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Iterative methods for large-scale problems

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Abstract

One linear bi-criterion mathematical program, which appears as a large-scale problem in practice, is considered. Problems related to the large size are usually solved with the help of methods based on the possibilities created by the zeros of the matrix of the problem. In this way, a large number of different separation schemes have been suggested in the scientific literature. However, the problems considered here have no such possibility due to their large size. In order to overcome the size problem during the solution of the problem, the possibility of reducing it to a smaller problem is investigated. The reduction is carried out without disturbing the original structure of the problem. The goal is to maintain the possibility of using the existing effective solution methods for the problems before the reduction, as well as for the problems received after the reduction. Suggested here method mainly uses sequential approximation schemes to fulfill.

Keywords. Linear programming, Dual problem, Dual theorem, Basic variable, Basic solution, Pareto bound.
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1. Introduction

The following bi-criterion linear programming is considered:

$$Ix \leq Ax - Bx + b, x \geq 0, y_1(x) = \left(c^1, x\right) \to max, y_2(x) = \left(c^2, x\right) \longrightarrow max. \tag{1.1}$$

Here $I,\ A,\ B \in R^{n\times n},\ x,\ b,\ \ c^1,\ \ c^2 \in R^n$ is a unit matrix, the coordinates of the vector b are positive and the elements of the matrices A and B are nonnegative ($b > 0, A, B \geq 0$). When $B = 0$ in the scalar objective case, the problem (1.1) is studied in [21], and a sequential decision-making method is suggested to solve it, and given extensive studies of a number of real practical problems to illustrate the possibility of the method [20]. The theoretical foundation of the method is given in [6]. All success obtained as a result of application of the method is due to the used sequential approximation being varied monotonically. Such a property depends on the special structure of the condition matrix of the problem. For example, if matrix A is M–matrix monotonicity property takes place. As mentioned in [7], in the large–scale case the problem like (1.1) cannot be solved by well-known separation schemes developed on the basis of various decomposition techniques [8, 9, 19]. The decomposition technique is used to reduce a large-scale problem into a number of smaller problems. In this way, we handle smaller tasks rather than handling a very large problem as a whole. A problem is divided into smaller sub-problems. Each sub-problem is then solved independently, and then solutions are combined to solve the original problem. The problem (1.1) is not investigated in [20, 21] when $B \neq 0$ and when the objective function is a scalar function. But we meet such a problem when we are going, for example, to investigate the stability of the solution ([20], p.164). Stability problems for non-dominated solutions in multi-decision-making in a more general case are studied in [22, 33]. As a solution to the problem (1.1),

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here we assume the set of all Pareto optimal criterion estimations (we call this set the Pareto bound (front)) of the problem and the minimal volume of the non-dominated solution set that defines this front. The multistage decision-making procedure is used to solve the problem. The procedure suggested here is rather simple and convenient from an algorithmic point of view. The solution defined by this procedure is a piecewise broken line curve. The number of broken points of the coordinate plane depends on the required accuracy of the Pareto front we want to set. Here, a regular approximation variant is used [12]. Another form of accuracy is the integral form used in [28]. Determining all efficient solutions to a linear vector maximization problem in a more general case is given in [11, 13]. Each of these methods in [12] and [28] solves the linear programming problem at each step of approximation. However, the first variant uses the same condition as it is in the original problem, unlike the second variant. Under the circumstantial, any method that solves the linear programming with the condition as in (1.1) can be successfully used when solving the problem (1.1). The effectiveness of the method we are going to suggest mainly depends on how efficiently each stage of approximation is performed. Many authors try to suggest new ideas and methods in order to solve large-scale problems [10, 16, 17, 23, 25, 30–32, 34–38]. Our paper attempts to implement an efficient algorithm that allows us to reduce the large-scale problem (1.1) into the problems of smaller dimension. We apply this idea at each step of decision-making. Therefore, the scheme, considered here, works as a two-stage procedure, where the first stage tries to solve the dimension problem for all sub-procedures and the second stage tries to solve the dimension problem for each of the sub-procedures, which defines the broken points of the Pareto front. Then the scheme is applied when the criteria are scalar fractional linear criteria and scalar linear parametric function. We suggest new methods to solve each of these two problems under the condition as in (1.1). As a result, we have the possibility to solve new large-scale bi-criterion linear fractional programming more efficiently in comparison with standard methods (for example [13]). We suggest the Pareto bound (front) method to solve the considered large-scale linear parametric programming, and such an approach allows us to use only simple iterative procedures to set up the solution of the problem.

