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Comprehensive analysis of three seventh-order KdV-type equations

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Abstract

In this paper, we employ the generalized $\exp(-\psi(\xi))$-expansion approach to derive analytical solutions for three specific cases of the generalized seventh-order KdV equation: the seventh-order Sawada-Kotera-Ito equation, the seventh-order Lax equation, and the seventh-order Kaup-Kupershmidt equation. These equations hold significant importance in the nonlinear sciences. By utilizing this approach, we obtain a variety of new exact traveling wave solutions for the aforementioned nonlinear models. Moreover, we showcase 2D, 3D, contour plots, and density plots to acquire comprehensive representations, using cutting-edge scientific instruments. Our results confirm the effectiveness and practicality of the proposed method in solving the aforementioned problems, as well as other nonlinear evolution equations encountered in the domains of engineering and mathematical physics.

Keywords. Generalized $\exp(-\psi(\xi))$-expansion method, gsKdV equation, Nonlinear evolution equation, Traveling wave solution, Bright soliton solution.

2010 Mathematics Subject Classification. 35C07, 35C08, 35G20, 35Q53.

1. Introduction

Nonlinear evolution equations (NLEEs) are extensively used for describing the evolution of nonlinear wave phenomena in various applied scientific fields, particularly in shallow water theory. To comprehend the physical mechanisms governed by NLEEs, it is crucial to explore exact traveling wave solutions.

Traveling wave solutions play a significant role in understanding the qualitative properties of diverse phenomena and processes in applied sciences and engineering. They enable researchers to design experiments and create suitable natural conditions for studying these phenomena.

Consequently, the search for exact solutions has become a fundamental and important task in studying nonlinear physical phenomena. Several techniques have been proposed by mathematicians and physicists [1, 4, 6, 7, 9, 10, 12, 14–24, 26], among others.

Each method has its own advantages and disadvantages, and there is no standardized and universally effective method capable of solving all types of NLEEs. Therefore, whenever an improvement is made to any of these methods, new solutions can be obtained.

Recently, the generalized $\exp(-\psi(\xi))$-expansion (GEE) method [2, 5, 8, 13] has been introduced to study NLEEs that model physical problems. This method was initially proposed by Hafez and Lu [8] considering the auxiliary equation

$$\psi'(\xi) = p\exp(-\psi(\xi)) + q\exp(\psi(\xi)) + r.$$

The primary objective of this work is to utilize the GEE method to obtain new exact solutions, including bright soliton solutions, dark solitary wave solutions, and multiple dark solitary wave solutions for specific cases of the generalized seventh-order KdV (gsKdV) equation.

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This equation reads

$$\phi_t + a\phi^3\phi_x + b\phi_x^3 + c\phi\phi_x\phi_{2x} + d\phi^2\phi_{3x} + e\phi_{2x}\phi_{3x} + f\phi_x\phi_{4x} + g\phi\phi_{5x} + \phi_{7x} = 0, \tag{1.1}$$

where a, b, c, d, e, f, and g are non-zero arbitrary parameters.

In this equation, $\phi(x,t)$ represents the unknown function dependent on the spatial variable x and the temporal variable t. The subscripts denote partial derivatives with respect to the corresponding variable.

The gsKdV equation finds applications in various fields, where it helps to understand wave propagation phenomena and their interactions in complex systems.

The specific cases of Eq. (1.1) considered in this work are:

1. Seventh-order Sawada-Kotera-Ito (sSKI) equation:
$$(a,b,c,d,e,f,g) = (252, 63, 378, 126, 63, 42, 21).$$
2. Seventh-order Lax (sLax) equation:
$$(a,b,c,d,e,f,g) = (140, 70, 280, 70, 70, 42, 14).$$
3. Seventh-order Kaup-Kupershmidt (sKK) equation:
$$(a,b,c,d,e,f,g) = (2016, 630, 2268, 504, 252, 147, 42).$$

Indeed, various analytical and numerical techniques have been proposed to solve these equations [4, 6, 7, 18, 20, 23]. In recent years, researchers have also explored new techniques to study these equations. For example: Zada et al. [25] applied the optimal auxiliary function method to approximate solutions for the sLax and sSKI equations, Aljahdaly et al. [3] discovered stable and analytical solutions for specific applications of the gsKdV equation using a modified auxiliary equation of the direct algebraic method, Kumar and Saxena [11] employed a new iterative method to obtain analytical solutions for various forms of the gsKdV equation. These studies highlight the continuous efforts to develop novel approaches for solving the gsKdV equation and its specific cases.

The remainder of this paper is organized as follows: Section 2 presents the GEE method, detailing the algorithm used to obtain analytical solutions. In section 3, we implement this method to solve the sSKI, sLax, and sKK equations, deriving various classes of exact solutions based on different parameter settings. Section 4 provides graphical representations (2D, 3D, contour, and density plots) of representative wave solutions to demonstrate the behavior of the solutions. Finally, section 5 concludes the study, emphasizing the effectiveness and novelty of the GEE approach in addressing complex nonlinear wave equations.

2. Algorithm of the GEE method

Let's consider the NLEE as follows:

$$M\left(\phi, \phi_t, \phi_x, \phi_{xx}, \phi_{xt}, \phi_{xxx}, \ldots\right) = 0, \tag{2.1}$$

where M is a polynomial and $\phi(x,t)$ is the unknown function.

The GEE method can be carried out using the following steps:

Step 1. Assume a traveling wave solution of the form $\phi(x,t) = \Phi(\xi)$, where $\xi = x \pm \varpi t$. By substituting this ansatz into Eq. (2.1), we can convert the NLEE into an ordinary differential equation (ODE) for $\Phi = \Phi(\xi)$:

$$N\left(\Phi, \Phi', \Phi'', \Phi''', \ldots\ldots.\right) = 0, \tag{2.2}$$

where N is a function of Φ and its derivatives.

Step 2. In this step, we seek the analytical solutions of Eq. (2.2) in the form:

$$\Phi(\xi) = \sum_{i=0}^{m} \eta_i (\exp(-\psi(\xi)))^i, \;\; \eta_m \neq 0, \tag{2.3}$$

where η_i's are constants to be determined, and $\psi = \psi(\xi)$ satisfies the following ODE:

$$\psi'(\xi) = p\exp(-\psi(\xi)) + q\exp(\psi(\xi)) + r. \tag{2.4}$$

It is worth mentioning that the ODE given by Eq. (2.4) has three general solution types, which depend on the constant values of p, q, and r.

Type 1: For $p = 1$,

$$\psi(\xi) = \begin{cases} \ln\left(\dfrac{-\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}(\xi+k)\right) - r}{2q}\right), & \\ \quad\text{or} & \Theta > 0,\ q \neq 0, \\ \ln\left(\dfrac{-\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}(\xi+k)\right) - r}{2q}\right), & \\ \ln\left(\dfrac{\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}(\xi+k)\right) - r}{2q}\right), & \\ \quad\text{or} & \Theta < 0,\ q \neq 0, \\ \ln\left(\dfrac{\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}(\xi+k)\right) - r}{2q}\right), & \\ \ln\left(\frac{\exp(r(\xi+k))-1}{r}\right), & q = 0,\ r \neq 0, \\ \ln\left(-\frac{2r(\xi+k)+4}{r^2(\xi+k)}\right), & \Theta = 0,\ qr \neq 0, \end{cases}$$

where $\Theta = r^2 - 4q$.

Type 2: For $r = 0$,

$$\psi(\xi) = \begin{cases} \ln\left(\sqrt{\frac{p}{q}}\tan\left(\sqrt{pq}(\xi+k)\right)\right), & p > 0,\ q > 0, \\ \ln\left(-\sqrt{\frac{p}{q}}\cot\left(\sqrt{pq}(\xi+k)\right)\right), & p < 0,\ q < 0, \\ \ln\left(\sqrt{-\frac{p}{q}}\tanh\left(\sqrt{-pq}(\xi+k)\right)\right), & p > 0,\ q < 0, \\ \ln\left(-\sqrt{-\frac{p}{q}}\coth\left(\sqrt{-pq}(\xi+k)\right)\right), & p < 0,\ q > 0. \end{cases}$$

Type 3: For $q = 0$ and $r = 0$,

$$\psi(\xi) = \ln\left(p(\xi+k)\right).$$

For all types, k is the integrating constant.

Step 3. The value of m can be obtained by balancing the higher-order derivative term with the highest-order nonlinearity term given in Eq. (2.2).

Step 4. Substituting Eq. (2.3) into Eq. (2.2) and using Eq. (2.4), we obtain an algebraic system of equations for η_i, p, q, r, and ϖ. By solving this system, we can find the exact solutions of the NLEEs.

