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Symmetry solutions of the fifth-order non-linear equation with temporal and spatial dispersion terms

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Abstract

The fifth-order non-linear partial differential equations (NPDEs) with temporal and spatial dispersion have been seen in the literature to model optical communication, fluid mechanics, condensed matter, electro-magnetic and the propagation of pulses in optical fibers. The considered model is not an easily solvable equation because of temporal and spatial dispersion. Our main aim is to obtain the explicit form of exact solutions via a combination of Lie group transformation and ansatz-based methods. Finally, the obtained results, which are novel solutions in explicit form, have been presented to imply the physical nature by means of three-dimensional plots with result discussion.

Keywords. Fifth-order non-linear equation, Exact solutions, Lie group transformation, Ansatz-based methods, Solitons.
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1. Introduction

Mathematical models are arising in engineering and applied science as non-linear partial differential equations (NPDEs) [1–6] . The exact solutions of them have a major role in explaining the models and whether satisfy the process. Most of the described models are integrable for which the exact solutions can be mentioned in various literatures [3, 5–9, 21–23].

Such models include the Korteweg–de Vries (KdV) equation, the nonlocal modified KdV equation, the non-linear derivative Schrödinger equation, Kadomtsev–Petviashvili (KP) equation, Camassa–Holm (CH) equation, and they have variety of application area [2–6, 10, 17, 19, 20, 23, 24]. Lately, in the literature, fifth-order NPDEs include the temporal and spatial distribution terms applied to model optical communication, fluid mechanics, condensed matter, and the propagation of pulses in optical fibers. For this reason, we have considered here the fifth-order NPDE [4] as follows:

$$u_{ttt} - \lambda_1 u_{txxxx} - \lambda_2 u_{xxt} - \lambda_3 (u_x u_t)_{xx} - \lambda_4 (u_x u_{tx})_x = 0, \tag{1.1}$$

where $\lambda_i \neq 0$ $(i = 1, 2, 3, 4)$ are parameters, u_{txxxx} is the spatial derivative and u_{ttt} is third-order dispersion causing a new form of impact instability [1]. Eq.(1.1) was considered for $\lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 4$ by Wazwaz [4, 25] where the Hirota's method is considered for solving the nonlinearity, additionally Wang et al.[3] considered Lie symmetry method. Besides Eq.(1.1) was considered for $\lambda_1 = 1$,$\lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4$ and additionally six fifth-order non-linear equations were considered by Wazwaz [5].

Moreover, for the general case, the number of fifth-order equations is limitless. It is known that Eq.(1.1) is not an easily solvable equation because of temporal and spatial dispersion but it has variety of travelling waves such as solitons, peakons, kinks etc[5]. The effect of the dispersion term on the behavior is that it has a kink solution that

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changes it from one asymptotic state to another. So, the soliton structure of Eq.(1.1) is more complicated because of the spatial and dispersion terms.

The general case of the fifth-order non-linear equation Eq.(1.1) is considered in this work. As mentioned, difficulty of both nonlinearity and dispersion, classical methods known in the literature are inadequate.

Our main contribution is to give the exact solution of the most general form of the equation Eq.(1.1) by a combination of the Lie group transformation and the ansatz-based method. To the best of our knowledge, these solutions are the most general solutions, those are presented here.

2. Lie transformations

One of the well-known and useful tools for reducing the non-linear differential equations is Lie transformation method [11–20]. Briefly, with the help of the Lie transformations, which are invariant transformations, the considered linear or non-linear differential equation can be reduced into solvable equation or lower order equation.

A one-parameter group transformations is considered as:

$$\tilde{x}^i = f^i(x, \varepsilon), \quad \epsilon \text{ is a canonical parameter}, \qquad i = 1, 2, ..., n. \tag{2.1}$$

When the Taylor's series of $f^i(x, \varepsilon)$ at $\varepsilon = 0$, where $O(\varepsilon)$ is neglected, is considered, the infinitesimal transformation is hold as follows:

$$\tilde{x}^i \approx x^i + \varepsilon \xi^i(x), \; x^i = f^i(x, 0), \; \xi^i(x) = \left. \frac{\partial f^i(x, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon = 0}, \quad i = 1, 2, ..., n \tag{2.2}$$

Geometrically, ξ^i determines the tangent field vector of the group infinitesimal generator of group generator, which is given as:

$$X_j = \sum_{i=1}^{n} \xi^i(x) \frac{\partial^i}{\partial x^i}, \quad j = 1, ..., n \tag{2.3}$$

The generalized infinitesimal transformation is $X = \sum\limits_{j=1}^{n} a_j X_j$, where $a_j (j = 1, ..., n)$ are parameters, each considered sum satisfies the group properties.

