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Singular Boundary Value Problems on Time Scales with Iterative Terms

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Abstract

This paper investigates the existence of multiple positive solutions for a class of singular two-point boundary value problems defined on time scales. Utilizing Hölder's inequality together with Krasnoselskii's fixed point theorem in the context of a Banach space, we establish new sufficient conditions that guarantee the existence of countably infinite positive solutions. To demonstrate the practical applicability of our theoretical findings, we provide a concrete example that illustrates the effectiveness of the proposed approach.

Keywords. Fractional derivative, Boundary value problem, Kernel, Fixed-point theorems, Positive solution.

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1. Introduction

The study of dynamic systems often necessitates a framework that can encompass both continuous and discrete behaviors. Traditional approaches rely on separate methodologies – differential equations for continuous systems and difference equations for discrete ones. This research delves into the theory of time scales, a powerful tool that unifies these analyses. Introduced by Stefan Hilger in 1988, time scales provide a versatile platform for modeling hybrid systems by treating time as a non-empty, closed subset of the real numbers. This flexibility allows us to capture the complexities of phenomena that exhibit characteristics of both continuous and discrete change. The theory of time scales has significantly advanced our understanding of boundary value problems (BVPs) [21]. Over the past decade, researchers have employed a diverse arsenal of mathematical techniques to investigate the existence and properties of solutions to these problems. These techniques include fixed-point theorems, upper and lower solution methods, degree theory, and variational methods [2–4, 10, 23, 24].

This unified framework transcends theoretical benefits, offering a powerful tool for modeling real-world phenomena across disciplines. Its strength lies in its ability to capture systems exhibiting both continuous and discrete dynamics, a common feature in fields like neural networks, heat transfer, and epidemiology [16, 20]. For instance, models for insect population dynamics or disease propagation necessitate a hybrid approach to accurately represent the interplay between continuous changes (e.g., population growth) and discrete events (e.g., birth events, transmission). The foundational aspects of this approach have been extensively documented in the literature[6, 7].

The study of heat transfer in porous structures is important for many research and engineering applications, including fluidized bed junctions, compact heat exchangers, chemical catalytic reactors, and high-temperature gas-cooled reactors. These applications require powerful mathematical models that can represent the complex behavior in porous structures. In response to these difficulties, Leibenson [14] put forward the fundamental equation:

$$\big(\phi_p(\mathtt{w}'(\mathtt{z}))\big)' = g\big(\mathtt{z}, \mathtt{w}(\mathtt{z}), \mathtt{w}'(\mathtt{z})\big),$$

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where $\phi_p(\mathtt{w}) = |\mathtt{w}|^{p-2}\mathtt{w},\ p > 1$, is the p-Laplacian operator and inverse expressed as $\phi_q(\mathtt{z}) = |\mathtt{z}|^{q-2}\mathtt{z}$ and p, q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. This model has been widely used in fields as diverse as filtration in porous media, hemodynamics, rheology, and materials science, demonstrating its importance in modeling viscoplasticity and other difficult phenomena. One of the key challenges in modeling these systems is the need for mathematical representations that capture the nonlinear and often singular behaviors within porous media. The p-Laplacian operator has been widely used in this context for modeling non-linear diffusion, yet it has limitations in handling certain singularities and boundary conditions, particularly in complex geometries. This work aims to extend and refine these models to better account for these complexities.

To address the limitations of the p-Laplacian operator, we introduce the Increasing Homeomorphism and Positive Homomorphism Operator (IHPHO), which extends the p-Laplacian for specific values of $p > 1$. Unlike the standard p-Laplacian, the IHPHO operator provides a more flexible framework for modeling highly nonlinear and singular behavior in porous structures, offering a broader range of applications. Liang and Zhang[15] previously used the IHPHO operator to study the existence of multiple optimal solutions for a nonlinear BVP,

$$\big(\phi(\mathtt{w}^{\Delta}(\mathtt{z}))\big)^{\nabla} + a(\mathtt{z})g\big(\mathtt{w}(\mathtt{z})\big) = 0,\ \mathtt{z} \in [0, \mathfrak{T}]_{\mathbb{T}}$$
$$\mathtt{w}(0) = \sum_{i=1}^{m-2} a_i \mathtt{w}(\xi_i),\ \mathtt{w}^{\Delta}(\mathfrak{T}) = 0.$$

A mapping $\phi : \mathbb{R} \to \mathbb{R}$ is referred to as an *increasing homeomorphism* and a *positive homomorphism* if it satisfies the following conditions:

(1) For all $x, y \in \mathbb{R}$, if $x \leq y$, then $\phi(x) \leq \phi(y)$ (monotonicity).
(2) Function ϕ is a continuous bijection and its inverse ϕ^{-1} is also continuous (homeomorphism).
(3) For all $x, y \in [0, \infty)$, the multiplicative property holds: $\phi(xy) = \phi(x)\phi(y)$.

Furthermore, condition (3) can be replaced by the following stronger condition:

(4) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$.

Remark 1.1. If conditions (1), (2), and (4) are fulfilled, then ϕ is homogeneous and corresponds to a p-Laplacian type operator. In this case, ϕ can be expressed as

$$\phi(x) = |x|^{p-2}x, \quad \text{for some } p > 1.$$

Dogan [9] studied the multiple solutions by using fixed-point index theory to the BVP,

$$\big(\phi_p(\mathtt{w}^{\Delta}(\mathtt{z}))\big)^{\nabla} + \omega(\mathtt{z})g\big(\mathtt{z}, \mathtt{w}(\mathtt{z})\big) = 0,\ \mathtt{z} \in [0, \mathfrak{T}]_{\mathbb{T}}$$
$$\mathtt{w}(0) = \sum_{i=1}^{m-2} a_i \mathtt{w}(\xi_i),\ \phi_p(\mathtt{w}^{\Delta}(\mathfrak{T})) = \sum_{i=1}^{m-2} b_i \phi_p(\mathtt{w}^{\Delta}(\xi_i)).$$

In light of recent developments, we examine a dynamic BVP characterized by iterations and subject to two-point boundary conditions, along with the influence of multiple singularities. By employing Krasnoselskii's fixed-point theorem within the context of Banach spaces, we establish the existence of a countable set of positive solutions to BVP:

$$\left.\begin{aligned} \phi(\mathtt{w}_i^{\Delta^2}(\mathtt{z})) + \varepsilon(\mathtt{z})\hbar_i\big(\mathtt{w}_{i+1}(\mathtt{z})\big) = 0,\ 1 \leq i \leq m,\ \mathtt{z} \in (0,1)_{\mathbb{T}} \\ \mathtt{w}_{i+1}(\mathtt{z}) = \mathtt{w}_1(\mathtt{z}),\ \mathtt{z} \in (0,1)_{\mathbb{T}}, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \mathtt{w}_i(0) - \mathtt{w}_i^{\Delta}(0) = 0,\ 1 \leq i \leq m, \\ \mathtt{w}_i(1) + \mathtt{w}_i^{\Delta}(1) = 0,\ 1 \leq i \leq m, \end{aligned}\right\} \tag{1.2}$$

where $m \in \mathbb{N}$, $\varepsilon(\mathtt{z}) = \prod_{i=1}^{n} \varepsilon_i(\mathtt{z})$ and $\varepsilon_i(\mathtt{z}) \in L_{\Delta}^{p_i}([0,1]_{\mathbb{T}})$ where $p_i \geq 1$ and for each $i = 1, 2, \cdots, m$, $\varepsilon_i(\mathtt{z})$ exhibits a singularity within $\left(0, \frac{1}{2}\right)_{\mathbb{T}}$. This provides new insights into the behavior of nonlinear systems with singularities, offering potential advancements in both theoretical and applied mathematics. Our model employs a nonlinear IHPHO acting on the second-order delta derivative, which significantly generalizes the dynamic structure. This is in contrast to the referenced works, where the differential systems involve standard second-order delta or nabla derivatives without such operator complexity. Secondly, our boundary conditions are dynamic and coupled two-point conditions which link both the function and its delta derivative at the endpoints in a symmetric and nontrivial manner. In contrast, the problems in the earlier works [13, 18] involve either Dirichlet-type conditions or linear expressions involving function values only, without incorporating derivative terms.

