Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-37
DOI:10.22034/cmde.2025.63609.2840

A fractional stochastic differential modeling of cancer cells with an application to ommune response of tumor dynamics

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Abstract

In this paper, we present a fractional stochastic model that examines the response of cancer cells to the immune system. The model combines the long-term memory dependence of fractional derivatives with the stochastic nature of cancer cell growth. The geometric Brownian motion is used to present the stochastic nature of this model. By applying the global derivative from different versions of Caputo, Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives, and converting them into the fractional integral version, we demonstrate the memory property of the model by maintaining the initial conditions. We also prove the stability of the model analytically in the two states of the ordinary differential equation and the fractional differential equation by obtaining the equilibrium points of the model in the disease-free state and the disease state. Additionally, we use the numerical method based on Lagrange polynomials, and Newton's polynomials, to examine and compare the approximate solution of the model in two different states of disease-free state and disease state. Finally, using numerical simulation, we examine the stability of the model in the fractional-random state. We show that using Newton's polynomial will preserve the stability condition better than Lagrange's polynomial. Further, we analyze that the solutions of the stochastic fractional model are positive and bounded, and we also prove their uniqueness and existence.

Keywords. Cancer model, Immune system, Fractional stochastic differential equations, fractional derivatives, Numerical approximations.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Mathematical models are useful tools that provide solutions to problems in various sciences, including medicine and biology [13, 14, 16]. References [10, 15, 19, 21] contain recent works on the application of mathematics to biological and epidemiological problems. One of the famous biological and epidemiological problems is cancer tumors. Cancer is a serious health concern that affects millions of people worldwide. According to the latest data from the World Health Organization, by 2050, there is a prediction of over 35 million new cancer cases, which is a 77% increase from the estimated 20 million cases in 2022. The rapid increase in global cancer cases is due to the aging and growth of the population, as well as changes in people's exposure to risk factors, many of which are linked to socioeconomic development. Key factors contributing to the rising cancer rates include tobacco use, alcohol consumption, and obesity, while air pollution remains a significant environmental risk factor[1] So, considering the increasing incidence of cancer in the world, in this paper, the application of mathematics by modeling cancerous tumors and their interaction and response to the immune system will be discussed as a type of treatment method. Tumors are formed as a result of uncontrolled cell division and can start and multiply from one of our body's cells. Depending on whether the tumor is malignant or benign, it spreads unexpectedly around the area where it is found. To destroy the patient's tumor cells, the treatment used must spread faster than the growth movement of the tumor cell. There are several

Received: 20 September 2024 ; Accepted: 02 July 2025.
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[1]This subject was published by the World Health Organization at the website https://www.who.int/news/item/01-02-2024-global-cancer-burden-growing-amidst-mounting-need-for-services and also, it is published on 1 February 2024.

methods available to treat cancer tumors, which include surgery, radiation therapy, chemotherapy, oxygen therapy, and immunotherapy. The type of treatment that is chosen depends on the type of cancer and the patient's conditions, such as their age, any underlying conditions, and any history of heredity. There is a new cancer treatment that uses viruses to target tumors without harming healthy cells. These viruses are called oncolytic viruses, which have been modified through genetic engineering to kill cancer cells and induce immune responses [18]. Treatment using oncolytic viruses has several important advantages over traditional approaches, such as selectivity, because only cancer cells will be damaged in this method, and it avoids exposure of normal tissues to excessive doses of chemotherapy and radiation therapy. Therefore, this treatment method can destroy cancer cells that have metastasized and provides the potential to induce an anti-cancer vaccine response [22]. Oncolytic viruses are used in the treatment of various types of tumors, such as liver and pancreatic carcinomas, mesothelioma, myeloma, and breast cancer, and have shown promise [26]. Many scientists have used mathematical models of complex structured tumors, tumor growth, and the interaction between tumors and the immune system. In these researches, safety has been studied. References [9, 12, 17, 20, 24] are among the research that investigated the relationship between tumor cells and the immune system using mathematical modeling. In 2011, Crivelli et al. [9] have presented cell dynamics and recommended control strategies. In 2020, Elaiw and Al Agha [12] have studied an ordinary differential equation (ODE) model to investigate the potent efficacy of the modified M1 virus. In 2020, Nouni et al. [17] have studied the dynamics of tumor cells and their response to the immune systems by a model of virus therapy for cancer therapy and have evaluated the effectiveness of combination therapy. Also, Due to the importance of the memory effect and heredity in mathematical models for understanding natural phenomena, the use of fractional calculus in mathematical models has been investigated in recent years. The reference [24] is one of the sources that show the effect of long-term memory, and heredity in mathematical modeling of the dynamics of tumor cells and their response to the immune system, and therefore to do this in 2020, Uçar and Özdemir [24] have developed a fractional model of tumor, and immune response with partial derivatives Caputo and Fabrizio Caputo analyzed. Also, regarding the random growth of cancer cells, in 2022, Raza et al. [20] have investigated the complex interaction between tumor cells, oncolytic viruses, and immune cell response.

Using the fractional stochastic model is one of the main ways to control the spread of cancer cells. Since it can show the randomness of the cancer cell expansion process and the response of the immune system by considering the random property in the model. It also expresses the effect of long memory dependence when passing from one process to another by applying the fractional property in the model. In recent years, stochastic fractional differential equations have been proposed to capture processes that simultaneously obey randomness and non-locality of memory. For example, in 2021, Alkahtani and Koca [1] have considered an SIR model and have analyzed it analytically and numerically for different values of fractional order and random density. In 2023, Zafar et al. [25] have investigated a stochastic HIV/AIDS model for different values of the fractional order. Now, our goal in this paper is to express a nonlinear model of cancer disease and its response to viral therapy in the form of a stochastic fractional equation. Therefore, we apply Caputo-Fabrizio and Atangana-Baleanu fractional derivatives in the stochastic differential equation that was developed by Raza et al. are presented in reference [20]. Hence, we will be able to convert the presented stochastic model into a stochastic fractional model. Finally, we approximate the solution of the equation using the numerical scheme by considering the Lagrange and Newton polynomials. This method was initially proposed by Atangana and his team in 2016, [11] to estimate the solution of nonlinear equations using a new concept of fractional differential equations with the Mittag-Leffer kernel. Additionally, the corresponding fractional integral was also presented in their work. In 2017, Toufik and Atangana [23] have studied a new numerical approximation of a non-local derivative and a non-singular kernel. In 2020, Atangana [7] alone in another study, carefully examined the concept of singular and non-singular kernels and was able to introduce a new fractional integral called Caputo's fractional integral. In 2020, Atangana and Araz [2] have introduced the Newton polynomial Atangana-Seda numerical scheme which is based on the Newton polynomial for interpreting and examining real-world problems. Also, in 2021 Atangana and Araz [3] have investigated the Newton polynomial method for solving partial and ordinary differential equations, as well as systems of ordinary and partial differential equations with different types of integral operators. In this paper, a nonlinear model of cancer disease and its response to virus therapy is considered further to investigate the application of this type of differential equation. By applying Caputo-Fabrizio and Atangana-Baleanu order to the stochastic differential equation, we will be able to convert the presented stochastic model into a stochastic fractional model. Finally, we

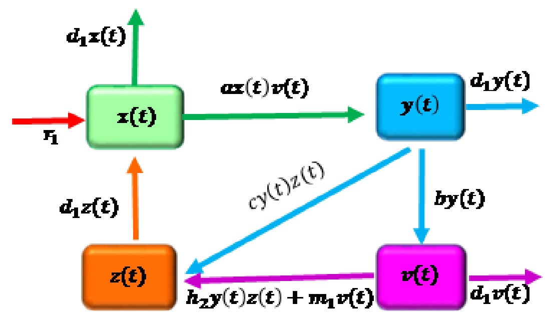

FIGURE 1. Dynamic flow map of tumor cell response to immune cells.

approximate the solution of the equation by using the numerical scheme. Some numerical simulations have been done for fractional orders with different values and some different random parameters. In the following, to express the stochastic fractional model proposed in the paper and to examine the numerical scheme based on it, we will introduce the stochastic model of virus therapy for cancer and the dynamics of the cell population presented by Reza et al. in [20]. They initially modeled their model as a set of nonlinear ordinary differential equations as follows:

$$\dot{x}(t) = r_1 - ax(t)v(t) - d_1x(t), \qquad t \geq 0, \tag{1.1}$$
$$\dot{y}(t) = ax(t)v(t) - cy(t)z(t) - d_1y(t) - by(t), \qquad t \geq 0, \tag{1.2}$$
$$\dot{v}(t) = by(t) - h_2y(t)z(t) - d_1v(t) - m_1v(t), \qquad t \geq 0, \tag{1.3}$$
$$w\dot{z}(t) = cy(t)z(t) - h_2y(t)z(t) - d_1z(t) + m_1v(t), \qquad t \geq 0. \tag{1.4}$$

Considering the initial non-negative conditions $x_0 = x(0) \geq 0$, $y_0 = y(0) \geq 0$, $v_0 = v(0) \geq 0$, $z_0 = z(0) \geq 0$, and $x + y + v + z = 1$.

