there, the approximate solution $\mathcal{V}(x,\tau)$ corresponding to (3.5) is obtained.

6. Checking the performance of the method

At this stage, we would like to check the effectiveness of the above optimization technique. In this regard, we examine the solution of some examples of equations in form (1.1) by implementing the above method step by step. The following formulas are used to measure the accuracy of the results:

$$E^{\infty}_{\mathcal{V}} = \max_{1<n<N} |\mathcal{V}^n(x,\tau) - \widetilde{\mathcal{V}}^n(x,\tau)|, \qquad E^{2}_{\mathcal{V}} = \max_{1<n<N} \left(\Delta x \sum_{n=1}^{N} \left(\mathcal{V}^n(x,\tau) - \widetilde{\mathcal{V}}^n(x,\tau) \right)^2 \right)^{\frac{1}{2}}.$$

Example 6.1. As an example of Equation (1.1), consider the following diffusion-wave equation for the $\mu_1 = 2,\ \mu_2 = 1$ and $\eta = 3$ under the given initial and boundary conditions:

$$\begin{cases} \mathcal{V}(x,0) = 0, & \mathcal{V}_\tau(x,0) = 0, \\ \mathcal{V}(0,\tau) = 0, & \mathcal{V}(1,\tau) = \dfrac{1}{2}\tau^2 \sin(1), \end{cases}$$

and

$$F(x,\tau) = \frac{\Gamma(3)}{2\Gamma(3-\alpha)} \tau^{2-\alpha} \sin(x) + 2\tau \sin(x) + \left(\frac{1}{2}\tau^2 \sin(x) \right)^3 + \frac{1}{2}\tau^2 \sin(x).$$

The analytical solution to the problem is $\mathcal{V}(x,\tau) = \frac{1}{2}\tau^2 \sin(x)$. The maximum calculated error resulting from the solving of this equation according to the proposed method, taking into account $m_1 = 3, m_2 = 2$ and the different values of α is reported in Table 1. To better understand the significant effectiveness of the above optimization method, the diagram of the approximate solution and the absolute error for $\alpha = 1.8$ is drawn in Figure 1. The analytical solution compared to the results presented in this research, which determines the solutions with the optimal choice of parameters, indicates that the structure of this method is very effective in solving non-linear fractional equations and the approximate solution corresponds with the analytical solution with high accuracy.

Table 1. The absolute error for various α, $m_1 = 3$, $m_2 = 2$ and fixed $\Delta x = \frac{1}{10000}$ for Example 6.1.

$\Delta\tau_i$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
1/40	8.3208E-09	1.1007E-10	4.4203E-13
1/80	2.9211E-09	3.2902E-11	1.0292E-13
1/160	1.0114E-09	9.6829E-12	2.3878E-14
1/320	3.4676E-10	2.8151E-12	5.5173E-15
1/640	1.1804E-10	8.1051E-13	1.2689E-15
1/1280	3.9978E-11	2.3149E-13	2.9029E-16
1/2560	1.3486E-11	6.5680E-14	6.6001E-17
1/5120	4.5363E-12	1.8530E-14	1.4897E-17
$CPUtime(s)$	432.091″	404.760″	489.032″

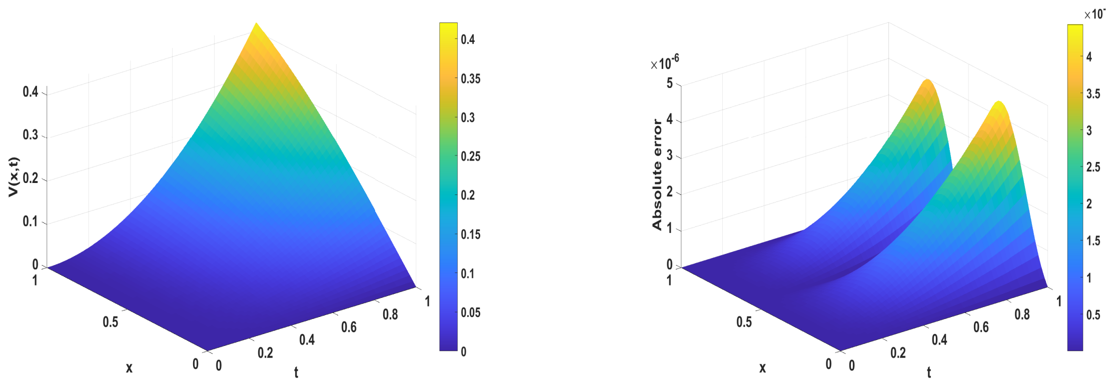

FIGURE 1. diagrams of the approximate solution (left) and the absolute error (right) with $\alpha = 1.8$ for Example 6.1.

