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Trannsformation of motion equations in the oil extraction process into Roesser-type equations and their solution using laplace transform

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Abstract

In the article, the method of solving Roesser-type equations with Laplace transformation for modeling and analysis of oil extraction processes is analyzed. Roesser-type equations are widely used to describe the dynamics of multidimensional systems, and their solution is important in improving the efficiency of oil extraction processes. First, the structure and properties of Roesser-type equations are presented. Later, the process of solving these equations is presented step by step by applying the Laplace transform method. By converting the special differential equations given by this method to simpler algebraic equations, both analytical and computer calculations can be significantly simplified. Based on the calculations and examples, it is shown that the proposed method provides high accuracy and efficiency. The research results enable the application of new approaches in the optimization and management of oil extraction processes.

Keywords. Gas-lift, Cauchy problem, Boundary problem, Differential equations of hyperbolic type, Roesser model, Laplace transform.
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1. Introduction

The oil industry is one of the most important sectors of the world economy, and the technologies and methods used in this sector are constantly updated and developed. Various mathematical methods and models are applied in order to increase efficiency and accuracy in oil extraction processes [3–5]. At the end of the 19th century and the beginning of the 20th century, the compressor-less gas-lift method began to be applied in oil wells using the ability of natural gas to create pressure [9, 10]. The use of naturally available gas in oil fields has shown that it can ensure the extraction of oil without spending additional energy [6, 9, 17].

Gas-lift processes are one of the widely used methods for bringing oil to the surface in oil wells. In this process, oil is brought to the surface by injecting gas into the bottom of the well. In this method, various mathematical models describing the movement have been developed [10, 13, 17], and with their help, various problems have been addressed, such as obtaining maximum oil with minimum gas [12, 13, 15], increasing the oil yield coefficient, etc.[5, 6]. In this study, a mixed problem for a system of Roesser-type differential equations was considered [7, 8, 14].

The system of special differential equations, which is a mathematical description of the given problem, was solved with the help of integral Laplace transform and inverse Laplace transform [11, 16]. Such linear notation allows for the replacement of systems of differential equations with algebraic equations.

The Laplace transform is a widely used method in mathematical analysis and engineering, used to analyze and control the dynamic behavior of systems. This transformation simplifies the solution of differential equations, allowing

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for more efficient and accurate analysis [16]. On the other hand, Roesser-type equations play an important role in the modeling of multidimensional systems, and the solution of these equations is important for the management of complex industrial processes.

Solving the Roesser-type equations in the gas-lift process by means of the Laplace transform mainly involves two steps: first, transforming these equations by means of the Laplace transform, and then solving the resulting algebraic equations. This method allows more accurate and effective analysis of the dynamic system of the gas-lift process.

This problem is described mathematically by a system of special derivative differential equations as follows.

2. Problem statement

It is known that [9, 13, 17], the system of special derivative hyperbolic type differential equations characterizing the movement in the gas-lift process is as follows.

$$\begin{cases} -F_i \frac{\partial P(x,t)}{\partial x} = \frac{\partial Q(x,t)}{\partial t} + 2a_i Q(x,t), \\ -F_i \frac{\partial P(x,t)}{\partial t} = c_i \frac{\partial Q(x,t)}{\partial x}, \end{cases} \quad x \in (0,l), \quad t \in (-\infty; +\infty). \tag{2.1}$$

$$2a_i = \frac{g}{\omega_i} + \frac{\sigma_i \omega_i}{2\delta_i} = const.$$

Here $P(x;t)-$, the pressure $Q(x;t)$ is the volume of the liquid in the tube. Here F_i- is the cross-sectional area of the pipe and is constant; ω_i- is average speed of flow; c_i- is the speed of sound in a suitable medium (for example, $i = 1$ in an annular space consisting of a gas, $i = 2$ if present, in a liquid-gas mixture in a lift); $\delta_i-$is the internal and effective diameter of the annular space and lifter depending on the direction of movement; g is the urgency of gravity; σ_i is the coefficient of hydraulic resistance [2, 7].

Each equation of the given system Eq. (2.1) includes derivatives of different functions $P(x;t)$ and $Q(x;t)$ with respect to different x and t variables, in order to obtain a compact model, the application of different discretization methods does not result successfully. For this reason, it is possible to get a suitable Roesser model [2, 8] by introducing a new unknown function into the system Eq. (2.1) in the following way.

$$P(x;t) = R(x;t) + \alpha Q(x;t). \tag{2.2}$$

Here $R(x;t)$ is a new unknown function replacing, $P(x;t)$ and the unit of α quantity should be chosen so that the units of the variables $R(x;t)$ and $\alpha Q(x;t)$ are the same. It should be noted that the hyperbolic-type partial differential equations corresponding to the Roesser model applied to the gas-lift method are constructed using the new variable introduced in Eq. (2.2), and the model is formulated based on the following two systems of equations [10].

$$\begin{cases} \frac{\partial Q(x,t)}{\partial x} = W(x,t), \\ \frac{\partial R(x,t)}{\partial t} = \chi(x,t) \end{cases} \quad t > 0,\ x > 0, \tag{2.3}$$

$$\begin{cases} \frac{\partial R(x,t)}{\partial x} = \left(\frac{c_i}{\alpha F_i^2} - \alpha\right) W(x,t) + \frac{1}{\alpha F_i}\chi(x,t) - \frac{2a_i}{F_i} Q(x,t), \\ \frac{\partial Q(x,t)}{\partial t} = -\frac{1}{\alpha}\chi(x,t) - \frac{c_i}{\alpha F_i} W(x,t), \end{cases} \quad t > 0,\ x > 0. \tag{2.4}$$