3. Methodology

Sometimes the reduced equation is not generally solvable analytically with theory of differential equations. So, in this work, modifications of the well-known AEM [6–10] is considered to obtain the exact solutions. Moreover, in the next section application of the methodology has be presented in detail.

3.1. **Solutions and Explanations:** In this section, firstly the Lie transformations of the generalized fifth-order NPDE Eq.(1.1) is obtained and then according to the reduced equation, it is solved via directly the classical theory of differential equations or modifications of the well-known AEM.

Applying the given procedure, the determining equations have one solution set where $\Phi(x)$ is an arbitrary function

$$\zeta_x = C_2, \; \zeta_t = C_1, \; \zeta_u = \Phi(x) \tag{3.1}$$

As seen from the solution set, Lie group transformations change according to arbitrary $F(x)$ function. Determination of $\Phi(x)$ is an open research area for physicists.

Depending on Eq.(3.1), two single parameter groups are obtained:

$$\begin{aligned} &\text{For } C_1 = 1, C_2 = 0, \\ &X_1 = \frac{\partial}{\partial t} + \Phi(x) \frac{\partial}{\partial u}. \end{aligned} \tag{3.2}$$

and

$$\text{For } C_1 = 0, C_2 = 1,$$
$$X_2 = \frac{\partial}{\partial x} + \Phi(x)\frac{\partial}{\partial u}. \tag{3.3}$$

Case 1. Considering $X_2 = \frac{\partial}{\partial x} + \frac{1}{x}\frac{\partial}{\partial u}$ for $\Phi(x) = \frac{1}{x}$, the general form of solution $u\left(x,t\right) = \Theta\left(t\right)x$ is obtained. When the general solution is substituted in Eq.(1.1) with the $\lambda_3 = \lambda_4 = 4, \lambda_2 = 0, \lambda_1 = 1$, $\Theta(t) = \frac{c_1}{2}t^2 + c_2 t + c_3$ is obtained. Hence, $u\left(x,t\right) = \left(\frac{c_1}{2}t^2 + c_2 t + c_3\right)x$ is the generalized solution.

Case 2. The generalized infinitesimal transformation is $X = a_1X_1 + a_2X_2$. For this case, $a_1 = a_2 = 1$ i.e. $X = X_1 + X_2$ for $\Phi(x) = \frac{1}{x}$ is considered for the generalized type of Eq.(1.1) with the $\lambda_3 = \lambda_4 \neq 0, \lambda_1 \neq 0, \lambda_2 \neq 0$. Therefore, the solution is $u\left(x,t\right) = xH(\zeta), \zeta = -x + t$ and the reduced equation is held as

$$\begin{aligned} -\lambda_1 x H^{(5)} + 4\lambda_1 H^{(4)} + 6\lambda_3 (H')^2 + \left(\lambda_2\left(2-x\right) - 16x\lambda_3 H' + 4\lambda_3 H\right)H'' + \\ x\left(1 - \lambda_2 + \lambda_3\left(3xH' - 2H\right)\right)H''' = 0 \end{aligned} \tag{3.4}$$

Firstly, to solve Eq.(3.4), the method of reducing order, which depends on the theory of differential equations, is considered. Therefore, the coefficient of third and second derivative of $H(\zeta)$ is assumed to be zero. As a result,

$$H(\zeta) = -\frac{\left(-2 + \lambda_2 x\right)\zeta}{10\lambda_3 x} - \frac{-16x + 10\lambda_2 x + 3\lambda_2 x^2 + 4\zeta - 2\zeta\lambda_2 x}{20\lambda_3 x} \tag{3.5}$$

is obtained. The solution of Eq.(1.1) is

$$u\left(x,t\right) = x\left(-\frac{\left(-2 + \lambda_2 x\right)\left(-x + t\right)}{10\lambda_3 x} - \frac{-16x + 10\lambda_2 x + 3\lambda_2 x^2 + 4\left(-x + t\right) - 2\left(-x + t\right)\lambda_2 x}{20\lambda_3 x}\right) \tag{3.6}$$

where λ_3 and λ_2 are parameters which have physical meanings.