At first, the model is divided into four main parts, which are x, as non-infected cancer; y, as infected cancer cells; v, as virus-free cells; and z, as immune cells. In addition, c is the carrying capacity; the value of d_1 indicates the number of cells that undergo natural death or die due to infection; b, where the emission rate of new particles, is equal to the explosion; a is the proportion of uninfected cells due to immune response; m_1 represents the number of infected cells that occur as a result of a weak immune response; h_2 is the speed of stimulation of non-infectious cells by the immune system. According to [24], Figure 1 illustrates the process by which cancer cells respond to immunotherapy. Further, in [24], a stochastic term is added to every part of the system (1.1)–(1.4). Then these equations became as follows:

$$\begin{aligned}
dx(t) &= r_1 - ax(t)v(t) - d_1x(t) + \sigma_1 x(t)\,dB(t), && t \geq 0,\\
dy(t) &= ax(t)v(t) - cy(t)z(t) - d_1y(t) - by(t) + \sigma_2 y(t)\,dB(t), && t \geq 0,\\
dv(t) &= by(t) - h_2y(t)z(t) - d_1v(t) - m_1v(t) + \sigma_3 v(t)\,dB(t), && t \geq 0,\\
dz(t) &= cy(t)z(t) - h_2y(t)z(t) - d_1z(t) + m_1v(t) + \sigma_4 z(t)\,dB(t), && t \geq 0,
\end{aligned} \tag{1.5}$$

where σ_i $(i = 1, 2, 3, 4)$ is the randomness of the model, and $B(t)$ is the geometric Brownian motion (or standard Brownian motion). In this paper, we add the feature of non-locality to the stochastic model (1.5) by introducing the global derivative of different versions of the Caputo, Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives and converting them to the fractional integral version. Since the derivative is defined in the interval $(0,\ t]$, the derivative cannot be calculated at the point $t_0 = 0$. That is, when the zero instant is considered as the origin, no memory can be recorded using the derivative to start the process. So, the initial conditions are removed. But by using the integral, we can maintain the initial conditions. Therefore, maintaining the initial conditions by the integral, we can apply the non-local effect in the model. As a result, we generalize the model presented in reference [24] to a stochastic fractional model. By doing this, we can simultaneously apply the effect of non-locality and the random nature of cancer cell growth in the model. Also, as another result, we can better show the complexity of actual models. To better understand the method used in the paper, we divide the structure of the paper into five essential parts as follows: Modeling the interaction of cancer cells and their response to the body's immune system as a stochastic differential equation with a global derivative and converting it to the integral version is given in Section 2. In Section 3, we will mention important issues: First, we express the global derivative of Caputo, Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu. Then, we mention converting them into the fractional integral version, which is used to fractionate the stochastic model with the derivative global. Second, we prove the boundedness and positivity of the model. Third, we obtain the two equilibrium points of the model in disease-free state and disease state. Fourth, we obtain the multiplication number by using the Jacobian matrix and by establishing the Routh-Hurwitz conditions to prove the stability of the model in the case of ODEs. Fifth, we prove the stability of the model in the fractional state, by establishing the condition $|arg\lambda\ | > \alpha\pi/2$. Sixth, we prove the existence and uniqueness of the random fractional model per the methodologies utilized in references [7] and [6].

In Section 4, we first approximate the solution of the stochastic fractional model using the numerical scheme with Newton and Lagrange polynomial interpolation. Then, we present their graphical results. It is worth mentioning that in the presented random fractional model, the random nature of the model is expressed using Brownian motion; and the preservation of the long-term-memory feature of the model is expressed using the Caputo-Fabrizio and Atangana-Baleanu fractional operators. Section 5 shows the summary and the final result of this work.

2. Model Formulation

In this section, we intend to convert the classical time derivative in the stochastic differential equation (1.5) into the global derivative.

It is important to mention that a global derivative of a differentiable function f is defined using a non-negative, increasing, and ascending continuous function g:

$$D_g f(t) = \lim_{t\to t_1} \frac{f(t) - f(t_1)}{g(t) - g(t_1)}, \quad \text{for} \quad t_1 \epsilon R.$$

Indeed, if g is differentiable, then:

$$D_g f(t) = \frac{f'(t)}{g'(t)}.$$

Now, we will consider the system of stochastic differential Equations (1.5) with a global derivative. Assuming $g(\mathrm{t})$ is a positive and increasing function, we write:

$$D_g x(t) = [r_1 - ax(t)v(t) - d_1 x(t)] + \sigma_1 x(t) dB(t), \tag{2.1}$$
$$D_g y(t) = [ax(t)v(t) - cy(t)z(t) - d_1 y(t) - by(t)] + \sigma_2 y(t) dB(t), \tag{2.2}$$
$$D_g v(t) = [by(t) - h_2 y(t)z(t) - d_1 v(t) - m_1 v(t)] + \sigma_3 v(t) dB(t), \tag{2.3}$$
$$D_g z(t) = [cy(t)z(t) - h_2 y(t)z(t) - d_1 z(t) + m_1 v(t)] + \sigma_4 z(t) dB(t), \tag{2.4}$$
$$x(0) = x_0, \quad y(0) = y_0, \quad v(0) = v_0, \quad z(0) = z_0.$$

Considering function g is differentiable, we rewrite Equations (2.1)–(2.4) as follows:

$$\begin{aligned}
dx(t) &= [r_1 - ax(t)v(t) - d_1 x(t)]\, g'(t)\, dt + \sigma_1 x(t) g'(t) dB(t),\\
dy(t) &= [ax(t)v(t) - cy(t)z(t) - d_1 y(t) - by(t)] g'(t)\, dt + \sigma_2 y(t) g'(t) dB(t),\\
dv(t) &= [by(t) - h_2 y(t)z(t) - d_1 v(t) - m_1 v(t)]\, g'(t)\, dt + \sigma_3 v(t)\, g'(t)\, dB(t),\\
dz(t) &= [cy(t)z(t) - h_2 y(t)z(t) - d_1 z(t) + m_1 v(t)]\, g'(t)\, dt + \sigma_4 z(t)\, g'(t)\, dB(t).
\end{aligned} \tag{2.5}$$

Getting the integral from both sides of the Equation (2.5) we rewrite above system to integral version with exponential kernel. So, we have nonlinear stochastic equations with classical global derivatives as follows:

$$x(t) = x(0) + \int_0^t g'(\tau)\, [r_1 - ax(\tau)v(\tau) - d_1 x(\tau)] d\tau + \int_0^t g'(\tau)\, [\sigma_1 x(\tau)] dB(\tau),$$

$$y\left(t\right)=y\left(0\right)+\int_{0}^{t}g'\left(\tau\right)[ax\left(\tau\right)v\left(\tau\right)-cy\left(\tau\right)z\left(\tau\right)-d_{1}y\left(\tau\right)-by\left(\tau\right)]d\tau+\int_{0}^{t}g'\left(\tau\right)\left[\sigma_{2}y(\tau)\right]dB\left(\tau\right),$$
$$v\left(t\right)=v\left(0\right)+\int_{0}^{t}g'\left(\tau\right)[by\left(\tau\right)-h_{2}y\left(\tau\right)z\left(\tau\right)-d_{1}v\left(\tau\right)-m_{1}v\left(\tau\right)]d\tau+\int_{0}^{t}g'\left(\tau\right)\left[\sigma_{3}v(\tau)\right]dB\left(\tau\right),$$
$$z\left(t\right)=z\left(0\right)+\int_{0}^{t}g'\left(\tau\right)[cy\left(\tau\right)z\left(\tau\right)-h_{2}y\left(\tau\right)z\left(t\right)-d_{1}z\left(\tau\right)+m_{1}v\left(\tau\right)]d\tau+\int_{0}^{t}g'\left(\tau\right)\left[\sigma_{4}z(\tau)\right]dB(\tau). \tag{2.6}$$

Here we take as:

$$f_{1}\left(\tau,x(\tau)\right)=\left[r_{1}-ax\left(\tau\right)v\left(\tau\right)-d_{1}x\left(\tau\right)\right],f_{2}\left(\tau,x(\tau)\right)=\ \sigma_{1}x\left(\tau\right),$$
$$g_{1}\left(\tau,y(\tau)\right)=\left[ax\left(\tau\right)v\left(\tau\right)-cy\left(\tau\right)z\left(\tau\right)-d_{1}y\left(\tau\right)-by\left(\tau\right)\right],g_{2}\left(\tau,y(\tau)\right)=\sigma_{2}y(\tau),$$
$$h_{1}\left(\tau,v(\tau)\right)=[by\left(\tau\right)-h_{2}y\left(\tau\right)z\left(\tau\right)-d_{1}v\left(\tau\right)-m_{1}v\left(\tau\right)],h_{2}\left(\tau,v(\tau)\right)=\ \sigma_{3}v\left(\tau\right),$$
$$I_{1}\left(\tau,z(\tau)\right))=\ \left[cy\left(\tau\right)z\left(\tau\right)-h_{2}y\left(\tau\right)z\left(t\right)-d_{1}z\left(\tau\right)+m_{1}v\left(\tau\right)\right],I_{2}\left(\tau,z(\tau)\right)=\sigma_{4}z\left(\tau\right).$$

Then the Equation (2.6) become:

$$x\left(t\right)=x\left(0\right)+\int_{0}^{t}g'\left(\tau\right)f_{1}\left(\tau,x(\tau)\right)d\tau+\int_{0}^{t}g'\left(\tau\right)f_{2}\left(\tau,x(\tau)\right)dB(\tau), \tag{2.7}$$
$$y\left(t\right)=y\left(0\right)+\int_{0}^{t}g'\left(\tau\right)g_{1}\left(\tau,y(\tau)\right)d\tau+\int_{0}^{t}g'\left(\tau\right)g_{2}\left(\tau,y(\tau)\right)dB\left(\tau\right), \tag{2.8}$$
$$v\left(t\right)=v\left(0\right)+\int_{0}^{t}g'\left(\tau\right)h_{1}\left(\tau,v(\tau)\right)]d\tau+\int_{0}^{t}g'\left(\tau\right)h_{2}\left(\tau,v(\tau)\right)dB\left(\tau\right), \tag{2.9}$$
$$z\left(t\right)=z\left(0\right)+\int_{0}^{t}g'\left(\tau\right)I_{1}\left(\tau,z(\tau)\right)d\tau+\int_{0}^{t}g'\left(\tau\right)I_{2}\left(\tau,z(\tau)\right)dB(\tau), \tag{2.10}$$

where B, is standard Brownian motion while σ_1, σ_2, σ_3, σ_4 are stochastic constant.