Let us accept the initial and boundary conditions of Equations (2.3) and (2.4) as follows.

$$\begin{cases} Q(0,t) = \bar{Q}(t), \\ R(0,t) = \bar{Q}(t),\ t > 0, \end{cases} \tag{2.5}$$

$$\begin{cases} Q(x,0) = \bar{\bar{Q}}(x), \\ R(x,0) = \bar{\bar{Q}}(x),\ x \in (0,l). \end{cases} \tag{2.6}$$

let's determine the mixed derivative $\frac{\partial}{\partial x}\left(\frac{\partial Q(x,t)}{\partial t}\right) = \frac{\partial W(x,t)}{\partial t}$ of the first equation of the system (2.3) corresponding to the parameter t and consider it in the second equation of the system (2.4).

$$\frac{1}{\alpha}\frac{\partial \chi(x,t)}{\partial x} + \frac{c_i}{\alpha F_i}\frac{\partial W(x,t)}{\partial x} + \frac{\partial W(x,t)}{\partial t} = 0.$$

Later we will differentiate the second equation of system (2.3) with respect to the parameter.

$$\frac{\partial^2 R(x,t)}{\partial t \partial x} = \frac{\partial \chi(x,t)}{\partial x},$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial R(x,t)}{\partial x}\right) = \frac{\partial \chi(x,t)}{\partial x}.$$

If we consider the first equation of the system Eq. (2.4) in accordance with the above-mentioned analogy in this expression, we get the following expression.

$$\left(\frac{c_i}{\alpha F_i^2} - \alpha\right)\frac{\partial W(x,t)}{\partial t} + \frac{1}{\alpha F_i}\frac{\partial \chi(x,t)}{\partial t} - \frac{2a_i}{F_i}\frac{\partial Q(x,t)}{\partial t} = \frac{\partial \chi(x,t)}{\partial x},$$

$$\frac{\partial \chi(x,t)}{\partial x} - \frac{1}{\alpha F_i}\frac{\partial \chi(x,t)}{\partial t} + \frac{2a_i}{F_i}\frac{\partial Q(x,t)}{\partial t} - \left(\frac{c_i}{\alpha F_i^2} - \alpha\right)\frac{\partial W(x,t)}{\partial t} = 0. \tag{2.7}$$

Consider the obtained expression Eq. (2.7) as the second equation of the system Eq. (2.4):

$$\frac{\partial \chi(x,t)}{\partial x} - \frac{1}{\alpha F_i}\frac{\partial \chi(x,t)}{\partial t} + \frac{2a_i}{F_i}\left(-\frac{1}{\alpha}\chi(x,t) - \frac{c_i}{\alpha F_i}W(x,t)\right) - \left(\frac{c_i}{\alpha F_i^2} - \alpha\right)\frac{\partial W(x,t)}{\partial t} = 0,$$

$$\frac{\partial \chi(x,t)}{\partial x} - \frac{1}{\alpha F_i}\frac{\partial \chi(x,t)}{\partial t} - \left(\frac{c_i}{\alpha F_i^2} - \alpha\right)\frac{\partial W(x,t)}{\partial t} - \frac{2a_i}{\alpha F_i}\chi(x,t) - \frac{2a_i c_i}{\alpha F_i^2}W(x,t) = 0.$$

Simplifying:

$$\frac{\partial W(x,t)}{\partial t} + \frac{1}{\alpha^2 F_i}\frac{\partial \chi(x,t)}{\partial t} + \left(\frac{c_i}{\alpha^2 F_i^2} - 1\right)\frac{\partial W(x,t)}{\partial t} + \frac{2a_i}{\alpha^2 F_i}\chi(x,t) + \frac{2a_i c_i}{\alpha^2 F_i}W(x,t) + \frac{c_i}{\alpha F_i}\frac{\partial W(x,t)}{\partial x} = 0.$$

In the end, it will be like this:

$$\frac{1}{\alpha}\frac{\partial \chi(x,t)}{\partial t} + \frac{c_i}{\alpha F_i}\frac{\partial W(x,t)}{\partial t} + \frac{2a_i}{\alpha}\chi(x,t) + \frac{2a_i c_i}{\alpha}W(x,t) + c_i\frac{\partial W(x,t)}{\partial x} = 0. \tag{2.8}$$

Some notation has been done for simplicity.

$$\begin{pmatrix} W(x,t) \\ \chi(x,t) \end{pmatrix} = Z(x,t). \tag{2.9}$$

Then expression Eq. (2.8) can be written as follows.

$$\begin{pmatrix} \frac{c_i}{\alpha F_i} & \frac{1}{\alpha} \\ 0 & 1 \end{pmatrix}\frac{\partial Z(x,t)}{\partial x} + \begin{pmatrix} 1 & 0 \\ \alpha - \frac{c_i}{\alpha F_i^2} & \frac{1}{\alpha F_i} \end{pmatrix}\frac{\partial Z(x,t)}{\partial t} + \begin{pmatrix} 1 & 0 \\ -\frac{2a_i c_i}{\alpha F_i^2} & -\frac{2a_i}{\alpha F_i} \end{pmatrix} Z(x,t) = 0. \tag{2.10}$$

