Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-15
DOI:10.22034/cmde.2025.66684.3141

On homogeneous Ricci-quadratic Randers metrics

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Abstract

In this paper, we find the necessary and sufficient conditions under which a left-invariant Randers metric is a Ricci-quadratic metric. Then, by using the Ha-Bumlee's classification of left invariant Riemannian metrics and our characterization of Ricci-quadratic left-invariant Randers metrics, we give the classification of left-invariant Ricci-quadratic Randers metrics on three dimensional Lie groups. As an application, we find an interesting Ricci-quadratic Randers metric on $SU(2)$ which is a generalized Berwald metric while it is not Berwaldian. Also, we prove that it is a generalized Douglas-Weyl metric which is neither Douglasian nor Weyl metric.

Keywords. Left-invariant metric, Randers metric, Ricci-quadratic metric, Berwald metric.
2010 Mathematics Subject Classification. 53B40, 53C60.

1. Introduction

The notion of Riemann curvature $\mathbf{R} = \mathbf{R}(x, y)$ is a basic concept in the Riemannian-Finsler geometry which was introduced by Riemann in 1854 for Riemannian manifolds and was extended by Berwald in 1926 for Finsler manifolds. The Riemannian curvature of every Riemannian metric is quadratic in y, while this fact does not hold for general Finsler metrics. Therefore, the concept of R-quadratic Finsler metrics was born, namely, a Finsler metric is called R-quadratic if its Riemannian curvature is quadratic in y. In other words, a Finsler metric is R-quadratic if and only if the h-curvature of the Berwald connection depends on position only in the sense of Bácsó-Matsumoto [2]. But in fact, it was Shen who chose such a name for these metrics. The class of R-quadratic Finsler metrics contains the classes of Berwald metrics and R-flat metrics as special cases [17–19].

As defined in Riemann geometry, in Finsler geometry, the Ricci curvature $\mathbf{Ric}(x, y)$ is obtained by taking the trace of Riemann curvature, namely, $\mathbf{Ric}(x, y) = trace(\mathbf{R}(x, y))$. A Finsler metric is called Ricci-quadratic if its Ricci curvature $\mathbf{Ric}(x, y)$ is quadratic in y. By definition, every R-quadratic metric is Ricci-quadratic [9]. It is remarkable that Shen proved that every closed R-quadratic metric is a Landsberg metric [19]. It is interesting to ask if the same position holds for the Ricci-quadratic Finsler metric. Then, a natural question in Finsler setting arises as follows:

Conjecture. *Dose a Ricci-quadratic Finsler metric on a closed manifold reduce to a Berwald metric?*

In Finsler geometry, to get the answer to such a problem, people usually first research the class of Randers metrics because such metrics are the simplest non-Riemannian Finsler metrics and also they are computable [6]. Recently, Bao and Robles investigated the Ricci curvature of Randers metrics and obtained the necessary and sufficient conditions for a Randers metric to be Einstein [5, 16]. In [24], Tayebi-Najafi proved that a homogeneous (α, β)-metric on a manifold M is an R-quadratic metric if and only if it is a Berwald metric. In [13], Li and Shen studied Randers metrics with quadratic Riemann and Ricci curvatures. They found the equations that characterize Ricci-quadratic Randers metrics. In [12], Hu-Deng proved that a homogeneous Randers metric is Ricci-quadratic if and only if it is a Berwald metric.

Received: 08 April 2025 ; Accepted: 18 July 2025.
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Suppose that G is a Lie group with a left-invariant Randers metric F defined by the underlying left-invariant Riemannian metric α and the left-invariant vector field U. Among the Finsler spaces, Lie groups equipped with left-invariant metrics are very important spaces because of interpolation between algebraic and geometric properties of Lie groups. S. Deng has achieved very important results on these spaces [8]. He introduced homogeneous Finsler spaces for the first time in [7]. Homogeneous spaces, including Lie groups as a special case, have many applications in physics. In [28], Zhang and Huang classified Lie groups of scalar Randers type. Therefore, it is natural to study and characterize the left-invariant Ricci-quadratic Randers metrics on Lie groups . In this paper, we can characterize the left-invariant Ricci-quadratic Randers metrics.

Theorem 1.1. *Let G be a Lie group equipped with a left-invariant Randers metric F defined by the underlying left-invariant Riemannian metric $\alpha = \sqrt{\alpha_p(v,v)}$ and the left-invariant vector field U, i.e., $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(U,v)$. Then, the Randers metric F is a Ricci-quadratic metric if and only if for any left-invariant vector fields X, Y, and Z the following holds*

$$\langle [U,X],Y\rangle + \langle X,[U,Y]\rangle - \langle [U,X],U\rangle\langle U,Y\rangle - \langle [U,Y],U\rangle\langle U,X\rangle = 0, \tag{1.1}$$

$$tr\Big(\widehat{\nabla d\beta}(F\ell,-)\Big) = (n-1)\alpha\Big(U,[U,(ad^*(F\ell))(U)]\Big), \tag{1.2}$$

where ℓ is the unit tangent vector in the direction of $y \in T_xM$, $\beta(p,v) := \alpha_p(U,v)$ and $\widehat{\nabla d\beta} := s^i{}_{j|k} dx^j \otimes dx^k \otimes \partial/\partial x^i$ is the horizontal covariant derivative of $d\beta$ with respect to α.

In [10], Ha-Bumlee studied 3-dimensional Lie algebras and classified the Left-invariant Riemannian metrics on simply connected 3-dimensional Lie groups. Let us consider $\{x, y, z\}$ as a basis for a 3-dimensional Lie algebra. Then, it is isometric isomorphic to one of the presented Lie algebras endowed with the given inner product in Table 1 (for more details see Table 1). In this paper, using Ha-Bumlee's classification, we classify Left-invariant Ricci-quadratic Randers metrics as follows.