3. Applications of the method

In this part, we will use the GEE method to solve three well-known nonlinear partial differential equations in shallow water. These equations include the sSKI equation, the sLax equation, and the sKK equation. All of these equations are special forms of the gsKdV equation, which is commonly used to describe physical phenomena in fluid mechanics.

3.1. **The sSKI equation.** Consider the sSKI equation that has the form

$$\phi_t + 252\phi^3\phi_x + 63\phi_x^3 + 378\phi\phi_x\phi_{2x} + 126\phi^2\phi_{3x} + 63\phi_{2x}\phi_{3x} + 42\phi_x\phi_{4x} + 21\phi\phi_{5x} + \phi_{7x} = 0. \tag{3.1}$$

Using $\phi(x,t) = \varPhi(\xi)$ and $\xi = x - \varpi t$ reduces this equation to a nonlinear ODE

$$-\varpi\varPhi' + 252\varPhi^3\varPhi' + 63\left(\varPhi'\right)^3 + 378\varPhi\varPhi'\varPhi'' + 126\varPhi^2\varPhi''' + 63\varPhi''\varPhi''' + 42\varPhi'\varPhi^{(4)} + 21\varPhi\varPhi^{(5)} + \varPhi^{(7)} = 0. \tag{3.2}$$

By homogeneous balance we get $m = 2$. The solution of (3.2) can be described as

$$\varPhi(\xi) = \sum_{i=0}^{2}\eta_i(\exp(-\psi(\xi)))^i, \ \ \eta_2 \neq 0, \tag{3.3}$$

where $\psi(\xi)$ satisfies the ODE (2.4), η_i's are unknown constants that need to be identified.

By substituting (3.3) into (3.2) and using (2.4), and then setting the coefficients of $(\exp(-\psi(\xi)))^i$ equal to zero, we obtain an algebraic system of equations. For the sake of simplicity, this system is overlooked. Solving this system yields the following solution sets:

Set 1.

$$\begin{aligned}\varpi &= -\frac{4}{3}(r^2 - 4pq)^3,\\ \eta_0 &= -\frac{1}{3}(r^2 + 8pq), \quad \eta_1 = -4pr, \quad \eta_2 = -4p^2.\end{aligned}$$

Set 2.

$$\begin{aligned}\varpi =& (r^2 - 4pq)^3 + 21(2pq + \eta_0)(r^2 - 4pq)^2 + 126(r^2 + 2\eta_0)(2pq + \eta_0)^2,\\ \eta_0 =& \eta_0, \quad \eta_1 = -2pr, \quad \eta_2 = -2p^2.\end{aligned}$$

According to set 1 and set 2, the solutions of the sSKI equation result in the following form:

For Set 1:

Case 1.1. When $p = 1$, $q \neq 0$, $\varTheta = r^2 - 4q > 0$,

$$\phi_1(x,t) = -\frac{1}{3}(r^2 + 8q) + \frac{8qr}{\sqrt{\varTheta}\tanh\left(\frac{\sqrt{\varTheta}}{2}\zeta\right) + r} - \frac{16q^2}{\left(\sqrt{\varTheta}\tanh\left(\frac{\sqrt{\varTheta}}{2}\zeta\right) + r\right)^2}, \tag{3.4}$$

or

$$\phi_2(x,t) = -\frac{1}{3}(r^2 + 8q) + \frac{8qr}{\sqrt{\varTheta}\coth\left(\frac{\sqrt{\varTheta}}{2}\zeta\right) + r} - \frac{16q^2}{\left(\sqrt{\varTheta}\coth\left(\frac{\sqrt{\varTheta}}{2}\zeta\right) + r\right)^2}, \tag{3.5}$$

where $\zeta = x + \frac{4}{3}(r^2 - 4q)^3 t + k$.

Case 1.2. When $p = 1$, $q \neq 0$, $\varTheta = r^2 - 4q < 0$,

$$\phi_3(x,t) = -\frac{1}{3}(r^2 + 8q) - \frac{8qr}{\sqrt{-\varTheta}\tan\left(\frac{\sqrt{-\varTheta}}{2}\zeta\right) - r} - \frac{16q^2}{\left(\sqrt{-\varTheta}\tan\left(\frac{\sqrt{-\varTheta}}{2}\zeta\right) - r\right)^2}, \tag{3.6}$$

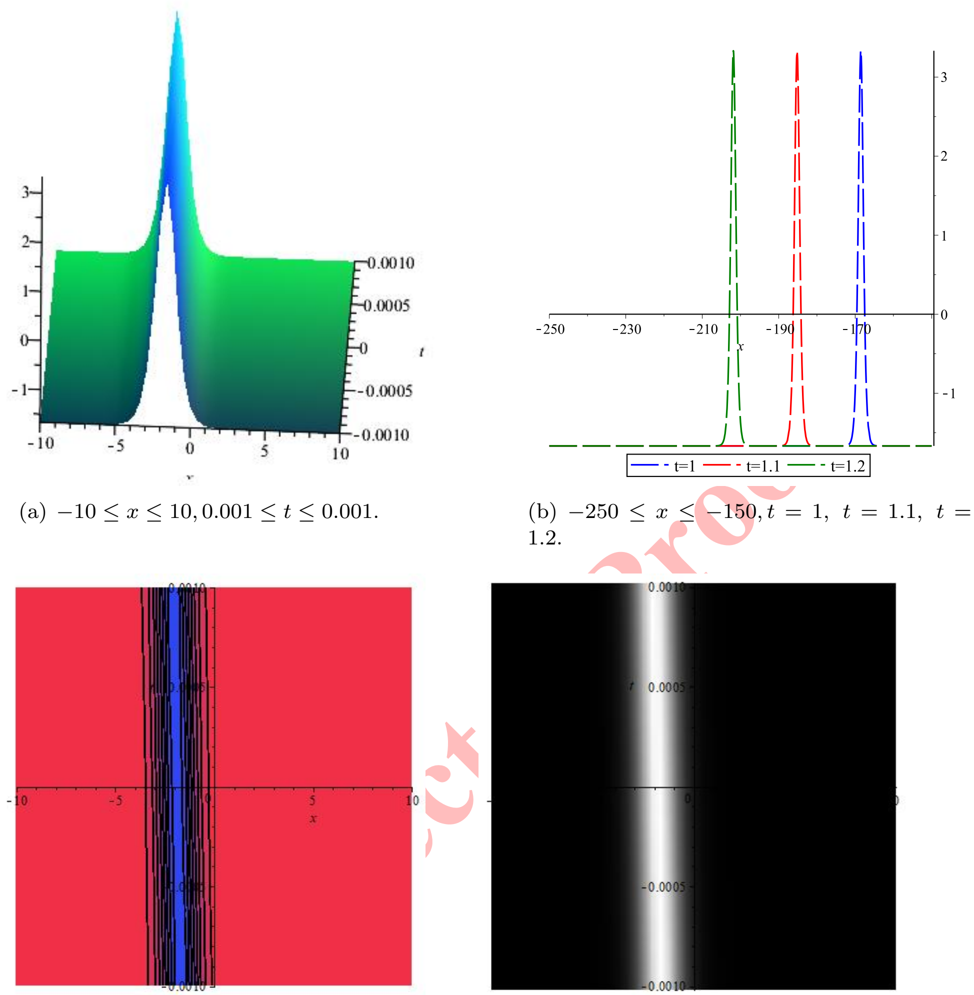
or

$$\phi_4(x,t) = -\frac{1}{3}(r^2 + 8q) - \frac{8qr}{\sqrt{-\varTheta}\cot\left(\frac{\sqrt{-\varTheta}}{2}\zeta\right) - r} - \frac{16q^2}{\left(\sqrt{-\varTheta}\cot\left(\frac{\sqrt{-\varTheta}}{2}\zeta\right) - r\right)^2}, \tag{3.7}$$

where $\zeta = x + \frac{4}{3}(r^2 - 4q)^3 t + k$.

Case 1.3. When $p = 1$, $q = 0$, $r \neq 0$,

$$\phi_5(x,t) = -\frac{1}{3}r^2 - \frac{4r^2\exp(r(x + \frac{4}{3}r^6 t + k))}{\left(\exp(r(x + \frac{4}{3}r^6 t + k)) - 1\right)^2}. \tag{3.8}$$



(a) $-10 \leq x \leq 10, 0.001 \leq t \leq 0.001$.

(b) $-250 \leq x \leq -150, t = 1,\ t = 1.1,\ t = 1.2$.