Let's solve the obtained equation Eq. (2.10) with the help of Laplace transform.

$$A=\begin{pmatrix} \frac{c_i}{\alpha F_i} & \frac{1}{\alpha} \\ 0 & 1 \end{pmatrix},\quad B=\begin{pmatrix} 1 & 0 \\ \alpha-\frac{c_i}{\alpha F_i^2} & \frac{1}{\alpha F_i} \end{pmatrix},\quad C=\begin{pmatrix} 1 & 0 \\ -\frac{2a_i c_i}{\alpha F_i^2} & -\frac{2a_i}{\alpha F_i} \end{pmatrix},$$

$$A\int\limits_0^\infty e^{-\lambda t}\frac{\partial Z(x,t)}{\partial x}dt+B\int\limits_0^\infty e^{-\lambda t}\frac{\partial Z(x,t)}{\partial t}dt+C\int\limits_0^\infty e^{-\lambda t}Z(x,t)dt=0,$$

$$A\frac{d\tilde{Z}(x,\lambda)}{dx}+Be^{-\lambda t}\left.Z(x,t)\right|_{t=0}^{\infty}+B\lambda\int\limits_0^\infty e^{-\lambda t}Z(x,t)dt+C\tilde{Z}(x,\lambda)=0,$$

$$A\tilde{Z}'(x,\lambda)+(B\lambda+C)\,\tilde{Z}(x,\lambda)=BZ(x,0). \tag{2.11}$$

Let's determine the expression $Z(x,0)$ in Eq. (2.11) from Eq. (2.3):

$$Z(x,0)=\begin{pmatrix} W(x,0) \\ \chi(x,0) \end{pmatrix}=\begin{pmatrix} \frac{\partial Q(x,0)}{\partial x} \\ 2a_i\alpha Q(x,0)+\left(\alpha^2F_i-\frac{c_i}{F_i}\right)\frac{\partial Q(x,0)}{\partial x}+\alpha F_i\frac{\partial R(x,0)}{\partial x} \end{pmatrix}$$
$$=\begin{pmatrix} \bar{\bar{Q}}'(x) \\ 2a_i\alpha\bar{\bar{Q}}(x)+\left(\alpha^2F_i-\frac{c_i}{F_i}\right)\bar{\bar{Q}}'(x)+\alpha F_i\frac{\partial R(x,0)}{\partial x} \end{pmatrix},$$

$$Z(x,0)=\begin{pmatrix} \bar{\bar{Q}}'(x) \\ 2a_i\alpha\bar{\bar{Q}}(x)+\left(\alpha^2F_i-\frac{c_i}{F_i}\right)\bar{\bar{Q}}'(x)+\alpha F_i\bar{\bar{R}}'(x) \end{pmatrix}. \tag{2.12}$$

Consider Eq. (2.12) in Eq. (2.11):

$$A\tilde{Z}'(x,\lambda)+(B\lambda+C)\,\tilde{Z}(x,\lambda)=B\begin{pmatrix} \bar{\bar{Q}}'(x) \\ 2a_i\alpha\bar{\bar{Q}}(x)+\left(\alpha^2F_i-\frac{c_i}{F_i}\right)\bar{\bar{Q}}'(x)+\alpha F_i\bar{\bar{R}}'(x) \end{pmatrix},$$

if we multiply the lass expression by the inverse of matrix B; we obtain:

$$B^{-1}A\tilde{Z}'(x,\lambda)+\left(\lambda I+B^{-1}C\right)\tilde{Z}(x,\lambda)=\begin{pmatrix} \bar{\bar{Q}}'(x) \\ 2a_i\alpha\bar{\bar{Q}}(x)+\left(\alpha^2F_i-\frac{c_i}{F_i}\right)\bar{\bar{Q}}'(x)+\alpha F_i\bar{\bar{R}}'(x) \end{pmatrix}\equiv M(x). \tag{2.13}$$

For simplicity, let's notate as follows.

$$B^{-1}A=\begin{pmatrix} 1 & 0 \\ \alpha^2F_i-\frac{c_i}{F_i} & -\alpha F_i \end{pmatrix}\begin{pmatrix} \frac{c_i}{\alpha F_i} & \frac{1}{\alpha} \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} \frac{c_i}{\alpha F_i} & \frac{1}{\alpha} \\ c_i\alpha-\frac{c_i^2}{\alpha F_i^2} & \alpha F_i-\frac{c_i}{\alpha F_i}-\alpha F_i \end{pmatrix} \tag{2.14}$$

$$=\begin{pmatrix} \frac{c_i}{\alpha F_i} & \frac{1}{\alpha} \\ c_i\alpha-\frac{c_i^2}{\alpha F_i^2} & -\frac{c_i}{\alpha F_i}, \end{pmatrix}=D, \tag{2.15}$$

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad B^{-1}C=\begin{pmatrix} 1 & 0 \\ \alpha^2F_i-\frac{c_i}{F_i} & -\alpha F_i \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -\frac{2a_ic_i}{\alpha F_i^2} & -\frac{2a_i}{\alpha F_i} \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \frac{2a_ic_i}{F_i} & 2a_i \end{pmatrix}=E. \tag{2.16}$$