Theorem 1.2. *A simply connected 3-dimensional Lie group G admits a Ricci-quadratic Randers metric $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(U,v)$, where α denotes the left invariant Riemannian metric induced by the inner product given in each case in the Table 1, if and only if G is isomorphism isometric with one of the following*

(i) *The 3-dimensional Euclidean space $\mathbb{R}^3$;*
(ii) *The Heisenberg group Nil;*
(iii) *The solvable Lie group $\tilde{E}_0(2)$;*
(iv) *The simple Lie group $SU(2)$;*
(v) *The non-unimodular Lie group G_c, where $0 < \mu \leq |c|$ and $v > 0$;*
(vi) *The non-unimodular Lie group G_c, where $0 < \mu \leq 1$, $v > 0$ and $c = 1$.*

In final section, we consider the Riemannian and non-Riemannian curvature properties of the Randers metric $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(z,v)$ defined on $SU(2)$, where it is the special case (iv) of Theorem 1.2. More precisely, in Proposition 5.1, we prove that: (i) F is not a Landsberg metric nor weakly Landsberg metric nor stretch metric; (ii) F is a generalized Douglas-Weyl metric which is not Douglasian nor Weyl metric; (iii) F is a generalized Berwald metric which is not Berwaldian; (iv) F is not projectively flat; (v) F satisfies $\mathbf{S} = 0$; (vi) F is Ricci-quadratic but not R-quadratic.

2. Preliminaries

Let (M,F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is called fundamental tensor

$$\mathbf{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

For a vector $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{M}_y(u, v, w) := \mathbf{C}_y(u, v, w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\}, \tag{2.1}$$

where $\mathbf{h}_y(u, v) := \mathbf{g}_y(u, v) - F^{-2}(y)\mathbf{g}_y(y, u)\mathbf{g}_y(y, v)$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. In [14], Matsumoto and Hōjō proved the following.

Lemma 2.1. *A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric or Kropina metric if and only if $\mathbf{M}_y = 0$ for all $y \in TM_0$.*

Let $c = c(t)$ be a C^∞ curve and $U(t) = U^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ be a vector field along c. Define the covariant derivative of $U(t)$ along c by

$$D_{\dot c}U(t) := \left\{\frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}\big(c(t), \dot c(t)\big)\right\}\frac{\partial}{\partial x^i}\Big|_{c(t)}.$$

$U(t)$ is said to be linearly parallel if $D_{\dot c}U(t) = 0$.

For a vector $y \in T_xM$, define

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big)\Big]|_{t=0},$$

where $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$, and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. F is called a Landsberg metric if $\mathbf{L} = \mathbf{0}$.

For $y \in T_xM$, define the mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

Here, $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y = 0$.

For a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k\partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\Big\}, \quad y \in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M, F). A Finsler metric is called a Douglas metric if

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i,$$

where $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ is a scalar function on M and $P = P(x, y)$ is a homogeneous function of degree one with respect to y on TM_0. If $P = 0$, then F is called a Berwald metric.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}\big(\mathbb{B}^n(1)\big)}{\mathrm{Vol}\Big\{(y^i) \in R^n \ \big|\ F\big(y^i \frac{\partial}{\partial x^i}|_x\big) < 1\Big\}}.$$

For $\mathbf{y} = y^i\partial/\partial x^i|_x \in T_xM$, the S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big]. \tag{2.2}$$

For a vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\partial/\partial x^i$, where

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.3}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature. Let us put

$$R^i_{j\ kl} := \frac{1}{3}\Big\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\Big\}. \tag{2.4}$$

Then

$$R^i_k = R^i_{j\ kl} y^j y^l. \tag{2.5}$$

Therefore R^i_k is quadratic in $y \in T_xM$ if and only if $R^i_{j\ kl}$ are functions of position alone.

Let (M, F) be an n-dimensional Finsler manifold. Put

$$\mathbf{Ric} := \sum_{i=1}^n g^{ij}\Big(\mathbf{R}_y(b_i), b_j\Big),$$

where $\{b_i\}$ is a basis for T_xM. **Ric** is a well-defined scalar function on TM_0. We call **Ric** the Ricci curvature. In a local coordinate system,

$$\mathbf{Ric} = g^{ij}R_{ij} = R^m_m.$$

3. Proof of Theorem 1.1

Let G be a smooth n-dimensional connected Lie group endowed with a Riemannian metric $\alpha = a_{ij}dx^i \otimes dx^j$. We denote the inverse of (a_{ij}) by (a^{ij}). It is well-known that the Riemannian metric α induces the musical bijection between 1-forms and vector fields on G, which is denoted by $\flat : T_pG \longrightarrow T^*_pG$ and given by $v \longrightarrow \alpha_p(v, -)$. The inverse of $\flat$ is denoted by $\sharp : T^*_pG \longrightarrow T_pG$. Suppose that $\beta = b_idx^i$ is a 1-form on G, in which we have used Einstein's convention for summation. Then $\beta^\sharp = b^i\partial/\partial x^i$, where $b^i = a^{ij}b_j$. Consider β such that $\|\beta\|_\alpha := \sqrt{a_{ij}b^ib^j} < 1$. In this case, one can define a Randers metric F on G which is defined as follows

$$F(p, v) = \alpha(p, v) + \beta(p, v), \qquad \forall p \in M,\ \ \forall v \in T_pM\ .$$

where we have

$$\alpha(p, v) = \sqrt{a_{ij}v^iv^j}, \qquad \beta(p, v) = (\beta^\sharp)^\flat(v) = \alpha_p(\beta^\sharp, v),$$

Put

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i})\ ,\ \ s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_i := b^m r_{im},\ \ s_i := b^m s_{im},\ \ r_{i0} := r_{ij}y^j\ ,\quad s_{i0} := s_{ij}y^j,\ \ r_0 := r_jy^j,\ \ s_0 := s_jy^j.$$

where "|" denotes the covariant derivative with respect to the Levi-Civita connection of α. We define the following two 1-forms

$$\tilde{s} = s_idx^i, \qquad \tilde{r} = r_idx^i.$$

Then for every vector field X on G, we have

$$\begin{aligned}\tilde{s}(X) = d\beta(X, \beta^\sharp) &= X \cdot \beta(\beta^\sharp) - \beta^\sharp.\beta(X) - \beta([X, \beta^\sharp]) \\ &= X.\alpha(\beta^\sharp, \beta^\sharp) - \beta^\sharp.\alpha(\beta^\sharp, X) - \alpha([X, \beta^\sharp], \beta^\sharp). \end{aligned} \tag{3.1}$$

$$r(X, Y) = r_{ij} X^i Y^j = (\pounds_{\beta^\sharp}\alpha)(X, Y) = \beta^\sharp.\alpha(X, Y) - \alpha([\beta^\sharp, X], Y) - \alpha(X, [\beta^\sharp, Y]), \tag{3.2}$$

where $\pounds_{\beta^\sharp}$ denotes the Lie derivative along $\beta^\sharp$.

