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Iterative methods for large scale problem

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Abstract

One linear bi-criterion mathematical program, which appears as a large-scale problem in practice, is considered. Problems, related to the large size, are usually solved with the help of the methods, based on the possibilities created by the zeros of the matrix of the problem. In this way, a large number of different separation schemes have been suggested in the scientific literature. However, the problems considered here have no such possibility due to its large size. In order to overcome the size problem during the solution of the problem, the possibility of reducing it to a smaller problem is investigated. The reduction is carried out without disturbing the original structure of the problem. The goal is to maintain the possibility of using the existing effective solution methods for the problems before the reduction also for the problems received after the reduction. Suggested here method mainly uses sequential approximation schemes in fulfilling.

Keywords. Linear programming, Dual problem, Dual theorem, Basic variable, Basic solution, Pareto bound.
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1. Introduction

The following bi-criterion linear programming is considered:

$$Ix \leq Ax - Bx + b, x \geq 0, y_1(x) = \left(c^1, x\right) \to max, y_2(x) = \left(c^2, x\right) \longrightarrow max. \tag{1.1}$$

Here $I,\ A,\ B \in R^{n\times n},\ x,\ b,\ \ c^1,\ \ c^2 \in R^n$. Is a unit matrix, the coordinates of the vector b are positive and the elements of the matrices A and B are nonnegative ($b > 0, A, B \geq 0$). When $B = 0$ in scalar objective case, the problem (1.1) is studied in [22] and sequential decision making method is suggested to solve it and given extensive studies a number of real practical problems to illustrate possibility of the method [21]. Theoretical foundation of the method is given in [6]. All success obtained as a result of application of the method is to due to the used sequential approximation being varied monotonically. Such property depends on special structure of the condition matrix of the problem. For example, if matrix A is M–matrix monotonicity property takes place. As mentioned in [7], in the large–scale case the problem like (1.1) cannot be solved by well known separation schemes developed on the base of various decomposition techniques [8, 9, 20]. Decomposition technique is used to reduce large–scale problem into a number of smaller problems. In this way we handle smaller tasks than to handle a very large problem as a whole. A problem is divided into smaller sub problems. Each sub problems is then solved independently, and then solutions are combined to solve the original problems. The problem (1.1) is not investigated in [21, 22] when $B \neq 0$ and when objective function is scalar function. But we meet such kind of problem when we are going, for example, to investigate of the stability of the solution ([21], p.164). Stability problems for non-dominated solutions in multi-decision–making in more general case are studied in [23, 33]. As a solution of the problem (1.1) here we assume the set of all Pareto optimal criterion estimations

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(we call this set as Pareto bound (front)) of the problem and minimal volume of non-dominated solution's set that defines this front. The multistage decision making procedure is used to solve the problem. Procedure suggested here is rather simple and convenient from algorithmic point of view. The solution defined by this procedure is of piece-wise broken line curve. The number of broken points of coordinate plane depends on required accuracy of the Pareto front we want to set. Here regular approximation variant is used [13]. Another form of accuracy is integral form used in [28]. Determining of all efficient solutions to a linear vector maximation problem in more general case are given in [12, 14]. Each of these methods in [13] and [28] solves linear programming problem at each step of approximation. However the first variant uses the same condition as it is in the original problem unlike the second variant. Under the circumstantial, existing any method that solves the linear programming with condition as in (1.1) can be successfully used when solving the problem (1.1). Effectiveness of the method we are going to suggest mainly depends on how efficiently each stage of approximation is performed. Many authors try to suggest new ideas and methods in order to solve large–scale problems [10, 11, 17, 18, 25, 30–32, 34–38]. Our paper attempts to implement an efficient algorithm that allows us to reduce the large-scale problem (1.1) into the problems of smaller dimension. We apply this idea at the each step of decision-making. Therefore the scheme, considered here works as two-stage procedure, where the first stage tries to solve the dimension problem on the whole for all sub-procedures and the second one stage tries to solve dimension problem for each of sub-procedures which defines broken points of Pareto front. Then the scheme is applied when criteria is scalar fractional linear criteria and scalar linear parametric function. We suggest new methods to solve each of two these problems under the condition as in (1.1). As a result we have possibility to solve new large–scale bi-criterion linear fractional programming more efficiently in comparison with standard methods (for example [14]). We suggest Pareto bound (front) method to solve considered large–scale linear parametric programming and such approach allows us to use only simple iterative procedures to realize setting up of the solution of the problem.