3. Model Analysis

In this section, we will introduce some essential concepts such as proof of positivity and boundedness, determination of equilibrium points, proof of stability, and proof of existence and uniqueness.

3.1. Preliminaries Definitions.

In this subsection, some necessary preliminary concepts that are needed in the next sections are reviewed from the reference [4, 5, 7].

Definition 3.1. Caputo fractional derivative of order $\alpha >\ 0$ of a continuous and differentiable function f : $(0,\ \infty)\to$R, is given as:

$${}_{0}^{C}\mathrm{D}^{\alpha}f\left(t\right)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\left(t-x\right)^{-\alpha}\frac{d}{dx}f\left(x\right)dx,\quad 0<\alpha\leq 1.$$

Definition 3.2. Let f : $(0,\ \infty)\to$R, The Riemann-Liouville fractional integral of a function f of order $\alpha >\ 0$ is defined as follows:

$${}_{0}^{C}\mathrm{I}^{\alpha}f\left(t\right)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\left(t-x\right)^{\alpha-1}f\left(x\right)dx,\quad 0<\alpha\leq 1.$$

Definition 3.3. Let f : $(0,\ \infty)\to$R, the Caputo's version of a global derivative of a function f of order $\alpha >\ 0$ is given as:

$${}_{0}^{C}D_{g}^{\alpha}f\left(t\right)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}D_{g}f\left(x\right)\left(t-x\right)^{-\alpha}dx,\quad 0<\alpha\leq 1.$$

Definition 3.4. Let f : (0, ∞)→R, the Riemann-Liouville version of the global derivative of a function f of order $\alpha>$ 0is given as:

$$I_g^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t g'(x) f(x) (t-x)^{\alpha-1} dx, \quad 0 < \alpha \leq 1.$$

Definition 3.5. Let f : (0, ∞)→R, g is a positive increasing function, we write the Riemann-Liouville version of the global derivative of a function f of order $\alpha >$ 0, is given as:

$${}_0^{RL}D_g^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} D_g \int_0^t f(x) (t-x)^{-\alpha} dx, \quad 0 < \alpha \leq 1.$$

Definition 3.6. Consider $f(t)$ as a continuous function and $g(t)$ as a non-constant and positive increasing function. Furthermore, assuming that $K(t)$ is a single or non-singular version kernel. For 0 <α≤1, we define a general derivative of fractional type with the Caputo operator as follows:

$${}_0^{C}D_g^\alpha f(t) = D_g f(t) * k(t).$$

The Riemann-Liouville version is also written as follows

$${}_0^{RL}D_g^\alpha f(t) = D_g (f(t) * k(t)),$$

where * means the convolution operator.

Definition 3.7. Consider $f(t)$ as a continuous function and $g(t)$ as a non-constant and positive increasing function. If the kernel $k(t) = \frac{\exp[\frac{-\alpha}{(1-\alpha)}]}{(1-\alpha)}$, then, the Caputo-Fabrizio version of a global derivative as a new version of the Caputo derivative of a fractional derivative is given by:

$${}_0^{CF}D_g^\alpha f(t) = \frac{1}{(1-\alpha)} \int_0^t D_g f(x) \exp\left[-\alpha \frac{(t-x)}{(1-\alpha)}\right] dx, \quad 0 < \alpha \leq 1.$$

Definition 3.8. Let f : (0, ∞)→R, The Riemann-Liouville fractional integral of a function f of fractional order $\alpha >$ 0 in the Caputo-Fabrizio version is presented as follows:

$${}_0^{CF}I_g^\alpha f(t) = \frac{1-\alpha}{M(\alpha)} g'(x) f(x) + \frac{\alpha}{M(\alpha)} \int_0^t g'(x) f(x) dx, \quad 0 < \alpha \leq 1.$$

Definition 3.9. Consider $f(t)$ as a differentiable and continuous function, and $g(t)$ as a non-constant and positive increasing function. If the kernel $k(t) = \frac{AB(\alpha)}{(1-\alpha)} \mathrm{E}_\alpha[\frac{-\alpha}{(1-\alpha)} t^\alpha]$, then the Atangana-Baleanu version of global derivatives is given by:

$${}_0^{ABC}D_g^\alpha f(t) = \frac{AB(\alpha)}{(1-\alpha)} \int_a^t D_g f(x) \mathrm{E}_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)}\right] dx, \quad 0 < \alpha \leq 1,$$

where ${}_0^{AB}\mathrm{D}^\alpha f(t)$ is a fractional operator with Mittag-Leffler kernel in the Caputo sense with order α with respect to t and with a normalization function that is defined as:

$$AB(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}.$$

When $f(t)$ is not differentiable, then the new fractional derivative is defined as follows:

$${}_0^{ABR}D_g^\alpha f(t) = \frac{AB(\alpha)}{(1-\alpha)} D_g \int_a^t f(x) \mathrm{E}_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)}\right] dx, \quad 0 < \alpha \leq 1.$$

Definition 3.10. Let f : (0, ∞)→R, The Riemann-Liouville fractional integral of a function f of fractional order α in the Atangana-Baleanu version of global derivatives are given by:

$${}_0^{AB}I_g^\alpha f(t) = \frac{1-\alpha}{AB(\alpha)} f(t) g'(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t g'(x) f(x) (t-x)^{\alpha-1} dx, \; 0 < \alpha \leq 1,$$

where $\alpha =$ 0, the initial function is obtained. Otherwise, if α= 1, the ordinary integral is obtained.

3.2. **Positivity and boundedness.** In this section, we discuss the positivity of the solutions for every $t \geq 0$. We will define the subsequent norm as follows:

$$\|f(t)\|_{\infty} = sup_{t\in[0,\tau]} |f(t)| .$$

Now, by considering equation (1.a), we have:

$$\frac{dx(t)}{dt} = r_1 - ax(t)v(t) - d_1x(t) \;=\; r_1 - (av(t) + d_1)x(t), \quad \forall t \geq 0,$$

$$\overset{if\ r_1\geq 0}{\Longrightarrow} \equiv \frac{dx(t)}{dt} \geq -(av(t) + d_1)x(t) \geq -(a|v(t)| + d_1)x(t)$$
$$\geq -\left(a\ sup_{t\in[0,\tau]}|v(t)| + d_1\right)x(t) = -(a\|v(t)\|_{\infty} + d_1)x(t) = -\gamma_{11}x(t), \quad \forall t \geq 0,$$

where γ_{11}=$(a\|v(t)\|_{\infty} + d_1)$. Then we obtained

$$x(t) \geq x_0e^{-\gamma_{11}t}, \quad \forall t \geq 0.$$

Likewise, by considering Equations (1.2), (1.3), and (1.4) we have:

$$y(t) \geq y_0e^{-\gamma_{12}t}, \qquad \forall t \geq 0,$$
$$v(t) \geq v_0e^{-\gamma_{13}t}, \qquad \forall t \geq 0,$$
$$z(t) \geq z_0e^{-\gamma_{14}t}, \qquad \forall t \geq 0,$$

where

$$\gamma_{12} = (c\|z(t)\|_{\infty} + d_1 + b), \qquad \text{if } ax(t)v(t) \geq 0,$$
$$\gamma_{13} = (m_1 + d_1), \qquad \text{if } (by(t) - h_2y(t)z(t)) \geq 0,$$
$$\gamma_{14} = ((h_2 - c)\|y(t)\|_{\infty} + d_1), \qquad \text{if } m_1v(t) \geq 0.$$

Now, we will discuss the boundedness of the (1.1)–(1.4), solutions for all $t \geq 0$.
Let $N(t) = (x(t) + y(t) + v(t) + z(t))$. So, we get:

$$\frac{dN(t)}{dt} = r_1 - d_1N(t).$$

By considering $N(0) = 0$, we can write:

$$N(t) = \frac{r_1}{d_1}\left(1 - e^{-d_1t}\right),$$

then for all $t \geq 0$, we have:

$$N(t) \leq \frac{r_1}{d_1},$$

thus, the region which is called the feasible region, where the solution to the model is invariant and biologically feasible, is defined by:

$$\varphi = \left\{(x(t), y(t), v(t), z(t))\epsilon R^{+4} \;\middle|\; N(t) \leq \frac{r_1}{d_1}, x_0 \geq 0, y_0 \geq 0, v_0 \geq 0, z_0 \geq 0\right\}.$$

So, the region is positive for the system. Also, for $t\to\infty$ we obtain:

$$\lim_{t\to\infty} \sup N(t) \;\leq \frac{r_1}{d_1}.$$

So, in the feasible region, all of the solutions are uniformly bounded.

3.3. **The equilibrium points.** In this section, we will obtain the equilibrium points for both the disease-free and endemic states.