(c) $-10 \leq x \leq 10, 0.001 \leq t \leq 0.001$.

(d) $-10 \leq x \leq 10, 0.001 \leq t \leq 0.001$.

FIGURE 1. Wave profile of ϕ_1 for $p = 1$, $q = 1$, $r = 3$, and $k = 1$.

Case 1.4. When $p = 1$, $q \neq 0$, $r \neq 0$, $r^2 - 4q = 0$,

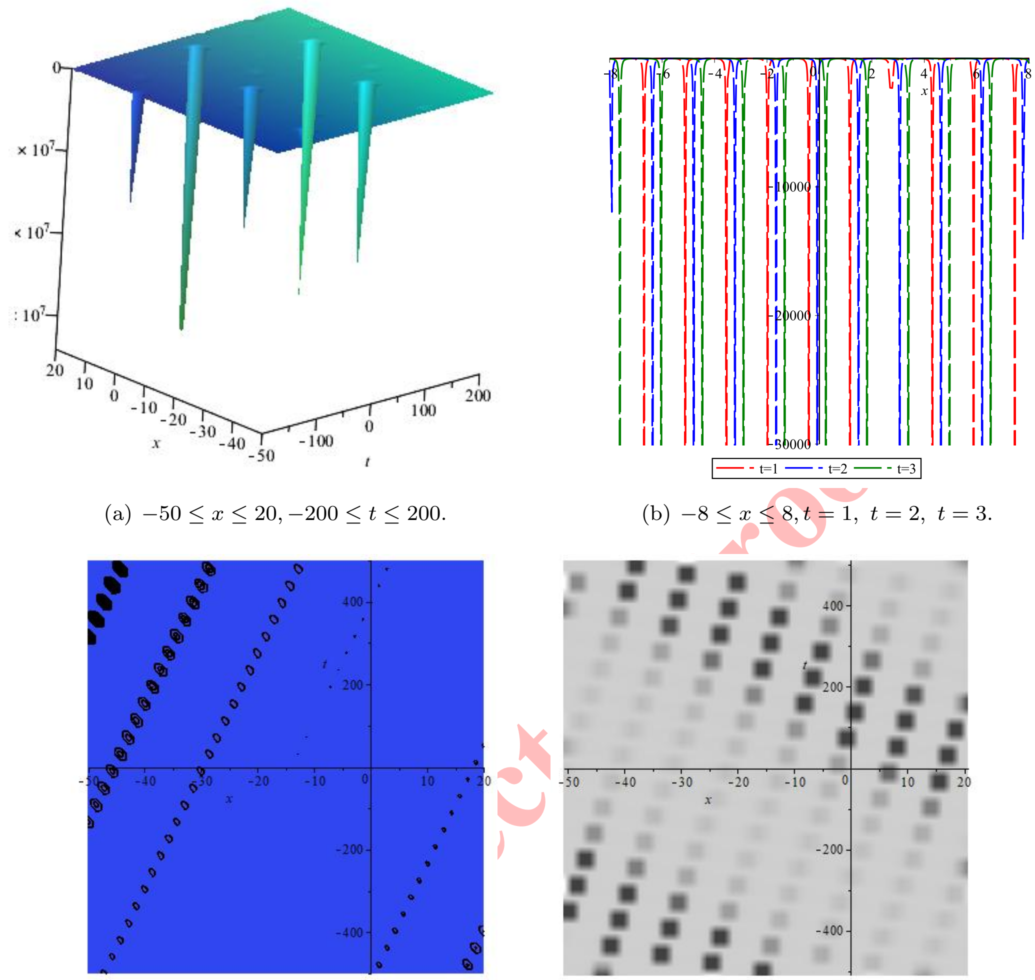
$$\phi_6(x,t) = -r^2 + \frac{2r^3(x+k)}{r(x+k)+2} - \frac{r^4(x+k)^2}{(r(x+k)+2)^2}. \tag{3.9}$$

Case 1.5. When $r = 0$, $p > 0$, $q > 0$,

$$\phi_7(x,t) = -\frac{8}{3}pq - \frac{4pq}{\tan^2\left(\sqrt{pq}(x - \frac{256}{3}p^3q^3t + k)\right)}. \tag{3.10}$$

Case 1.6. When $r = 0$, $p < 0$, $q < 0$,

$$\phi_8(x,t) = -\frac{8}{3}pq - \frac{4pq}{\cot^2\left(\sqrt{pq}(x - \frac{256}{3}p^3q^3t + k)\right)}. \tag{3.11}$$



(a) $-50 \leq x \leq 20, -200 \leq t \leq 200$.
(b) $-8 \leq x \leq 8, t = 1,\ t = 2,\ t = 3$.
(c) $-50 \leq x \leq 20, -500 \leq t \leq 500$.
(d) $-50 \leq x \leq 20, -500 \leq t \leq 500$.

FIGURE 2. Wave profile of ϕ_7 for $p = 2$, $q = 2$, $r = 0$, and $k = 1$.

Case 1.7. When $r = 0$, $p > 0$, $q < 0$,

$$\phi_9(x,t) = -\frac{8}{3}pq + \frac{4pq}{\tanh^2\left(\sqrt{-pq}(x - \frac{256}{3}p^3q^3t + k)\right)}. \tag{3.12}$$

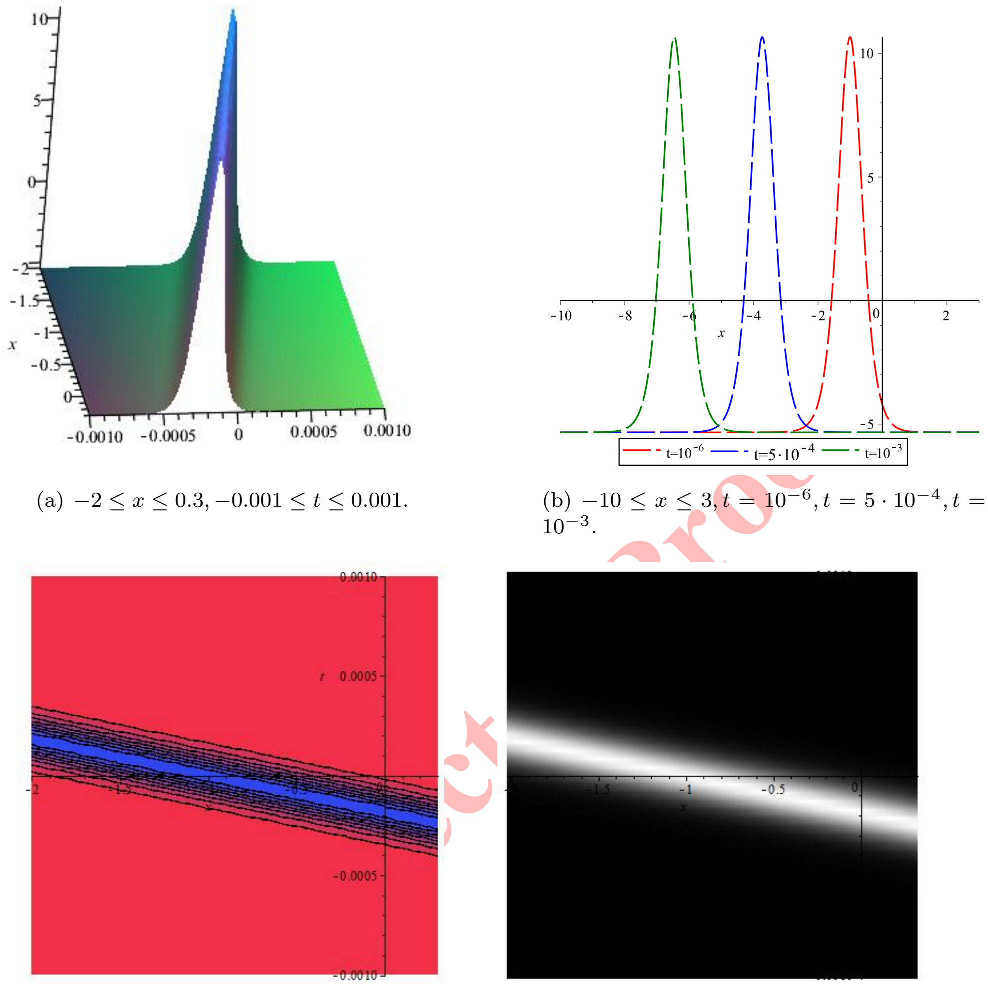
Case 1.8. When $r = 0$, $p < 0$, $q > 0$,

$$\phi_{10}(x,t) = -\frac{8}{3}pq + \frac{4pq}{\coth^2\left(\sqrt{-pq}(x - \frac{256}{3}p^3q^3t + k)\right)}. \tag{3.13}$$

Case 1.9. When $r = 0$, $q = 0$,

$$\phi_{11}(x,t) = -\frac{4}{(x+k)^2}. \tag{3.14}$$

For Set 2:



(a) $-2 \leq x \leq 0.3, -0.001 \leq t \leq 0.001$.