Moreover, for every vector field X on G, we have

$$\tilde{r}(X) = (\pounds_{\beta^\sharp}\alpha)(X, \beta^\sharp) = \beta^\sharp.\alpha(X, \beta^\sharp) - \alpha([\beta^\sharp, X], \beta^\sharp). \tag{3.3}$$

Hence

$$\tilde{r}(X) + \tilde{s}(X) = X \cdot \alpha(\beta^\sharp, \beta^\sharp). \tag{3.4}$$

It follows from (3.4), the norm of β with respect to α is constant if and only if $r_i + s_i = 0$.

A vector field X on a Lie group G is said to be left-invariant if it is invariant under every left translation of G. Similarly, a Riemannian metric α on G is called left-invariant if every left translation of G is an isometry of α. Suppose that X and $\beta^\sharp$ are left-invariant vector fields, and α is left-invariant Riemannian metric. Then, it is well known that $\alpha(\beta^\sharp, \beta^\sharp)$ and $\alpha(\beta^\sharp, X)$ are constant functions. Thus, from (3.1) and (3.3), we have

$$\tilde{s}(X) = -\alpha([X, \beta^\sharp], \beta^\sharp), \quad \tilde{r}(X) = \alpha([X, \beta^\sharp], \beta^\sharp). \tag{3.5}$$

Let us define

$$t_{ij} := s_{im} s^m{}_j, \quad t_j := b^i t_{ij} = s_m s^m{}_j.$$

In [13], Li-Shen characterized Ricci-quadratic Randers metrics and proved the following.

Theorem 3.1. ([13]) Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. Then F is Ricci-quadratic if and only if the following hold

$$e_{00} = 2c(\alpha^2 - \beta^2), \tag{3.6}$$

$$s^k{}_{0|k} = (n-1)A_0, \tag{3.7}$$

where $c = c(x)$ is a scalar function on M, $c_k := c_{x^k}$, $A_0 := A_k y^k$ and $A_k := 2cs_k + c^2 b_k + t_k + 1/2c_k$. In this case

$$\mathbf{Ric} = \overline{\mathbf{Ric}} - 2t_{00} - t^k{}_k \alpha^2 + (n-1)\Psi_0, \tag{3.8}$$

where $\Psi_0 := \Psi_k y^k$ and $\Psi_k := 3c^2 y_k - c^2 \beta b_k + 2\beta c_k - c_0 b_k + s_0 s_k + 2s_{0|k} - s_{k|0} - 6cs_{k0}$.

Let G be a connected Lie group, $\mathfrak{g} = T_e G$ its Lie algebra identified with the tangent space at the identity element. Suppose $\langle\ ,\ \rangle$ is an inner product on $\mathfrak{g}$ and α is the left Riemannian metric induced by $\langle\ ,\ \rangle$ on G, i.e.,

$$\alpha_p(u, v) = \Big\langle (L_{p^{-1}})_{*_p}(u), (L_{p^{-1}})_{*_p}(v) \Big\rangle.$$

Suppose U is a non-zero vector in $\mathfrak{g}$ and $\beta^\sharp$ is the left-invariant vector field induced by U on G, i.e.,

$$\beta^\sharp(p) = (L_p)_{*_e}(U).$$

It is easy to see that on the Lie algebra $\mathfrak{g}$ the relation (3.5) becomes

$$\tilde{s}(X) = \langle [U, X], U \rangle, \qquad r(X, Y) = -\langle [U, X], Y \rangle - \langle X, [U, Y] \rangle, \qquad \forall X, Y \in \mathfrak{g}.$$

The relation (3.6) is equivalent to this fact that for all $X, Y \in \mathfrak{g}$ the following holds

$$\big\langle [U, X], Y \big\rangle + \big\langle X, [U, Y] \big\rangle - \big\langle [U, X], U \big\rangle \big\langle U, Y \big\rangle - \big\langle [U, Y], U \big\rangle \big\langle U, X \big\rangle = -2c\Big(\big\langle X, Y \big\rangle - \big\langle U, X \big\rangle \big\langle U, Y \big\rangle \Big). \tag{3.9}$$

By Ming's paper [27], we have $c = 0$. Hence, for all $X, Y \in \mathfrak{g}$ the following holds

$$\big\langle [U, X], Y \big\rangle + \big\langle X, [U, Y] \big\rangle - \big\langle [U, X], U \big\rangle \big\langle U, Y \big\rangle - \big\langle [U, Y], U \big\rangle \big\langle U, X \big\rangle = 0. \tag{3.10}$$

Now, we are ready to characterize Ricci-quadratic left-invariant Randers metrics.