By considering $(x(t) \neq 0,\ y(t) = 0,\ v(t) = 0,\ z(t) = 0)$ in the Equations (1.1)–(1.4), the equilibrium point in the disease-free states of the model is obtained as follows:

$$E_0 = (\frac{r_1}{d_1}, 0, 0, 0).$$

Then, by considering $(x(t) \neq 0,\ y(t) \neq 0,\ v(t) \neq 0,\ z(t) \neq 0)$ in the Equations (1.1)–(1.4), which is solved simultaneously, we get the equilibrium point for the endemic state, shown by E_1 as follows:

$$\begin{cases} \dot{x}\,(t) = r_1 - ax(t)v(t) - d_1x(t) = 0, \\ \dot{y}\,(t) = ax(t)v(t) - cy(t)z(t) - d_1y(t) - by(t) = 0, \\ \dot{v}\,(t) = by(t) - h_2y(t)z(t) - d_1v(t) - m_1v(t) \ = 0, \\ \dot{z}\,(t) = cy(t)z(t) - h_2y(t)z(t) - d_1z(t) + m_1v(t) = 0. \end{cases} \tag{3.1}$$

The system (3.1), yields

$$x_1(t) = \frac{r_1}{av_1(t) + d_1}, \qquad y_1\,(t) = \frac{ar_1v_1\,(t)}{(av_1\,(t) + d_1)\,(cz_1\,(t) + d_1 + b)}, \qquad \text{and} \qquad v_1\,(t) = \frac{h_2\beta z_1\,(t) - b\beta}{d_1 + m_1}.$$

So that

$$\beta = -\frac{ar_1v_1(t)}{(av_1(t) + d_1)\,(cz_1(t) + d_1 + b)}, \qquad z_1(t) = \frac{m_1\gamma}{(c\beta + h_2\beta - d_1)}, \quad \text{and} \qquad \gamma = \frac{b\beta - h_2\beta z_1(t)}{d_1 + m_1}.$$

Thus, we have:

$$E_1 = (x_1(t), y_1(t), v_1(t), z_1(t))\,.$$

3.4. **The Stability.** In this part of the article, to be able to prove that the obtained equilibrium points are locally and asymptotically stable, we first obtain the reproduction number of the Equations (1.1)–(1.4) using the next-generation matrix. Note that threshold quantity plays an important role in the epidemiology of the disease, and the condition of the disease can be determined based on its numerical value.

3.4.1. *Reproduction number.* Let $\dot{x}\,(t) = 0$, and $E\,(\mathrm{x\,(t) \neq 0,\ y\,(t) = 0,\ v\,(t) = 0,\ z\,(t) = 0})$. So, we get:

$$E_0 = (\frac{r_1}{d_1}, 0, 0, 0).$$

So, we can write equations (1.1)–(1.4) as a matrix form:

$$\begin{bmatrix} \dot{y} \\ \dot{\mathrm{v}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & \frac{r_1}{d_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \\ z \end{bmatrix} - \begin{bmatrix} d_1 + b & 0 & 0 \\ -b & d_1 + m & 0 \\ 0 & -m_1 & d_1 \end{bmatrix} \begin{bmatrix} y \\ v \\ z \end{bmatrix}.$$

By considering

$$\mathrm{A}= \begin{bmatrix} 0 & \frac{r_1}{d_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathrm{B}= \begin{bmatrix} d_1 + b & 0 & 0 \\ -b & d_1 + m & 0 \\ 0 & -m_1 & d_1 \end{bmatrix},$$

we obtained

$$AB^{-1} = \begin{bmatrix} \frac{abr_1}{d_1(d_1+b)(d_1+m)} & \frac{ar_1}{d_1(d_1+m)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

And now, to calculate the reproduction number, we get the eigenvalue of the above matrix:

$$AB^{-1} - \lambda I = \begin{bmatrix} \frac{abr_1}{d_1(d_1+b)(d_1+m)} - \lambda & \frac{ar_1}{d_1(d_1+m)} & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix},$$

therefore, the reproduction number is obtained in follows:

$$R_0 = \frac{abr_1}{d_1(d_1+b)(d_1+m)} \ .$$

3.4.2. *Local stability of the system of ordinary differential equations.*

Theorem 3.11. *For the system of linear ODEs, if all the eigenvalues,λ, have negative real parts, then the solution is stable. If at least one eigenvalue, λ, has a positive real part, then the solution is unstable.*

3.4.3. *Routh-Hurwitz Conditions.* One of the methods to prove stability is to use the Roth-Horwitz condition, and consider the nth-order ordinary differential equation as follows:

$$\frac{d^n}{dt^n}x = Jx.$$

Then, the characteristic equation is as follows:

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0 \ ,$$

where a_i , $i = 1, \ldots, n$ are real number. The necessary and sufficient conditions on the a_i such that the roots of $P(\lambda)$ have $Re\ \lambda < 0$ are the Routh-Hurwitz conditions,

$$D_1 = a_1 > 0 \ , \ D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0, \ D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0,$$

$$D_i = \begin{vmatrix} a_1 & a_3 & a_5 & \ldots \\ 1 & a_2 & a_4 & \ldots \\ 0 & a_1 & a_3 & \ldots \\ 0 & 1 & a_2 & \ldots \\ 0 & 0 & \ldots & a_i \end{vmatrix} > 0,$$

where $a_i = 0$ for $i > n$.

Now, to prove the stability of the mentioned model, first, we obtain the following Jacobian matrix of Equations (1.1)–(1.4):

$$J(x(t), y(t), v(t), z(t)) = \begin{bmatrix} -av(t) - d_1 & 0 & -ax(t) & 0 \\ av(t) & -cz(t) - d_1 - b & ax(t) & -cy(t) \\ 0 & b - h_2z(t) & -d_1 - m_1 & -h_2y(t) \\ 0 & cz + h_2z(t) & m_1 & cz(t) + h_2y(t) - d_1 \end{bmatrix}. \tag{3.2}$$

By replacing the equilibrium point $E_0 = (\frac{r_1}{d_1}, 0, 0, 0)$in the Jacobi matrix (3.2), we check the establishment of stability conditions.

$$\left| J\left(\frac{r_1}{d_1}, 0, 0, 0\right) - \lambda I \right| = \begin{vmatrix} -d_1 - \lambda & 0 & -a\left(\frac{r_1}{d_1}\right) \\ 0 & -d_1 - b - \lambda & a\left(\frac{r_1}{d_1}\right) \\ 0 & 0 & -d_1 - m_1 \ - \lambda \\ 0 & 0 & m_1 \end{vmatrix} = 0.$$

So, we obtain the eigenvalues:

$$\lambda_1 = -d_1 < 0 \ , \ \lambda_2 = -d_1 < 0 \ ,$$

$$|J(E_0) - \lambda I| = \begin{vmatrix} -d_1 - b - \lambda & \frac{ar_1}{d_1} \\ b & -d_1 - m_1 \ - \lambda \end{vmatrix} = 0,$$

$$\lambda^2 + (2d_1 + b + m_1 \ )\lambda + \left({d_1}^2 + m_1d_1 + bm_1 + bd_1 - \frac{abr_1}{d_1} \right) = 0,$$

where $A_1 = 2d_1 + b + m_1$,and $A_2 = {d_1}^2 + m_1d_1 + bm_1 + bd_1 - \frac{abr_1}{d_1}$, so, we can see:
$A_1 > 0$. And also, if $R_0 = \frac{abr_1}{d_1(d_1+b)(d_1+m)} < 0$, we have $A_2 > 0$.

By using the Routh-Hurwitz condition, we can establish stability for a second-degree polynomial when all coefficients are positive. So, as a result, E_0 is a stable equilibrium point.
Now, to prove the stability in the endemic state, we insert $E_1 = (x_1, y_1, v_1, z_1)$ in the Jacobian matrix (3.2).

$$|J(E_1) - \lambda I| = \begin{vmatrix} -av_1 - d_1 - \lambda & 0 & -ax_1 & 0 \\ av_1 & -cz_1 - d_1 - b - \lambda & ax_1 & ax_1 \\ 0 & b - h_2 z_1 & -d_1 - m_1 - \lambda & -d_1 - m_1 - \lambda - h_2 y_1 \\ 0 & cz_1 + h_2 z_1 & m_1 & cz + h_2 y - d_1 - \lambda \end{vmatrix} = 0,$$

by obtaining the characteristic equation as follows:

$$\begin{aligned} &\lambda^4 + (A + d_1 + F - I - B)\lambda^3 + (AF - AB - AI - d_1F - DF + BI - FI - CF - DH)\lambda^2 \\ &- (ABI - ABF - AFI - ACE - ADH + d_1BI - d_1BF - d_1FI - ACE)\lambda \\ &+ (ABFI + AGm_1 + ACEI - ACGH - ADEm_1 - ADHF + BFTd_1 + Gd_1m_1 + d_1CEI \\ &- CGHd_1 - DEm_1d_1 - DHFd_1 - ACEI + ACHG) = 0, \end{aligned}$$

where,

$$\begin{aligned} &\mathrm{A} = \mathrm{av}_1, && B = -d_1 - b_1 - cy_1, && C = c\,x_1, && D = -cy_1, && E = b - h_2 z_1, \\ &F = d_1 + m_1, && G = -h_2 y_1, && H = cz_1 + h_2 z_1, && I = cy_1 + h_2 y - d_1, \end{aligned}$$

and by considering the coefficients of the given characteristic equation as a fourth-order polynomial, we obtain:

$$\lambda^4 + \mathrm{m}_1\lambda^3 + \mathrm{m}_2\lambda^2 - \mathrm{m}_3\lambda + \mathrm{m}_4 = 0.$$