One specific application of these equations can be found in heat transfer in porous media, where temperature evolution may not be continuous due to sudden changes in environmental conditions or operational modes. For instance, consider a thermal system in a porous structure subject to periodic heating and cooling, where the process occurs in both continuous phases (when heat flows steadily through the medium) and discrete phases (when heat is applied in pulses or intervals).

- The time scale $\mathbb{T}$ could represent both discrete heating intervals (e.g., every 10 minutes when heat is applied) and continuous cooling during non-heating periods. The time scale framework allows for the modeling of these processes as a unified approach, combining discrete and continuous behavior.
- The function $\varepsilon(\mathtt{z})$ represents the spatially varying thermal conductivity of the porous material, which could change depending on its position within the system (e.g., different layers of the material may conduct heat differently).
- The function $\hbar_i\big(\mathtt{w}_{i+1}(\mathtt{z})\big)$ can represent an external heat source, which is applied intermittently, corresponding to each heating cycle. This term reflects how the temperature profile is updated at each interval as the system responds to the applied heat.
- The iterative term $\mathtt{w}_i(\mathtt{z})$ indicates the step-by-step evolution of the temperature. Each iteration i represents the temperature profile at a specific time step, with each new profile $\mathtt{w}_{i+1}(\mathtt{z})$ depending on the previous one. This iterative process models how the heat accumulates and spreads through the medium over successive heating and cooling cycles.

We assume the following conditions hold throughout the paper:

(H_1) The function $\hbar_i : [0, +\infty) \to [0, +\infty)$ is assumed to be continuous.

(H_2) We consider a sequence $\{\mathtt{z_j}\}_{r=1}^{\infty}$ such that $0 < \mathtt{z_{j+1}} < \mathtt{z_j} < \frac{1}{2}$ and $\lim_{r\to\infty} \mathtt{z_j} = \mathtt{z}^* < \frac{1}{2}$. Moreover, as t approaches $\mathtt{z_j}$, $\varepsilon_i(\mathtt{z})$ approaches $+\infty$ for $i = 1, 2, \ldots, n$. Additionally, there exist constants $\delta_i > 0$ such that $\phi^{-1}(\varepsilon_i(\mathtt{z})) > \delta_i$, and $\varepsilon_i(\mathtt{z})$ does not vanish identically on any subinterval of $[0, 1]_{\mathbb{T}}$.

2. Preliminaries

In this section, we present fundamental definitions and lemmas that will be beneficial for our subsequent discussions. Please see [5, 6, 12, 19, 22] for additional information. A time scale $\mathbb{T}$ is characterized as a closed, non-empty subset of the real numbers $\mathbb{R}$. For any $\mathtt{z} < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, we can define the forward jump operator σ and the backward jump operator ρ as follows:

$$\sigma(\mathtt{z}) = \inf\{\tau \in \mathbb{T} \mid \tau > \mathtt{z}\}, \quad \rho(\mathtt{z}) = \sup\{\tau \in \mathbb{T} \mid \tau < \mathtt{z}\}$$

for each $\mathtt{z} \in \mathbb{T}$. If $\sigma(\mathtt{z}) > \mathtt{z}$, we refer to $\mathtt{z}$ as right scattered; if $\sigma(\mathtt{z}) = \mathtt{z}$, it is described as right dense. In the same vein, if $\rho(\mathtt{z}) < \mathtt{z}$, we call $\mathtt{z}$ left scattered, while if $\rho(\mathtt{z}) = \mathtt{z}$, it is identified as left dense.

A function f is considered left-dense continuous if it is continuous at every left-dense point within $\mathbb{T}$ and has right-sided limits at each right-dense point in $\mathbb{T}$.

Let's take $u : \mathbb{T} \to \mathbb{R}$ and $\mathtt{z} \in \mathbb{T}$. The delta-derivative of $u(\mathtt{z})$, denoted $u^{\Delta}(\mathtt{z})$, is defined as a number (if it exists) possessing the following property: for every $\epsilon > 0$, there exists a neighborhood U surrounding $\mathtt{z}$ such that for all $s \in U$:

$$|u(\sigma(\mathtt{z})) - u(\mathtt{y}) - u^{\Delta}(\mathtt{z})(\sigma(\mathtt{z}) - \mathtt{y})| \le \epsilon|\sigma(\mathtt{z}) - \mathtt{y}|.$$

In a similar manner, the nabla-derivative of $u(\mathtt{z})$, represented as $u^{\nabla}(\mathtt{z})$, is defined as a number (when it exists) that satisfies this condition: for every $\epsilon > 0$, there exists a neighborhood U around $\mathtt{z}$ such that for all $\mathtt{y} \in U$:

$$|u(\rho(\mathtt{z})) - u(\mathtt{y}) - u^{\nabla}(t)(\rho(\mathtt{z}) - \mathtt{y})| \le \epsilon|\sigma(\mathtt{z}) - \mathtt{y}|.$$

Lastly, we will briefly revisit some notations and an overarching existence theorem.

Definition 2.1. [8] Let μ_Δ and μ_∇ represent the Δ-measure and ∇-measure, respectively, on the time scale $\mathbb{T}$. A set $A \subset \mathbb{T}$ qualifies as measurable, denoted by $\mu(A)$, if it satisfies $\mu_\Delta(A) = \mu_\nabla(A)$. This common value is recognized as the Lebesgue measure of A. Consider a statement P concerning an element $\mathtt{z} \in \mathbb{T}$:

(i) If a subset $\Gamma_1 \subset A$ exists such that $\mu_\Delta(\Gamma_1) = 0$ and P holds true on $A \setminus \Gamma_1$, we say that P holds Δ-almost everywhere (a.e.) on A.

(ii) If a subset $\Gamma_2 \subset A$ can be found with $\mu_\nabla(\Gamma_2) = 0$ such that P is valid on $A \setminus \Gamma_2$, we claim that P holds ∇-almost everywhere (a.e.) on A.

Lemma 2.2. *[1] Let $\{t_i\}_{i\in I}$, $I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$ and $E \subset \mathbb{T}$ be a Δ-measurable set. If $u : \mathbb{T} \to \mathbb{R}$ is Δ-integrable on E, then*

$$\int_E u(s)\Delta s = \int_E u(s)ds + \sum_{i\in I_E} (\sigma(t_i) - t_i)\cdot f(t_i) + r(u, E),$$

where

$$r(u, E) = \begin{cases} \mu_M(E)\cdot f(M), & \text{if } M \in \mathbb{T},\\ 0, & \text{if } M \notin T,\end{cases}$$

$I_E := \{i \in I : t_i \in E\}$.

Lemma 2.3. *For any $\mathtt{y}(\mathtt{z}) \in \mathcal{C}([0,1]_\mathbb{T})$, the boundary value problem,*

$$-\phi(\mathtt{w}_1^{\Delta^2}(\mathtt{z})) = \mathtt{y}(\mathtt{z}),\ \mathtt{z} \in (0,1)_\mathbb{T}, \tag{2.1}$$

$$\mathtt{w}_1(0) = \mathtt{w}_1^{\Delta}(0),\ \mathtt{w}_1(1) = -\mathtt{w}_1^{\Delta}(1) \tag{2.2}$$

has a unique solution

$$\mathtt{w}_1(\mathtt{z}) = \int_0^1 \mathrm{Q}(\mathtt{z},\mathtt{y})\mathtt{y}(\mathtt{y})\Delta\mathtt{y}, \tag{2.3}$$

where

$$\mathrm{Q}(\mathtt{z},\mathtt{y}) = \frac{1}{3}\begin{cases} (2-\sigma(\mathtt{y}))(1+\mathtt{z}), & if\ \mathtt{z} \le \mathtt{y},\\ (2-\mathtt{z})(1+\sigma(\mathtt{y})), & if\ \sigma(\mathtt{y}) \le \mathtt{z},\end{cases} \tag{2.4}$$