The results obtained here can be applied to a number of practical problems [1–5, 14, 24, 29].

2. Basic notations and definitions

We solve the problem (1.1) under the conditions:

(i) the spectral radius of the matrix $A+B$ is less than one.
(ii) $b > B(I-A)b$.

Denote by X the set of all feasible solutions of the problem (1.1). And consider the set of all criterion valuations $Y = \left\{ y\left(x\right) \in R^2 \mid y_1(\mathrm{x}) = \left(c^1, \mathrm{x}\right), y_2(\mathrm{x}) = \left(c^2, \mathrm{x}\right), \mathrm{x} \in \mathrm{X} \right\}$. $Y^P = \left\{ y^0 \in Y | \mathrm{y} \in \mathrm{Y}, \mathrm{y} \geq y^0 \Longrightarrow \mathrm{y} = y^o \right\}$ is Pareto set (or Pareto front of the problem (1.1)).

The set $X^P = \left\{ x \in X \mid \left(\left(c^1, x \right), \left(c^2, x \right) \right) \in Y^P \right\}$ is called the Pareto optimal solution set of the problem.

Lemma 2.1. *The set X is bounded.*

From condition (i) we have to justify the inequality $(I-A)^{-1} \geq 0$. Therefore we can write:

$$(\mathrm{I}-\mathrm{A})\mathrm{x}+\mathrm{y}=\mathrm{b}, \mathrm{x}\geq 0, \mathrm{y}\geq 0, \mathrm{x}+(\mathrm{I}+\mathrm{A})^{-1}\mathrm{y}=(\mathrm{I}-\mathrm{A})^{-1}\mathrm{b}.$$

From here we have:

$$x \leq (\mathrm{I}-\mathrm{A})^{-1}\mathrm{b}. \tag{2.1}$$

Consider the canonical form of the problem (1.1):

$$\mathrm{x}+\mathrm{y}=\mathrm{Ax}-\mathrm{Bx}+\mathrm{b}, \mathrm{x}\geq 0\mathrm{y}\geq 0, (\mathrm{c}^1,\mathrm{x})\to\max, (\mathrm{c}^2,\mathrm{x})\to\max. \tag{2.2}$$

Let pair (x^0, y^0) be any feasible solution of (2.2).

Lemma 2.2.

$$x_i^0 + y_i^0 > 0, i = \overline{1,n}. \tag{2.3}$$

Indeed, from (2.2) and from (2.1) we can write:

$$x_i^0 - (Ax^0)_i + y_i^0 \geq -(B(I-A)b)_i + b_i.$$

From here and from condition (ii) we have:

$$x_i^0 + y_i^0 > 0, i = \overline{1, n}. \tag{2.4}$$

From the inequalities (2.4) and the definition of basis, the following propositions can be easily proved.

Proposition 2.3. *The feasible solution $(x^0,\ \ y^0)$ is a basis solution if and only if the condition $x_i^0\ .\ y_i^0 = 0,\ \ i = \overline{1,n}$ is true.*

Proposition 2.4. *When we solve the problem by the simplex method then only one of columns $(I - A + B)_i,\ (I)_i$ of matrices $I - (A - B)$ and I can be basis columns.*

From (2.4) we also have:

Corollary 2.5. *All basis solutions of the problem (1.1) are non-degenerate extreme points.*

Consider the points $y^i = \left(y_1^i\ , y_2^i\right) \in Y^P,\ i = \overline{1,l}$ such that $y_1^i < y_1^{i+1}\ \ i = \overline{1,\ l-1}$ and define the sets $Y^i = [y^i\ \ y^{i+1}\ ],\ \ i = \overline{1, l-1}$ as line segments and the set

$$Y_P^0 = \bigcup_{i=1}^{l-1} Y^i. \tag{2.5}$$

The distance between the sets Y^P and Y_P^0 we call d.