If $\mathrm{R}_0 = \frac{\mathrm{abr}_1}{\mathrm{d}_1(\mathrm{d}_1+\mathrm{b})(\mathrm{d}_1+\mathrm{m})} > 0$,

$$m_0, m_1 > 0\ ,\quad m_1m_2 - m_0m_3 > 0, \quad (m_1m_2 - m_0m_3)(m_3) - {m_1}^2 m_4 > 0, \quad m_4 > 0,$$

where

$$\mathrm{m}_0 = 1, \mathrm{m}_1 = (\mathrm{A} + \mathrm{d}_1 + \mathrm{F} - \mathrm{I} - \mathrm{B}), \mathrm{m}_2 = (\mathrm{AF} - \mathrm{AB} - \mathrm{AI} - \mathrm{d}_1\mathrm{F} - \mathrm{DF} + \mathrm{BI} - \mathrm{FI} - \mathrm{CF} - \mathrm{DH}),$$

$$\mathrm{m}_3 = (\mathrm{ABI} - \mathrm{ABF} - \mathrm{AFI} - \mathrm{ACE} - \mathrm{ADH} + \mathrm{d}_1\mathrm{BI} - \mathrm{d}_1\mathrm{BF} - \mathrm{d}_1\mathrm{FI} - \mathrm{ACE}),$$

$$\begin{aligned} \mathrm{m}_4 = &(\mathrm{ABFI} + \mathrm{AGm}_1 + \mathrm{ACEI} - \mathrm{ACGH} - \mathrm{ADEm}_1 - \mathrm{ADHF} + \mathrm{BFTd}_1 + \mathrm{Gd}_1\mathrm{m}_1 + \mathrm{d}_1\mathrm{CEI} - \mathrm{CGHd}_1 \\ &- \mathrm{DEm}_1\mathrm{d}_1 - \mathrm{DHFd}_1 - \mathrm{ACEI} + \mathrm{ACHG}). \end{aligned}$$

Therefore, the Routh-Hurwitz condition is valid for the stability of the 4th-degree equation, and as a result, E_1 is a stable equilibrium point.

3.4.4. *Local stability of the system of fractional differential equations.* In this section, we examine the stability of the fractional differential equation system of the cancer virus therapy model.
Lemma 3.1. Let x^* be an equilibrium of the nonlinear system [8] $D_t^\alpha x = f(x)$,
Then, x^* is locally asymptotically stable if for all eigenvalues λ of the Jacobian matrix:

$$|\arg\lambda\ | > \alpha\frac{\pi}{2},$$

or in other words, their real parts are negative.
We have already checked in the asymptotic stability section for ordinary differential equations: when, Re-$\lambda < 0$, the Routh-Hurwitz condition is satisfied, and since, according to the calculations made in the previous section, the Routh-Hurwitz condition was established by accepting conditions of R_0 Therefore, the roots of characteristic equations mentioned in the section of fractional differential equations also apply in stability conditions.

3.5. **Uniqueness and Existence.**

3.5.1. *Uniqueness.* Let's by Theorem 3.12, present the conditions for finding a unique solution to the nonlinear state of a stochastic equation with a global derivative, referencing Atangana's papers [7] and [6].

Theorem 3.12. *Assuming that there exist two positive constants* $\overline{K}$ *and* K *such that:*

(1) *(Lipschitz condition)* $\forall x,\ x_1 \in R$ *and* $\forall t \in [t_0, T]$, $|f(t,\ x(t)) - f(t,\ x_1(t))|^2 \le \overline{K}|x(t) - x_1(t)|^2$,
(2) *(Linear growth condition)* $\forall\,(x,\ t) \in R \times [t_0, T]$, $|f(t,\ x(t))|^2 \le K(1 + |x(t)|^2)$.

By considering $g^{'}(t)$ is bounded and continuous. Then the nonlinear stochastic equation has a unique solution in $M^2(\ [t_0,\ T], R)$.
Now, the linear growth condition holds, then $\forall n \ge 1$, we define the stopping periodic:

$$\lambda_n = \ inf\{T,\ inf\{t\ \in\ [t_0,\ T]\ :\ |x(t)|\ >\ m\ \}\ \}.$$

We can see clearly $\lim_{n\to\infty} \lambda_n\ =\ T$.
Also, we define the sequence $x_n(t) = x(inf(t,\ \lambda_m)),\ \forall t\ \in\ [t_0,\ T]$.
Indeed $x_n(t)$ meets the requirements that:

$$x_n\,(t) = x\,(0) + \int_{t_0}^{t} g'\,(\tau)\, f_1\,(\tau, x(\tau))d\tau + \int_{t_0}^{t} g'\,(\tau)\, f_2\,(\tau, x\,(\tau))\, dB\,(\tau)\,.$$

Thus

$$|x_n\,(t)|^2 = \left| x\,(0) + \int_{t_0}^{t} g'\,(\tau)\, f_1\,(\tau, x(\tau))d\tau + \int_{t_0}^{t} g'\,(\tau)\, f_2\,(\tau, x(\tau))\, dB(\tau) \right|^2.$$

We enjoy the result of the following inequality $|a\, +\, b\, +\, c|^2 \le 3|a|^2 + 3|b|^2 + 3|c|^2$.
So, we have:

$$|x_n\,(t)|^2 \le 3|\ x\,(0)|^2 + 3\left|\int_{t_0}^{t} g'\,(\tau)\, f_1\,(\tau, x\,(\tau))d\tau\right|^2 + 3\left|\int_{t_0}^{t} g'\,(\tau)\, f_2\,(\tau, x\,(\tau))\, dB\,(\tau)\right|^2.$$

By applying the growth linear condition and the Holder-inequality, we get:

$$|x_n\,(t)|^2 \le 3|\ x\,(0)|^2 + 3\mathrm{K}(g(t) - g(t_0))(1 + |x_n\,(\tau)|)^2 d\tau + 3\left|\int_{t_0}^{t} g'\,(\tau)\, f_2\,(\tau, x\,(\tau))\, dB\,(\tau)\right|^2.$$

Thus

$$sup_{t\ \in\ [t_0 l]}\, |x_n\,(l)|^2 \le 3|\ x\,(0)|^2 + 3\mathrm{K}(g(T) - g(t_0))(1 + |x_n\,(\tau)|)^2 d\tau + 3\left|\int_{t_0}^{t} g'\,(\tau)\, f_2\,(\tau, x\,(\tau))\, dB\,(\tau)\right|^2.$$

Using the expectation formula, we get:

$$E(sup_{t\ \in\ [t_0 l]}\, |x_n\,(l)|^2) \le 3E|\ x\,(0)|^2 + 3K(g(T) - g(t_0))(1 + E\,|x_n\,(\tau)|)^2 d\tau +\ 3E\left|\int_{t_0}^{t} g'\,(\tau)\, f_2\,(\tau, x\,(\tau))\, dB\,(\tau)\right|^2.$$

Thus

$$E\left(|x_n\,(l)|^2\right) \le 3E|\ x\,(0)|^2 + 3K\,(g\,(T) - g\,(t_0)) \int_{t_0}^{t} (1 + E\,|x_n\,(\tau)|)^2 d\tau + 12E \int_{t_0}^{t} |g'\,(\tau)\, f_2\,(\tau, x\,(\tau))\, dB\,(\tau)|^2.$$

By considering the condition of linear growth, then we have:

$$\begin{aligned} E\left(sup_{t\ \in\ [t_0,\ l]}|x_n\,(l)|^2\right) &\le 3E|\ x\,(0)|^2 + 3K\,(g\,(T) - g\,(t_0)) \int_{t_0}^{t} \left(1 + E\ |x_n\,(\tau)|^2\right) d\tau \\ &+ 12K \int_{t_0}^{t} E|g'\,(\tau)|^2 (1 + |x_n\,(\tau)|^2) d\tau \\ &\le 3E|\ x\,(0)|^2 + 3K\,(g\,(T) - g\,(t_0)) \int_{t_0}^{t} (1 + Esup_{t\ \in\ [t_0,\ l]}\ |x_n\,(l)|^2) d\tau \end{aligned}$$

$$+12K\int_{t_0}^{t}E\Big|sup_{t\,\in\,[t_0 l]}|g'(l)|^2\,\Big|^2(1+sup_{t\,\in\,[t_0,\,l]}|x_n(l\,)|^2)d\tau$$

$$\leq 3E|\,x(0)|^2+3K\left(g(T)-g(t_0)+E\|g'\|_\infty^2\right)\int_{t_0}^{t}\left(1+Esup_{t\,\in\,[t_0,\,l]}\,|x_n(l)|^2\right)d\tau.$$

Adding 1 on both sides, we get:

$$E\left(sup_{t\,\in\,[t_0,\,T]}|x_n(l)|^2\right)<1+E\left(sup_{t\,\in\,[t_0,T]}|x_n(t)|^2\right)$$

$$\leq 1+3E|\,x(0)|^2+3K\left(g(T)-g(t_0)+E\|g'\|_\infty^2\right)\int_{t_0}^{t}\left(1+Esup_{t\,\in\,[t_0,\,l]}\,|x_n(l)|^2\right)d\tau.$$

By using the Gronwall inequality, we obtain:

$$E\left(sup_{t\,\in\,[t_0,\,T]}|x_n(t)|^2\right)\leq(1+3E|\,x(0)|^2)\exp(3K(\,\left(g(T)-g(t_0)+4E\|g'\|_\infty^2\right)\,(T-t_0))).$$

Finally taking $\lim_{n\to\infty}$ on both sides, we get:

$$E\left(sup_{t\,\in\,[t_0,\,T]}|x(t)|^2\right)\leq(1+3E|\,x(0)|^2)\exp(3K(\,\left(g(T)-g(t_0)+4E\|g'\|_\infty^2\right)\,(T-t_0))).$$

Which provides the requested result.