Case 3. The generalized infinitesimal transformation is $X = a_1X_1 + a_2X_2$. For this case, $a_1 = -1, a_2 = 1$ i.e. $X = X_1 + X_2$ for $\Phi(x) = \frac{1}{x}$ is considered for the generalized type of Eq.(1.1) with the $\lambda_3 \neq \lambda_4, \lambda_1, \lambda_2$ and all are different from zero. Therefore, $u\left(x,t\right) = xH(\zeta), \zeta = x + t$ is considered and the reduced equation is held as

$$\begin{aligned} -\lambda_1 x H^{(5)} - 4\lambda_1 H^{(4)} + x\left(1 - \lambda_2 - x\left(2\lambda_3 + \lambda_2\right)H' - \left(\lambda_3 + \lambda_4\right)H\right)H''' + \\ \left(-2\lambda_2 - 2x^2\lambda_3 - x\left(11\lambda_3 + 5\lambda_4\right)H' - \left(2\lambda_3 + 2\lambda_4\right)H - \left(4\lambda_3 + \lambda_4\left(x^2+2\right)\right)H''\right)H'' = 0 \end{aligned} \tag{3.7}$$

For the simplicity of Eq.(3.7), $\lambda_4 = -2\lambda_3$ is considered. To solve Eq.(3.7), the Hermite approximation method is considered. First step is to determine the ansatz via balancing principle. Hence, the ansatz is $H\left(\zeta\right) = \sum\limits_{i=0}^{N} a_i z(\zeta)^i$, where N is determined via balancing principle, a_i are the parameters that will be determined by solving algebraic system and $z(\zeta)$ is the solution of the Hermite differential (HD) equation $z'' - 2\zeta z' - \lambda z = 0$ where it is the auxiliary equation. For the second step, the ansatz and the HD equation are substituted to Eq.(3.7), and a system of equations is obtained by polynomial equivalence, and the parameters are its solutions.

For Eq.(3.7), $H\left(\zeta\right) = \sum\limits_{i=0}^{1} a_i z(\zeta)^i$ is determined by using balancing principle. Applying the given procedure, the solution sets satisfying our goals for the parameters are proposed in Table 1 and others gives trivial or meaningless solutions.

For the first set, the solution is

Table 1. The solution sets of parameters for Case 3.

Set 1	Set 2
$\lambda = -2,$	$\lambda = \frac{2x^2-3x\zeta+1}{x^2}, a_1 = \frac{x\zeta+1}{2x^2-3x\zeta+1},$
$a_1 = \frac{x\zeta+1}{2x^2},$	$\lambda_2 = \frac{2\left(x^2\lambda_1\zeta(2x-\zeta)-\lambda_1(6\zeta x+1)+x\right)}{x^4(x\zeta+1)},$
$\lambda_3 = -2x\lambda_2\zeta,$	$\lambda_3 = \frac{x^2\lambda_1\zeta(2x-\zeta)-\lambda_1(6\zeta x+1)+x}{x^4}$
$a_0 = \frac{2\zeta x^3\lambda_2-8\lambda_1\zeta^2-6\lambda_1\zeta x-4\lambda_1-4\lambda_1\zeta^3 x+x\zeta-x\lambda_2\zeta-\lambda_2}{\zeta(x\zeta+1)\lambda_2 x}$	$a_0 = -\frac{4x^5\lambda_1\zeta\left(\zeta^2+1\right)-x^4(\zeta x+1)+x^3\lambda_1\left(14\zeta+4x+15x\zeta^2\right)}{(x\zeta+1)(x^2\lambda_1\zeta(2x-\zeta)-\lambda_1(6\zeta x+1)+x^2)}+\frac{-x^2\lambda_1\left(1+2\zeta^2\right)-2\lambda_1(6\zeta x+1)+2x^2}{(x\zeta+1)(x^2\lambda_1\zeta(2x-\zeta)-\lambda_1(6\zeta x+1)+x^2)}$

$$
\begin{aligned}
u(x,t) = & \frac{2\,(x+t)\,x^3\lambda_2 - 8\lambda_1(x+t)^2 - 6\lambda_1\,(x+t)\,x - 4\lambda_1 - 4\lambda_1(x+t)^3 x + x\,(x+t) - x\lambda_2\,(x+t) - \lambda_2}{(x+t)\,(x\,(x+t)+1)\,\lambda_2} \\
& + \frac{(x\,(x+t)+1)\left(c_1\,(x+t)\,KummerM\left(0,\frac{3}{2},(x+t)^2\right) + c_2\,(x+t)\,KummerU\left(0,\frac{3}{2},(x+t)^2\right)\right)}{2x}
\end{aligned} \tag{3.8}
$$

Case 4. The generalized infinitesimal transformation is $X = X_1 + X_2$ and $\Phi(x) = 0$ is considered for the generalized type of Eq.(1.1) with the $\lambda_3 = \lambda_4 \neq 0, \lambda_1 \neq 0, \lambda_2 \neq 0$. Therefore, the solution is $u\,(x,t) = H(\zeta), \zeta = x + t$ and the reduced equation is held as

$$
-\lambda_1 H^{(5)} - 3\lambda_3 (H'')^2 + (1 - \lambda_2 - 3\lambda_3 H')\,H''' = 0. \tag{3.9}
$$