(b) $-10 \leq x \leq 3, t = 10^{-6}, t = 5 \cdot 10^{-4}, t = 10^{-3}$.

(c) $-2 \leq x \leq 0.3, -0.001 \leq t \leq 0.001$.

(d) $-2 \leq x \leq 0.3, -0.001 \leq t \leq 0.001$.

FIGURE 3. Wave profile of ϕ_{10} for $p = -2$, $q = 2$, $r = 0$, and $k = 1$.

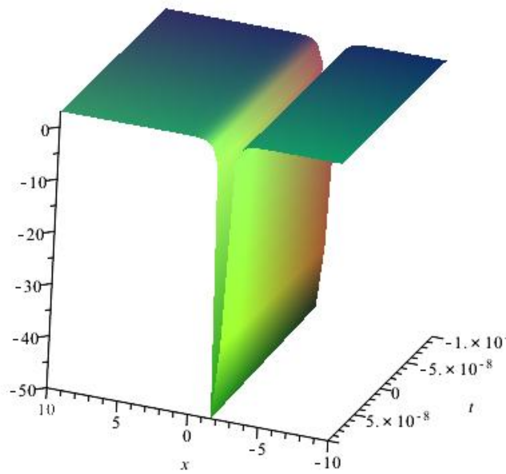
Case 2.1. When $p = 1$, $q \neq 0$, $\Theta = r^2 - 4q > 0$,

$$\phi_{12}(x,t) = \eta_0 + \frac{4qr}{\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r} - \frac{8q^2}{\left(\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r\right)^2}, \tag{3.15}$$

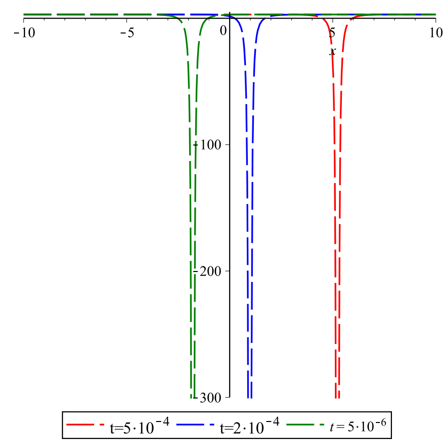
or

$$\phi_{13}(x,t) = \eta_0 + \frac{4qr}{\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r} - \frac{8q^2}{\left(\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r\right)^2}, \tag{3.16}$$

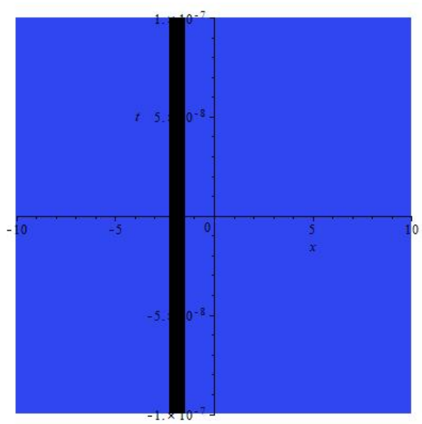
where $\zeta = x - \varpi t + k$, $\varpi = \Theta^3 + 21(2q + \eta_0)\Theta^2 + 126(r^2 + 2\eta_0)(2q + \eta_0)^2$.



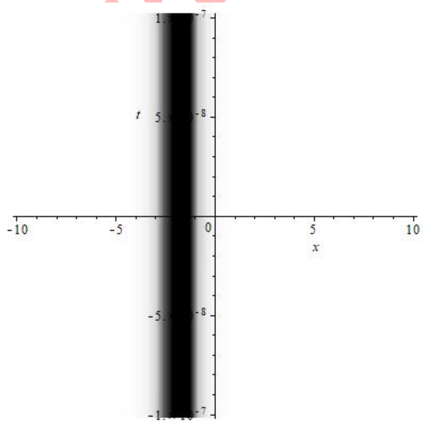
(a) $-10 \leq x \leq 10, -10^{-7} \leq t \leq 10^{-7}$.



(b) $-10 \leq x \leq 10$, $t = 5 \cdot 10^{-4}, t = 2 \cdot 10^{-4}, t = 5 \cdot 10^{-6}$.



(c) $-10 \leq x \leq 10, -10^{-7} \leq t \leq 10^{-7}$.



(d) $-10 \leq x \leq 10, -10^{-7} \leq t \leq 10^{-7}$.

FIGURE 4. Wave profile of ϕ_{13} for $p = 1$, $q = 1$, $r = 3$, $\eta_0 = 1$, and $k = 1$.

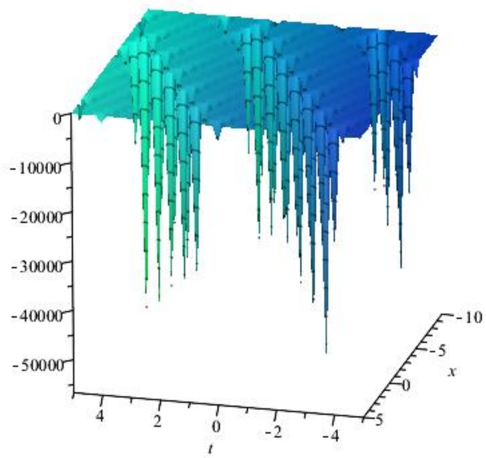
Case 2.2. When $p = 1$, $q \neq 0$, $\Theta = r^2 - 4q < 0$,

$$\phi_{14}(x,t) = \eta_0 - \frac{4qr}{\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r} - \frac{8q^2}{\left(\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r\right)^2}, \tag{3.17}$$

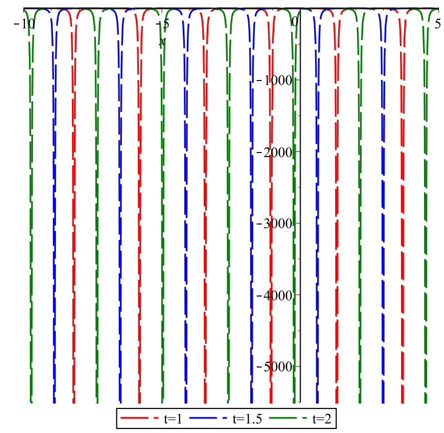
or

$$\phi_{15}(x,t) = \eta_0 - \frac{4qr}{\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r} - \frac{8q^2}{\left(\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r\right)^2}, \tag{3.18}$$

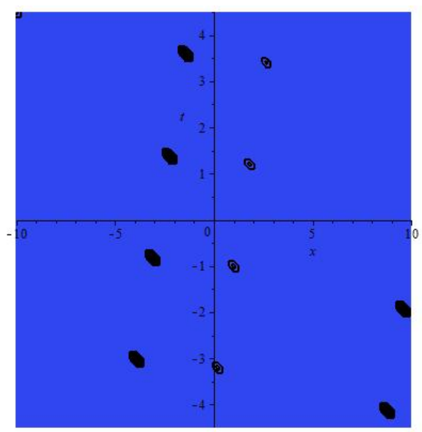
where $\zeta = x - \varpi t + k$, $\varpi = \Theta^3 + 21(2q + \eta_0)\Theta^2 + 126(r^2 + 2\eta_0)(2q + \eta_0)^2$.



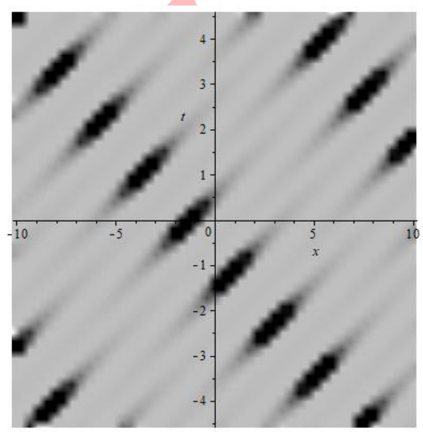
(a) $-10 \leq x \leq 5, -5 \leq t \leq 5$.



(b) $-10 \leq x \leq 5, t = 1,\ t = 1.5,\ t = 2$.