We now present the uniqueness. Let $x(t)$ and $x_1(t)$ be two solutions of Equation (2.7). Thus, by the above, inequality, $(t),\ x_1(t)\ \in M^2(\,[t_0,\ T],R)$. We have that:

$$x(t)-x_1(t)=\int_{t_0}^{t}g'(\tau)\left(f_1(\tau,x(\tau))-f_1(\tau,x_1(\tau))\right)d\tau+\int_{t_0}^{t}g'(\tau)\left(f_2(\tau,x(\tau))-f_2(\tau,x_1(\tau))\right)dB(\tau).$$

Then, by Holder-inequality, we get:

$$|x(t)-x_1(t)|^2\leq 2\left(g(t)-g(t_0)\right)\int_{t_0}^{t}|f_1(\tau,x(\tau))-f_1(\tau,x_1(\tau))|^2d\tau$$

$$+2\left|\int_{t_0}^{t}g'(\tau)\left|f_2(\tau,x(\tau))-f_2(\tau,x_1(\tau))\right|dB(\tau)\right|^2.$$

Using the Lipschitz condition, we have:

$$sup_{t_0\leq t\leq T}|x(t)-x_1(t)|^2\leq 2\left(g(t)-g(t_0)\right)\overline{K}\int_{t_0}^{t}|x(\tau)-x_1(\tau)|^2d\tau+2\left|\int_{t_0}^{t}g'(\tau)\left|f_2(\tau,x(\tau))-f_2(\tau,x_1(\tau))\right|dB(\tau)\right|^2.$$

We have:

$$sup_{t_0\leq t\leq T}|x(t)-x_1(t)|^2\leq 2\left(g(T)-g(t_0)\right)\overline{K}\int_{t_0}^{t}sup|x(\tau)-x_1(\tau)|^2d\tau$$

$$+2\left|\int_{t_0}^{t}g'(\tau)\left|f_2(\tau,x(\tau))-f_2(\tau,x_1(\tau))\right|dB(\tau)\right|^2.$$

Thus

$$E(sup_{t_0\leq t\leq T}|x(t)-x_1(t)|^2)\leq 2\left(g(T)-g(t_0)\right)\overline{K}\int_{t_0}^{t}sup|x(\tau)-x_1(\tau)|^2d\tau$$

$$+2E\left|\int_{t_0}^{t}g'(\tau)\left|f_2(\tau,x(\tau))-f_2(\tau,x_1(\tau))\right|dB(\tau)\right|^2$$

$$\leq(2\left(g(T)-g(t_0)\right)+8E\|g'\|_\infty{}^2\overline{K})\int_{t_0}^{t}sup|x(\tau)-x_1(\tau)|^2d\tau$$

$$+2E\left|\int_{t_0}^{t}g'(\tau)\left|f_2(\tau,x(\tau))-f_2(\tau,x_1(\tau))\right|dB(\tau)\right|^2,$$

by the Gronwall inequality, we get:

$$E\left(sup_{t_0\le t\le T}\left|x\left(t\right)-x_1\left(t\right)\right|^2\right)=0,\quad \forall t\ \in\ [t_0,\ T].$$

Thus, we have $x\left(t\right)=x_1\left(t\right), \forall t\ \in\ [t_0,\ T]$ almost surely. Therefore, we can conclude this yields the uniqueness. Now, we present the existence using the Picard iterative approach. In case, we set $x(t_0)\ = x_0$ but $n\ge 1$, the iteration is defined as follows:

$$x_n\left(t\right)=x\left(0\right)+\int_{t_0}^{t} g'\left(\tau\right)f_1\left(\tau,x(\tau)\right)d\tau+\int_{t_0}^{t} g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)dB\left(\tau\right).$$

We have:

$$E(\int_{t_0}^{t}|x_0|^2 d\tau)=E(|x_0|^2(T{-}t_0))\le\infty.$$

Therefore $x_0\ \in M^2([t_0,\ T],R)$. We assume for $n\ge 1$, $x_n\left(t\right)\ \in M^2([t_0,\ T],\ R\ )$, we now prove that $x_{n+1}\left(t\right)\ \in M^2([t_0,\ T],R\ )$. Then, we evaluate:

$$E\int_{t_0}^{T}\left|x_{n+1}\left(t\right)\right|^2 d\tau.$$

Thus, we have:

$$\begin{aligned}\left|x_{n+1}\left(t\right)\right|^2&=\left|x\left(0\right)+\int_{t_0}^{t} g'\left(\tau\right)f_1\left(\tau,x(\tau)\right)d\tau+\int_{t_0}^{t} g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)dB(\tau)\right|^2\\&\le 3|\ x\left(0\right)|^2+3\left(g\left(T\right)-g\left(t_0\right)\right)\left(T-t_0\right)\mathrm{K}(1{+}x_n\left(\tau\right)^2)+\ 3|\int_{t_0}^{t} g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)dB\left(\tau\right)|^2.\end{aligned}$$

By considering $B(t)$ being a Brownian motion, and also, assume $B(t)$ is bounded variation on $[t_0,\ T]$, thus by denoting $\psi_{t_0}^{t}(B)$, the total variation of $B(t)$ on $[t_0,\ T]$, we have that:

$$\begin{aligned}&|\int_{t_0}^{t} g'\left(\tau\right)f_2\left(\tau,x_n\left(\tau\right)\right)dB\left(\tau\right)|^2\le max_{t_0\le l\le t}|g'\left(l\right)f_2\left(l,x_n\left(l\right)\right)|^2(\psi_{t_0}^{t}(B))^2\\&\le max_{t_0\le l\le t}|f_2\left(l,x_n\left(l\right)\right)|^2 max_{t_0\le l\le t}|g'\left(l\right)|^2\left(\psi_{t_0}^{t}\left(B\right)\right)^2.\end{aligned}$$

Using the linear growth condition of f_2, we get:

$$\begin{aligned}|\int_{t_0}^{t} g'\left(\tau\right)f_2\left(\tau,x_n\left(\tau\right)\right)dB\left(\tau\right)|^2&\le(1+max_{t_0\le l\le t}|x_n\left(l\right)|^2)max_{t_0\le l\le t}|g'\left(l\right)|^2\left(\psi_{t_0}^{t}\left(B\right)\right)^2\\&\le(1+max_{t_0\le t\le T}|x_n\left(t\right)|^2)max_{t_0\le t\le T}|g'\left(t\right)|^2\left(\psi_{t_0}^{t}\left(B\right)\right)^2.\end{aligned}$$

So, we obtain:

$$\begin{aligned}\left|x_{n+1}\left(t\right)\right|^2&\le 3|\ x\left(0\right)|^2+3\mathrm{K}\left(g\left(T\right)-g\left(t_0\right)\right)\left(T-t_0\right)(1+\left|x_n\left(\tau\right)\right|)^2\\&+3(1+max_{t_0\le l\le t}|x_n\left(l\right)|^2)max_{t_0\le l\le t}|g'\left(l\right)|^2\left(\psi_{t_0}^{t}\left(B\right)\right)^2.\end{aligned}$$

Then

$$\begin{aligned}E\int_{t_0}^{T}\left|x_{n+1}\left(t\right)\right|^2 dt&\le 3E\int_{t_0}^{T}|\ x\left(0\right)|^2 dt+3\mathrm{K}\left(g\left(T\right)-g\left(t_0\right)\right)\left(T-t_0\right)\times\left\{E\left(T-t_0\right)+E\int_{t_0}^{T}|\ x_n\left(t\right)|^2 dt)\right\}\\&+max_{t_0\le t\le T}|g'\left(t\right)|^2\left(\psi_{t_0}^{t}\left(B\right)\right)^2\left\{E\left(T-t_0\right)+Emax_{t_0\le t\le T}|\ x_n\left(t\right)|^2\right\}.\end{aligned}$$

$x_0, x_n\ \in M^2([t_0,\ T],R\ )$, thus:

$$E\int_{t_0}^{T}|x_0|^2 dt+E\int_{t_0}^{T}|x_n(t)|^2 dt<\infty.$$

Thus $x_{n+1}(t) \in M^2([t_0,\ T],\ R\ )$, which shows that $\forall n \geq 0,\ x_n(t) \in M^2([t_0,\ T], R\ )$, with this in hand, we can evaluate:

$$|x_1(t) - x_0(t)|^2 \leq 2\left|\int_{t_0}^{t} g'(\tau) f_1(\tau, x_0(\tau)) d\tau\right|^2 + 2\left|\int_{t_0}^{t} g'(\tau) f_1(\tau, x_0(\tau)) dB(\tau)\right|^2.$$

Therefore, using Holder-inequality and linear growth conditions, we obtain:

$$|x_1(t) - x_0(t)|^2 \leq 2(g(T) - g(t_0))(T - t_0) k(1 + |x_0(t)|^2) + 2\left|\int_{t_0}^{t} g'(\tau) |f_2(\tau, x_0(\tau))| dB(\tau)\right|^2.$$

And

$$E|x_1(t) - x_0(t)|^2 \leq 2(g(T) - g(t_0))(T - t_0) k(1 + E|x_0(t)|^2) + 8E\int_{t_0}^{t} |g'(\tau)|^2 |f_2(\tau, x_0(\tau))|^2 d\tau$$

$$\leq 2(g(T) - g(t_0))(T - t_0)\left(1 + E|x_0(t)|^2\right) + 8K\left(1 + E|x_0(t)|^2\right)(T - t_0)\|g'\|_{\infty}^2 \leq \aleph,$$

where

$$\aleph = 2(T - t_0)\left(1 + E|x_0(t)|^2\right)\left\{(g(T) - g(t_0)) + 4\|g'\|_{\infty}^2\right\}.$$