The given procedure of Hermite approximation method is applied and as a result the parameters are obtained

$$
a_1 = \frac{-4\lambda_1\left(4\zeta^2-1\right)}{3\lambda_3\left((c_1\kappa_1 + c_2\kappa_2)\left(7+10\zeta^2\right) - c_1\kappa_3\left(9+12\zeta^2\right) + c_2\kappa_4\right)},\ \lambda_2 = -4\lambda_1\zeta^2 - 2\lambda_1 + 1, \tag{3.10}
$$

where $\kappa_1 = KummerM\left(-2\zeta^2, \frac{3}{2}, \zeta^2\right)$, $\kappa_2 = KummerU\left(-2\zeta^2, \frac{3}{2}, \zeta^2\right)$, $\kappa_3 = KummerM\left(1-2\zeta^2, \frac{3}{2}, \zeta^2\right)$ and $\kappa_4 = KummerU\left(1-2\zeta^2, \frac{3}{2}, \zeta^2\right)$.
Substituting the parameters and $z(\zeta)$ into the ansatz, the general solution is obtained.

4. Result and Discussion

The dynamics of wave propagation in the pertinent media have been covered in this section. The aforementioned solutions were thoroughly investigated, discussed, and shown using three-dimensional graphs and numerical values applied to the unknown parameters.

4.1. Graphical representation of the solutions. Here, the three dimensional graphical representation the plot has been plotted for Case 3 and Case 4 is given by Figures 1 and 2 as follows:

4.2. Dynamics of the solutions. This section explains the physical meaning of the several solution graphs that were successfully achieved using numerical simulation of the unknown components. Moreover, the dynamic characteristics of the distinct waveforms that were acquired and their application to the pertinent media are covered in this section. The graphical picturization of Case 3 and Case 4 under distinct parametric conditions has been shown in Figures 1 and 2 shows the soliton which propagated through out the medium with assigned different values of unknown parameters.

In the above section, the graphical analysis illustrates how the waves proceed without any information loss at each point in time, and these wave patterns arise from assigning discrete numerical values to the unknowns. Furthermore, it is evident from the description above how these waves are applicable in diverse environments and how they will act in a medium with distinct intensity backgrounds.Furthermore, these solutions retain their shapes during propagation in the medium, resulting their durability and stability.

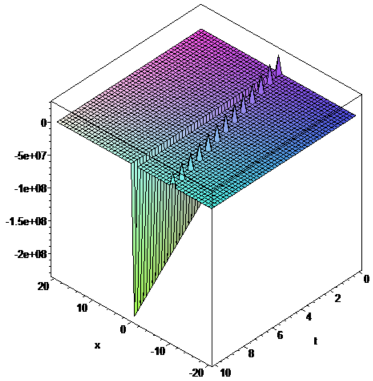

FIGURE 1. The solution of Case 3 considering the parameters of Set 1 for $c_2 = 1$, $c_1 = 1$, $\lambda_2 = -3$, $\lambda_1 = -1$.

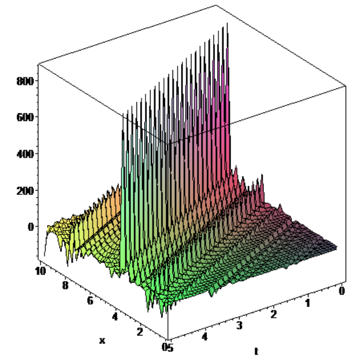

FIGURE 2. The solution of Case 4 for $a_0 = 1$, $c_1 = 1$, $c_2 = -1$, $\lambda = 1$.

5. Conclusion

In this paper, the exact and explicit solutions of Eq.(1.1) via Lie symmetry analysis and combination with modification of the well-known AEM, where the Hermite approximation method, are obtained. The determination of the Lie group generators has an arbitrary function $F(x)$ that may have physical meaning. For the considered model is not an easy solvable equation because of temporal and spatial dispersion. Our main contribution is to give the exact solution of the most general form of the Eq.(1.1) by a combination of the Lie group transformation and the ansatz-based method, and supported by Figures. Finally, the newly obtained results have been presented in the explicit form with physical interpretation in three dimensional plots to show the nature of the solutions. The scientific contribution of this work is related to the research area in solitary waves theory where generalized new fifth-order integrable equation is introduced and investigated.

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Conflict of Interest

The authors declare that they have no conflict of interest.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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