The results obtained here can be applied to a number of practical problems [1–5, 15, 24, 29].

2. Basic notations and definitions

We solve the problem (1.1) under the conditions:

(1) the spectral radius of the matrix $A + B$ is less than one.
(2) $b > B(I - A)b$.

Denote by X the set of all feasible solutions of the problem (1.1). And consider the set of all criterion valuations $Y = \{y(x) \in R^2 \mid y_1(\mathrm{x}) = (c^1, \mathrm{x}), y_2(\mathrm{x}) = (c^2, \mathrm{x}), \mathrm{x} \in \mathrm{X}\}$. $Y^P = \{y^0 \in Y | \mathrm{y} \in \mathrm{Y}, \mathrm{y} \geq y^0 \Longrightarrow \mathrm{y} = y^o\}$ is Pareto set (or Pareto front of the problem (1.1)).
The set $X^P = \{x \in X \mid ((c^1, x), (c^2, x)) \in Y^P\}$ is called Pareto optimal solutions set of the problem.

Lemma 2.1. *The set X is bounded.*

From the condition i) we have justify the inequality $(I - A)^{-1} \geq 0$. Therefore we can write:

$$(\mathrm{I} - \mathrm{A})\mathrm{x} + \mathrm{y} = \mathrm{b}, \mathrm{x} \geq 0, \mathrm{y} \geq 0, \mathrm{x} + (\mathrm{I} + \mathrm{A})^{-1}\mathrm{y} = (\mathrm{I} - \mathrm{A})^{-1}\mathrm{b}.$$

From here we have:

$$x \leq (\mathrm{I} - \mathrm{A})^{-1}\mathrm{b}. \tag{2.1}$$

Consider the canonical form of the problem (1.1):

$$\mathrm{x} + \mathrm{y} = \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \mathrm{x} \geq 0 \mathrm{y} \geq 0, (\mathrm{c}^1, \mathrm{x}) \to \max, (\mathrm{c}^2, \mathrm{x}) \to \max. \tag{2.2}$$

Let pair (x^0, y^0) be any feasible solution of (2.2).

Lemma 2.2.

$$x_i^0 + y_i^0 > 0, i = \overline{1, n}. \tag{2.3}$$

Indeed, from (2.2) and from (2.1) we can write:

$$x_i^0 - (Ax^0)_i + y_i^0 \geq -(B(I - A)b)_i + b_i.$$

From here and from the condition ii) we have:

$$x_i^0 + y_i^0 > 0, i = \overline{1, n}. \tag{2.4}$$

From the inequalities (2.4) and the definition of basis the following propositions easily can be proved.

Proposition 2.3. *The feasible solution $(x^0, \ y^0)$ is basis solution if and only if the condition $x_i^0 \ . \ y_i^0 = 0, \ \ i = \overline{1, n}$ is true.*

Proposition 2.4. *When we solve the problem by simplex method then only one of columns $(I - A + B)_i$, $(I)_i$ of matrices $I - (A - B)$ and I can be basis column.*

From (2.4) we also have:

Corollary 2.5. *All basis solutions of the problem (1.1) are non-degenerate extreme points.*

Consider the points $y^i = \left(y_1^i \ , y_2^i\right) \in Y^P$, $\ i = \overline{1, l}$ such that $y_1^i < y_1^{i+1}$ $\ i = \overline{1, \ \ l-1}$ and define the sets $Y^i = [y^i \ \ y^{i+1} \ ]$, $\ i = \overline{1, l-1}$ as line segments and the set

$$Y_P^0 = \bigcup_{i=1}^{l-1} Y^i. \tag{2.5}$$

The distance between the sets Y^P and Y_P^0 we call as d.

Definition 2.6. Y_P^0 will be called $\varepsilon-$ solution of the problem (1.1) if $d \leq \varepsilon$.