Proof of Theorem 1.1: Suppose that "|" denotes the covariant derivative with respect to the Levi-Civita connection of α. Let us express $s_{ij|k}$ in the index-free form. It is easy to see that $s_{ij|k}$ are the components of the following

$$\nabla d\beta(X, Y, Z) \tag{3.11}$$

where X, Y, and Z are arbitrary vector fields on G. It is easy to see that $\nabla d\beta$ is left-invariant [11]. On the other hand, we have

$$\nabla d\beta(X, Y, Z) := (\nabla_X d\beta)(Y, Z) = X.d\beta(Y, Z) - d\beta(\nabla_X Y, Z) - d\beta(Y, \nabla_X Z). \tag{3.12}$$

Now, suppose that X, Y, and Z are left-invariant vector fields. Thus, we have

$$X.d\beta(Y, Z) = 0.$$

From $\beta^\sharp = U$, it follows

$$\begin{aligned} d\beta(\nabla_X Y, Z) &= (\nabla_X Y).\alpha(U, Z) - Z.\alpha(U, \nabla_X Y) - \alpha(U, [\nabla_X Y, Z]) \\ &= -\alpha(U, [\nabla_X Y, Z]). \end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned} d\beta(Y, \nabla_X Z) &= Y.\alpha(U, \nabla_X Z) - (\nabla_X Z).\alpha(U, Y) - \alpha(U, [Y, \nabla_X Z]) \\ &= -\alpha(U, [Y, \nabla_X Z]). \end{aligned} \tag{3.14}$$

Therefore, one can see that for any left-invariant vector fields X, Y, and Z the following holds

$$\nabla d\beta(X, Y, Z) = \alpha\Big(U, [\nabla_X Y, Z] + [Y, \nabla_X Z]\Big). \tag{3.15}$$

We need to express $s^k_{\ 0|k}$ in the index-free form.

We know that

$$d\beta = s_{ij} dx^i \wedge dx^j,$$

and we set $\widehat{d\beta} = s^i_{\ j} dx^j \otimes \partial/\partial x^i$. Then we have

$$\alpha(\widehat{d\beta}(X), Y) = d\beta(X, Y), \qquad \forall X, Y \in \mathfrak{X}_L(G).$$

By definition, we get

$$d\beta(X, Y) = X.\beta(Y) - Y.\beta(X) - \beta([X, Y]).$$

We know that X and Y are left invariant vector fields, so $\beta(Y)$and $\beta(X)$ are constant functions. In this case, we obtain

$$Y.\beta(X) = X.\beta(Y) = 0$$

which implies that

$$\begin{aligned} d\beta(X, Y) &= -\beta([X, Y]) \\ &= -\alpha(U, [X, Y]). \end{aligned} \tag{3.16}$$

On the other hand, the following holds

$$\alpha(\widehat{d\beta}(X), Y) = d\beta(X, Y). \tag{3.17}$$

By (3.16) and (3.17), we have

$$\begin{aligned} \alpha(\widehat{d\beta}(X), Y) &= -\alpha(U, [X, Y]) \\ &= -\alpha(U, ad(X)Y) \\ &= -\alpha((adX)^*(U), Y). \end{aligned}$$

Since Y is arbitrary vector field, so we get

$$\widehat{d\beta}(X) = -ad^*(X)(U). \tag{3.18}$$

Now, we express $t_{ij} = s_{im}s^m{}_j$ in the index-free form.

$$\begin{aligned} T(X,Y) &:= d\beta(X, \widehat{d\beta}(Y)) \\ &= \alpha\left(\widehat{d\beta}(X), \widehat{d\beta}(Y)\right) \\ &= \alpha\Big(-ad^*(X)(U), -ad^*(Y)(U)\Big) \\ &= \alpha\Big(U, ad(X)(ad^*(Y)(U)\Big), \end{aligned}$$

and also we say $t_j = b^i t_{ij}$ in the index-free form

$$\begin{aligned} \hat{t}(X) &= T(\beta^\sharp, X), \\ &= d\beta(\beta^\sharp, \widehat{d\beta}(X)) \\ &= \alpha\Big(U, ad(U)(ad^*(X))(U)\Big) \\ &= \alpha\Big(U, [U, (ad^*(X))(U)]\Big). \end{aligned}$$

By Ming's paper, a homogeneous Finsler space has isotropic S-curvature if and only if it has vanishing S-curvature [27]. Hence, we have

$$A_k = t_k.$$

Therefore, we have

$$A_0 = \alpha\Big(U, \big[U, (ad^*(F\ell))(U)\big]\Big). \tag{3.19}$$

Then, we obtain

$$\alpha(\widehat{\nabla d\beta}(X,Y), Z) = \nabla d\beta(X,Y,Z),$$

and by (3.15) we get

$$\nabla d\beta(X,Y,Z) = \alpha\Big(U, [\nabla_X Y, Z] + [Y, \nabla_X Z]\Big),$$

Thus, one can deduce that

$$\alpha\Big(\widehat{\nabla d\beta}(X,Y), Z\Big) = \alpha\Big(U, [\nabla_X Y, Z] + [Y, \nabla_X Z]\Big). \tag{3.20}$$

On the other hand, for every left invariant vector fields such as X, Z, the vector field $\nabla_X Z$ is also left invariant. It is easy to see that

$$\nabla_X Z = \frac{1}{2}\Big([X,Z] - (adX)^* Z - (adZ)^* X\Big). \tag{3.21}$$

By (3.21), we can rewritten (3.20) as follows:

$$\begin{aligned} \alpha(\widehat{\nabla d\beta}(X,Y), Z) &= \alpha\Big(\big(ad(\nabla_X Y)\big)^*(U), Z\Big) \\ &\quad + \frac{1}{2}\Big\{\alpha\Big(\big(adY \circ\ adX\big)^*(U), Z\Big) - \alpha\Big(\big(adX \circ\ (adY)^*\big)(U), Z\Big) \\ &\quad + \alpha\Big(\big(ad\big((adY)^*(U)\big)\big)^*(X), Z\Big)\Big\}. \end{aligned} \tag{3.22}$$

Since Z is an arbitrary left invariant vector field and additionally, then we have

$$ad(\nabla_X Y)^*(U) = \frac{1}{2}\Big\{ad([X,Y])^*(U) - ad(ad(X)^*Y)^*(U) - ad(ad(Y)^*X)^*(U)\Big\}.$$

and

$$\widehat{\nabla d}\beta(X,Y) = \frac{1}{2}\Big\{ad([X,Y])^*(U) - ad(ad(X)^*Y)^*(U) - ad(ad(Y)^*X)^*(U)\Big\} + \frac{1}{2}\Big\{\big(adY \circ\ adX\big)^*(U) - \big(adX \circ\ (adY)^*\big)(U) + \Big(ad\big((adY)^*(U)\big)\Big)^*(X)\Big\}. \tag{3.23}$$

Then $s^k{}_{0|k}$ is the trace of $\widehat{\nabla d}\beta(F\ell,-)$, i.e.,

$$s^k{}_{0|k} = tr\Big(\widehat{\nabla d}\beta(F\ell,-)\Big), \tag{3.24}$$

which gives us (1.2). □

As a consequence of Theorem 1.1, one can get the following.