Proof. Suppose $\mathtt{w}_1$ is a solution of (2.1), then

$$\begin{aligned}\mathtt{w}_1(\mathtt{z}) &= -\int_0^{\mathtt{z}}\int_0^{\mathtt{y}} \phi^{-1}(\mathtt{y}(\mathtt{y}_1))\Delta\mathtt{y}_1\Delta\mathtt{y} + B\mathtt{z} + C\\ &= -\int_0^{\mathtt{z}} (\mathtt{z}-\sigma(\mathtt{y}))\phi^{-1}(\mathtt{y}(\mathtt{y}))\Delta\mathtt{y} + B\mathtt{z} + C,\end{aligned}$$

where $B = \mathtt{w}_1^{\Delta}(0)$ and $C = \mathtt{w}_1(0)$. By the conditions (2.2), we get

$$B = C = \frac{1}{3}\int_0^1 (2-\sigma(\mathtt{y}))\phi^{-1}(\mathtt{y}(\mathtt{y}))\Delta\mathtt{y}.$$

So, we have

$$\begin{aligned}\mathtt{w}_1(\mathtt{z}) &= \int_0^{\mathtt{z}} (\mathtt{z}-\sigma(\mathtt{y}))\phi^{-1}(\mathtt{y}(\mathtt{y}))\Delta\mathtt{y} + \frac{1}{3}\int_0^1 (2-\sigma(\mathtt{y}))(1+\mathtt{z})\phi^{-1}(\mathtt{y}(\mathtt{y}))\Delta\mathtt{y}\\ &= \int_0^1 \mathrm{Q}(\mathtt{z},\mathtt{y})\phi^{-1}(\mathtt{y}(\mathtt{y}))\Delta\mathtt{y}.\end{aligned}$$

This completes the proof. □

Lemma 2.4. *Suppose (H_1)-(H_2) hold. For $\gamma \in (0, \frac{1}{2})_\mathbb{T}$, let $\aleph(\gamma) = \dfrac{\gamma+1}{2}$. Then $\mathrm{Q}(\mathtt{z},\mathtt{y})$ have the following properties:*

(i) $0 \le \mathrm{Q}(\mathtt{z},\mathtt{y}) \le \mathrm{Q}(\sigma(\mathtt{y}),\mathtt{y})$ *for all* $\mathtt{z},\mathtt{y} \in [0,1]_\mathbb{T}$,

(ii) $\aleph(\gamma)Q(y,y) \leq Q(z,y)$ *for all* $z \in [\gamma, 1-\gamma]_{\mathbb{T}}$ *and* $y \in [0,1]_{\mathbb{T}}$.

Proof. (i) is evident. To prove (ii), let $z \in [\gamma, 1-\gamma]_{\mathbb{T}}$ and $z \leq y$. Then

$$\frac{Q(z,y)}{Q(\sigma(y),y)} = \frac{1+z}{1+\sigma(y)} \geq \frac{1+\gamma}{2} = \aleph(\gamma).$$

For $\sigma(y) \leq z$,

$$\frac{Q(z,y)}{Q(y,y)} = \frac{2-z}{2-\sigma(y)} \geq \frac{\gamma+1}{2} = \aleph(\gamma).$$

This completes the proof. □

It can be observed that an m-tuple $(w_1(z), w_2(z), w_3(z), \dots, w_m(z))$ constitutes a solution to the BVP (1.1)–(1.2) when

$$\begin{aligned} w_i(z) &= \int_0^1 Q(z,y)\phi^{-1}\big(\varepsilon(y)\hbar_i(w_{i+1}(y))\big)\Delta y, \ z \in (0,1)_{\mathbb{T}}, \ 1 \leq i \leq m, \\ w_{i+1}(z) &= w_1(z), \ z \in (0,1)_{\mathbb{T}}, \end{aligned}$$

i.e.,

$$\begin{aligned} w_1(z) = \int_0^1 & Q(z,y_1)\phi^{-1}\Bigg[\varepsilon(y_1)\hbar_1\Bigg(\int_0^1 Q(y_1,y_2)\phi^{-1}\Bigg[\varepsilon(y_2)\hbar_2\Bigg(\int_0^1 Q(y_2,y_3)\cdots \\ & \times \hbar_{m-1}\Bigg(\int_0^1 Q(y_{m-1},y_m)\phi^{-1}\big[\varepsilon(y_m)\hbar_m(w_1(y_m))\big]\Delta y_m\Bigg)\cdots\Delta y_3\Bigg)\Delta y_2\Bigg)\Bigg]\Delta y_1. \end{aligned}$$

Definition 2.5. [11] A cone C in a Banach space A is a subset with the following properties:

- **Positivity:** For any element $z \in C$ and any non-negative scalar $\gamma \geq 0$, the product γz also belongs to C.
- **Non-negativity:** The cone does not contain any negative multiples of its elements other than zero, meaning $C \cap (-C) = \{0\}$. This ensures that C and its reflection about the origin only intersect at the zero vector.

Cones are essential in fixed-point theory as they introduce a structure that restricts solutions to a positive or "non-negative" region within the space.

Definition 2.6. [11] An operator F on a Banach space A is said to be completely continuous if:

- It is continuous, meaning small changes in input yield small changes in output.
- It maps bounded subsets of A to relatively compact subsets. In other words, the image of any bounded set under F has a compact closure, which implies that every sequence within this image has a convergent subsequence.

Let Y be the Banach space $C_{rd}((0,1)_{\mathbb{T}}, \mathbb{R})$ with the norm $\|w\| = \max_{z \in (0,1)_{\mathbb{T}}} |w(z)|$. For $\gamma \in \left(0, \frac{1}{2}\right)_{\mathbb{T}}$, we define the cone $L_\gamma \subset Y$ as

$$L_\gamma = \left\{ w \in Y : w(z) \geq 0 \ \text{ and } \min_{z \in [\gamma,\, 1-\gamma]_{\mathbb{T}}} w(z) \geq \aleph(\gamma)\|w(z)\| \right\},$$

For any $w_1 \in L_\gamma$, define an operator $\mho : L_\gamma \to Y$ by

$$\begin{aligned} (\mho w_1)(z) = \int_0^1 & Q(z,y_1)\phi^{-1}\Bigg[\varepsilon(y_1)\hbar_1\Bigg(\int_0^1 Q(y_1,y_2)\phi^{-1}\Bigg[\varepsilon(y_2)\hbar_2\Bigg(\int_0^1 Q(y_2,y_3)\cdots \\ & \times \hbar_{m-1}\Bigg(\int_0^1 Q(y_{m-1},y_m)\phi^{-1}\big[\varepsilon(y_m)\hbar_m(w_1(y_m))\big]\Delta y_m\Bigg)\cdots\Delta y_3\Bigg)\Delta y_2\Bigg)\Bigg]\Delta y_1. \end{aligned}$$

Lemma 2.7. *Let* (H_1) *and* (H_2) *be assumed to hold. For every* $\gamma \in \left(0, \frac{1}{2}\right)_{\mathbb{T}}, \mho(L_\gamma) \subseteq L_\gamma$, *and the mapping* $\mho : L_\gamma \to L_\gamma$ *is completely continuous.*

Proof. It is evident $\mathsf{Q}(\mathtt{z},\mathtt{y}) \geq 0$ for all $\mathtt{z},\mathtt{y} \in (0,1)_{\mathbb{T}}$ from Lemma 2.4. So, $(\mho\mathtt{w}_1)(\mathtt{z}) \geq 0$. Let $\mathtt{w}_1 \in \mathrm{L}_\gamma$. Then

$$\begin{aligned}\|\mho\mathtt{w}_1\| = &\max_{\mathtt{z}\in(0,1)_{\mathbb{T}}} \int_0^1 \mathsf{Q}(\mathtt{z},\mathtt{y}_1)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_1)\hbar_1\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_1,\mathtt{y}_2)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_2)\hbar_2\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_2,\mathtt{y}_3)\cdots\\
&\times \hbar_{m-1}\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_{m-1},\mathtt{y}_m)\phi^{-1}\big[\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big]\Delta\mathtt{y}_m\Bigg)\cdots\Delta\mathtt{y}_3\Bigg)\Delta\mathtt{y}_2\Bigg)\Bigg]\Delta\mathtt{y}_1\\
\leq &\int_0^1 \mathsf{Q}(\sigma(\mathtt{y}_1),\mathtt{y}_1)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_1)\hbar_1\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_1,\mathtt{y}_2)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_2)\hbar_2\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_2,\mathtt{y}_3)\cdots\\
&\times \hbar_{m-1}\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_{m-1},\mathtt{y}_m)\phi^{-1}\big[\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big]\Delta\mathtt{y}_m\Bigg)\cdots\Delta\mathtt{y}_3\Bigg)\Delta\mathtt{y}_2\Bigg)\Bigg]\Delta\mathtt{y}_1.\end{aligned}$$