Consider Eq. (2.14) and Eq. (2.16) expressions in Eq. (2.13):

$$D\tilde{Z}'(x,\lambda)+(\lambda I+E)\,\tilde{Z}(x,\lambda)=M(x),\quad x\in(0,l),$$

$$\tilde{Z}(x,\lambda) = \sum_{k=1}^{\infty} \lambda^{-k} \tilde{Z}_k(x),$$

$$D\sum_{k=1}^{\infty} \lambda^{-k} \tilde{Z}'_k(x) + \lambda \sum_{k=1}^{\infty} \lambda^{-k} \tilde{Z}_k(x) + E\sum_{k=1}^{\infty} \lambda^{-k} \tilde{Z}_k(x) = M(x). \tag{2.17}$$

Now let's look at the coefficients of λ^0

$$\tilde{Z}_1(x) = M(x).$$

The coefficients of λ^{-1} are as follows.

$$D\tilde{Z}'_1(x) + \tilde{Z}_2(x) + E\tilde{Z}_1(x) = 0, \qquad \tilde{Z}_2(x) = -DM'(x) - EM(x) \; x \in (0;l)\,.$$

Similarly the coefficients of λ^{-2} can be determined as follows:

$$D\tilde{Z}'_2(x) + \tilde{Z}_3(x) + E\tilde{Z}_2(x) = 0, \qquad \tilde{Z}_3(x) = D^2M''(x) + (DE+ED)M'\,(x) + E^2M(x).$$

$$D^2 = \begin{pmatrix} c_i & 0 \\ 0 & c_i \end{pmatrix} = C\cdot I, DE+ED = \begin{pmatrix} \frac{2a_ic_i}{\alpha F_i} & \frac{2a_i}{\alpha} \\ -\frac{2a_ic_i^2}{\alpha F_i^2} - 2a_i\alpha c_i & -\frac{2a_ic_i}{\alpha F_i} \end{pmatrix}, E^2 = \begin{pmatrix} 0 & 0 \\ \frac{4a_i^2c_i}{F_i} & 4a_i^2 \end{pmatrix}.$$

Continuing this process, all $\tilde{Z}'_K(x)$ limits given by formula Eq. (2.17) can be found in a single-valued way. Using the found $\tilde{Z}'_K(x)$ values, the function $Z\,(x;t)$ is determined through the inverse Laplace transformation.

$$Z(x,t) = \frac{1}{2\pi i}\sum_{\sigma-i}^{\sigma+i} e^{\lambda t}\tilde{Z}(x,\lambda)d\lambda, \;\sigma > 0,$$

or

$$Z(x,t) = \frac{1}{2\pi i}\int\limits_L e^{\lambda t}\sum_{k=1}^{\infty}\lambda^{-k}\tilde{Z}_k(x)d\lambda = \frac{1}{2\pi i}\sum_{k=1}^{\infty}\tilde{Z}_k(x)\int\limits_L \frac{e^{\lambda t}}{\lambda^k}d\lambda$$

$$= \frac{1}{2\pi i}\;2\pi i\sum_{k=1}^{\infty}\tilde{Z}_k(x)\frac{t^{k-1}}{(k-1)!} = \sum_{k=1}^{\infty}\tilde{Z}_k(x)\frac{t^{k-1}}{(k-1)!}. \tag{2.18}$$

Considering the function $Z\,(x;t)$ determined from Eq. (2.18) in expression Eq. (2.9) , $W\,(x;t)$ and $\chi\,(x;t)$ the unknown functions can be determined as follows.

$$W(x,t) = \sum_{k=1}^{\infty}\tilde{W}_k(x)\frac{t^{k-1}}{(k-1)!}\; x\in[0,l]\,, \tag{2.19}$$

$$\chi(x,t) = \sum_{k=1}^{\infty}\tilde{\chi}_k(x)\frac{t^{k-1}}{(k-1)!}, \tag{2.20}$$

and functions $W\,(x;t)$ and $\chi\,(x;t)$ determined from expressions Eq. (2.19) and Eq. (2.20) in the first and second equations of the system Eq. (2.3), respectively, for the functions $Q\,(x;t)$ and $R\,(x;t)$ we get the following expressions:

$$Q(x,t) = Q(0,t) + \int\limits_0^x W(\xi,t)d\xi = \bar{Q}(t) + \int_0^x W(\xi,t)d\xi = \bar{\bar{Q}}(t) + \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \int\limits_0^x \tilde{W}_k(\xi)d\xi,$$

$$R(x,t) = R(x,0) + \int\limits_0^t \chi(x,\tau)d\tau = \bar{\bar{R}}(x) + \sum_{k=1}^{\infty} \tilde{\chi}_k(x)\frac{t^k}{k!}.$$

In this, it is possible to determine the functions $R\,(x;t)$ $Q\,(x;t)$ $W\,(x;t)$ $\chi\,(x;t)$ in the segment $x \in [0;\,l]\,, t \geq .0$

It should be noted that these relationships allow us to find a solution to the gas-lift problem in general, but the system of differential equations (2.1) given by conditions Eq. (2.5) and Eq. (2.6) at the beginning and the end of annular part of the lift, lift, as in [2], should be taken into account.