3. Method of finding the solution Y_P^0

The Pareto front of the problem (1.1) is of the graphic of the concave and strictly decreasing function on the coordinate plane [26].

Step 1. Define two Pareto points $y^1\left(x^1\right), y^{\ 2}(x^2)$ and Pareto optimal solutions $x^1, \ \ x^2$ from the solutions of the following linear programming problems:

$$y_2^1 \ \left(\ x^1\right) = \max_{x \in X} y_2\left(x\right), \qquad y_1^2 \ \left(x^2\right) = \max_{x \in X} y_1(x)\ ,$$

$$y^1\left(x^1\right) = \left(\ y_1 \ \ \left(\ x^1\right) \ , y_2 \ \left(\ x^1\right) \ \right), \qquad y^{\ 2}\left(x^2\right) = \ \left(\ y_1 \ \left(\ x^2\right), \ y_2 \ \left(\ x^2\right)\right).$$

Further, the number of Pareto defined points recall as l. At the first step $l = 2, y^1, \ y^l$ are the left-end point and the right- end point of the Pareto front correspondingly. Set two new vectors $\alpha = (\alpha_1, \ \alpha_2) = (y_2^2 - y_2^1, y_1^1 - y_1^2)$, $c = \alpha_1 c^1 \ + \ \ \alpha_2 c^2$ then form the following linear programming:

$$\mathrm{x} \leq \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \mathrm{x} \geq 0, (\mathrm{c}, \mathrm{x}) \to \max. \tag{3.1}$$

Step 2. Take the optimal solution x^* of the problem and define the new Pareto point $y^3 = y(x^*)$. Then take the triangular with the vertices y^1, y^3, y^2. Now take the altitude of the triangular drawn through the vertex y^2. Let h be the length of it. If the inequality $h < \varepsilon$ is true then (2.5) defines the solution of the problem (1.1) and go to the Step End. Otherwise go to the next step. Here ε is the advanced given required precision.

Step 3. We have two pare of Pareto points: $(y^1, y^3), (y^3, y^2)$. We apply step1 to each of them and define new two Pareto points. We call them z^1, z^2 correspondingly and make new notations

$$y^5 = y^2, \ y^4 = \ z^2, \ y^3 = y^2, \ y^2 = \ z^1.$$

In the new case $l = 5$. Now verify just how these five Pareto points approximates the Pareto front. For take two pair triangular $y^1y^2y^3, y^3y^4y^5$ and evaluate theirs altitudes as it was at the step 2. The following situations can be occurred:

Case 1. All altitudes satisfy accuracy. It means that we have solved the problem and go to the Step End.

Case 2. There is an altitude such that doesn't satisfy accuracy. In this case remember the triangular which satisfies accuracy and eliminate it from further consideration. Then return to step1 and apply it to the triangular that doesn't satisfy accuracy and define new Pareto points. In this way the set of Pareto points is extended until the requisite accuracy is satisfied.

Step End.

Remark 3.1. Calculation volume used to evaluate a Pareto front mainly depends on how efficiently the problems (3.1) are solved at each step of iterations.

4. Reduction of the problem (3.1)

In this section we study haw the problem (3.1) can be reduced to the problem with the same structure and with the smaller dimension.

Take the problem dual to the problem (3.1):

$$\mathrm{y} \leq \mathrm{yA} - \mathrm{yB} + \mathrm{c}, \mathrm{y} \geq 0, \mathrm{y}, \mathrm{b} \to \min. \tag{4.1}$$

Solve the problem as limit of the approximations $\psi^1, \psi^2, ..., \psi^k, ...$ constructed according to the following scheme:

$$\psi^1 = 0, \psi^{k+1} = \max\left(0, c + \psi^k A - \psi^k B\right), k = 1, 2, ... \tag{4.2}$$

The limit $\lim_{k\to\infty} \psi^k = \psi^*$ gives the optimal solution to the problem (4.1) ([6], p.16). Define the sets:

$$G^* = \{i \mid i \in [1:n], \psi_i^\star \neq 0\}, \overline{G^*} = \{1, 2, \ldots, n\} nG^*$$

.