Corollary 3.2. *Suppose that G is a 3-dimensional Lie group equipped with a left-invariant Randers metric F defined by the underlying left-invariant Riemannian metric α and the left-invariant vector field U, i.e., $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(U,v)$. Then, F is a Ricci-quadratic metric if and only if for any left-invariant vector fields X, Y, and Z the following hold*

$$\big\langle[U,X],Y\big\rangle + \big\langle X,[U,Y]\big\rangle + \big\langle[U,X],U\big\rangle\langle U,Y\rangle + \big\langle[U,Y],U\big\rangle\langle U,X\rangle = 0, \tag{3.25}$$

$$tr\Big(\widehat{\nabla d}\beta(F\ell,-)\Big) = 0. \tag{3.26}$$

For the abelian Lie groups, we have $d\beta = 0$. Clearly (1.1) and (1.2) hold. So, one can conclude the following result.

Corollary 3.3. *Any left invariant Randers metric on an abelian Lie group is Ricci-quadratic.*

Remark 3.4. As an observation if $U \in [\mathfrak{g},\mathfrak{g}]^\perp$ and $ad(U)$ is skew-adjoint with respect to α, then (1.1) and (1.2) hold. Therefore, in this case $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(U,v)$ is Ricci-quadratic. This observation verifies Theorem 1.1 in [19].

A Randers metric $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(U,v)$ on Lie group G is said $\mathcal{Z}$-Randers metric if U belongs to the center of Lie algebra $\mathfrak{g}$. By this definition, we get the following result.

Corollary 3.5. *Any $\mathcal{Z}$-Randers metric $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(U,v)$ is Ricci-quadratic if and only if $tr(\widehat{\nabla d}\beta(F\ell,-)) = 0$ holds.*

In [25], Tóth and Kovács considered the class of left invariant Randers metrics on the 3-dimensional Heisenberg group. They gave a complete description of the Chern connection defined by a left invariant Randers metric on the 3 dimensional Heisenberg group. Here, we prove the following.

Proposition 3.6. *Consider the Heisenberg group Nil with the basis $\{x,y,z\}$ and the following Lie bracket*

$$[x,y] = z,\ \ [x,z] = 0,\ \ [y,z] = 0. \tag{3.27}$$

Then the $\mathcal{Z}$-Randers metric $F(p,v) = \sqrt{\alpha_p(v,v)} + \alpha_p(U,v)$ with $U = z$ is Ricci-quadratic. In this case, the following holds

$$\boldsymbol{Ric} = \overline{\boldsymbol{Ric}}.$$

Proof. According to Corollary 3.5, $\mathcal{Z}$-Randers metric F is Ricci-quadratic if and only if

$$tr\Big(\widehat{\nabla d}\beta(F\ell,-)\Big) = 0.$$

We know that $F\ell$ is a tangent vector and at the identity element of G, i.e., e it is a linear combination of $\{x, y, z\}$. Suppose that $F\ell = ax + by + cz$, for some real constants a, b, c. By linearity of $\widehat{\nabla d\beta}(F\ell, -)$ in its first argument, we have

$$\widehat{\nabla d\beta}(F\ell, -) = a\widehat{\nabla d\beta}(x, -) + b\widehat{\nabla d\beta}(y, -) + c\widehat{\nabla d\beta}(z, -),$$

Thus, it is sufficient to consider $\widehat{\nabla d\beta}(x, -)$, $\widehat{\nabla d\beta}(y, -)$ and $\widehat{\nabla d\beta}(z, -)$. Using (3.27), we easily obtain the ad and ad^* operators for the Heisenberg group Nil as follows

$$ad(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad ad(x)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad ad(y)^* = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.28}$$

and

$$ad(z) = ad(z)^* = 0. \tag{3.29}$$

By using (3.28) and (3.29), we have:

$$\widehat{\nabla d\beta}(x, x) = 0, \quad \widehat{\nabla d\beta}(x, y) = 0, \quad \widehat{\nabla d\beta}(x, z) = \frac{x}{2},$$

and consequently

$$\widehat{\nabla d\beta}(x, -) = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that

$$tr\left(\widehat{\nabla d\beta}(x, -)\right) = 0. \tag{3.30}$$

Similarly,

$$\widehat{\nabla d\beta}(y, -) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \end{pmatrix}, \quad \widehat{\nabla d\beta}(z, -) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and consequently

$$tr\left(\widehat{\nabla d\beta}(y, -)\right) = 0, \quad tr\left(\widehat{\nabla d\beta}(z, -)\right) = 0. \tag{3.31}$$

Therefore $tr\left(\widehat{\nabla d\beta}(F\ell, -)\right) = 0$. This completes the proof. □

Corollary 3.7. *The Heisenberg group Nil with the $\mathcal{Z}$-Randers metric $F(p, v) = \sqrt{\alpha_p(v, v)} + \alpha_p(U, v)$ with $U = z$ and α given in Table 1 is not an Einsteinian manifold.*

Proof. According to Proposition 3.6, The Heisenberg group Nil has $\mathcal{Z}$-Randers metric F and by (3.8), we have $\mathbf{Ric} = \overline{\mathbf{Ric}}$. By [15] there are directions U and V such that $r(U) > 0$ and $r(V) < 0$. Thus F is not an Einstein metric. □

4. Proof of Theorem 1.2

Case (i) is obvious and case (ii) is proved in Proposition 3.6. Then, we prove the statement for simple Lie group $SU(2)$ and the non-unimodular Lie group G_c, where $0 < \mu \leq |c|$ and $v > 0$. The other cases are similar to the mentioned two cases. For this aim, we have to show that (3.25) and (3.26) hold.