From Lemma 2.4, we can deduce that

$$\begin{aligned}\min_{\mathtt{z}\in[\gamma,1-\gamma]_{\mathbb{T}}}\big\{(\mho\mathtt{w}_1)(\mathtt{z})\big\} \geq \aleph(\gamma)&\int_0^1 \mathsf{Q}(\sigma(\mathtt{y}_1),\mathtt{y}_1)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_1)\hbar_1\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_1,\mathtt{y}_2)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_2)\hbar_2\cdots\\
&\times \hbar_{m-1}\Bigg(\int_0^1 \mathsf{Q}(\mathtt{y}_{m-1},\mathtt{y}_m)\phi^{-1}\big[\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big]\Delta\mathtt{y}_m\Bigg)\cdots\Delta\mathtt{y}_3\Bigg)\Delta\mathtt{y}_2\Bigg)\Bigg]\Delta\mathtt{y}_1.\end{aligned}$$

From the inequalities above, it follows that

$$\min_{\mathtt{z}\in[\gamma,1-\gamma]_{\mathbb{T}}}\big\{(\mho\mathtt{w}_1)(\mathtt{z})\big\} \geq \aleph(\gamma)\|\mho\mathtt{w}_1\|.$$

Thus, we conclude $\mho\mathtt{w}_1 \in \mathrm{L}_\gamma$, implying $\mho(\mathrm{L}_\gamma) \subseteq \mathrm{L}_\gamma$. Moreover, utilizing standard techniques and the Arzelà-Ascoli theorem, one can easily show that $\mho$ is completely continuous. This affirms the conclusion. □

3. Main Results

We will employ the following theorems to prove the existence of countable set of positive solutions for the iterative system defined by BVP (1)-(2).

Theorem 3.1 (Krasnoselskii Fixed Point Theorem on a Cone)**.** *[11] Let A be a real Banach space, and let $C \subset A$ be a cone in A. Suppose V_1 and V_2 are bounded open subsets of A with $0 \in V_1 \subset \overline{V_1} \subset V_2$. Assume there exists an operator*

$$F : C \cap (V_2 \setminus V_1) \to C$$

that is completely continuous.

The theorem states that if the following two conditions hold:

- ***Condition 1:** For each $z \in C \cap \partial V_1$, we have*
  $$\|F(z)\| \geq \|z\|.$$
- ***Condition 2:** For each $z \in C \cap \partial V_2$, we have*
  $$\|F(z)\| \leq \|z\|.$$

Then, there exists a point $z \in C \cap (V_2 \setminus V_1)$ such that

$$F(z) = z.$$

Theorem 3.2. *[7, 17] Let u be a function in $L^p_\nabla(J)$ for $p > 1$, and let v be in $L^q_\Delta(J)$ with $q > 1$, satisfying the relation $\frac{1}{p} + \frac{1}{q} = 1$. Then the product uv belongs to $L^1_\Delta(J)$, and the following inequality holds:*

$$\|uv\|_{L^1_\Delta} \leq \|u\|_{L^p_\Delta}\|v\|_{L^q_\Delta},$$

where

$$\|u\|_{L^p_\Delta} := \begin{cases}\left[\displaystyle\int_J |u(s)|^p\Delta s\right]^{\frac{1}{p}}, & \text{if } p \in \mathbb{R},\\ \inf\left\{M \in \mathbb{R} \mid |u| \leq M \ \Delta\text{-almost everywhere on } J\right\}, & \text{if } p = \infty.\end{cases}$$

Here, $J = [a, b)_{\mathbb{T}}$.

Theorem 3.3. *(Hölder's Inequality [5]) Let* θ_i *belong to* $L^{p_i}_{\Delta}(J)$ *with* $p_i > 1$ *for* $i = 1, 2, \ldots, n$*, and suppose that* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$*. Then the product* $\prod_{i=1}^{n} \theta_i$ *is an element of* $L^1_{\Delta}(J)$*, and the following inequality is valid:*

$$\left\| \prod_{i=1}^{n} \theta_i \right\|_1 \leq \prod_{i=1}^{n} \|\theta_i\|_{p_i}.$$

Moreover, if $u \in L^1_{\Delta}(J)$ *and* $v \in L^{\infty}_{\Delta}(J)$*, then their product* uv *is in* $L^1_{\Delta}(J)$*, and the inequality holds:*

$$\|uv\|_1 \leq \|u\|_1 \|v\|_{\infty}.$$

Three cases for $\varepsilon_i \in L^{p_i}_{\Delta}(0,1)_{\mathbb{T}}$ are as follows:

$$\sum_{i=1}^{n} \frac{1}{p_i} < 1, \ \sum_{i=1}^{n} \frac{1}{p_i} = 1, \ \sum_{i=1}^{n} \frac{1}{p_i} > 1.$$

Firstly, we seek countably many positive solutions for the case $\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

Theorem 3.4. *Assume* (H_1)–(H_3) *hold and the sequence* $\{\gamma_j\}_{r=1}^{\infty}$ *satisfies* $z_{j+1} < \gamma_j < z_j$*. Let* $\{\chi_j\}_{r=1}^{\infty}$ *and* $\{\lambda_j\}_{r=1}^{\infty}$ *be such that*

$$\chi_{j+1} < \aleph(\gamma_j)\lambda_j < \lambda_j < \kappa\lambda_j < \chi_j, \ r \in \mathbb{N},$$

where

$$\kappa = \max\left\{ \left[\aleph(\gamma_1) \prod_{i=1}^{n} \delta_i \int_{\gamma_1}^{1-\gamma_1} Q(y,y)\Delta y \right]^{-1}, \ 1 \right\}.$$

Assume that $\hbar_i$ *satisfies*

(C_1) $\hbar_i(w) \leq \phi(\mathfrak{L}_1 \chi_j) \ \forall \ z \in (0,1)_{\mathbb{T}}, \ 0 \leq w \leq \chi_j$,
where

$$\mathfrak{L}_1 < \left[\|Q\|_{L^q_{\Delta}} \prod_{i=1}^{n} \left\|\phi^{-1}(\varepsilon_i)\right\|_{L^{p_i}_{\Delta}} \right]^{-1},$$

(C_2) $\hbar_i(w) \geq \kappa\lambda_j \ \forall \ z \in [\gamma_j, 1-\gamma_j]_{\mathbb{T}}, \ \aleph(\gamma_j)\lambda_j \leq w \leq \lambda_j$.

The iterative boundary value problem (1.1)-(1.2) then has a countable set of solutions $\{(\mathbf{w}_1^{[r]}, \mathbf{w}_2^{[r]}, \ldots, \mathbf{w}_m^{[r]})\}_{r=1}^{\infty}$ *such that* $\mathbf{w}_i^{[r]}(z) \geq 0$ *on* $(0,1)_{\mathbb{T}}$ *for* $i = 1, 2, \ldots, m$ *and* $r \in \mathbb{N}$*.*

Proof. Let $M_{1,r} = \{w \in Y : \|w\| < \chi_r\}$, $M_{2,r} = \{w \in Y : \|w\| < \lambda_j\}$ be open subsets of the space Y. Consider the sequence $\{\gamma_j\}_{r=1}^{\infty}$ given in the hypothesis, where it is noted that

$$z^* < z_{j+1} < \gamma_j < z_j < \frac{1}{2}$$

for all integers $r \in \mathbb{N}$. For each natural number r, define the cone L_{γ_j} by

$$L_{\gamma_j} = \left\{ w \in Y : w(z) \geq 0, \min_{z \in [\gamma_j, 1-\gamma_j]_{\mathbb{T}}} w(z) \geq \aleph(\gamma_j)\|w(z)\| \right\}.$$