(c) $-10 \leq x \leq 10, -4.5 \leq t \leq 4.5$.



(d) $-10 \leq x \leq 10, -4.5 \leq t \leq 4.5$.

FIGURE 5. Wave profile of ϕ_{15} for $p = 1,\ q = 2,\ r = 1,\ \eta_0 = 1$, and $k = 1$.

Case 2.3. When $p = 1,\ q = 0,\ r \neq 0$,

$$\phi_{16}(x,t) = \eta_0 - \frac{2r^2 \exp(r\zeta)}{(\exp(r\zeta) - 1)^2}, \tag{3.19}$$

where $\zeta = x - \left(r^6 + 21\eta_0 r^4 + 126{\eta_0}^2(r^2 + 2\eta_0)\right) t + k$.

Case 2.4. When $p = 1,\ q \neq 0,\ r \neq 0,\ r^2 - 4q = 0$,

$$\phi_{17}(x,t) = \eta_0 + \frac{r^3\zeta}{r\zeta + 2} - \frac{r^4\zeta^2}{2(r\zeta + 2)^2}, \tag{3.20}$$

where $\zeta = x - \frac{63}{2}(r^2 + 2\eta_0)^3 t + k$.

Case 2.5. When $r = 0$, $p > 0$, $q > 0$,

$$\phi_{18}(x,t) = \eta_0 - \frac{2pq}{\tan^2\left(\sqrt{pq}\zeta\right)}, \tag{3.21}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 336p^2q^2(2pq + \eta_0) + 252\eta_0(2pq + \eta_0)^2$.

Case 2.6. When $r = 0$, $p < 0$, $q < 0$,

$$\phi_{19}(x,t) = \eta_0 - \frac{2pq}{\cot^2\left(\sqrt{pq}\zeta\right)}, \tag{3.22}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 336p^2q^2(2pq + \eta_0) + 252\eta_0(2pq + \eta_0)^2$.

Case 2.7. When $r = 0$, $p > 0$, $q < 0$,

$$\phi_{20}(x,t) = \eta_0 + \frac{2pq}{\tanh^2\left(\sqrt{-pq}\zeta\right)}, \tag{3.23}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 336p^2q^2(2pq + \eta_0) + 252\eta_0(2pq + \eta_0)^2$.

Case 2.8. When $r = 0$, $p < 0$, $q > 0$,

$$\phi_{21}(x,t) = \eta_0 + \frac{2pq}{\coth^2\left(\sqrt{-pq}\zeta\right)}, \tag{3.24}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 336p^2q^2(2pq + \eta_0) + 252\eta_0(2pq + \eta_0)^2$.

Case 2.9. When $r = 0$, $q = 0$,

$$\phi_{22}(x,t) = \eta_0 - \frac{2}{(x - 252\alpha_0^3 t + k)^2}. \tag{3.25}$$

3.2. **The sLax equation.** Let's consider the sLax equation

$$\phi_t + 140\phi^3\phi_x + 70\phi_x^3 + 280\phi\phi_x\phi_{2x} + 70\phi^2\phi_{3x} + 70\phi_{2x}\phi_{3x} + 42\phi_x\phi_{4x} + 14\phi\phi_{5x} + \phi_{7x} = 0. \tag{3.26}$$

Using $\phi(x,t) = \Phi(\xi)$ and $\xi = x - \varpi t$ reduces this equation to a nonlinear ODE

$$-\varpi\Phi' + 140\Phi^3\Phi' + 70\left(\Phi'\right)^3 + 280\Phi\Phi'\Phi'' + 70\Phi^2\Phi''' + 70\Phi''\Phi''' + 42\Phi'\Phi^{(4)} + 14\Phi\Phi^{(5)} + \Phi^{(7)} = 0. \tag{3.27}$$

The balancing rule in (3.27) gives $m = 2$, then the general solution is given by

$$\Phi(\xi) = \sum_{i=0}^{2}\eta_i(\exp(-\psi(\xi)))^i, \;\; \eta_2 \neq 0. \tag{3.28}$$

Substituting (3.28) into (3.27) and using (2.4) we get a system of algebraic equations. If we solve the conserving system, we get the following solution sets:

Set 3.

$$\varpi = \pm\frac{1}{5}(21I\sqrt{5} \pm 5)(r^2 - 4pq)^3,$$

$$\eta_0 = \pm\frac{I\sqrt{5}}{10}(r^2 - 4pq) - \frac{1}{2}(r^2 + 8pq),$$

$$\eta_1 = -6pr, \quad \eta_2 = -6p^2.$$

Set 4.

$$\varpi = (r^2 - 4pq)^3 + 14(2pq + \eta_0)(r^2 - 4pq)^2 + 70(r^2 + 2\eta_0)(2pq + \eta_0)^2,$$

$$\eta_0 = \eta_0, \quad \eta_1 = -2pr, \quad \eta_2 = -2p^2.$$

According to set 3 and set 4, the solutions of the sLax equation result in the following form:

For Set 3.

Case 3.1. When $p = 1,\ q \neq 0,\ \Theta = r^2 - 4q > 0$,

$$\phi_{23,24}(x,t) = \eta_0 + \frac{12qr}{\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r} - \frac{24q^2}{\left(\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r\right)^2}, \tag{3.29}$$

or

$$\phi_{25,26}(x,t) = \eta_0 + \frac{12qr}{\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r} - \frac{24q^2}{\left(\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r\right)^2}, \tag{3.30}$$

where $\eta_0 = \pm\frac{I\sqrt{5}}{10}\Theta - \frac{1}{2}(r^2 + 8q)$, $\zeta = x \mp (\frac{21}{5}I\sqrt{5} \pm 1)(r^2 - 4q)^3 t + k$.

Case 3.2. When $p = 1,\ q \neq 0,\ \Theta = r^2 - 4q < 0$,

$$\phi_{27,28}(x,t) = \eta_0 - \frac{12qr}{\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r} - \frac{24q^2}{\left(\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r\right)^2}, \tag{3.31}$$

or

$$\phi_{29,30}(x,t) = \eta_0 - \frac{12qr}{\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r} - \frac{24q^2}{\left(\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r\right)^2}, \tag{3.32}$$

where $\eta_0 = \pm\frac{I\sqrt{5}}{10}\Theta - \frac{1}{2}(r^2 + 8q)$, $\zeta = x \mp (\frac{21}{5}I\sqrt{5} \pm 1)(r^2 - 4q)^3 t + k$.

Case 3.3. When $p = 1,\ q = 0,\ r \neq 0$,

$$\phi_{31,32}(x,t) = \frac{-5 \pm I\sqrt{5}}{10}r^2 - \frac{6r^2\exp(r\zeta)}{\left(\exp(r\zeta) - 1\right)^2}, \tag{3.33}$$

where $\zeta = x \mp (\frac{21}{5}I\sqrt{5} \pm 1)r^6 t + k$.

Case 3.4. When $p = 1,\ q \neq 0,\ r \neq 0,\ r^2 - 4q = 0$,

$$\phi_{33}(x,t) = -\frac{3}{2}r^2 + \frac{3r^3(x+k)}{r(x+k)+2} - \frac{3r^4(x+k)^2}{2(r(x+k)+2)^2}. \tag{3.34}$$

Case 3.5. When $r = 0,\ p > 0,\ q > 0$,

$$\phi_{34,35}(x,t) = \frac{-20 \mp 2I\sqrt{5}}{5}pq - \frac{6pq}{\tan^2\left(\sqrt{pq}\zeta\right)}, \tag{3.35}$$

where $\zeta = x \pm \frac{64}{5}(21I\sqrt{5} \pm 5)p^3q^3 t + k$.

Case 3.6. When $r = 0,\ p < 0,\ q < 0$,

$$\phi_{36,37}(x,t) = \frac{-20 \mp 2I\sqrt{5}}{5}pq - \frac{6pq}{\cot^2\left(\sqrt{pq}\zeta\right)}, \tag{3.36}$$

where $\zeta = x \pm \frac{64}{5}(21I\sqrt{5} \pm 5)p^3q^3 t + k$.