For the simple Lie group $SU(2)$ the (3.25) holds if $\mu = \lambda$ and for (3.26) we have:

$$ad(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad ad(x)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad ad(y) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad ad(y)^* = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

and

$$ad(z) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad(z)^* = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

By (4.1) and (4.2) we get

$$\widehat{\nabla d\beta}(x,x) = -z, \;\; \widehat{\nabla d\beta}(x,y) = 0, \;\; \widehat{\nabla d\beta}(x,z) = \frac{x}{2},$$

so

$$\widehat{\nabla d\beta}(x,-) = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

which implies that

$$tr\left(\widehat{\nabla d\beta}(x,-)\right) = 0. \tag{4.3}$$

Similarly,

$$\widehat{\nabla d\beta}(y,-) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{-3}{2} & 0 \end{pmatrix}, \quad \widehat{\nabla d\beta}(z,-) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$tr\left(\widehat{\nabla d\beta}(y,-)\right) = 0, \quad tr\left(\widehat{\nabla d\beta}(z,-)\right) = 0. \tag{4.4}$$

Therefore $tr\left(\widehat{\nabla d\beta}(F\ell,-)\right) = 0$, which means (3.26) also holds.

For non-unimodular Lie group G_c, where $0 < \mu \leq |c|$ and $v > 0$, the (3.25) holds if $\mu = c$ and for (3.26), we have:

$$ad(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad(x)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \; ad(y) = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad(y)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c & -2 & 0 \end{pmatrix}, \tag{4.5}$$

and

$$ad(z) = \begin{pmatrix} 0 & -c & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad(z)^* = \begin{pmatrix} 0 & 1 & 0 \\ -c & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

By (4.1) and (4.2), we have

$$\widehat{\nabla d\beta}(x,x) = 0, \quad \widehat{\nabla d\beta}(x,y) = 0, \quad \widehat{\nabla d\beta}(x,z) = 0.$$

So

$$\widehat{\nabla d\beta}(x,-) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that $tr\big(\widehat{\nabla d\beta}(x,-)\big) = 0$. Similarly,

$$\widehat{\nabla d\beta}(y,-) = \widehat{\nabla d\beta}(z,-) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $tr\big(\widehat{\nabla d\beta}(y,-)\big) = 0$ and $tr\big(\widehat{\nabla d\beta}(z,-)\big) = 0$. Therefore $tr\big(\widehat{\nabla d\beta}(F\ell,-)\big) = 0$, and (3.26) also holds for this case. □

5. Applications of the Classification Theorem 1.2

The well-known Wallach Theorem in Riemannian geometry explains that the 3-sphere group $SU(2)$, consisting of 2×2 unitary matrices of determinant 1, is the only simply connected Lie group which admits a left invariant metric of strictly positive sectional curvature. This motivates us to consider the Riemannian and non-Riemannian curvature properties of the Randers metric $F(p,v)=\sqrt{\alpha_p(v,v)}+\alpha_p(z,v)$ defined on $SU(2)$, where it is the special case (iv) of Theorem 1.2.

5.1. In [19], Shen showed that every R-quadratic Finsler metric on a closed manifold is a Landsberg metric. In other words, on a compact Finsler manifolds, we have the following

$$\big\{\text{Berwald metrics}\big\}\subseteq\big\{\text{R-quadratic metrics}\big\}\subseteq\big\{\text{Landsberg metrics}\big\}.$$

On the other hand, the class of Ricci-quadratic Finsler metrics are a natural extension of the class of R-quadratic Finsler metrics. Then, on arbitrary Finsler manifolds, the following holds

$$\big\{\text{Berwald metrics}\big\}\subseteq\big\{\text{R-quadratic metrics}\big\}\subseteq\big\{\text{Ricci-quadratic metrics}\big\}.$$

It is natural to ask if every Ricci-quadratic Finsler metric on a closed manifold is Landsbergian? We have an example that shows this theorem is no longer true for Ricci-quadratic Finsler metrics. For this aim, let us consider the simple Lie group $SU(2)$ equipped with the following Randers metric

$$F(p,v)=\sqrt{\alpha_p(v,v)}+\alpha_p(z,v), \tag{5.1}$$

where α is the Reimaninan metric introduced in Table 1. By Theorem 1.2, F is a Ricci-quadratic metric. On the other hand, the related 1-form $\alpha_p(z,.)$ is not parallel with respect to α which shows that F is not Berwaldian nor Landsbergian. On the other hand, by Lemma 2.1, every Randers metric is C-reducible

$$\mathbf{C}_y(u,v,w)=\frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w)+\mathbf{I}_y(v)\mathbf{h}_y(u,w)+\mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\}, \tag{5.2}$$

By taking horizontal derivation of (5.2), we get

$$\mathbf{L}_y(u,v,w)=\frac{1}{n+1}\Big\{\mathbf{J}_y(u)\mathbf{h}_y(v,w)+\mathbf{J}_y(v)\mathbf{h}_y(u,w)+\mathbf{J}_y(w)\mathbf{h}_y(u,v)\Big\}, \tag{5.3}$$

According to (5.3), a Randers metric is Lansberg metric $\mathbf{L}=0$ if and only if it is a weakly Landsberg metric $\mathbf{J}=0$. Thus, the Randers metric (5.1) is not a weakly Landsberg metric.