The feasible solution y of the problem (4.1) will be optimal if it satisfies the conditions:

$$y_i = (yA - yB + c)_i, \quad i \in G^*, \; y_j = 0, \quad j \in \overline{G^*}.$$

From here and from the duality theorem we come to the conclusion that the feasible solution x of the primal problem (4.1) will be optimal if it satisfies the conditions:

$$x_i = (Ax - Bx + b)_i, \quad i \in G^*, \quad x_j = 0, \quad j \in \overline{G^*}.$$

Therefore, to have the set G^* is sufficient to identify the optimal basis variables of the problem (1.1). The more we have information about this set the more we have chance to solve the dimension problem for the problem (3.1). Consider the case when $B = 0$ and construct the following approximations:

$$\mathrm{t}^1 = 0, \quad t^{k+1} = \max\left(0, \; c + t^k A\right) \quad k = 1, 2, \ldots$$

Sequence $\mathrm{t}^1, t^2, \cdots, t^k, \cdots$, is bounded and increasing sequence. Denote

$$\operatorname*{Lim}_{k\to\infty} t^k = t^*.$$

Lemma 4.1. $\psi^k \leq t^k, \quad k = 1, 2, \ldots$

Validity of the lemma easily obtained from the definition of sequences. From the lemma we have

Corollary 4.2. $\psi_i^\star = 0.i \in T^* = \{i \mid i \in [1:n], t_i^\star = 0\}$. *Take* $b^0 = c - t^0 B$ *and consider the approximations:*

$$\mathrm{g}^1 = 0, \; g^{k+1} = \max\left(0, \; c + g^k A + b^0\right) \; k = 1, 2, \cdots$$

Denote $\mathrm{Lim}_{k\to\infty} g^k = g^*$.

Lemma 4.3. $g^k \leq \psi^k, \quad k = 1, 2, \ldots$

Corollary 4.4. $\psi_i^\star > 0, \quad i \in G^* = \{i \mid i \in [1:n], g_i^\star > 0\}$.

The following theorem is proved.

Theorem 4.5. *The problem* (3.1) *and the following problem are equivalent:*

$$\mathrm{x} + \mathrm{y} = \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \mathrm{x} > 0, \mathrm{y} > 0,$$
$$y_i = 0, i \in G^*, \quad x_i = 0, \; i \in T^*.$$
$$(c, x) \to max.$$

From the theorem as a result we have:

Corollary 4.6. *The dimension of the basis simplex table of the problem (3.1) is* $(n \times 2n)$*, but the dimension of the new problem is* $(m \times 2m)$*, where* $m = n - [G^* \cup T^*]$*.*

Corollary 4.7. *Monotonous of the sequences which we use to define the sets* G^*, T^* *allows us to interrupt further iterations when desire reduction of dimension is reached.*

5. Application of reduction scheme

Now we are going to demonstrate the possibility of reduction scheme based on the problem like (1.1) on the parametric and fractional-linear programming problems.

Consider the problem under the condition as in (1.1):

$$\mathrm{Ix} \leq \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, 0 \leq \mathrm{x} \leq \mathrm{d}^1 + \lambda \mathrm{d}^2, 0 \leq \lambda \leq 1, (\mathrm{c}, \mathrm{x}) \to \max \tag{5.1}$$

The following additional conditions are also assumed:

$$a) c > 0, d^1, d^2 > 0; b) b - Bd^1 - Bd^2 > 0.$$

Condition b) makes the problem (5.1) is non-degenerate at the each mean of the parameter. Really:

$$x + z = A\,x - B\,x + b\,,\; x \;\geq 0, \;\; z \geq 0,$$

$$(I - A)\,x + z = \; -Bx + b,$$

$$x + (I - A)^{-1} z = (\,I - A\,)^{\,-1} (\,b - B\,x)\,,$$

$$x \;\geq\; (\,I - A\,)^{-1} \;\left(b - \; Bd^1 - Bd^2\right) > 0.$$

Take the dual problem to the problem (4.1):

$$y\;\left(\,I - A + B\,\right) + t \;\geq c, y \geq 0, t \geq 0, yb + t\left(d^1 + \lambda d^2\right) \to min, 0 \leq \lambda \leq 1. \tag{5.2}$$