Definition 2.6. Y_P^0 will be called $\varepsilon-$ solution of the problem (1.1) if $d \le \varepsilon$.

3. Method of finding the solution Y_P^0

The Pareto front of the problem (1.1) is the graphic of the concave and strictly decreasing function on the coordinate plane [26].

Step 1. Define two Pareto points $y^1\left(x^1\right), y^{\,2}(x^2)$ and Pareto optimal solutions $x^1,\ \ x^2$ from the solutions of the following linear programming problems:

$$y_2^1\left(x^1\right) = \max_{x \in X} y_2\left(x\right), \qquad y_1^2\left(x^2\right) = \max_{x \in X} y_1(x),$$

$$y^1\left(x^1\right) = \left(y_1\left(x^1\right), y_2\left(x^1\right)\right), \qquad y^2\left(x^2\right) = \left(y_1\left(x^2\right), y_2\left(x^2\right)\right).$$

Further, the number of Pareto defined points is recalled as l. At the first step $l = 2, y^1,\ y^l$ are the left-end point and the right- end point of the Pareto front correspondingly. Set two new vectors $\alpha = (\alpha_1,\ \alpha_2) = (y_2^2 - y_2^1, y_1^1 - y_1^2)$, $c = \alpha_1 c^1\ +\ \alpha_2 c^2$ then form the following linear programming:

$$\mathrm{x} \le \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \mathrm{x} \ge 0, (\mathrm{c}, \mathrm{x}) \to \max. \tag{3.1}$$

Step 2. Take the optimal solution x^* of the problem and define the new Pareto point $y^3 = y(x^*)$. Then take the triangle with the vertices y^1, y^3, y^2. Now take the altitude of the triangle drawn through the vertex y^2. Let h be the length of it. If the inequality $h < \varepsilon$ is true then (2.5) defines the solution of the problem (1.1) and go to the Step End. Otherwise, go to the next step. Here ε is the advanced given required precision.

Step 3. We have two pare of Pareto points: $(y^1, y^3), (y^3, y^2)$. We apply step 1 to each of them and define two new Pareto points. We call them z^1, z^2 correspondingly and make new notations

$$y^5 = y^2,\ y^4 =\ z^2,\ y^3 = y^2,\ y^2 =\ z^1.$$

In the new case $l = 5$. Now verify just how these five Pareto points approximate the Pareto front. Take two pairs triangular $y^1y^2y^3, y^3y^4y^5$ and evaluate their altitudes as it was at step 2. The following situations can occur:

Case 1. All altitudes satisfy accuracy. It means that we have solved the problem and have gone to the step end.

Case 2. There is an altitude that doesn't satisfy accuracy. In this case, remember the triangle which satisfies accuracy and eliminate it from further consideration. Then return to step 1 and apply it to the triangle that doesn't satisfy the accuracy and define new Pareto points. In this way, the set of Pareto points is extended until the requisite accuracy is satisfied.

Step End.

Remark 3.1. Calculation volume used to evaluate a Pareto front mainly depends on how efficiently the problems (3.1) are solved at each step of iterations.

4. Reduction of the problem (3.1)

In this section, we study how the problem (3.1) can be reduced to the problem with the same structure and with the smaller dimension.

Take the problem dual to the problem (3.1):

$$\mathrm{y} \leq \mathrm{yA} - \mathrm{yB} + \mathrm{c}, \mathrm{y} \geq 0, \mathrm{y}, \mathrm{b} \to \min. \tag{4.1}$$

Solve the problem as limit of the approximations $\psi^1, \psi^2, ..., \psi^k, ...$ constructed according to the following scheme:

$$\psi^1 = 0, \psi^{k+1} = \max\left(0, c + \psi^k A - \psi^k B\right), k = 1, 2, ... \tag{4.2}$$

The limit $\lim_{k\to\infty} \psi^k = \psi^*$ gives the optimal solution to the problem (4.1) ([6], p.16). Define the sets:

$$G^* = \{i \mid i \in [1:n], \psi^\star_i \neq 0\}, \overline{G^*} = \{1, 2, \ldots, n\}\, nG^*.$$