Now let's look at the solution of the problem the lifting in the $x \in [l;2\,l]\,, t \geq 0$. Let's assume that the movement of the liquid-gas mixture in the bottom zone is described by the following impulse system.

$$\begin{cases} Q(l+0,t) = F_\delta^{'} Q(l-0,t) + F_\phi^{'} R(l-0,t), \\ R(l+0,t) = F_\delta^{''} Q(l-0,t) + F_\phi^{''} Q(l-0,t). \end{cases} \tag{2.21}$$

It is known that $i = 1$ when the equations Eq. (2.1) describe the movement of the gas and with the help of the Eq. (2.21) momentum system the values $R\,(l-0,t)\,,\ Q\,(l-0,t)$ are brought to, $R\,(l+0,t)\,,\ Q\,(l+0,t)$.

Using the analogy mentioned above, in $x \in [l;2\,l]\,, t \geq 0$ the joint solution of equations Eq. (2.3) and Eq. (2.4) can be written as follows according to the riser pipe.

$$\begin{cases} \frac{1}{\alpha}\frac{\partial \chi(x,t)}{\partial x} + \frac{c_i}{\alpha F_i}\frac{\partial W(x,t)}{\partial x} + \frac{\partial W(x,t)}{\partial t} = 0, \\ \frac{\partial \chi(x,t)}{\partial x} - \frac{1}{\alpha F_i}\frac{\partial \chi(x,t)}{\partial t} - \left(\frac{c_i}{\alpha F_i^2} - \alpha\right)\frac{\partial W(x,t)}{\partial t} - & t \geq 0,\ x \in [l, 2l]\,. \\ -\frac{2a_i}{\alpha F_i}\chi(x,t) - \frac{2a_i c_i}{\alpha F_i^2} W(x,t) = 0, \end{cases}$$

Applying the Laplace transform to the last expression, we get:

$$\begin{cases} \frac{1}{\alpha}\int\limits_0^\infty e^{-\lambda t}\frac{\partial \chi(x,t)}{\partial x}dt + \frac{c_i}{\alpha F_i}\int\limits_0^\infty e^{-\lambda t}\frac{\partial W(x,t)}{\partial x}dt + \int\limits_0^\infty e^{-\lambda t}\frac{\partial W(x,t)}{\partial t}dt = 0 \\ \int\limits_0^\infty e^{-\lambda t}\frac{\partial \chi(x,t)}{\partial x}dt - \frac{1}{\alpha F_i}\int\limits_0^\infty e^{-\lambda t}\frac{\partial \chi(x,t)}{\partial t}dt - \left(\frac{c_i}{\alpha F_i^2} - \alpha\right)\int\limits_0^\infty e^{-\lambda t}\frac{\partial W(x,t)}{\partial t}dt - \\ -\frac{2a_i}{\alpha F_i}\int\limits_0^\infty e^{-\lambda t}\chi(x,t)dt - \frac{2a_i c_i}{\alpha F_i^2}\int\limits_0^\infty e^{-\lambda t}W(x,t)dt = 0, t \geq 0,\ x \in [l, 2l]\,. \end{cases} \tag{2.22}$$

Then for the functions $W\,(x,t)$ and $\chi\,(x,t)$ can be written.

$$\begin{cases} \int\limits_0^\infty e^{-\lambda t}W(x,t)dt = \tilde{W}(x,\xi), \\ \int\limits_0^\infty e^{-\lambda t}\chi(x,t)dt = \tilde{\chi}(x,\xi), \end{cases} \tag{2.23}$$

or

$$\begin{cases} \int\limits_0^\infty e^{-\lambda t}\frac{\partial W(x,t)}{\partial x}dt = \tilde{W}'(x,\xi), \\ \int\limits_0^\infty e^{-\lambda t}\frac{\partial \chi(x,t)}{\partial x}dt = \tilde{\chi}'(x,\xi), \end{cases} \tag{2.24}$$

$$\begin{cases} \int\limits_0^\infty e^{-\lambda t} \frac{\partial W(x,t)}{\partial t} dt = e^{-\lambda t}\ W(x,t)|_{t=0}^\infty + \lambda \int\limits_0^\infty e^{-\lambda t} W(x,t) dt = -W(x,0) + \lambda \tilde{W}(x,\xi), \\ \int\limits_0^\infty e^{-\lambda t} \frac{\partial \chi(x,t)}{\partial t} dt = e^{-\lambda t}\ \chi(x,t)|_{t=0}^\infty + \lambda \int\limits_0^\infty e^{-\lambda t} \chi(x,t) dt = -\chi(x,0) + \lambda \tilde{\chi}(x,\xi). \end{cases} \tag{2.25}$$

Consider Eq. (2.23), Eq. (2.24), Eq. (2.25) in Eq. (2.22):

$$\begin{cases} -W(x,0) + \lambda \tilde{W}(x,\xi) + \frac{1}{\alpha} \tilde{\chi}'(x,\xi) + \frac{c_i}{\alpha F_i} \tilde{W}'(x,\xi) = 0, \\ \tilde{\chi}'(x,\xi) - \frac{1}{\alpha F_i} \left[-\chi(x,0) + \lambda \tilde{\chi}(x,\xi) \right] - \left(\frac{c_i}{\alpha F_i^2} - \alpha \right) \left[-W(x,0) + \tilde{W}(x,\xi) \right], \\ -\frac{2a_i}{\alpha F_i} \tilde{\chi}(x,\xi) - \frac{2a_i c_i}{\alpha F_i^2} \tilde{W}(x,\xi) = 0\ x \in [l; 2l]. \end{cases} \tag{2.26}$$