Let $\mathtt{w}_1 \in \mathrm{L}_{\gamma_{\mathrm{j}}} \cap \partial\mathtt{M}_{1,r}$. Then, $\mathtt{w}_1(\mathtt{y}) \leq \chi_{\mathrm{j}} = \|\mathtt{w}_1\|$ for all $\mathtt{y} \in (0,1)_{\mathbb{T}}$. By (C_1) and for $\mathtt{y}_{m-1} \in (0,1)_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \mathrm{Q}(\mathtt{y}_{m-1}, \mathtt{y}_m)\phi^{-1}\big(\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big)\Delta\mathtt{y}_m &\leq \int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_m), \mathtt{y}_m)\phi^{-1}\big(\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big)\Delta\mathtt{y}_m \\ &\leq \mathfrak{L}_1\chi_{\mathrm{j}} \int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_m), \mathtt{y}_m)\phi^{-1}\left(\prod_{i=1}^{n}\varepsilon_i(\mathtt{y}_m)\right)\Delta\mathtt{y}_m \\ &\leq \mathfrak{L}_1\chi_{\mathrm{j}} \int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_m), \mathtt{y}_m)\prod_{i=1}^{n}\phi^{-1}(\varepsilon_i(\mathtt{y}_m))\Delta\mathtt{y}_m.\end{aligned}$$

We can find a value q greater than 1 such that

$$\frac{1}{q} + \sum_{i=1}^{n}\frac{1}{p_i} = 1.$$

Consequently, it follows that

$$\begin{aligned}\int_0^1 \mathrm{Q}(\mathtt{y}_{m-1}, \mathtt{y}_m)\phi^{-1}\big(\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big)\Delta\mathtt{y}_m &\leq \mathfrak{L}_1\chi_{\mathrm{j}}\|\mathrm{Q}\|_{L^q_\Delta}\left\|\prod_{i=1}^{n}\phi^{-1}(\varepsilon_i)\right\|_{L^{p_i}_\Delta} \\ &\leq \mathfrak{L}_1\chi_{\mathrm{j}}\|\mathrm{Q}\|_{L^q_\Delta}\prod_{i=1}^{n}\big\|\phi^{-1}(\varepsilon_i)\big\|_{L^{p_i}_\Delta} \leq \chi_{\mathrm{j}}.\end{aligned}$$

In a comparable way, for $\mathtt{y}_{m-2} \in (0,1)_{\mathbb{T}}$, we conclude

$$\begin{aligned}&\int_0^1 \mathrm{Q}(\mathtt{y}_{m-2}, \mathtt{y}_{m-1})\phi^{-1}\left(\varepsilon(\mathtt{y}_{m-1})\hbar_{m-1}\left(\int_0^1 \mathrm{Q}(\mathtt{y}_{m-1}, \mathtt{y}_m)\phi^{-1}\big(\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big)\Delta\mathtt{y}_m\right)\right)\Delta\mathtt{y}_{m-1} \\ &\leq \int_0^1 \mathrm{Q}(\mathtt{y}_{m-2}, \mathtt{y}_{m-1})\phi^{-1}\big(\varepsilon(\mathtt{y}_{m-1})\hbar_{m-1}(\chi_{\mathrm{j}})\big)\Delta\mathtt{y}_{m-1} \\ &\leq \int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_{m-1}), \mathtt{y}_{m-1})\phi^{-1}\big(\varepsilon(\mathtt{y}_{m-1})\hbar_{m-1}(\chi_{\mathrm{j}})\big)\Delta\mathtt{y}_{m-1} \\ &\leq \mathfrak{L}_1\chi_{\mathrm{j}}\int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_{m-1}), \mathtt{y}_{m-1})\phi^{-1}\left(\prod_{i=1}^{n}\varepsilon_i(\mathtt{y}_{m-1})\right)\Delta\mathtt{y}_{m-1} \\ &\leq \mathfrak{L}_1\chi_{\mathrm{j}}\int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_{m-1}), \mathtt{y}_{m-1})\prod_{i=1}^{n}\phi^{-1}(\varepsilon_i(\mathtt{y}_{m-1}))\Delta\mathtt{y}_{m-1} \\ &\leq \mathfrak{L}_1\chi_{\mathrm{j}}\|\mathrm{Q}\|_{L^q_\Delta}\prod_{i=1}^{n}\big\|\phi^{-1}(\varepsilon_i)\big\|_{L^{p_i}_\Delta} \leq \chi_{\mathrm{j}}.\end{aligned}$$

Proceeding further with the bootstrapping technique, we obtain

$$\begin{aligned}(\mho\mathtt{w}_1)(\mathtt{z}) = &\int_0^1 \mathrm{Q}(\mathtt{z}, \mathtt{y}_1)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_1)\hbar_1\left(\int_0^1 \mathrm{Q}(\mathtt{y}_1, \mathtt{y}_2)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_2)\hbar_2\left(\int_0^1 \mathrm{Q}(\mathtt{y}_2, \mathtt{y}_3)\cdots\right.\right.\right. \\ &\left.\left.\left.\times\hbar_{m-1}\left(\int_0^1 \mathrm{Q}(\mathtt{y}_{m-1}, \mathtt{y}_m)\phi^{-1}\big[\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big]\Delta\mathtt{y}_m\right)\cdots\Delta\mathtt{y}_3\right)\Delta\mathtt{y}_2\right)\Bigg]\Delta\mathtt{y}_1 \\ &\leq \chi_{\mathrm{j}}.\end{aligned}$$

Since $\chi_{\mathrm{j}} = \|\mathtt{w}_1\|$ for $\mathtt{w}_1 \in \mathrm{L}_{\gamma_{\mathrm{j}}} \cap \partial\mathtt{M}_{1,r}$, we get

$$\|\mho\mathtt{w}_1\| \leq \|\mathtt{w}_1\|. \tag{3.1}$$

Let $z \in [\gamma_j, 1-\gamma_j]_{\mathbb{T}}$. Then,

$$\lambda_j = \|w_1\| \geq w_1(z) \geq \min_{z\in[\gamma_j,1-\gamma_j]_{\mathbb{T}}} w_1(z) \geq \aleph(\gamma_j)\,\|w_1\| \geq \aleph(\gamma_j)\lambda_j.$$

By (C_2) and for $y_{m-1} \in [\gamma_j, 1-\gamma_j]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 Q(y_{m-1}, y_m)\phi^{-1}\big(\varepsilon(y_m)\hbar_m(w_1(y_m))\big)\Delta y_m &\geq \int_{\gamma_j}^{1-\gamma_j} Q(y_{m-1}, y_m)\phi^{-1}\big(\varepsilon(y_m)\hbar_m(w_1(y_m))\big)\Delta y_m\\ &\geq \aleph(\gamma_j)\kappa\lambda_j \int_{\gamma_j}^{1-\gamma_j} Q(\sigma(y_m), y_m)\phi^{-1}(\varepsilon(y_m))\Delta y_m\\ &\geq \aleph(\gamma_j)\kappa\lambda_j \int_{\gamma_j}^{1-\gamma_j} Q(\sigma(y_m), y_m)\prod_{i=1}^{n}\phi^{-1}(\varepsilon_i(y_m))\Delta y_m\\ &\geq \aleph(\gamma_1)\kappa\lambda_j \prod_{i=1}^{n}\delta_i \int_{\gamma_1}^{1-\gamma_1} Q(\sigma(y_m), y_m)\Delta y_m\\ &\geq \lambda_j.\end{aligned}$$

Proceeding further with the bootstrapping technique, we obtain

$$\begin{aligned}(\mho w_1)(z) &= \int_0^1 Q(z, y_1)\phi^{-1}\bigg[\varepsilon(y_1)\hbar_1\bigg(\int_0^1 Q(y_1, y_2)\phi^{-1}\bigg[\varepsilon(y_2)\hbar_2\bigg(\int_0^1 Q(y_2, y_3)\cdots\\ &\times\hbar_{m-1}\bigg(\int_0^1 Q(y_{m-1}, y_m)\phi^{-1}\big[\varepsilon(y_m)\hbar_m(w_1(y_m))\big]\Delta y_m\bigg)\cdots\Delta y_3\bigg)\Delta y_2\bigg)\bigg]\Delta y_1\\ &\geq \lambda_j.\end{aligned}$$