According to the second duality theorem, the condition of the problem can be written as following:

$$y\,(I - A + B) + t = c.$$

Then, after simple transformation the problem (5.2) can be written as:

$$y\,(I - A + B) \leq c, \;\; y \geq 0,$$

$$y\,((I - A + B) - \mathrm{b}) + \lambda(y\,(I - A + B)\,d^2 + \lambda c d^2 + c d^1 \to max, \qquad 0 \;\leq\; \lambda \;\leq\; 1. \tag{5.3}$$

Denote

$$(I - A + B)\,d^1 - b = \; e^1\;,\;\; (\,I - A + B\,)\;\, d^2 = \; e^2.$$

The problem (5.3) in new notation has such form:

$$y \leq yA - yB + c, \quad y \geq 0, \quad y\;e^1 + \;\; \lambda\;y\;e^2 - \lambda c d^2 - c d^1 \to max, \qquad 0 \leq \lambda \leq 1. \tag{5.4}$$

The solution of the problem (5.4) is presented as the Pareto bound of the following two-criterion problem:

$$y \;\; \leq yA - yB + c\;,\;\; y \geq 0,\;\, y\;r^1 + c l^1 \;\; \to max, \quad y e^1 + c\;d^1 \to \;\, max. \tag{5.5}$$

Here $r^1 = e^1 + e^2,\; l^1 = \; d^1 + d^2$. Let $x^i \;\in\; X^P$ such that $y_1 \;\left(\,x^i\right) \;< y_1 \;\left(x^{i+1}\right),\, i \;\in\; \overline{1,l}$ and assume that

$$Y_P^0 = \bigcup_{i=1}^{l} \;[(\;y_1 \;\;(\;x^i)\;,\;\,y_2 \;\;(\;x^i)]$$

piece-wise linear approximates Pareto front of the problem (5.5).

Define the vectors

$$n^i = (y_2^i {-} y_2^{i+1}, y_1^{i+1} - y_1^i),$$

$$n^{i+1} = \;(\;y_2^{i+1} {-} y_2^{i+2}, y_1^{i+2} - y_1^{i+1}), \qquad i \;\in\; [1 : l - 2]\,.$$

Then construct the following sets, by using them:

$$\Lambda^i = \mu \quad \frac{n^i}{n_1^i + n_2^i} \quad + (1-\mu)\frac{n^{i+1}}{n_1^{i+1} + n_2^{i+1}}, 0 \le \mu \le 1 i \ \in \ [1 : l-2] \, .$$

Proposition 5.1. *$\left(\bigwedge\ ^i, x^i\ \right)$, $i \in [\ 1 :\ \ l-2]$ is $\varepsilon-$ approximation of the solution of the problem (5.1).*

Now we are going to illustrate how reducing scheme can be used to solve the fractional linear programming:

$$\mathrm{Ix} \le \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \mathrm{x} \ge 0,$$

$$\frac{(c\ , x\ ) + d}{(e,\ x) + f} \quad \to max. \tag{5.6}$$

Assumption: $(e, x) + f > 0$.

Based on this assumption without loss of the generality, we can also assume that the numerator of the fraction is positive. Then

$$Y = \left\{ (y_1, y_2) \,|\ \ y_1 = l0 * -(c, x) + d\ , y_2 = (e, x) + f\ ,\ \ \ x \in X \right\}$$

will be convex, closed and bounded set of the first quadrant. Let x^0 be optimal solution to the problem (5.6) and $y^0\left(x^0\right) = \left(y_1^0(x^0)\, , y_2^{0}\left(x^0\right)\right) \in\ Y$. $y^0(x^0)$ has the following property: the straight line, connecting this point with the coordinate origin is the support line of Y and keeps the set Y to the left side of the line. Our aim is to obtain the optimal solution by solving a finite number of problems such as (3.1) that differs only by their criteria.