The feasible solution y of the problem (4.1) will be optimal if it satisfies the conditions:

$$y_i = (yA - yB + c)_i, \quad i \in G^*, \ y_j = 0, \quad j \in \overline{G^*}.$$

From here and from the duality theorem, we conclude that the feasible solution x of the primal problem (4.1) will be optimal if it satisfies the conditions:

$$x_i = (Ax - Bx + b)_i, \quad i \in G^*, \quad x_j = 0, \quad j \in \overline{G^*}.$$

Therefore, to have the set G^* is sufficient to identify the optimal basis variables of the problem (1.1). The more we have information about this set, the more we have a chance to solve the dimension problem for the problem (3.1). Consider the case when $B = 0$ and construct the following approximations:

$$\mathrm{t}^1 = 0, \ t^{k+1} = \max\left(0, \ c + t^k A\right), \quad k = 1, 2, \cdots.$$

Sequence $\mathrm{t}^1, t^2, \cdots, t^k, \cdots$, is bounded and increasing sequence. Denote $\mathrm{Lim}_{k\to\infty} t^k = t^*$.

Lemma 4.1. $\psi^k \leq t^k, \quad k = 1, 2, \cdots$. *Validity of the lemma a easily obtained from the definition of sequences. From the lemma we have*

Corollary 4.2. $\psi^\star_i = 0.i \in T^* = \{i \mid i \in [1:n], t^\star_i = 0\}$. *Take* $b^0 = c - t^0 B$ *and consider the approximations:*

$$\mathrm{g}^1 = 0, \ g^{k+1} = \max\left(0, \ c + g^k A + b^0\right), k = 1, 2, \cdots.$$

Denote $\mathrm{Lim}_{k\to\infty} g^k = g^*$.

Lemma 4.3. $g^k \leq \psi^k, \quad k = 1, 2, \cdots$.

Corollary 4.4. $\psi^\star_i > 0, \quad i \in G^* = \{i \mid i \in [1:n], g^\star_i > 0\}$.

The following theorem is proved.

Theorem 4.5. *The problem* (3.1) *and the following problem are equivalent:*

$$\mathrm{x} + \mathrm{y} = \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \quad \mathrm{x} > 0, \mathrm{y} > 0,$$

$$y_i = 0, i \in G^*, \quad x_i = 0, \ i \in T^*, \quad (c, x) \to max.$$

From the theorem, as a result, we have:

Corollary 4.6. *The dimension of the basis simplex table of the problem* (3.1) *is* $(n \times 2n)$, *but the dimension of the new problem is* $(m \times 2m)$, *where* $m = n - [G^* \cup T^*]$.

Corollary 4.7. *Monotonicity of the sequences which we use to define the sets* G^*, T^* *allows us to interrupt further iterations when desired reduction of dimension is reached.*

5. Application of reduction scheme

Now we are going to demonstrate the possibility of a reduction scheme based on the problem like (1.1) on the parametric and fractional-linear programming problems.

Consider the problem under the condition as in (1.1):

$$\mathrm{Ix} \leq \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \quad 0 \leq \mathrm{x} \leq \mathrm{d}^1 + \lambda \mathrm{d}^2, \quad 0 \leq \lambda \leq 1, \quad (\mathrm{c}, \mathrm{x}) \to \max. \tag{5.1}$$

The following additional conditions are also assumed:

$$a) c > 0, d^1, d^2 > 0, \qquad b) b - Bd^1 - Bd^2 > 0.$$

Condition b) makes the problem (5.1) is non-degenerate at the each mean of the parameter. Really:

$$x + z = A\,x - B\,x + b\,,\ x \ \geq 0, \quad z \geq 0,$$
$$(I - A)\,x + z = \ -Bx + b,$$
$$x + (I - A)^{-1} z = (\ I - A\ )^{\ -1} (\ b - B\ x)\,,$$
$$x \ \geq \ \ (\ I - A\ )^{-1} \ \left(b - \ Bd^1 - Bd^2\right) > 0.$$