Thus, if $w_1 \in L_{\gamma_j} \cap \partial L_{2,r}$, then

$$\|\mho w_1\| \geq \|w_1\|. \tag{3.2}$$

It is clear that $0 \in M_{2,k} \subset \overline{M}_{2,k} \subset M_{1,k}$. From equations (3.4) and (3.2), Theorem 3.1 implies that the mapping $\mho$ possesses a fixed point $w_1^{[r]} \in L_{\gamma_j} \cap (\overline{M}_{1,r} \backslash M_{2,r})$ such that $w_1^{[r]}(z) \geq 0$ for $(0,1)_{\mathbb{T}}$, $r \in \mathbb{N}$. Subsequently, by defining $w_{m+1} = w_1$, we derive countably many positive solutions $\{(w_1^{[r]}, w_2^{[r]}, \ldots, w_m^{[r]})\}_{r=1}^{\infty}$ of (1.1)-(1.2) obtained as

$$w_i(z) = \int_0^1 Q(z, y)\phi^{-1}\big(\varepsilon(y)\hbar_i(w_{i+1}(y))\big)\Delta y,\ z \in (0,1)_{\mathbb{T}},\ i = m, m-1, \cdots, 1.$$

The proof is completed. □

For the condition $\sum_{i=1}^{n}\frac{1}{p_i} = 1$, the following theorem can be stated.

Theorem 3.5. *Assume that conditions (H_1) through (H_3) are satisfied. Let $\{\gamma_j\}_{r=1}^{\infty}$ represent a sequence where γ_j lies in the interval (z_{j+1}, z_j). Furthermore, let the sequences $\{\chi_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ be defined as*

$$\chi_{j+1} < \aleph(\gamma_j)\lambda_j < \lambda_j < \kappa\lambda_j < \chi_j,\ r \in \mathbb{N},$$

where

$$\kappa = \max\left\{\left[\aleph(\gamma_1)\prod_{i=1}^{n}\delta_i\int_{\gamma_1}^{1-\gamma_1} Q(y, y)\Delta y\right]^{-1},\ 1\right\}.$$

Assume that $\hbar_i$ satisfies (C_2) and

(C_3) $\hbar_j(\mathtt{w}) \le \phi(\mathfrak{L}_2\chi_{\mathrm{j}})\ \forall\ \mathtt{z} \in (0,1)_{\mathbb{T}},\ 0 \le \mathtt{w} \le \chi_{\mathrm{j}}$, *where*

$$\mathfrak{L}_2 < \min\left\{\left[\|\mathrm{Q}\|_{L^\infty_\Delta}\prod_{i=1}^{n}\left\|\phi^{-1}(\varepsilon_i)\right\|_{L^{p_i}_\Delta}\right]^{-1}, \kappa\right\}.$$

The iterative boundary value problem (1.1)-(1.2) *possesses a countable collection of solutions* $\{(\mathbf{w}_1^{[r]}, \mathbf{w}_2^{[r]}, \dots, \mathbf{w}_m^{[r]})\}_{r=1}^{\infty}$ *with* $\mathbf{w}_i^{[r]}(z) \ge 0$ *for* $(0,1)_{\mathbb{T}}$, *where* $i = 1, 2, \dots, m$ *and* $r \in \mathbb{N}$.

Proof. For a given r, let $\mathtt{M}_{1,r}$ be defined as outlined in the demonstration of Theorem 3.4 and consider $\mathtt{w}_1 \in \mathrm{L}_{\gamma_{\mathrm{j}}} \cap \partial\mathtt{M}_{2,r}$. Again

$$\mathtt{w}_1(\mathtt{y}) \le \chi_{\mathrm{j}} = \|\mathtt{w}_1\|,$$

for all $\mathtt{y} \in (0,1)_{\mathbb{T}}$. By (C_3) and for $\mathtt{y}_{m-1} \in (0,1)_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \mathrm{Q}(\mathtt{y}_{m-1},\mathtt{y}_m)\phi^{-1}\big(\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big)\Delta\mathtt{y}_m &\le \int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_m),\mathtt{y}_m)\phi^{-1}\big(\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big)\Delta\mathtt{y}_m\\
&\le \mathfrak{L}_2\chi_{\mathrm{j}}\int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_m),\mathtt{y}_m)\phi^{-1}\left(\prod_{i=1}^{n}\varepsilon_i(\mathtt{y}_m)\right)\Delta\mathtt{y}_m\\
&\le \mathfrak{L}_2\chi_{\mathrm{j}}\int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_m),\mathtt{y}_m)\prod_{i=1}^{n}\phi^{-1}(\varepsilon_i(\mathtt{y}_m))\Delta\mathtt{y}_m\\
&\le \mathfrak{L}_2\chi_{\mathrm{j}}\|\mathrm{Q}\|_{L^\infty_\Delta}\left\|\prod_{i=1}^{n}\phi^{-1}(\varepsilon_i)\right\|_{L^{p_i}_\Delta}\\
&\le \mathfrak{L}_2\chi_{\mathrm{j}}\|\mathrm{Q}\|_{L^\infty_\Delta}\prod_{i=1}^{n}\left\|\phi^{-1}(\varepsilon_i)\right\|_{L^{p_i}_\Delta} \le \chi_{\mathrm{j}}.\end{aligned}$$

By a comparable reasoning, for $\mathtt{y}_{m-2} \in [0,1]_{\mathbb{T}}$, it can be derived that

$$\begin{aligned}&\int_0^1 \mathrm{Q}(\mathtt{y}_{m-2},\mathtt{y}_{m-1})\phi^{-1}\left(\varepsilon(\mathtt{y}_{m-1})\hbar_{m-1}\left(\int_0^1 \mathrm{Q}(\mathtt{y}_{m-1},\mathtt{y}_m)\phi^{-1}\big(\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big)\Delta\mathtt{y}_m\right)\right)\Delta\mathtt{y}_{m-1}\\
&\le \int_0^1 \mathrm{Q}(\mathtt{y}_{m-2},\mathtt{y}_{m-1})\phi^{-1}\big(\varepsilon(\mathtt{y}_{m-1})\hbar_{m-1}(\chi_{\mathrm{j}})\big)\Delta\mathtt{y}_{m-1}\\
&\le \int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_{m-1}),\mathtt{y}_{m-1})\phi^{-1}\big(\varepsilon(\mathtt{y}_{m-1})\hbar_{m-1}(\chi_{\mathrm{j}})\big)\Delta\mathtt{y}_{m-1}\\
&\le \mathfrak{L}_2\chi_{\mathrm{j}}\int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_{m-1}),\mathtt{y}_{m-1})\phi^{-1}\left(\prod_{i=1}^{n}\varepsilon_i(\mathtt{y}_{m-1})\right)\Delta\mathtt{y}_{m-1}\\
&\le \mathfrak{L}_2\chi_{\mathrm{j}}\int_0^1 \mathrm{Q}(\sigma(\mathtt{y}_{m-1}),\mathtt{y}_{m-1})\prod_{i=1}^{n}\phi^{-1}(\varepsilon_i(\mathtt{y}_{m-1}))\Delta\mathtt{y}_{m-1}\\
&\le \mathfrak{L}_2\chi_{\mathrm{j}}\|\mathrm{Q}\|_{L^\infty_\Delta}\prod_{i=1}^{n}\left\|\phi^{-1}(\varepsilon_i)\right\|_{L^{p_i}_\Delta} \le \chi_{\mathrm{j}}\end{aligned}$$

Proceeding further with the bootstrapping technique, we obtain

$$\begin{aligned}(\mho\mathtt{w}_1)(\mathtt{z}) = &\int_0^1 \mathrm{Q}(\mathtt{z},\mathtt{y}_1)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_1)\hbar_1\Bigg(\int_0^1 \mathrm{Q}(\mathtt{y}_1,\mathtt{y}_2)\phi^{-1}\Bigg[\varepsilon(\mathtt{y}_2)\hbar_2\Bigg(\int_0^1 \mathrm{Q}(\mathtt{y}_2,\mathtt{y}_3)\cdots\\
&\times\hbar_{m-1}\left(\int_0^1 \mathrm{Q}(\mathtt{y}_{m-1},\mathtt{y}_m)\phi^{-1}\big[\varepsilon(\mathtt{y}_m)\hbar_m(\mathtt{w}_1(\mathtt{y}_m))\big]\Delta\mathtt{y}_m\right)\cdots\Delta\mathtt{y}_3\Bigg)\Delta\mathtt{y}_2\Bigg)\Bigg]\Delta\mathtt{y}_1\\
&\le \chi_{\mathrm{j}}.\end{aligned}$$