From the system Eq. (2.23), it can be seen that if $t = 0$, for the function $W(x,t)$ we get:

$$W(x,0) = \left. \frac{\partial Q(x,t)}{\partial x} \right|_{t=0} = \frac{\partial Q(x,0)}{\partial x} = \bar{\bar{Q}}'(x), \tag{2.27}$$

in the first equation of system Eq. (2.4) if $t = 0$, for the function $\chi(x,t)$ we get:

$$\chi(x,0) = \alpha F_i \left[\left. \frac{\partial R(x,t)}{\partial x} \right|_{t=0} - \left(\frac{c_i}{\alpha F_i^2} - \alpha \right) W(x,0) + \frac{2a_i}{F_i} Q(x,0) \right],$$

or

$$\chi(x,0) = \alpha F_i \bar{\bar{R}}'(x) - \left(\frac{c_i}{F_i} - \alpha^2 F_i \right) \bar{\bar{Q}}'(x) + 2a_i \alpha \bar{\bar{Q}}(x). \tag{2.28}$$

Considering expressions Eq. (2.27) and Eq. (2.28) in system Eq. (2.26), we can write

$$\begin{cases} \lambda \tilde{W}(x,\xi) + \frac{1}{\alpha} \tilde{\chi}'(x,\xi) + \frac{c_i}{\alpha F_i} \tilde{W}'(x,\xi) = \bar{\bar{Q}}'(x), \\ \tilde{\chi}'(x,\xi) - \left(\frac{\lambda}{\alpha F_i} + \frac{2a_i}{\alpha F_i} \right) \tilde{\chi}(x,\xi) - \left[\left(\frac{c_i}{\alpha F_i^2} - \alpha \right) \lambda + \frac{2a_i c_i}{\alpha F_i^2} \right] \tilde{W}(x,\xi) = \\ = -\bar{\bar{R}}'(x) + \left(\frac{c_i}{\alpha F_i^2} - \alpha \right) \bar{\bar{Q}}'(x) - \frac{2a_i}{F_i} \bar{\bar{Q}}(x) - \left(\frac{c_i}{\alpha F_i^2} - \alpha \right) \bar{\bar{Q}}'(x),\ x \in [l; 2l]. \end{cases}$$

To put it simply,

$$\begin{cases} \tilde{W}'(x,\xi) + \frac{\lambda + 2a_i}{\alpha F_i} \tilde{W}(x,\xi) + \frac{\lambda + 2a_i}{\alpha_i c_i} \tilde{\chi}(x,\xi) = \frac{\alpha F_i}{c_i} \bar{\bar{Q}}'(x) + \frac{F_i}{c_i} \bar{\bar{R}}'(x) + \frac{2a_i}{c_i} \bar{\bar{Q}}(x), \\ \tilde{\chi}'(x,\xi) - \frac{\lambda + 2a_i}{\alpha F_i} \tilde{\chi}(x,\xi) - \frac{(c_i - \alpha^2 F_i^2)\lambda + 2ac}{\alpha F_i} \tilde{W}(x,\xi) = \bar{\bar{R}}'(x) - \frac{2a_i}{F_i} \bar{\bar{Q}}(x). \end{cases} \tag{2.29}$$

Let's adopt some notation for simplicity.

$$\tilde{\eta}(x,\lambda) = \begin{pmatrix} \tilde{W}(x,\xi) \\ \tilde{\chi}(x,\xi) \end{pmatrix}.$$

Then system Eq. (2.29) can be written in the following form.

$$\tilde{\eta}(x,\lambda) + \begin{pmatrix} \frac{\lambda + 2a_i}{\alpha F_i} & \frac{\lambda + 2a_i}{\alpha c_i} \\ -\frac{(c_i - \alpha^2 F_i^2)\lambda + 2a_i c_i}{\alpha F_i^2} & -\frac{\lambda + 2a_i}{\alpha F_i} \end{pmatrix} \tilde{\eta}(x,\lambda) = \begin{pmatrix} \frac{\alpha F_i}{c_i} \bar{\bar{Q}}'(x) + \frac{F_i}{c_i} \bar{\bar{R}}'(x) + \frac{2a_i}{c_i} \bar{\bar{Q}}(x) \\ -\bar{\bar{R}}'(x) - \frac{2a_i}{F_i} \bar{\bar{Q}}(x) \end{pmatrix}. \tag{2.30}$$

The expressions on the left and right sides of the equality can be written as follows.

$$\begin{pmatrix} \frac{\lambda+2a_i}{\alpha F_i} & \frac{\lambda+2a_i}{\alpha c_i} \\ -\frac{\left(c_i-\alpha^2F_i^2\right)\lambda+2a_ic_i}{\alpha F_i^2} & -\frac{\lambda+2a_i}{\alpha F_i}\end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha F_i} & \frac{1}{\alpha_i c_i} \\ \frac{c_i-\alpha^2F_i^2}{\alpha F_i^2} & -\frac{1}{\alpha F_i}\end{pmatrix}\lambda + \begin{pmatrix} \frac{2a_i}{\alpha F_i} & \frac{2a_i}{\alpha c_i} \\ \frac{2a_ic_i}{\alpha F_i^2} & -\frac{2a_i}{\alpha F_i}\end{pmatrix} \equiv S_1\lambda + S_0,$$

$$\begin{pmatrix} \frac{\alpha F_i}{c_i}\overline{\overline{Q'}}\left(x\right) + \frac{F_i}{c_i}\overline{\overline{R'}}\left(x\right) + \frac{2a_i}{c_i}\overline{\overline{Q}}\left(x\right) \\ -\overline{\overline{R'}}\left(x\right) - \frac{2a_i}{F_i}\overline{\overline{Q}}\left(x\right)\end{pmatrix} = S\left(x\right).$$