Step 1. Take any vector $y^(1) = \left(y_1^1, y_2^1\right)$? y as initial approximation and vector $n = (n_1, n_2) = \left(-y_2^1\ ,\ y_1^1\right)$. Then solve the following problem:

$$\mathrm{n}_1((\mathrm{c}, \mathrm{x}) + \mathrm{d}) + \mathrm{n}_2((\mathrm{e}, \mathrm{x}) + \mathrm{f}) \to \max, \mathrm{x} \in \mathrm{X}. \tag{5.7}$$

Let x_2 be the optimal solution to the problem (5.7). By using this solution form the vectors.

$$y^{(2)} = \left(y_1\left(x^2\ \right),\ y_2\left(x^2\ \right)\right), n^{(2)} = (-y_1^2,\ y_2^2).$$

Step 2. Take $y^1\ = y^2$, $\ n^1\ = n^2$, return to the step 1. Continue the process in this way. The process is finite. The value of the objective function increases at the each step. That is, the process is monotonous. It ends when next approximation coincides with the previous one.

The proposed solution to problem (5.6) is different from the existing standard way of solution (see [18]). The standard way of solution doesn't leave the initial structure of the problem and does not allow us to take advantage of the reduction proposed here.

6. Numerical example

We demonstrate the reduction process on the example for the problem (5.1) when $\lambda = 0$:

$$x \ \ \le\ x\ A\ \ - xB\ + b\ ,\ \ \ 0\ \le\ \ \ x\ \ \ \le\ \ \ d\ ,\ \ \ \ (\ c,\ x)\ \ \to\ \ \ max). \tag{6.1}$$

$$A = \begin{pmatrix} 0.20 & 0.00 & 0.00 & 0.06 & 0.00 \\ 0.00 & 0.30 & 0.00 & 0.00 & 0.10 \\ 0.30 & 0.00 & 0.10 & 0.00 & 0.20 \\ 0.00 & 0.10 & 0.20 & 0.30 & 0.00 \\ 0.00 & 0.00 & 0.06 & 0.00 & 0.20 \end{pmatrix}, \quad B = \begin{pmatrix} 0.00 & 0.10 & 0.30 & 0.00 & 0.00 \\ 0.10 & 0.00 & 0.00 & 0.20 & 0.00 \\ 0.00 & 0.20 & 0.00 & 0.08 & 0.00 \\ 0.04 & 0.00 & 0.00 & 0.00 & 0.30 \\ 0.00 & 0.05 & 0.00 & 0.10 & 0.00 \end{pmatrix},$$

$$x\ = (x_1,\ \ x_2,\ \ x_3,\ \ x_4,\ \ x_5)\, ,\ \ b\ = (15,\ \ 8,\ \ 4,\ \ 10,\ \ 6)\, ,\ \ d\ = (9,\ \ 12,\ \ 6,\ \ 8,\ \ 10)\, ,\ \ c\ = (5,\ \ 2,\ \ 1,\ \ 1,\ \ 1)\, .$$

Step 1.

Set the problem (5.5):

In our case the problem (5.5) has such form:
$y \leq Ay - By + c, \quad ey \longrightarrow max$, where

$$e = d\,(I - A + B) - b.$$

$$e = (9, 12, 6, 8, 10)\begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.10 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.8 \end{pmatrix} - (15, 8, 4, 10, 6) =$$

$$= (-8.08,\ \ 2.20,\ \ 1.90,\ \ -1.06,\ 2.00)$$

Step 2.
Construct the approximations:

$$\eta^{(1)} = 0, \quad \eta^{(i)+1} = \max\left(0\ ,\ e^T +\ A\eta^{(i)}\right)\ ,\ i\ = 1,\ 2\ , \quad . \quad . \quad .$$

$$\eta^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \eta^{(2)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 2.20 \\ 1.90 \\ 0 \\ 2.00 \end{pmatrix},$$

$$\eta^{(3)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.10 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\begin{pmatrix} 0 \\ 2.20 \\ 1.90 \\ 0 \\ 2.00 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 3.06 \\ 2.49 \\ 0 \\ 2.51 \end{pmatrix},$$