For $n =\ 0$, the inequality holds. We assume $\forall t \in\ [t_0,\ T]$ the inequality holds, then we prove at $n + 1$

$$E|x_{n+2}(t) - x_{n+1}(t)|^2 \leq N\int_{t_0}^{t} E|x_{n+1}(t) - x_n(t)|^2 d\tau$$

$$\leq N\int_{t_0}^{t} \frac{\aleph N^n |T - t_0|^n}{n!} d\tau \leq \frac{\aleph(N(t - t_0))^{n+1}}{(n + 1)!}.$$

Therefore, at $n\ -\ 1$, also, we have

$$E\left(sup_{t_0 \leq t \leq T}|x_{n+1}(t) - x_n(t)|^2\right) \leq \frac{4\aleph(N(t - t_0))^n}{n!}.$$

Thus, by the probability

$$P\left\{sup_{t_0 \leq t \leq T}|x_{n+1}(t) - x_n(t)|^2 > \frac{1}{2^n}\right\} \leq \frac{4\aleph(4N(t - t_0))^n}{n!}.$$

Indeed,

$$\sum_{n=0}^{\infty} \frac{4^{n+1} C(T - t_0)^n}{(n)!} < \infty.$$

The Borel-Cantelli lemma helps to find a positive integer number $n_0 = n_0(w)$, $\forall\ w \in\ \Omega$ that

$$sup_{t_0 \leq t \leq T}|x_{n+1}(t) - x_n(t)|^2 \leq \frac{1}{2^n}, \forall x \geq x_0.$$

It follows that the sum

$$x_0(t) = \sum_{j=0}^{n+1} (x_{j+1}(t) - x_j(t)) = x_n(t),$$

converges uniformly in $[0,\ T]$. Now we take:

$$\lim_{n \to \infty} x_n(t) = x(t)\ .$$

For some $x(t)$ is continuous. Also, the sequence $\forall t \in [0,\ T]$, $\{x_n(t)\}_{n\geq 1}$ is a Cauchy sequence in L_2 also, it follows that $x_n(t) \to x(t)$ in L_2. With the growth property, we can conclude that $x(t) \in M^2([t_0,\ T], R)$. We now have to show that $x(t)$ satisfies the equation:

$$E\left|\int_{t_0}^{t} g'(\tau)\left(f_1(\tau, x_n(\tau)) - f_1(\tau, x(\tau))\right)d\tau\right|^2 + E\left|\int_{t_0}^{t} g'(\tau)\left(f_2(\tau, x_n(\tau)) - f_2(\tau, x(\tau))\right)dB(\tau)\right|^2$$

$$\leq \left(K\left(g(T) - g(t_0)\right) + 4\|g'\|_\infty^2\right)\int_{t_0}^{t} E|x_n(t) - x(t)|^2 d\tau,$$

and $d\tau \to 0,\ as\ n \to \infty$, which satisfies the equation. Therefore, our equation has a unique solution $x(t)$. This completes the proof.

Similarly, by performing the above operations on equations 6.b, 6.c, 6.d. We can obtain the same results. Therefore, we have unique solutions $y(t)$, $v(t)$, and $z(t)$.

With the Theorem 3.12, we can now present the uniqueness.

At first, we want to prove the Lipschitz condition for the stochastic model with the global derivative in Equations (1.1)–(1.4). So, let us $\alpha_1, \overline{\alpha}_1$, $\alpha_2, \overline{\alpha}_2$, α_3, $\overline{\alpha}_3$, α_4 , $\overline{\alpha}_4$ are four positive constants, such that:

$$|f_1(t,x) - f_1(t,x_1)|^2 \leq \alpha_1|x - x_1|^2, \qquad |f_2(t,x) - f_2(t,x_1)|^2 \leq \overline{\alpha}_1|x - x_1|^2,$$
$$|g_1(t,y) - g_1(t,y_1)|^2 \leq \alpha_2|y - y_1|^2, \qquad |g_2(t,y) - g_2(t,y_1)|^2 \leq \overline{\alpha}_2|y - y_1|^2,$$
$$|h_1(t,v) - h_1(t,v_1)|^2 \leq \alpha_3|v - v_1|^2, \qquad |h_2(t,v) - h_2(t,v_1)|^2 \leq \overline{\alpha}_3|v - v_1|^2,$$
$$|I_1(t,z) - I_1(t,z_1)|^2 \leq \alpha_4|z - z_1|^2, \qquad |I_2(t,z) - I_2(t,z_1)|^2 \leq \overline{\alpha}_4|z - z_1|^2,$$

and there are four positive constants β_1, $\overline{\beta}_1$, β_2, $\overline{\beta}_2$, β_3, $\overline{\beta}_3$, β_4, $\overline{\beta}_4$, such that:

$$|f_1(t,x)|^2 \leq \beta_1\left(1 + |x|^2\right), |f_2(t,x)|^2 \leq \overline{\beta}_1\left(1 + |x|^2\right),$$
$$|g_1(t,y)|^2 \leq \beta_2\left(1 + |y|^2\right), |g_2(t,y)|^2 \leq \overline{\beta}_2\left(1 + |y|^2\right),$$
$$|h_1(t,v)|^2 \leq \beta_3\left(1 + |v|^2\right), |h_2(t,v)|^2 \leq \overline{\beta}_3\left(1 + |v|^2\right),$$
$$|I_1(t,z)|^2 \leq \beta_4\left(1 + |z|^2\right), |I_2(t,z)|^2 \leq \overline{\beta}_4\left(1 + |z|^2\right).$$

We have to define the following norm

$$\|\theta\|_\infty = sup_{t\in[0,\tau]}|\theta|^2, \qquad \forall t \in [t_0,\ T].$$

then we have $\forall x(t), x_1(t) \in R^2$and $t \in [0,\ T]$ we have:

$$\begin{aligned} |f_1(t,x(t)) - f_1(t,x_1(t))|^2 &= |(-av(t) - d_1)(x(t) - x_1(t))|^2 \\ &\leq \left\{2a^2|v(t)|^2 + 2{d_1}^2\right\}|x(t) - x_1(t)|^2 \\ &\leq \left\{2{d_1}^2 + 2a^2 sup_{t\in[0,\tau]}|v(t)|^2\right\}|x(t) - x_1(t)|^2 \\ &= \left\{2{d_1}^2 + 2a^2{\|v(t)\|_\infty}^2\right\}|x(t) - x_1(t)|^2 \\ &= \alpha_1|x(t) - x_1(t)|^2, \end{aligned}$$

where $\alpha_1 = \left\{2{d_1}^2 + 2a^2{\|v(t)\|_\infty}^2\right\}$.

Similarity we have,

$$|g_1(t,y(t)) - g_1(t,y_1(t))|^2 \leq \alpha_2|y(t) - y_1(t)|^2,$$

where $\alpha_2 = \left\{ 2{d_1}^2 + 2b^2 + 2c^2{\|z(t)\|_\infty}^2 \right\}$.

$$|h_1(t, v(t)) - h_1(t, v_1(t))|^2 \le \alpha_3 |v(t) - v_1(t)|^2,$$

where $\alpha_3 = \left\{ (d_1 + m_1)^2 + \varepsilon \right\}$.

$$|I_1(t, z(t)) - I_1(t, z_1(t))|^2 \le \alpha_4 |z(t) - z_1(t)|^2,$$

where $\alpha_4 = \left\{ 2(c - h_2)^2 {\|y(t)\|_\infty}^2 + 2{d_1}^2 \right\}$.

And also, we have

$$\begin{aligned} |f_2(t, x(t)) - f_2(t, x_1(t))|^2 &= |(\sigma_1)(x(t) - x_1(t))|^2 \\ &\le \left\{ 2{\sigma_1}^2 \right\} |x(t) - x_1(t)|^2 \\ &\le \overline{\alpha}_1 |x(t) - x_1(t)|^2, \end{aligned}$$

where $\overline{\alpha}_1 = \left\{ 2{\sigma_1}^2 \right\}$. Likewise, we have:

$$|g_2(t, y(t)) - g_2(t, y_1(t))|^2 \le \overline{\alpha}_2 |y(t) - y_1(t)|^2,$$

where $\overline{\alpha}_2 = \left\{ 2{\sigma_2}^2 \right\}$.

$$|h_2(t, v(t)) - h_2(t, v_1(t))|^2 \le \overline{\alpha}_3 |v(t) - v_1(t)|^2,$$

where $\overline{\alpha}_3 = \left\{ 2{\sigma_3}^2 \right\}$.

$$|I_2(t, z(t)) - I_2(t, z_1(t))|^2 \le \overline{\alpha}_4 |z(t) - z_1(t)|^2,$$

where $\overline{\alpha}_4 = \left\{ 2{\sigma_4}^2 \right\}$. Therefore, the Lipschitz condition is proved.

Now, by applying the second condition (Linear growth condition), we get:

$$\begin{aligned} |f_1(t, x)|^2 &= |r_1 - (av(t) + d_1)x(t)|^2 \\ &\le 2|r_1|^2 + (av(t) + d_1)^2 |x(t)|^2 \\ &\le 2|r_1|^2 (1 + 2\left({d_1}^2 + a^2 sup|v(t)|^2 \right) |x(t)|^2) \\ &\le \left(\frac{2{d_1}^2}{|r_1|^2} + \frac{2a^2{\|v(t)\|_\infty}^2}{|r_1|^2} \right) (1 + |x(t)|^2) \\ &\le \beta_1 \left(1 + |x(t)|^2 \right), \end{aligned}$$

under condition $\beta_1 = \left(\frac{2{d_1}^2}{|r_1|^2} + \frac{2a^2{\|v(t)\|_\infty}^2}{|r_1|^2} \right) < 1$.