Then for Eq. (2.30) we get,

$$\tilde{\eta}'(x,\lambda) + \left(S_1\lambda + S_0\right)\tilde{\eta}(x,\lambda) = S\left(x\right). \tag{2.31}$$

Let's look for the expression Eq. (2.31) in series form.

$$\sum_{k=1}^{\infty}\lambda^{-k}\eta_k'(x) + \left(S_1\lambda + S_0\right)\sum_{k=1}^{\infty}\lambda^{-k}\eta_k(x) = S(x).$$

It is clear that $\lambda^0$for the case that is $S_1\eta_1\left(x\right) = S\left(x\right)$ or $\eta_1\left(x\right) = S_1^{-1}\cdot S\left(x\right)$ is received.
In the same order, λ^{-1} while $\eta_1^{'}(x) + S_1\eta_2(x) + S_0\eta_1(x) = S(x)$ or $\eta_2(x) = S_1^{-1}\left(\eta_1^{'}(x) - S_0\eta_1(x)\right)$ is taken. Thus, by continuing the process, η_n the function can be determined with very small errors.
Summarizing the above, we can propose the following calculation algorithm for solving the gas-lift issue.

Algoritm 1

Step 1. According to the given equations characterizing pressure and gas volume, the boundary conditions are provided as $P(0,t) = \bar{P}(t)$ and $Q(0,t) = \bar{Q}(t)$. The initial conditions will be in the form $P(x,0) = \bar{\bar{P}}(x)$ and $Q(x,0) = \bar{\bar{Q}}(x)$.

Step 2. The function $\tilde{Z}_K'(x)$ is uniquely determined using formula Eq. (2.17).

Step 3. Using the terms $\tilde{Z}_K'(x)$, the function $Z\left(x;t\right)$ is determined by the following formula via the inverse Laplace transform :

$$Z(x,t) = \frac{1}{2\pi i}\int_L e^{\lambda t}\sum_{k=1}^{\infty}\lambda^{-k}\tilde{Z}_k(x)d\lambda = \frac{1}{2\pi i}\sum_{k=1}^{\infty}\tilde{Z}_k(x)\int_L\frac{e^{\lambda t}}{\lambda^k}d\lambda = \sum_{k=1}^{\infty}\tilde{Z}_k(x)\frac{t^{k-1}}{(k-1)!}$$

Step 4. Considering the function $Z\left(x;t\right)$ in the annular space as expressed in Eq. (2.9), the unknown functions $W\left(x;t\right)$ and $\chi\left(x;t\right)$ are calculated using the formulas Eq. (2.19) and Eq. (2.20), respectively.

Step 5. In the region $x \in \left[l;2\,l\right], t \geq 0$ the motion of the liquid-gas mixture is described by the impulse system:

$$\begin{cases} Q(l+0,t) = F_\delta^{'}Q(l-0,t) + F_\phi^{'}R(l-0,t), \\ R(l+0,t) = F_\delta^{''}Q(l-0,t) + F_\phi^{''}Q(l-0,t)\end{cases}$$

Step 6. The functions $W\left(x,t\right)$ and $\chi\left(x,t\right)$ corresponding to the lift pipe are determined using the following equations:

$$\begin{cases} \tilde{W}'(x,\xi) + \frac{\lambda+2a_i}{\alpha F_i}\tilde{W}(x,\xi) + \frac{\lambda+2a_i}{\alpha_i c_i}\tilde{\chi}(x,\xi) = \frac{\alpha F_i}{c_i}\bar{\bar{Q}}'(x) + \frac{F_i}{c_i}\bar{\bar{R}}'(x) + \frac{2a_i}{c_i}\bar{\bar{Q}}(x), \\ \tilde{\chi}'(x,\xi) - \frac{\lambda+2a_i}{\alpha F_i}\tilde{\chi}(x,\xi) - \frac{\left(c_i-\alpha^2F_i^2\right)\lambda+2ac}{\alpha F_i}\tilde{W}(x,\xi) = \bar{\bar{R}}'(x) - \frac{2a_i}{F_i}\bar{\bar{Q}}(x),\end{cases}$$

Step 7. Finally, considering the functions $W\left(x;t\right)$ and $\chi\left(x;t\right)$ determined according to the system Eq. (2.3), and taking into account the first and second equations, we obtain the following expressions for the functions $Q\left(x;t\right)$ and $R\left(x;t\right)$:

$$Q(x,t) = Q(0,t) + \int_0^x W(\xi,t)d\xi = \bar{Q}(t) + \int_0^x W(\xi,t)d\xi = \bar{\bar{Q}}(t) + \sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}\int_0^x\tilde{W}_k(\xi)d\xi$$