Take the dual problem to the problem (4.1):

$$y\ \ (\ I - A + B\ ) + t \ \ \geq c, y \geq 0, t \geq 0, yb + t\left(d^1 + \lambda d^2\right) \to min, 0 \leq \lambda \leq 1. \tag{5.2}$$

According to the second duality theorem, the condition of the problem can be written as follows:

$$y\,(I - A + B) + t = c.$$

Then, after a simple transformation, the problem (5.2) can be written as:

$$y\,(I - A + B) \leq c, \quad y \geq 0,$$
$$y\,((I - A + B)\,-\mathrm{b}) + \lambda(y\,(I - A + B)\,d^2 + \lambda cd^2 + cd^1 \to max, \qquad 0 \ \leq \ \lambda \ \leq \ 1. \tag{5.3}$$

Denote

$$(I - A + B)\,d^1 - b = \ e^1\ , \quad (\ I - \ A + B\ )\ d^2 = \ e^2.$$

The problem (5.3) in the new notation has the form:

$$y \leq yA - yB + c, \quad y \geq 0, \quad y\ e^1 + \ \ \lambda\ y\ e^2 - \lambda cd^2 - cd^1 \to max, \qquad 0 \leq \lambda \leq 1. \tag{5.4}$$

The solution of the problem (5.4) is presented as the Pareto bound of the following two-criterion problem:

$$y \ \ \leq yA - yB + c\ , \quad y \geq 0, \ y\ r^1 + cl^1 \ \to max, \quad ye^1 + c\ d^1 \to \ max. \tag{5.5}$$

Here $r^1 = e^1 + e^2$, $l^1 = \ d^1 + d^2$. Let $x^i \ \in \ X^P$ such that $y_1 \ \ \left(x^i\right) \ < y_1 \ \left(x^{i+1}\right)$, $i \ \in \ \overline{1, l}$ and assume that

$$Y_P^0 = \bigcup_{i=1}^{l} \ [(\ y_1 \ \ (\ x^i)\ ,\ y_2 \ \ (\ x^i)],$$

piece-wise linear approximates the Pareto front of the problem (5.5).

Define the vectors

$$n^i = (y_2^i {-} y_2^{i+1}, y_1^{i+1} - y_1^i),$$
$$n^{i+1} = \ (\ y_2^{i+1} {-} y_2^{i+2}, y_1^{i+2} - y_1^{i+1}), \qquad i \ \in \ [1 : l - 2]\,.$$

Then construct the following sets by using them:

$$\Lambda^i = \ \mu \quad \frac{n^i}{n_1^i + n_2^i} \ + (1 - \mu) \frac{n^{i+1}}{n_1^{i+1} + n_2^{i+1}}, \quad 0 \leq \mu \leq 1 i \ \in \ [1 : l - 2]\,.$$

Proposition 5.1. $\left(\bigwedge{}^{\ i}, x^i \quad\right)$, $i \in [\ 1 : \ l - 2]$ *is* $\varepsilon-$ *approximation of the solution of the problem* (5.1).

Now we are going to illustrate how a reducing scheme can be used to solve the fractional linear programming:

$$\mathrm{Ix} \leq \mathrm{Ax} - \mathrm{Bx} + \mathrm{b}, \mathrm{x} \geq 0,$$
$$\frac{(c\ , x\ ) + d}{(e,\ x) + f} \quad \to max. \tag{5.6}$$

Assumption: $(e, x) + f > 0$.

Based on this assumption, without loss of generality, we can also assume that the numerator of the fraction is positive. Then

$$Y = \left\{ (y_1, y_2) \mid\ y_1 = l0 * -(c, x) + d\ , y_2 = (e, x) + f\ ,\ \ x \in X \right\},$$

will be a convex, closed, and bounded set of the first quadrant. Let x^0 be optimal solution to the problem (5.6) and $y^0\left(x^0\right) = \left(y_1^0(x^0)\ , y_2^0\left(x^0\right)\right) \in\ Y$. $y^0(x^0)$ has the following property: the straight line connecting this point with the coordinate origin is the support line of Y and keeps the set Y to the left side of the line. We aim is to obtain the optimal solution by solving a finite number of problems such as (3.1), which differs only in their criteria.