Given that $\chi_j = \|w_1\|$ for $w_1 \in L_{\gamma_j} \cap \partial M_{1,r}$, so

$$\|\mho w_1\| \le \|w_1\|. \tag{3.3}$$

Next, define $M_{2,r} = \{w_1 \in Y : \|w_1\| < \lambda_j\}$. Let $w_1 \in L_{\gamma_j} \cap \partial M_{2,r}$ and let $y \in [\gamma_j,\, 1-\gamma_j]_{\mathbb{T}}$. The reasoning that leads to (3.2) can be applied here as well. Therefore, the theorem is established. $\square$

Finally, consider the scenario where $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Theorem 3.6. *Assume that conditions (H_1) through (H_3) are satisfied. Let $\{\gamma_j\}_{r=1}^{\infty}$ denote a sequence such that $\gamma_j \in (z_{j+1}, z_j)$. Furthermore, let the sequences $\{\chi_j\}_{r=1}^{\infty}$ and $\{\lambda_j\}_{r=1}^{\infty}$ be defined such that*

$$\chi_{j+1} < \aleph(\gamma_j)\lambda_j < \lambda_j < \kappa\lambda_j < \chi_j, \ r \in \mathbb{N},$$

where

$$\kappa = \max\left\{ \left[\aleph(\gamma_1) \prod_{i=1}^{n} \delta_i \int_{\gamma_1}^{1-\gamma_1} Q(y,y)\Delta y \right]^{-1}, 1 \right\}.$$

Assume that $\hbar_i$ satisfies (C_2) and

(C_4) $\hbar_j(w) \le \phi(\mathfrak{L}_3\chi_j)$ $\forall$ $z \in (0,1)_{\mathbb{T}}$, $0 \le w \le \chi_j$,
where

$$\mathfrak{L}_3 < \min\left\{ \left[\|Q\|_{L^\infty_\Delta} \prod_{i=1}^{n} \left\|\phi^{-1}(\varepsilon_i)\right\|_{L^1_\Delta} \right]^{-1}, \kappa \right\}.$$

The BVP (1.1)-(1.2) yields a countably infinite collection of solutions $\{(\mathbf{w}_1^{[r]}, \mathbf{w}_2^{[r]}, \ldots, \mathbf{w}_m^{[r]})\}_{r=1}^{\infty}$ such that $\mathbf{w}_i^{[r]}(z) \ge 0$ on $(0,1)_{\mathbb{T}}$ for $i = 1, 2, \ldots, m$ and $r \in \mathbb{N}$.

Proof. For a given r, define $M_{1,r}$ as stated in the proof of Theorem 3.4, and consider $w_1 \in L_{\gamma_j} \cap \partial M_{2,r}$. Again

$$w_1(y) \le \chi_j = \|w_1\|,$$

for all $y \in (0,1)_{\mathbb{T}}$. By (C_3) and for $y_{m-1} \in (0,1)_{\mathbb{T}}$, we have

$$\begin{aligned}
\int_0^1 Q(y_{m-1}, y_m)\phi^{-1}\big(\varepsilon(y_m)\hbar_m(w_1(y_m))\big)\Delta y_m &\le \int_0^1 Q(\sigma(y_m), y_m)\phi^{-1}\big(\varepsilon(y_m)\hbar_m(w_1(y_m))\big)\Delta y_m \\
&\le \mathfrak{L}_3\chi_j \int_0^1 Q(\sigma(y_m), y_m)\phi^{-1}\left(\prod_{i=1}^{n} \varepsilon_i(y_m)\right)\Delta y_m \\
&\le \mathfrak{L}_3\chi_j \int_0^1 Q(\sigma(y_m), y_m)\prod_{i=1}^{n} \phi^{-1}(\varepsilon_i(y_m))\Delta y_m \\
&\le \mathfrak{L}_3\chi_j \|Q\|_{L^\infty_\Delta} \left\| \prod_{i=1}^{n} \phi^{-1}(\varepsilon_i) \right\|_{L^1_\Delta} \\
&\le \mathfrak{L}_3\chi_j \|Q\|_{L^\infty_\Delta} \prod_{i=1}^{n} \left\|\phi^{-1}(\varepsilon_i)\right\|_{L^1_\Delta} \le \chi_j.
\end{aligned}$$

By a comparable reasoning, for $y_{m-2} \in [0,1]_{\mathbb{T}}$, it can be derived that

$$\begin{aligned}
&\int_0^1 Q(y_{m-2}, y_{m-1})\phi^{-1}\bigg(\varepsilon(y_{m-1})\hbar_{m-1}\bigg(\int_0^1 Q(y_{m-1}, y_m)\phi^{-1}\big(\varepsilon(y_m)\hbar_m(w_1(y_m))\big)\Delta y_m\bigg)\bigg)\Delta y_{m-1}\\
&\quad\le \int_0^1 Q(y_{m-2}, y_{m-1})\phi^{-1}\big(\varepsilon(y_{m-1})\hbar_{m-1}(\chi_j)\big)\Delta y_{m-1}\\
&\quad\le \int_0^1 Q(\sigma(y_{m-1}), y_{m-1})\phi^{-1}\big(\varepsilon(y_{m-1})\hbar_{m-1}(\chi_j)\big)\Delta y_{m-1}\\
&\quad\le \mathfrak{L}_3\chi_j \int_0^1 Q(\sigma(y_{m-1}), y_{m-1})\phi^{-1}\bigg(\prod_{i=1}^n \varepsilon_i(y_{m-1})\bigg)\Delta y_{m-1}\\
&\quad\le \mathfrak{L}_3\chi_j \int_0^1 Q(\sigma(y_{m-1}), y_{m-1})\prod_{i=1}^n \phi^{-1}(\varepsilon_i(y_{m-1}))\Delta y_{m-1}\\
&\quad\le \mathfrak{L}_3\chi_j \|Q\|_{L^\infty_\Delta}\prod_{i=1}^n \big\|\phi^{-1}(\varepsilon_i)\big\|_{L^1_\Delta} \le \chi_j
\end{aligned}$$

Proceeding further with the bootstrapping technique, we obtain

$$\begin{aligned}
(\mho w_1)(z) &= \int_0^1 Q(z, y_1)\phi^{-1}\bigg[\varepsilon(y_1)\hbar_1\bigg(\int_0^1 Q(y_1, y_2)\phi^{-1}\bigg[\varepsilon(y_2)\hbar_2\bigg(\int_0^1 Q(y_2, y_3)\cdots\\
&\times\hbar_{m-1}\bigg(\int_0^1 Q(y_{m-1}, y_m)\phi^{-1}\big[\varepsilon(y_m)\hbar_m(w_1(y_m))\big]\Delta y_m\bigg)\cdots\Delta y_3\bigg)\Delta y_2\bigg)\bigg]\Delta y_1\\
&\le \chi_j.
\end{aligned}$$

Given that $\chi_j = \|w_1\|$ for $w_1 \in L_{\gamma_j} \cap \partial M_{1,r}$, so

$$\|\mho w_1\| \le \|w_1\|. \tag{3.4}$$

Next, let us define $M_{2,r} = \{w_1 \in Y : \|w_1\| < \lambda_j\}$. Consider $w_1 \in L_{\gamma_j} \cap \partial M_{2,r}$ and let $y \in [\gamma_j, 1-\gamma_j]_{\mathbb{T}}$. The reasoning that leads to (3.2) can be applied in this situation as well. Therefore, the theorem is concluded. □

4. Examples

In order to validate the theoretical findings, we present an example of a boundary value problem on the interval $\mathbb{T} = [0,1]$.