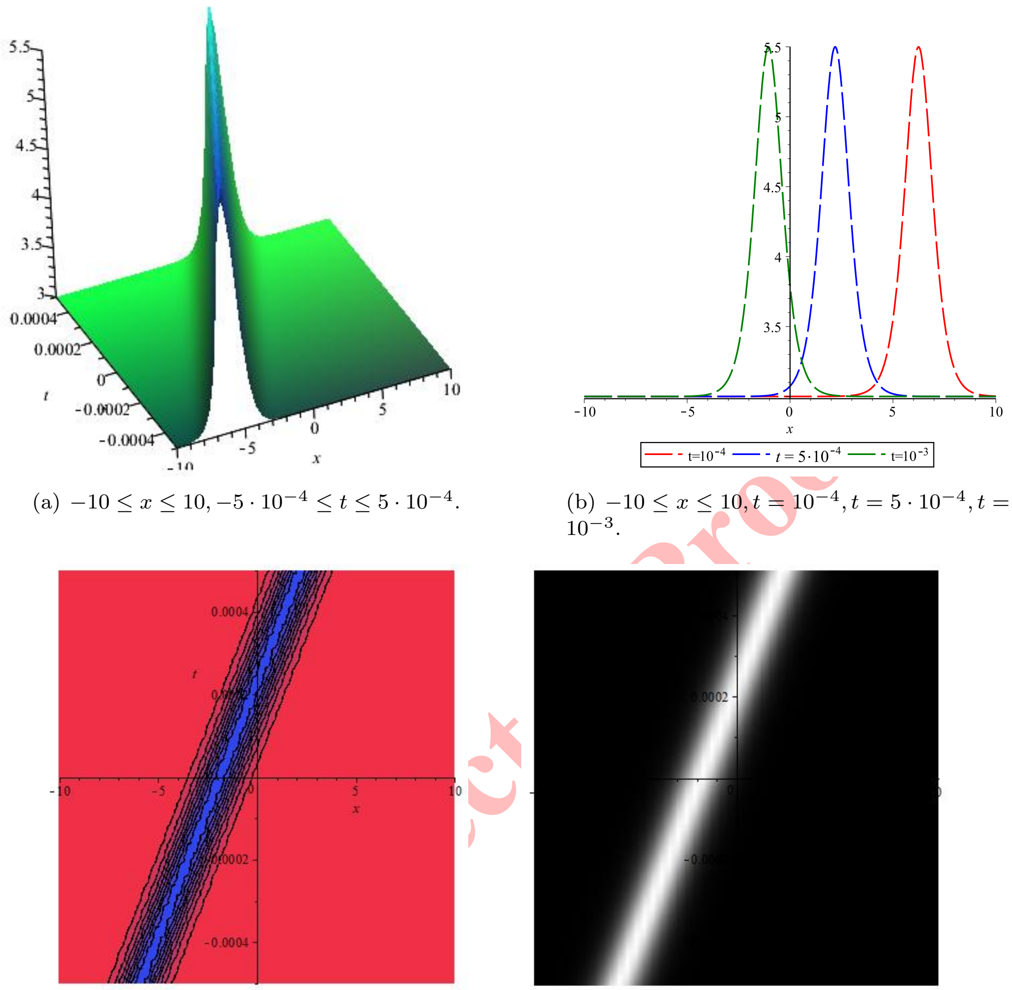
Case 3.7. When $r = 0,\ p > 0,\ q < 0$,

$$\phi_{38,39}(x,t) = \frac{-20 \mp 2I\sqrt{5}}{5}pq + \frac{6pq}{\tanh^2\left(\sqrt{-pq}\zeta\right)}, \tag{3.37}$$

where $\zeta = x \pm \frac{64}{5}(21I\sqrt{5} \pm 5)p^3q^3 t + k$.

Case 3.8. When $r = 0,\ p < 0,\ q > 0$,

$$\phi_{40,41}(x,t) = \frac{-20 \mp 2I\sqrt{5}}{5}pq + \frac{6pq}{\coth^2\left(\sqrt{-pq}\zeta\right)}, \tag{3.38}$$



(a) $-10 \leq x \leq 10, -5 \cdot 10^{-4} \leq t \leq 5 \cdot 10^{-4}$. (b) $-10 \leq x \leq 10, t = 10^{-4}, t = 5 \cdot 10^{-4}, t = 10^{-3}$.

(c) $-10 \leq x \leq 10, -5 \cdot 10^{-4} \leq t \leq 5 \cdot 10^{-4}$. (d) $-10 \leq x \leq 10, -5 \cdot 10^{-4} \leq t \leq 5 \cdot 10^{-4}$.

FIGURE 6. Wave profile of ϕ_{43} for $p = 1$, $q = 1$, $r = 3$, $\eta_0 = 1$, and $k = 1$.

where $\zeta = x \pm \frac{64}{5}(21I\sqrt{5} \pm 5)p^3q^3t + k$.

Case 3.9. When $r = 0$, $q = 0$,

$$\phi_{42}(x,t) = -\frac{6}{(x+k)^2}. \tag{3.39}$$

For Set 4.

Case 4.1. When $p = 1$, $q \neq 0$, $\Theta = r^2 - 4q > 0$,

$$\phi_{43}(x,t) = \eta_0 + \frac{4qr}{\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r} - \frac{8q^2}{\left(\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r\right)^2}, \tag{3.40}$$

or

$$\phi_{44}(x,t) = \eta_0 + \frac{4qr}{\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right)+r} - \frac{8q^2}{\left(\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right)+r\right)^2}, \tag{3.41}$$

where $\zeta = x - \varpi t + k$, $\varpi = \Theta^3 + 14(2q+\eta_0)\Theta^2 + 70(r^2+2\eta_0)(2pq+\eta_0)^2$.

Case 4.2. When $p=1$, $q \neq 0$, $\Theta = r^2 - 4q < 0$,

$$\phi_{45}(x,t) = \eta_0 - \frac{4qr}{\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right)-r} - \frac{8q^2}{\left(\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right)-r\right)^2}, \tag{3.42}$$

or

$$\phi_{46}(x,t) = \eta_0 - \frac{4qr}{\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right)-r} - \frac{8q^2}{\left(\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right)-r\right)^2}, \tag{3.43}$$

where $\zeta = x - \varpi t + k$, $\varpi = \Theta^3 + 14(2q+\eta_0)\Theta^2 + 70(r^2+2\eta_0)(2pq+\eta_0)^2$.

Case 4.3. When $p=1$, $q=0$, $r\neq 0$,

$$\phi_{47}(x,t) = \eta_0 - \frac{2r^2\exp(r\zeta)}{(\exp(r\zeta)-1)^2}, \tag{3.44}$$

where $\zeta = x - \left(r^6 + 14\eta_0 r^4 + 70\eta_0^2(r^2+2\eta_0)\right)t + k$.

Case 4.4. When $p=1$, $q\neq 0$, $r\neq 0$, $r^2-4q=0$,

$$\phi_{48}(x,t) = \eta_0 + \frac{r^3\zeta}{r\zeta+2} - \frac{r^4\zeta^2}{2(r\zeta+2)^2}, \tag{3.45}$$

where $\zeta = x - \frac{35}{2}(r^2+2\eta_0)^3 t + k$.

Case 4.5. When $r=0$, $p>0$, $q>0$,

$$\phi_{49}(x,t) = \eta_0 - \frac{2pq}{\tan^2\left(\sqrt{pq}\zeta\right)}, \tag{3.46}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 224p^2q^2(2pq+\eta_0) + 140\eta_0(2pq+\eta_0)^2$.

Case 4.6. When $r=0$, $p<0$, $q<0$,

$$\phi_{50}(x,t) = \eta_0 - \frac{2pq}{\cot^2\left(\sqrt{pq}\zeta\right)}, \tag{3.47}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 224p^2q^2(2pq+\eta_0) + 140\eta_0(2pq+\eta_0)^2$.

Case 4.7. When $r=0$, $p>0$, $q<0$,

$$\phi_{51}(x,t) = \eta_0 + \frac{2pq}{\tanh^2\left(\sqrt{-pq}\zeta\right)}, \tag{3.48}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 224p^2q^2(2pq+\eta_0) + 140\eta_0(2pq+\eta_0)^2$.