5.2. As a meaningful extension of Landsberg curvature, Berwald introduced the non-Riemannian quantity called the stretch curvature. He showed that a Finsler metric has vanishing stretch curvature if and only if the length of an arbitrary vector is unchanged under the parallel displacement along an infinitesimal parallelogram. A Finsler metric with vanishing stretch curvature is called stretch metric. For $y\in T_xM_0$, define $\mathbf{\Sigma}_y: T_xM\times T_xM\times T_xM\times T_xM\to\mathbb{R}$ by $\mathbf{\Sigma}_y(q,u,v,w):=\Sigma_{\ ijkl}(y)q^iu^jv^kw^l$, where $\Sigma_{ijkl}:=L_{ijk||l}-L_{ijl||k}$. The family $\mathbf{\Sigma}:=\{\mathbf{\Sigma}_y\}_{y\in TM_0}$ is called the stretch curvature. F is called a stretch metric if $\mathbf{\Sigma}=0$. It is proved that every R-quadratic metric is a stretch metric, namely, $\mathbf{\Sigma}=0$ (see [24]). In [24], Tayebi-Najafi showed that every homogeneous (α,β)-metric of stretch-type is a Berwald metric. Since the Finsler metric (5.1) is not Berwaldian then it is not a stretch metric.

5.3. In [3], Bácsó-M. Matsumoto proved that a Randers metric $F=\alpha+\beta$ is a Douglas metric if and only if β is a closed 1-form. On the other hand, for the Randers metric (5.1) on $SU(2)$, we have

$$\big\langle[x,y],z\big\rangle=\langle z,z\rangle\neq 0.$$

This shows that the related 1-form of Randers metric $F(p,v)=\sqrt{\alpha_p(v,v)}+\alpha_p(z,v)$ defined on $SU(2)$ is not closed. Thus F is not a Douglas metric.

TABLE 1. Euclidean 3-dimensional Lie algebras.

Cases	algebra structure	Associated simply connected Lie group	Left invariant Riemannian metric
Case 1	$[x,y]=0$, $[x,z]=0$, $[y,z]=0$	$\mathbb{R}^3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Case 2	$[x,y]=z$, $[x,z]=0$, $[y,z]=0$	The Heisenberg group Nil	$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ $\gamma>0$
Case 3	$[x,y]=0$, $[x,z]=-x$, $[y,z]=y$	The solvable Lie group Sol	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\upsilon>0$
Case 4	$[x,y]=0$, $[x,z]=-x$, $[y,z]=y$	The solvable Lie group Sol	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\kappa>1$, $\upsilon>0$
Case 5	$[x,y]=0$, $[x,z]=y$, $[y,z]=-x$	The solvable Lie group $\tilde{E}_0(2)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$, $\upsilon>0$
Case 6	$[x,y]=2z$, $[x,z]=-2y$, $[y,z]=-2x$	The simple Lie group $P\tilde{S}L(2,\mathbb{R})$	$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\kappa\geq\upsilon>0$, $\gamma>0$
Case 7	$[x,y]=z$, $[x,z]=-y$, $[y,z]=x$	The simple Lie group $SU(2)$	$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\gamma\geq\kappa\geq\upsilon$, $\upsilon>0$
Case 8	$[x,y]=0$, $[x,z]=-x$, $[y,z]=-y$	The non-unimodular Lie group G_I	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\upsilon>0$
Case 9	$[x,y]=0$, $[x,z]=-y$, $[y,z]=cx-2y$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq\lvert c\rvert$, $\upsilon>0$
Case 10	$[x,y]=0$, $[x,z]=-y$, $[y,z]=cx-2y$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\kappa>0$, $\upsilon>0$, $c=0$
Case 11	$[x,y]=0$, $[x,z]=-y$, $[y,z]=cx-2y$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\upsilon>0$, $c=0$
Case 12	$[x,y]=0$, $[x,z]=-y$, $[y,z]=cx-2y$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$, $c=1$, $\upsilon>0$
Case 13	$[x,y]=0$, $[x,z]=-y$, $[y,z]=cx-2y$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$, $c=1$, $\upsilon>0$, $0<\gamma<1$
Case 14	$[x,y]=0$, $[x,z]=-y$, $[y,z]=cx-2y$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq c$, $c>0$, $\upsilon>0$
Case 15	$[x,y]=0$, $[x,z]=-y$, $[y,z]=cx-2y$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & \kappa & 0 \\ \kappa & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$, $\upsilon>0$, $\gamma\pi=\sqrt{1-c}$

5.4. Let $F = F(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then, F is said to be projectively flat if its geodesics are straight line segment in $\mathcal{U}$ [21]. In [4], it is proved that a Randers metric $F = \alpha + \beta$ is projectively flat if and only if α is locally projectively flat (or constant sectional curvature) and β is closed. However, as we prove in above, the 1-form of (5.1) is not closed. Then, the Randers metric (5.1) is not projectively flat.

5.5. Now, we consider if the Randers metric (5.1) is of scalar flag curvature or not. Suppose that F is of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. Since F has vanishing S-curvature $\mathbf{S} = 0$ and $dim(M) = 3$, then by Akbar-Zadeh theorem we get $\mathbf{K} = \mathbf{K}(x)$. This is equal to following

$$R^i_j = \mathbf{K}(x)F^2 h^i_j, \tag{5.4}$$

where $h^i_j := \delta^i_j - F^{-2} y_j y^i$ denotes the angular metric. If $\mathbf{K} = 0$, then F is locally Minkowskian which is a contradiction. Suppose that $\mathbf{K} \neq 0$. Taking the trace of (5.4) yields

$$\mathbf{Ric} = 2\mathbf{K}(x)F^2. \tag{5.5}$$

On the other hand, F is Ricci-quadratic. Then the Ricci curvature of F can be written as follows

$$\mathbf{Ric} = f_{ij}(x)y^i y^j. \tag{5.6}$$

Comparing (5.5) and (5.6) gives us

$$F = \sqrt{\frac{1}{2\mathbf{K}(x)} f_{ij}(x)y^i y^j}. \tag{5.7}$$

(5.7) implies that F is Riemannian while it is impossible. Therefore, F is not of scalar flag curvature [28].