Example 4.1.

$$\begin{cases}\phi\big(w_i^{\Delta^2}(z)\big) + \varepsilon(z)\hbar_i(w_{i+1}(z)) = 0, i = 1, 2,\\ w_3(z) = w_1(z),\end{cases} \tag{4.1}$$

$$\begin{cases} w_i(0) - w_i'(0) = 0,\\ w_i(1) + w_i'(1) = 0,\end{cases} \tag{4.2}$$

where $\phi(z) = \frac{z^3}{1+z^2}$, if $z \le 0$ and $\phi(z) = z^2$, otherwise, and let

$$\varepsilon(z) = \varepsilon_1(z) = \frac{1}{|z - \frac{1}{4}|^{\frac{1}{2}}},$$

$$\begin{cases} \hbar_1(\mathtt{w}) = \hbar_2(\mathtt{w}) = \frac{21}{20}\times 10^{-4}, & \mathtt{w}\in(10^{-4},+\infty),\\ \frac{44\times 10^{-(4r+3)}-\frac{21}{20}\times 10^{-4r}}{10^{-(4r+3)}-10^{-4r}}(\mathtt{w}-10^{-4r})+\frac{21}{20}\times 10^{-8r}, & \mathtt{w}\in\left[10^{-(4r+3)},10^{-4r}\right],\\ 44\times 10^{-(4r+3)}, & \mathtt{w}\in\left(\frac{21}{20}\times 10^{-(4r+3)},10^{-(4r+3)}\right),\\ \frac{44\times 10^{-(4r+3)}-\frac{21}{20}\times 10^{-8r}}{\frac{21}{20}\times 10^{-(4r+3)}-10^{-(4r+4)}}(\mathtt{w}-10^{-(4r+4)})+\frac{21}{20}\times 10^{-8r}, & \mathtt{w}\in\left(10^{-(4r+4)},\frac{21}{20}\times 10^{-(4r+3)}\right],\\ 0, & \mathtt{w}=0,\end{cases}$$

Let

$$\mathtt{z}_\mathtt{j} = \frac{16}{33} - \sum_{k=1}^{r}\frac{1}{2(k+1)^4},\ \gamma_\mathtt{j} = \frac{1}{2}(\mathtt{z}_\mathtt{j}+\mathtt{z}_{\mathtt{j}+1}),\ r = 1,2,3,\cdots,$$

then

$$\gamma_1 = \frac{12845}{28512},\ \mathtt{z}_{\mathtt{j}+1} < \gamma_\mathtt{j} < \mathtt{z}_\mathtt{j},\ \gamma_\mathtt{j} > \frac{1}{2}.$$

Therefore,

$$\aleph(\gamma_\mathtt{j}) = \frac{\gamma_\mathtt{j}+1}{2} > \frac{1}{2},\ r = 1,2,3,\cdots.$$

It is clear that

$$\mathtt{z}_1 = \frac{479}{1056} < \frac{1}{2}.$$

Since $\sum_{x=1}^{\infty}\frac{1}{x^4} = \frac{\pi^4}{90}$ and $\sum_{x=1}^{\infty}\frac{1}{x^2} = \frac{\pi^2}{6}$, it follows that

$$\mathtt{z}^* = \lim_{r\to\infty}\mathtt{z}_\mathtt{j} = \frac{16}{33} - \sum_{k=1}^{\infty}\frac{1}{2(r+1)^4} = \frac{65}{66} - \frac{\pi^4}{180} = 0.4436868677,$$

$$\phi^{-1}(\varepsilon_1)\in L^p[0,1]\ \text{ for all }\ 0<p<2,\ \text{ and }\ \delta_1 = \frac{1}{\sqrt{3}},$$

$$\aleph(\gamma_1) = \frac{\gamma_1+1}{2} = 0.7252560326.$$

$$\begin{aligned}\int_{\gamma_1}^{1-\gamma_1}\mathrm{Q}(\sigma(\mathrm{y}),\mathrm{y})\Delta\mathrm{y} &= \int_{\frac{12845}{28512}}^{1-\frac{12845}{28512}}\frac{(2-\sigma(\mathrm{y}))(1+\sigma(\mathrm{y}))}{3}d\mathrm{y}\\ &= \int_{\frac{12845}{28512}}^{1-\frac{12845}{28512}}\frac{(1-\mathrm{y})(2+\mathrm{y})}{3}d\mathrm{y}\\ &= 0.05481140313.\end{aligned}$$

Thus, we get

$$\begin{aligned}\kappa &= \max\left\{\left[\aleph(\gamma_1)\prod_{i=1}^{n}\delta_i\int_{\gamma_1}^{1-\gamma_1}\mathrm{Q}(\sigma(\mathrm{y}),\mathrm{y})\nabla\mathrm{y}\right]^{-1},\ 1\right\}\\ &= \max\left\{\frac{1}{0.02295100154},1\right\}\\ &= 43.57108330.\end{aligned}$$

Next, let $0<\mathfrak{a}<1$ be fixed. Then $\varepsilon_1\in L^{1+\mathfrak{a}}[0,1]$ and

$$\prod_{i=1}^{n}\left\|\phi^{-1}(\varepsilon_i)\right\|_{1+\mathfrak{a}} = \left[\frac{1}{3-\mathfrak{a}}\left(3^{\frac{3-a}{4}}+1\right)2^{\frac{1+a}{2}}\right]^{\frac{1}{1+a}}.$$

and also $\|\mathrm{Q}\|_\infty = \frac{5}{9}$. So, for $0 < \mathfrak{a} < 1$, we have

$$1.089000601 \leq \left[\|\mathrm{Q}\|_\infty \prod_{i=1}^{n} \left\|\phi^{-1}(\varepsilon_i)\right\|_{L_\Delta^{p_i}} \right]^{-1} \leq 1.164314195.$$

Taking $\mathfrak{L}_1 = \frac{27}{25}$. In addition if we take

$$\chi_j = 10^{-4r},\ \lambda_j = 10^{-(4r+3)},$$

then

$$\chi_{j+1} = 10^{-(4r+4)} < \frac{1}{2} \times 10^{-(4r+3)} < \aleph(\gamma_j)\lambda_j$$
$$< \lambda_j = 10^{-(4r+3)} < \chi_j = 10^{-4r},$$

$\kappa\lambda_j = 43.57108330 \times 10^{-(4r+3)} < \frac{27}{25} \times 10^{-4r} = \mathfrak{L}_1\chi_j$, $r = 1, 2, 3, \cdots$, and $\hbar_1, \hbar_2$ satisfies the following growth conditions:

$$\hbar_1(\mathtt{w}) = \hbar_2(\mathtt{w}) \leq \mathfrak{L}_1\chi_j = \frac{27}{25} \times 10^{-4r},\ \mathtt{w} \in \left[0, 10^{-4r}\right]$$

$$\hbar_1(\mathtt{w}) = \hbar_2(\mathtt{w}) \geq \kappa\lambda_j = 43.57108330 \times 10^{-(4r+3)},\ \mathtt{w} \in \left[\frac{27}{25} \times 10^{-(4r+3)}, 10^{-(4r+3)}\right].$$

Thus, all the requirements of Theorem 3.4 are met. Consequently, according to Theorem 3.4, the iterative boundary value problem (1.1) possesses countably many solutions $\{(\mathtt{w}_1^{[r]}, \mathtt{w}_2^{[r]})\}_{r=1}^{\infty}$ such that $\mathtt{w}_i^{[r]}(\mathtt{z}) \geq 0$ on $[0, 1]$, where $i = 1, 2$ and $r \in \mathbb{N}$.

5. Conclusion and Future Work

This paper presents a dynamic model for heat transfer in porous media using the Increasing Homeomorphic and Positive Homomorphism Operator (IHPHO) on time scales. The proposed model captures both continuous and discrete thermal processes, providing a unified approach for temperature evolution in systems like thermal storage and cyclic heat exchangers. The iterative process efficiently describes how heat accumulates and spreads over successive cycles, making the model applicable to real-world engineering problems. Future research could focus on:

- Developing numerical methods for solving the model and simulating real-world applications.
- Extending the framework to handle nonlinear heat sources and more complex thermal behaviors.
- Exploring the model's application in optimizing energy systems and heat exchanger designs.

This work sets the foundation for more advanced modeling of heat transfer in porous structures and opens avenues for further improvements and practical applications.

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