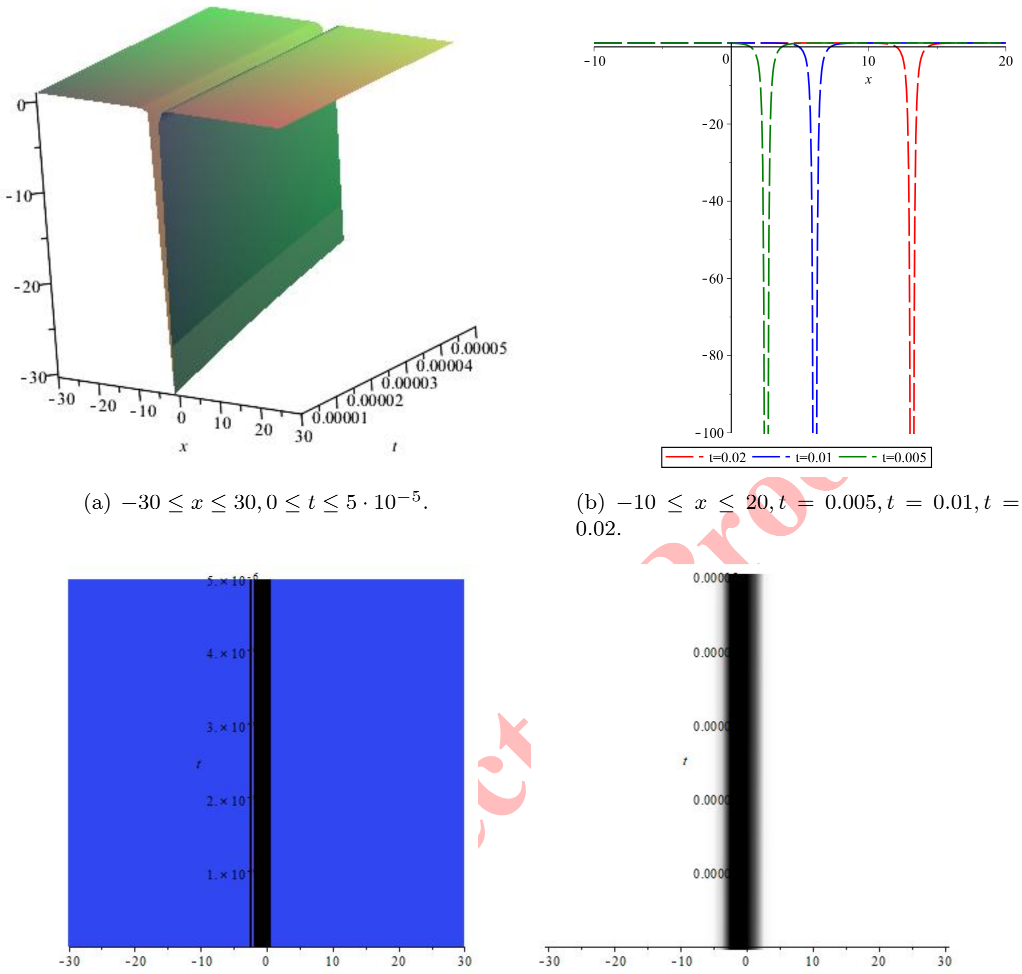
Case 4.8. When $r=0$, $p<0$, $q>0$,

$$\phi_{52}(x,t) = \eta_0 + \frac{2pq}{\coth^2\left(\sqrt{-pq}\zeta\right)}, \tag{3.49}$$

where $\zeta = x - \varpi t + k$, $\varpi = -64p^3q^3 + 224p^2q^2(2pq+\eta_0) + 140\eta_0(2pq+\eta_0)^2$.

Case 4.9. When $r=0$, $q=0$,

$$\phi_{53}(x,t) = \eta_0 - \frac{2}{(x-140\eta_0^3 t+k)^2}. \tag{3.50}$$



(a) $-30 \leq x \leq 30, 0 \leq t \leq 5 \cdot 10^{-5}$.

(b) $-10 \leq x \leq 20, t = 0.005, t = 0.01, t = 0.02$.

(c) $-30 \leq x \leq 30, 0 \leq t \leq 5 \cdot 10^{-5}$.

(d) $-30 \leq x \leq 30, 0 \leq t \leq 5 \cdot 10^{-5}$.

FIGURE 7. Wave profile of ϕ_{47} for $p = 1$, $q = 2$, $r = 1$, $\eta_0 = 1$, and $k = 1$.

3.3. **The sKK equation.** Consider the sKK equation that has the form

$$\phi_t + 2016\phi^3\phi_x + 630\phi_x^3 + 2268\phi\phi_x\phi_{2x} + 504\phi^2\phi_{3x} + 252\phi_{2x}\phi_{3x} + 147\phi_x\phi_{4x} + 42\phi\phi_{5x} + \phi_{7x} = 0. \tag{3.51}$$

Using $\phi(x,t) = \varPhi(\xi)$ and $\xi = x - \varpi t$ reduces this equation to a nonlinear ODE

$$-\varpi\varPhi' + 2016\varPhi^3\varPhi' + 630\left(\varPhi'\right)^3 + 2268\varPhi\varPhi'\varPhi'' + 504\varPhi^2\varPhi''' + 252\varPhi''\varPhi''' + 147\varPhi'\varPhi^{(4)} + 42\varPhi\varPhi^{(5)} + \varPhi^{(7)} = 0. \tag{3.52}$$

The balancing rule in (3.52) gives $m = 2$, then the general solution is given by

$$\varPhi(\xi) = \sum_{i=0}^{2}\eta_i(\exp(-\psi(\xi)))^i, \ \eta_2 \neq 0. \tag{3.53}$$

Substituting (3.53) into (3.52) and using (2.4) we get a system of algebraic equations that can be solved to find the solution:

Set 5.

$$\varpi = -\frac{1}{48}(r^2 - 4pq)^3,$$
$$\eta_0 = -\frac{1}{24}(r^2 + 8pq), \quad \eta_1 = -\frac{1}{2}pr, \quad \eta_2 = -\frac{1}{2}p^2.$$

Depending on set 5, the solutions of the sKK equation result in the following form:

Case 5.1. When $p = 1$, $q \neq 0$, $\Theta = r^2 - 4q > 0$,

$$\phi_{54}(x,t) = -\frac{1}{24}(r^2 + 8q) + \frac{qr}{\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r} - \frac{2q^2}{\left(\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r\right)^2}, \tag{3.54}$$

or

$$\phi_{55}(x,t) = -\frac{1}{24}(r^2 + 8q) + \frac{qr}{\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r} - \frac{2q^2}{\left(\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\zeta\right) + r\right)^2}, \tag{3.55}$$

where $\zeta = x + \frac{1}{48}(r^2 - 4q)^3 t + k$.

Case 5.2. When $p = 1$, $q \neq 0$, $\Theta = r^2 - 4q < 0$,

$$\phi_{56}(x,t) = -\frac{1}{24}(r^2 + 8q) - \frac{qr}{\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r} - \frac{2q^2}{\left(\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r\right)^2}, \tag{3.56}$$

or

$$\phi_{57}(x,t) = -\frac{1}{24}(r^2 + 8q) - \frac{qr}{\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r} - \frac{2q^2}{\left(\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\zeta\right) - r\right)^2}, \tag{3.57}$$

where $\zeta = x + \frac{1}{48}(r^2 - 4q)^3 t + k$.

Case 5.3. When $p = 1$, $q = 0$, $r \neq 0$,

$$\phi_{58}(x,t) = -\frac{1}{24}r^2 - \frac{r^2\exp(r(x + \frac{1}{48}r^6 t + k))}{2\left(\exp(r(x + \frac{1}{48}r^6 t + k)) - 1\right)^2}. \tag{3.58}$$

Case 5.4. When $p = 1$, $q \neq 0$, $r \neq 0$, $r^2 - 4q = 0$,

$$\phi_{59}(x,t) = -\frac{1}{8}r^2 + \frac{r^3(x+k)}{4r(x+k)+8} - \frac{r^4(x+k)^2}{8(r(x+k)+2)^2}. \tag{3.59}$$

Case 5.5. When $r = 0$, $p > 0$, $q > 0$,

$$\phi_{60}(x,t) = -\frac{1}{3}pq - \frac{pq}{2\tan^2\left(\sqrt{pq}(x - \frac{4}{3}p^3q^3 t + k)\right)}. \tag{3.60}$$

Case 5.6. When $r = 0$, $p < 0$, $q < 0$,

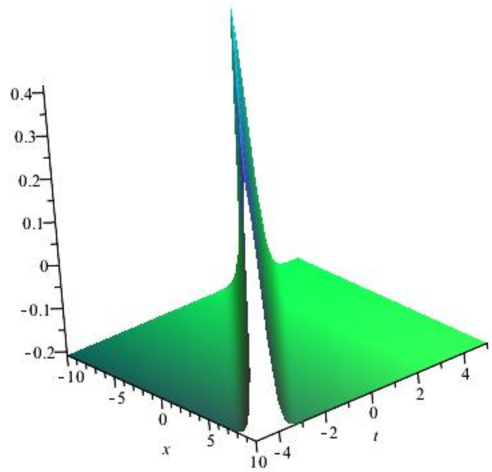
$$\phi_{61}(x,t) = -\frac{1}{3}pq - \frac{pq}{2\cot^2\left(\sqrt{pq}(x - \frac{4}{3}p^3q^3 t + k)\right)}. \tag{3.61}$$