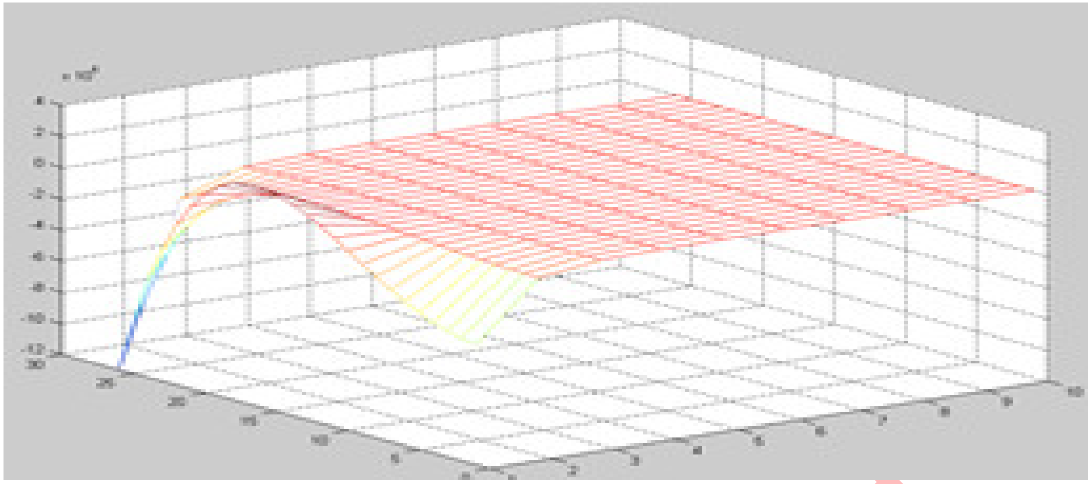

FIGURE 1. $R(2l, t)$.

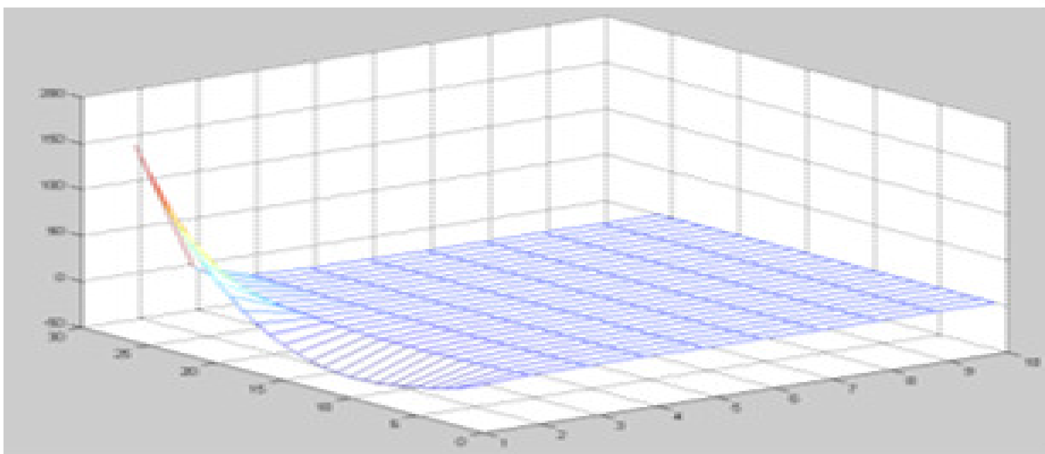

FIGURE 2. $Q(2l, t)$.

$$R(x,t) = R(x,0) + \int_0^t \chi(x,\tau)d\tau = \bar{\bar{R}}(x) + \sum_{k=1}^{\infty} \tilde{\chi}_k(x)\frac{t^k}{k!}$$

Thus, in the entire segment $x \in [l;\, 2l]\,, t \geq 0$ the sought functions $R\,(x;t)$, $Q\,(x;t)$, $W\,(x;t)$ and $\chi\,(x;t)$ are uniquely determined.

Using the proposed algorithm, calculations have been performed, and graphs representing the process for $R\,(x,t)$ and$Q\,(x,t)$ are provided (Figure 1, Figure 2).

The parameters have been adopted as follows, suitable for practical applications:

(1) In the annular space (i.e, when $0 < x < l$): $c_1 = 331\ m/s$, $\tilde{n}_1 = 0.717\ kg/m^3$, $d_1 = 1.05 \cdot 10^{-3}\,m$,$\lambda_1 = 0.01$.
(2) In the lift pipe (i.e, when $l < x < 2l$): $c_2 = 850\ m/s$, $\tilde{n}_2 = 700\ kg/m^3$, $d_2 = 0.073\,m$,$\lambda_2 = 0.23$.

From both graphs (Figures 1 and 2) it can be seen that the received and values correspond to the results obtained from the practice [2], [3].

3. Conclusions

It should be noted that the solution of the Roesser-type equation with the help of the Laplace transform was investigated for the first time for the system of differential equations describing the movement in the gas-lift process using this method. The results of the study showed that the solution of Roesser-type equations with Laplace transformation provides high accuracy and efficiency in modeling and controlling oil extraction processes. In particular, the solution of Roesser-type equations in oil extraction processes provides new approaches for the optimization and control of technological processes. This method can be considered useful in terms of increasing production and efficient use of resources.

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