Example 6.2. In this example, consider another type of Equation (1.1) with $\mu_1 = \mu_2 = 1,\ \eta = 3$ corresponding to the following boundary and initial conditions:

$$ {}_a^C D_\tau^\alpha \mathcal{V}(x,\tau) + \mathcal{V}_\tau(x,\tau) + \mathcal{V}^3(x,\tau) = \mathcal{V}_{xx}(x,\tau) + \left(\frac{4\tau^{2-\alpha}}{\Gamma(3-\alpha)}\right) x^2 + 4x^2\tau + \left(2x^2\tau^2\right)^3 - 4\tau^2,\ (x,\tau) \in [0,1]\times[0,1], $$

$$ \begin{cases} \mathcal{V}(x,0) = 0, & \mathcal{V}_\tau(x,0) = 0, \\ \mathcal{V}(0,\tau) = 0, & \mathcal{V}(1,\tau) = 2\tau^2, \end{cases} $$

where $\mathcal{V}(x,\tau) = 2x^2\tau^2$ is the exact solution of the problem. By solving the problem and finding an approximate solution, the maximum absolute error is obtained for different values of α the results of which are reported in Table 2. The high accuracy of these results shows the effectiveness of the mentioned method. Based on this, it should be said that this method is very accurate among other methods presented for solving nonlinear fractional equations, so it is very desirable to use this proposed method to achieve the desired numerical solution. In confirmation of the above, the diagram related to the approximate solution and the error of the method for $\alpha = 1.8$ is drawn in Figure 2.

TABLE 2. The absolute error for various α, $m_1 = 3,\ \ m_2 = 2$ and fixed $\Delta x = \frac{1}{10000}$ for Example 6.2.

$\Delta\tau_i$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
1/40	6.2898E-09	3.9941E-18	8.3801E-23
1/80	1.7943E-09	2.1520E-18	3.7900E-23
1/160	5.0562E-10	1.1543E-18	1.4835E-23
1/320	1.4106E-10	6.1783E-19	5.4039E-24
1/640	3.9032E-11	3.3036E-19	1.8870E-24
1/1280	1.0726E-11	1.7657E-19	6.4128E-25
1/2560	2.9303E-12	9.4352E-20	2.1393E-25
1/5120	7.9652E-13	5.0412E-20	7.0437E-26
$CPUtime(s)$	545.707″	384.136″	218.729″

7. CONCLUSION

The method presented in this work can be used to solve various types of fractional problems, including the aforementioned nonlinear fractional problems. In fact, this optimization method is based on Generalized Laguerre Polynomials

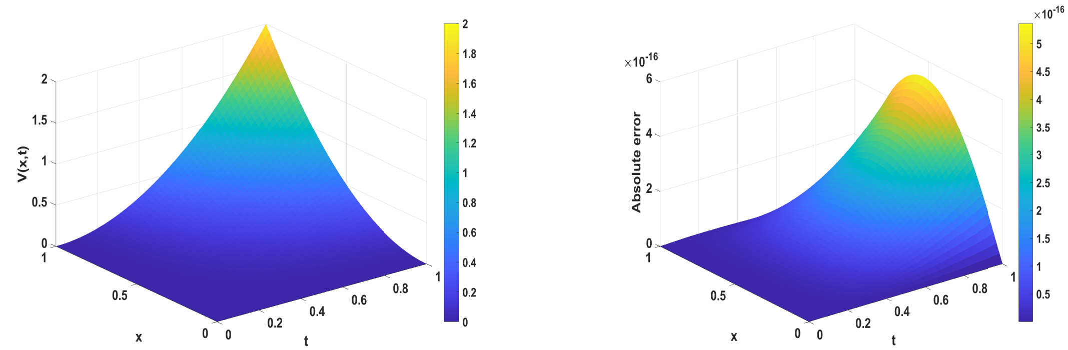

FIGURE 2. diagrams of the approximate solution (left) and the absolute error (right) with $\alpha = 1.8$ for Example 6.2.

and operational matrices, which approximate the numerical solution of the problem by minimizing the system of nonlinear equations resulting from the implementation of the method. This method is among the most efficient and common methods for approximating numerical solutions, because it works very powerfully and and effectively, both theoretically and practically, even with a small number of basic sentences.

In a recent study titled Application of Generalized Laguerre Polynomials in Solving Fractional Differential Equations, a comparison between this method and other methods has been made and the results of this study have been reported. In the future, we plan to use the proposed method to solve some practical problems such as the wave problem, the telegraph problem, nonlinear fractional differential equations of variable order and optimal control problems.

8. Declarations

Data availability statement: All data generated or analyzed during this study are included in this published article.

Conflict of interest: Authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants performed by any of the authors.

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