Case 5.7. When $r = 0$, $p > 0$, $q < 0$,

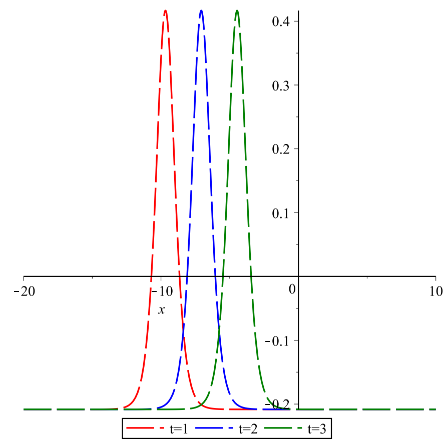
$$\phi_{62}(x,t) = -\frac{1}{3}pq + \frac{pq}{2\tanh^2\left(\sqrt{-pq}(x - \frac{4}{3}p^3q^3 t + k)\right)}. \tag{3.62}$$

Case 5.8. When $r = 0$, $p < 0$, $q > 0$,

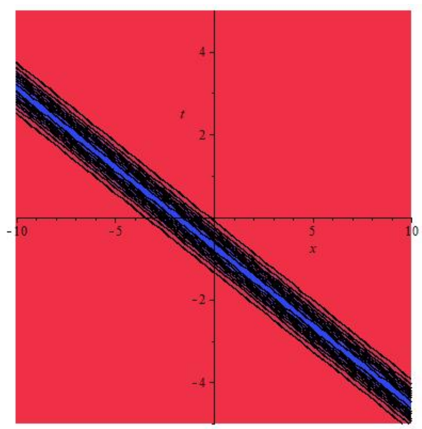
$$\phi_{63}(x,t) = -\frac{1}{3}pq + \frac{pq}{2\coth^2\left(\sqrt{-pq}(x - \frac{4}{3}p^3q^3 t + k)\right)}. \tag{3.63}$$



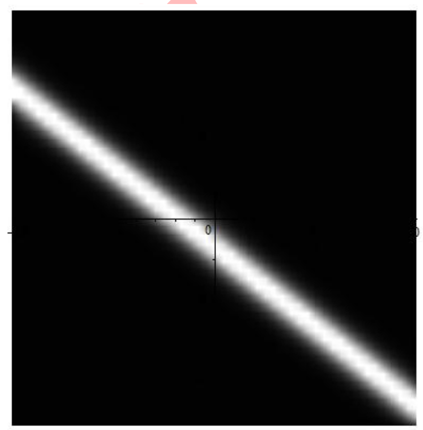
(a) $-10 \leq x \leq 10, -5 \leq t \leq 5$.



(b) $-20 \leq x \leq 10, t = 1,\ t = 2,\ t = 3$.



(c) $-10 \leq x \leq 10, -5 \leq t \leq 5$.



(d) $-10 \leq x \leq 10, -5 \leq t \leq 5$.

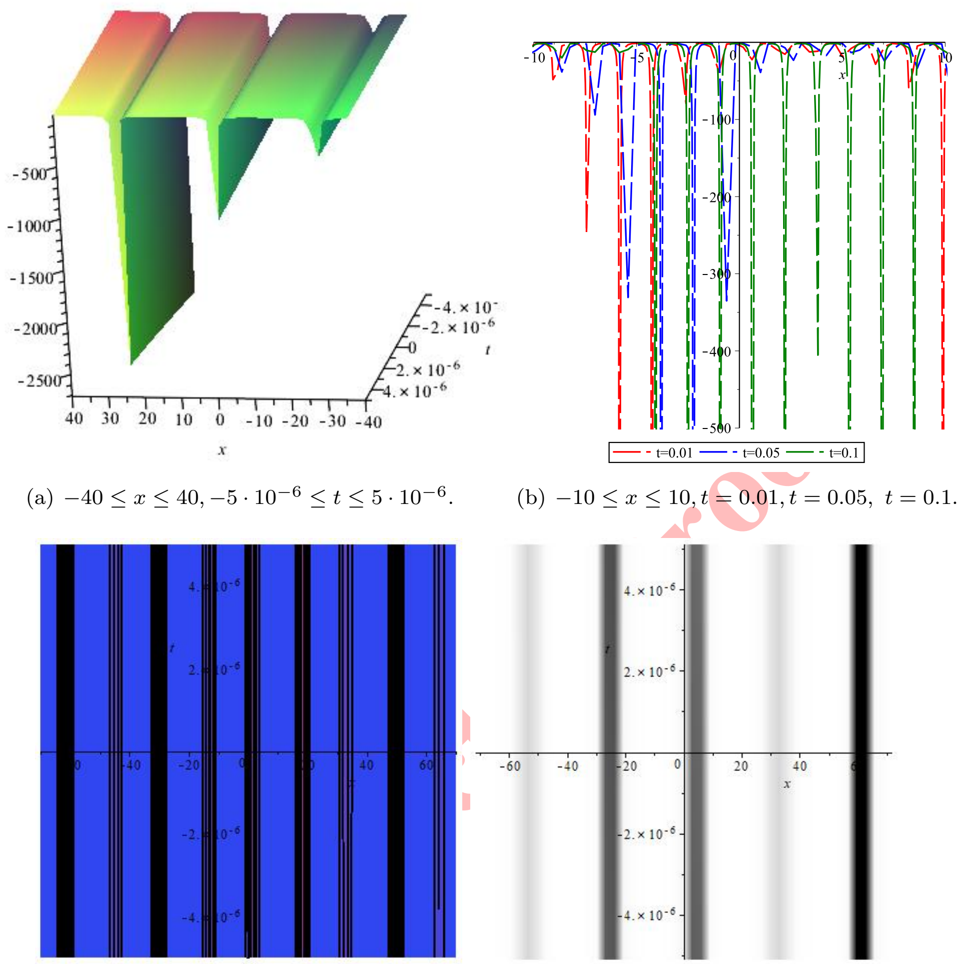
FIGURE 8. Wave profile of ϕ_{54} for $p = 1$, $q = 1$, $r = 3$, and $k = 1$.

Case 5.9. When $r = 0$, $q = 0$,

$$\phi_{64}(x, t) = -\frac{1}{2(x + k)^2}. \tag{3.64}$$

4. Illustrative Graphics

In this section, we present graphical representations in 2D, 3D, contour plots, and density plots, showcasing three distinct types of traveling wave solutions relevant to solitary wave theory. The Figures 1, 3, 6, and 8 correspond to the bright soliton solution associated with ϕ_1, ϕ_{10}, ϕ_{43}, and ϕ_{54}, respectively. These figures depict fixed parameter values as specified in their captions. Additionally, Figures 2, 5, and 9 illustrate multiple dark solitary wave solutions corresponding to ϕ_7, ϕ_{15}, and ϕ_{61}, respectively, with fixed parameters. Figures 4 and 7 display the dark solitary wave solution linked to ϕ_{13} and ϕ_{47}, respectively, with parameters indicated in the figure captions.



(a) $-40 \leq x \leq 40, -5 \cdot 10^{-6} \leq t \leq 5 \cdot 10^{-6}$. (b) $-10 \leq x \leq 10, t = 0.01, t = 0.05,\ t = 0.1$.

(c) $-70 \leq x \leq 70, -5 \cdot 10^{-6} \leq t \leq 5 \cdot 10^{-6}$. (d) $-70 \leq x \leq 70, -5 \cdot 10^{-6} \leq t \leq 5 \cdot 10^{-6}$.

FIGURE 9. Wave profile of ϕ_{61} for $p = -2$, $q = -2$, $r = 0$, and $k = 1$.

5. CONCLUSIONS

This article investigates novel traveling wave solutions for three specific instances of the gsKdV equation: the sSKI equation, the sLax equation, and the sKK equation. Employing the GEE method, we successfully identify bright soliton solutions, dark solitary wave solutions, and multiple dark solitary wave solutions. Significantly, the GEE method has not previously unveiled innovative solutions for the gsKdV problem. Hence, the exact solutions obtained in this study can be regarded as novel. These solutions, expressed in terms of hyperbolic, trigonometric, exponential, and rational functions, are visually represented through 2D, 3D, contour plots, and density plots illustrations generated using Maple computational tools within specific finite domains.

Ultimately, our proposed method demonstrates efficiency, reliability, and potency in delivering numerous consistent solutions for NLEEs encountered across various disciplines such as applied mathematics, mathematical physics, and engineering.

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