Step 1. Take any vector $y^{(1)} = \left(y_1^1, y_2^1\right)$, y as initial approximation and vector $n = (n_1, n_2) = \left(-y_2^1\ ,\ y_1^1\right)$. Then solve the following problem:

$$\mathrm{n}_1((\mathrm{c}, \mathrm{x}) + \mathrm{d}) + \mathrm{n}_2((\mathrm{e}, \mathrm{x}) + \mathrm{f}) \to \max, \mathrm{x} \in \mathrm{X}. \tag{5.7}$$

Let x_2 be the optimal solution to the problem (5.7). By using this solution, form the vectors.

$$y^{(2)} = \left(y_1\left(x^2\right),\ y_2\left(x^2\right)\right), n^{(2)} = (-y_1^2,\ y_2^2).$$

Step 2. Take $y^1 = y^2$, $\ n^1 = n^2$, return to the step 1. Continue the process in this way. The process is finite. The value of the objective function increases at each step. That is, the process is monotonous. It ends when the next approximation coincides with the previous one.

The proposed solution to problem (5.6) is different from the existing standard way of solution (see [18]). The standard way of solving doesn't leave the initial structure of the problem and does not allow us to take advantage of the reduction proposed here.

6. Numerical example

We demonstrate the reduction process on the example of the problem (5.1) when $\lambda = 0$:

$$x\ \leq\ x\ A\ - xB\ + b\ ,\ \ 0\ \leq\ \ x\ \ \leq\ \ d\ ,\ \ \ (c,\ x)\ \to\ \ max). \tag{6.1}$$

$$A = \begin{pmatrix} 0.20 & 0.00 & 0.00 & 0.06 & 0.00 \\ 0.00 & 0.30 & 0.00 & 0.00 & 0.10 \\ 0.30 & 0.00 & 0.10 & 0.00 & 0.20 \\ 0.00 & 0.10 & 0.20 & 0.30 & 0.00 \\ 0.00 & 0.00 & 0.06 & 0.00 & 0.20 \end{pmatrix},\quad B = \begin{pmatrix} 0.00 & 0.10 & 0.30 & 0.00 & 0.00 \\ 0.10 & 0.00 & 0.00 & 0.20 & 0.00 \\ 0.00 & 0.20 & 0.00 & 0.08 & 0.00 \\ 0.04 & 0.00 & 0.00 & 0.00 & 0.30 \\ 0.00 & 0.05 & 0.00 & 0.10 & 0.00 \end{pmatrix},$$

$$x\ = (x_1,\ \ x_2,\ \ x_3,\ \ x_4,\ \ x_5),\ b\ = (15,\ \ 8,\ \ 4,\ \ 10,\ \ 6),\ d\ = (9,\ \ 12,\ \ 6,\ \ 8,\ \ 10),\ c\ = (5,\ \ 2,\ \ 1,\ \ 1,\ \ 1).$$

Step 1.

Set the problem (5.5):

In our case the problem (5.5) has such form:

$y \leq Ay - By + c,\quad ey\ \ \longrightarrow max$, where

$$e = d\,(I - A + B) - b,$$

$$e = (9, 12, 6, 8, 10)\begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.10 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.8 \end{pmatrix} - (15, 8, 4, 10, 6)$$

$$= (-8.08,\quad 2.20,\quad 1.90,\quad -1.06,\ 2.00).$$

Step 2.
Construct the approximations:

$$\eta^{(1)} = 0,\quad \eta^{(i)+1} = \max\left(0\ ,\ e^T +\ A\eta^{(i)}\right)\ ,\ i\ = 1,\ 2\ ,\quad .\quad .\quad .$$

$$\eta^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \eta^{(2)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 2.20 \\ 1.90 \\ 0 \\ 2.00 \end{pmatrix},$$

$$\eta^{(3)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.10 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\begin{pmatrix} 0 \\ 2.20 \\ 1.90 \\ 0 \\ 2.00 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 3.06 \\ 2.49 \\ 0 \\ 2.51 \end{pmatrix},$$