$$\eta^{(4)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.10 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.37 \\ 2.65 \\ 0 \\ 2.65 \end{pmatrix},$$

$$\eta^{(5)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.37 \\ 2.65 \\ 0 \\ 2.65 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.48 \\ 2.70 \\ 0 \\ 2.69 \end{pmatrix},$$

$$\eta^{(6)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.48 \\ 2.70 \\ 0 \\ 2.69 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.51 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix},$$

$$\eta^{(7)} = \ max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.51 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.52 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix},$$

$$\eta^{(8)} = \ max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.52 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.53 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix},$$

$$\eta^{(9)} = max\left(\left(\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-8.08\\2.20\\1.90\\-1.06\\2.00\end{pmatrix} + \begin{pmatrix}0.80 & 0.10 & 0.30 & -0.06 & 0.00\\0.10 & 0.70 & 0.00 & 0.20 & -0.01\\-0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20\\0.04 & -0.10 & -0.20 & 0.70 & 0.30\\0.00 & 0.05 & -0.06 & 0.10 & 0.80\end{pmatrix}\right)\right)\begin{pmatrix}0\\3.53\\2.71\\0\\2.70\end{pmatrix} = \begin{pmatrix}0\\3.53\\2.71\\0\\2.70\end{pmatrix}.$$

Based on the last seven iterations we get: $T^* = \{1,\ 4\}$. That is, $x_1^{op} = d_1 = 9, x_4^{op} = d_4 = 8$. The given problem turns from the five-variable problem into the three-variable new problem.

Step 3.

Setting the new problem:

$y^1 \ \le\ A^1y^1 -\ B^1 +\ b^1, \qquad y^1 \ge 0, \qquad e^1y^1 \ \longrightarrow max$. Here $y^1 = (\ \ y\ _2,\ \ y_3\ ,\ y_5)$,

$b_i^1\ =\ b_i, -(9,\ 0\ ,0\ ,8\ ,0)\,(\ I - A + B)_i\ I = 2,\ 3\ ,\ 5.$ From here we have:

$$b^1 = (7.90,\ \ 2.90,\ \ 3.60)\,,$$

$$A^1 = \begin{pmatrix}0.30 & 0.00 & 0.10\\0.00 & 0.10 & 0.20\\0.00 & 0.06 & 0.20\end{pmatrix}\ ,\ B^{\ 1} = \begin{pmatrix}0.00 & 0.00 & 0.00\\0.20 & 0.00 & 0.00\\0.05 & 0.00 & 0.00\end{pmatrix}$$

$e^1 = d^1\left(\ I - A^1\ + B^1\right) -\ b^1,\ \ d^1 = (12,\ \ 6,\ 10)$. Calculate the value of the vector $e^1\ = (2.20,\ \ 1.90,\ 2.00)$. For the new problem $G^* = (2, 3, 5)$. As a result of the Theorem 1, the remaining coordinates of optimal solution of the new problem can be found from the system of equations:

$$0.70\ x_2 + 0.20\ x_3 + 0.05\ x_5 = 7.9,$$

$$0.90\ x_3 - 0.06\ x_5 = 2.9,$$

$$-0.10x_2 - 0.20\ x_3 + 0.8\ \ x_5 = 3.6,$$

$$x_2 = 9.76, \quad x_3\ = 3.66,\ x_5 = 6.64.$$

Finally, the optimal solution for the considered example will be the vector $(9.00, 9.76, 3.66, 8.00, 6.64)$.

Numeric calculation of the given example is performed by MATLAB.

7. Conclusion

Many problems have been considered and are being considered related to large size and different separation schemes have been suggested to solve them. However, existing separation schemes essentially have been based on the possibilities created by the zeros of matrix of the problems. The problem considered here has no such possibility due to its large size. But if the dimension reason it is shown that the problem considered successfully can be solved by existing iterative procedure. Here we try to solve the problems: i) how to identify Pareto front problem considered by using iterative procedure; ii) how to use iterative procedure to reduce the dimension of the problem: iii) how to carry out i) and ii) without disturbing the original structure of the problem. We demonstrate usability of the suggested method for parametric and fractional programming. Numerical example is solved.

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