5.6. Suppose that G is a Lie group equipped with a left-invariant Randers metric F defined by the underlying left-invariant Riemannian metric α and the left-invariant vector field U, i.e., $F(x, y) = \sqrt{\alpha_x(y, y)} + \alpha_x(U, y)$. In [1], Atashafrouz-Najafi proved that F is a generalized Douglas-Weyl metric if and only if for any left-invariant vector fields X, Y, and Z the following holds

$$\begin{aligned}\alpha\big(U, [\nabla_X Y, Z] + [Y, \nabla_X Z]\big) = {} & \frac{1}{n-1}\Big\{\alpha(X, Y)\sum_{i=1}^{n}\big\{\alpha\big(U, [\nabla_{X_i} X_i, Z] + [X_i, \nabla_{X_i} Z]\big)\big\}\Big\} \\ & - \frac{1}{n-1}\Big\{\alpha(X, Z)\sum_{i=1}^{n}\big\{\alpha\big(U, [\nabla_{X_i} X_i, Y] + [X_i, \nabla_{X_i} Y]\big)\big\}\Big\},\end{aligned} \tag{5.8}$$

where $\{X_1, ..., X_n\}$ is an orthonormal basis of the Lie algebra $\mathfrak{g} = T_e G$. They showed that the Randers metric $F(p, v) = \sqrt{\alpha_p(v, v)} + \alpha_p(z, v)$ defined on $SU(2)$ satisfies (5.8) and then it is a generalized Douglas-Weyl metric.

5.7. A Finsler metric F on a manifold M is called a generalized Berwald metric if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F (see [22] and [26]). In this case, F is called a generalized Berwald metric on M and (M, F) is called a generalized Berwald manifold. If the covariant derivative ∇ is also torsion-free, then F reduces to a Berwald metric. Therefore, the class of Berwald metrics belongs to the class of generalized Berwald metrics. Now, let X and $\beta^\sharp$ be a left-invariant vector fields and α be a left-invariant Riemannian metric on a Lie group G. It is easy to see that $\alpha(\beta^\sharp, \beta^\sharp)$ and $\alpha(\beta^\sharp, X)$ are constant functions. By (3.1) and (3.3), we get

$$\tilde{s}(X) = -\alpha([X, \beta^\sharp], \beta^\sharp), \quad \tilde{r}(X) = \alpha([X, \beta^\sharp], \beta^\sharp). \tag{5.9}$$

Hence, we obtain

$$r_i + s_i = 0. \tag{5.10}$$

In [26], Vincze proved that a Randers metric $F = \alpha + \beta$ is a generalized Berwald metric if and only if dual vector field $\beta^\sharp$ is of constant Riemannian length, namely, it satisfies (5.10). Consequently, the left-invariant Randers metric (5.1) is a generalized Berwald metric.

5.8. Let $F=\alpha+\beta$ be a Randers metric on an n-dimensional manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta=b_i(x)y^i$. Then, we have $dV_F=e^{(n+1)\rho(x)}dV_\alpha$, where $\rho:=\ln\sqrt{1-b^2}$. The S-curvature of F is given by

$$\mathbf{S}=(n+1)\Big\{\frac{e_{00}}{2F}-(s_0+\rho_0)\Big\},$$

where $\rho_0=\rho_{x^i}y^i$, $e_{ij}:=r_{ij}+b_is_j+b_js_i$ and $e_{00}=e_{ij}y^iy^j$. In [20], Shen proved that a Randers metric has vanishing S-curvature $\mathbf{S}=0$ if and only if the following hold $e_{ij}=0$. In [23], Tayebi-Eslami proved that a generalized Berwald (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, on an n-dimensional manifold M with $\phi'(0)\neq 0$ satisfies $\mathbf{S}=0$ if and only if β is a Killing form with constant length, namely, $r_{ij}=0$ and $s_i=0$. Then, the left-invariant Randers metric $F(p,v)=\sqrt{\alpha_p(v,v)}+\alpha_p(z,v)$ defined on $SU(2)$ satisfies $\mathbf{S}=0$.

Here, we give another simple argument to show that F has vanishing S-curvature. Since F is Ricci-quadratic then it satisfies (3.6). It is proved that a Randers metric has isotropic S-curvature $\mathbf{S}=(n+1)cF$ if and only if the 1-form β satisfies (3.6), where $c=c(x)$ is a scalar function on M (see [6]). On the other hand, in [27] it is proved that every homogeneous Randers metric of isotropic S-curvature has vanishing S-curvature $\mathbf{S}=0$.

Summarizing the above explanations, we conclude the following.

Proposition 5.1. *The left invariant Randers metric $F(p,v)=\sqrt{\alpha_p(v,v)}+\alpha_p(z,v)$ in (5.1) defined on $SU(2)$ has the following curvature properties:*

1. *F is not a Landsberg metric nor weakly Landsberg metric nor stretch metric;*
2. *F is a generalized Douglas-Weyl metric which is not Douglasian nor Weyl metric;*
3. *F is a generalized Berwald metric which is not Berwaldian;*
4. *F is not projectively flat;*
5. *F satisfies $\mathbf{S}=0$;*
6. *F is Ricci-quadratic but not R-quadratic.*

6. Conclusion

In this paper, we have established the necessary and sufficient conditions for a left-invariant Randers metric to be Ricci-quadratic. By combining these conditions with Ha-Bumlee's classification of left-invariant Riemannian metrics, we have provided a complete classification of left-invariant Ricci-quadratic Randers metrics on three-dimensional Lie groups. As an application, we discovered a particularly interesting Ricci-quadratic Randers metric on SU(2)SU(2). This metric exhibits notable properties: it is a generalized Berwald metric but not Berwaldian, and it is a generalized Douglas-Weyl metric that is neither Douglasian nor a Weyl metric. These findings highlight the rich geometric structure of left-invariant Randers metrics and contribute to a deeper understanding of their curvature properties. Our results open up potential avenues for further research, particularly in exploring higher-dimensional cases or investigating other special Finsler metrics with similar curvature characteristics. The interplay between Ricci-quadratic conditions and left-invariant structures provides a promising direction for future studies in Finsler geometry and its applications.

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