$$\eta^{(4)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.10 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.37 \\ 2.65 \\ 0 \\ 2.65 \end{pmatrix},$$

$$\eta^{(5)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.37 \\ 2.65 \\ 0 \\ 2.65 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.48 \\ 2.70 \\ 0 \\ 2.69 \end{pmatrix},$$

$$\eta^{(6)} = max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.48 \\ 2.70 \\ 0 \\ 2.69 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.51 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix},$$

$$\eta^{(7)} = \ max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.51 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.52 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix},$$

$$\eta^{(8)} = \ max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.52 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.53 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix},$$

$$\eta^{(9)} = \ max\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8.08 \\ 2.20 \\ 1.90 \\ -1.06 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.80 & 0.10 & 0.30 & -0.06 & 0.00 \\ 0.10 & 0.70 & 0.00 & 0.20 & -0.01 \\ -0.30 & 0.20 & 0.90 & 0.08 & -0.\ 20 \\ 0.04 & -0.10 & -0.20 & 0.70 & 0.30 \\ 0.00 & 0.05 & -0.06 & 0.10 & 0.80 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 3.53 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.53 \\ 2.71 \\ 0 \\ 2.70 \end{pmatrix}.$$

Based on the last seven iterations, we get: $T^* = \{1,\ 4\}$. That is, $x_1^{op} = d_1 = 9, x_4^{op} = d_4 = 8$. The given problem turns from the five-variable problem into the three-variable new problem.

Step 3.

Setting the new problem:

$y^1 \ \le\ A^1y^1 -\ B^1 +\ b^1, \qquad y^1 \ge 0, \qquad e^1y^1 \ \longrightarrow max$. Here $y^1 = (\ \ y\ _2,\ \ y_3\ ,\ y_5)$,

$b_i^1 \ =\ b_i, -(9,\ 0\ ,0\ ,8\ ,0)\,(\ I - A + B)_i\ I = 2,\ 3\ ,\ 5$. From here we have:

$$b^1 = (7.90, \quad 2.90, \quad 3.60)\,,$$

$$A^1 = \begin{pmatrix} 0.30 & 0.00 & 0.10 \\ 0.00 & 0.10 & 0.20 \\ 0.00 & 0.06 & 0.20 \end{pmatrix}\quad , \quad B^{\ 1} = \begin{pmatrix} 0.00 & 0.00 & 0.00 \\ 0.20 & 0.00 & 0.00 \\ 0.05 & 0.00 & 0.00 \end{pmatrix},$$

$e^1 = d^1\left(\ I - A^1\ + B^1\right) -\ b^1,\ \ d^1 = (12,\ \ 6,\ 10)$. Calculate the value of the vector $e^1\ \ = (2.20,\ \ 1.90,\ 2.00)$. For the new problem $G^* = (2, 3, 5)$. As a result of the Theorem 4.5, the remaining coordinates of the optimal solution of the new problem can be found from the system of equations:

$$0.70\ x_2 + 0.20\ x_3 + 0.05\ x_5 = 7.9,$$

$$0.90\ x_3 - 0.06\ x_5 = 2.9,$$

$$-0.10x_2 - 0.20\ x_3 + 0.8\ \ x_5 = 3.6,$$

$$x_2 = 9.76, \quad\ x_3\ \ = 3.66,\ x_5 = 6.64.$$

Finally, the optimal solution for the considered example will be the vector $(9.00, 9.76, 3.66, 8.00, 6.64)$.

The numeric calculation of the given example is performed by MATLAB.

7. Conclusion

Many problems have been considered and are being considered related to large size, and different separation schemes have been suggested to solve them. However, existing separation schemes essentially have been based on the possibilities created by the zeros of the matrix of the problems. The problem considered here has no such possibility due to its large size. But if the dimension reason is shown, it is shown that the problem considered successfully can be solved by the existing iterative procedure. Here we try to solve the problems: i) how to identify the Pareto front problem by using an iterative procedure; ii) how to use an iterative procedure to reduce the dimension of the problem; iii) how to carry out i) and ii) without disturbing the original structure of the problem. We demonstrate the usability of the suggested method for parametric and fractional programming.The numerical example is solved.

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