Similarity we have,

$$\begin{aligned} &|g_1(t, y)|^2 = |ax(t)\, v(t) - (cz(t) + d_1 + b)y(t)|^2 \\ &\le \beta_2 \left(1 + |y(t)|^2 \right), \end{aligned}$$

where $\beta_2 = (a^2 {\|x(t)\|_\infty}^2 {\|v(t)\|_\infty}^2 + (cz(t) + d_1 + b)^2)$.

$$\begin{aligned} &|h_1(t, v)|^2 = |(b - h_2 z(t))\, y(t) - (d_1 + m_1)v(t)|^2 \\ &\le \beta_3 \left(1 + |v(t)|^2 \right), \end{aligned}$$

where $\beta_3 = \left\{ 2(b - h_2\|z(t)\|_\infty)^2 {\|y(t)\|_\infty}^2 + (d_1 + m_1) \right\}$.

$$\begin{aligned} &|I_1(t, z)|^2 = |m_1 v(t) + (cy(t) - h_2 y(t) - d_1)z(t)|^2 \\ &\le \beta_4 \left(1 + |z(t)|^2 \right), \end{aligned}$$

where $\beta_4 = \left\{2({m_1}^2\|v\,(t)\|_\infty{}^2 + \left((c+h_2)\,\|y\,(t)\|_\infty{}^2 - d_1\right)^2\right\}$.

And also, we have

$$|f_2\,(t,x)|^2 = |\sigma_2 x(t)|^2 \le (2{\sigma_2}^2)|x(t)|^2 \le \overline{\beta}_1\left(1 + |x\,(t)|^2\right),$$

where $\overline{\beta}_1 = (2{\sigma_1}^2)$.

Likewise, we have:

$$|g_2\,(t,y)|^2 = |\sigma_2 y(t)|^2 \le \overline{\beta}_2\left(1 + |y\,(t)|^2\right),$$

where $\overline{\beta}_2 = (2{\sigma_2}^2)$.

$$|h_2\,(t,v)|^2 = |\sigma_3 v(t)|^2 \le \overline{\beta}_3\left(1 + |v\,(t)|^2\right),$$

where $\overline{\beta}_3 = (2{\sigma_3}^2)$

$$|I_2\,(t,y)|^2 = |\sigma_4 z(t)|^2 \le \overline{\beta}_4\left(1 + |z\,(t)|^2\right),$$

where $\overline{\beta}_4 = (2{\sigma_4}^2)$.

Both conditions of the theorem are confirmed. So, it can be proved, the system has a unique solution.

3.5.2. *Extinction.* In this subsection, we will prove the existence solution of the system by definition of extinction space class. We present the definition of space class extinction as follows:

$$\langle D\,(t)\rangle = \lim_{t\to\infty}\frac{1}{t}\int_0^t D\,(\tau)\,d\tau.$$

First, we define the class $x(t)$. By applying the integral to both sides of the Equation (1.5), $x(t)$ can be obtained:

$$x\,(t) - x\,(0) = \int_0^t [r_1 - ax\,(t)\,v\,(t) - d_1 x\,(t)]d\tau + \int_0^t [\sigma_1 x\,(\tau)]\,dB\,(\tau)\,, \tag{3.3}$$

by dividing both sides of the Equation (3.3) by t and taking the limit from it when the parameter t tends to ∞, we have:

$$\lim_{t\to\infty}\langle x\,(t)\rangle = \frac{r_1}{d_1}\;.$$

Similarity we obtain:

$$\lim_{t\to\infty}\langle y\,(t)\rangle = 0, \lim_{t\to\infty}\langle v\,(t)\rangle = 0, \lim_{t\to\infty}\langle z\,(t)\rangle = 0.$$

Therefore, the existence of the answer is also proved.

4. Numerical Schemes for the Model

The following numerical approach has been given to solve our proposed model numerically. We use Brownian motion as well as different random numbers to show the random nature of the model. Also, to preserve the memory feature of the model, we use the Caputo-Fabrizio and Atangana-Baleanu fractional operators. Finally, we will use the numerical scheme with Newton and Lagrange polynomial interpolation to approximate the solution of the random fractional equation.

4.1. **Numerical scheme for stochastic equations with global derivative by considering Lagrange interpolation polynomial.** In this subsection, we present a numerical method for solving the stochastic fractional-order model with a global derivative. To facilitate a more comprehensive analysis, we will consider kernels' exponential decay, and Mittag-Leffler rules. The numerical rules [12] will be employed to develop the numerical scheme.

To begin, we assuming $B(t)$ as a Brownian motion in the Equations (2.7)–(2.10). Also, they are differentiable. By considering Equation (2.7) at two points $t_{\omega+1} = (\omega+1)\Delta t$ and $t_\omega = (\omega)\Delta t$, and by taking the difference between them we can get the following equation:

$$x\left(t_{\omega+1}\right) - x\left(t_\omega\right) = \int_{t_\omega}^{t_{\omega+1}} g'(\tau) f_1\left(\tau, x\left(\tau\right)\right) d\tau + \int_{t_\omega}^{t_{\omega+1}} g'\left(\tau\right) f_2\left(\tau, x\left(\tau\right)\right) B'(\tau) d\tau. \tag{4.1}$$

Let us do some simplifications

$$\begin{aligned} &g'\left(\tau\right) f_1\left(\tau, x\left(\tau\right)\right) = \emptyset_1\left(\tau, x\left(\tau\right)\right), \\ &g'\left(\tau\right) f_2\left(\tau, x\left(\tau\right)\right) B'\left(\tau\right) = \emptyset_2\left(\tau, x\left(\tau\right)\right). \end{aligned} \tag{4.2}$$

So, we write

$$x\left(t_{\omega+1}\right) - x\left(t_\omega\right) = \int_{t_\omega}^{t_{\omega+1}} \emptyset_1\left(\tau, x\left(\tau\right)\right) d\tau + \int_{t_\omega}^{t_{\omega+1}} \emptyset_2\left(\tau, x\left(\tau\right)\right) d\tau. \tag{4.3}$$

We consider the interpolation:

$$P_i\left(\tau\right) = \emptyset_i\left(t_\omega, x_\omega\right) \frac{\tau - t_{\omega-1}}{t_\omega - t_{\omega-1}} - \emptyset_i\left(t_{\omega-1}, x_{\omega-1}\right) \frac{\tau - t_\omega}{t_\omega - t_{\omega-1}}. \tag{4.4}$$

Then, we replace the functions $\emptyset_1\left(t_\omega, x_\omega\right)$ and $\emptyset_2\left(t_\omega, x_\omega\right)$, by its Lagrange interpolation polynomial in Equation (4.4) and putting their values. Also, we consider $g'\left(t_i\right) = g\left(t_i\right) - g\left(t_{i-1}\right)$, $B'\left(t_i\right) = B\left(t_{i-1}\right) - B\left(t_{i-2}\right)$, so we have the following scheme:

$$\begin{aligned} x\left(t_{\omega+1}\right) - x\left(t_\omega\right) = &\frac{3}{2}\left(g\left(t_\omega\right) - g\left(t_{\omega-1}\right)\right) f_1\left(t_\omega, x_\omega\right) - \frac{1}{2}\left(g\left(t_{\omega-1}\right) - g\left(t_{\omega-2}\right)\right) f_1\left(t_{\omega-1}, x_{\omega-1}\right) \\ &+ \frac{3}{2}\left(g\left(t_\omega\right) - g\left(t_{\omega-1}\right)\right)\left(B\left(t_\omega\right) - B\left(t_{\omega-1}\right)\right) f_2\left(t_\omega, x_\omega\right) \\ &- \frac{1}{2}\left(g\left(t_{\omega-1}\right) - g\left(t_{\omega-2}\right)\right)\left(B\left(t_{\omega-1}\right) - B\left(t_{\omega-2}\right)\right) f_2\left(t_{\omega-1}, x_{\omega-1}\right). \end{aligned} \tag{4.5}$$

Similarly, by performing the above operations on the functions $y(t)$, $v(t)$, and $z\left(t\right)$. we can obtain the same result.

4.2. **Numerical scheme for Caputo-Fabrizio order stochastic equation with global derivative version.** First, we introduce the Equation (2.7) with the Caputo-Fabrizio version:

$${}_0^{CF}D_g^a x\left(t\right) = f_1\left(t, x\left(t\right)\right) + f_2\left(t, x\left(t\right)\right), \qquad x\left(t_0\right) = x_0. \tag{4.6}$$

We assume that the function $g(t)$ is a continuous and differentiable function, then we will write

$${}_0^{CF}D_g^a x\left(t\right) = g'(t) f_1\left(t, x\left(t\right)\right) + g'(t) f_2\left(t, x\left(t\right)\right). \tag{4.7}$$

Using the Caputo-Fabrizio integral definition, the Equation (4.7) can be rewritten as follows:

$$\begin{aligned} x\left(t\right) - x\left(0\right) = &\frac{\left(1-\alpha\right)}{M\left(\alpha\right)} g'\left(t\right) f_1\left(t, x\left(t\right)\right) + \frac{\alpha}{M\left(\alpha\right)} \int_0^t g'\left(\tau\right) f_1\left(\tau, x\left(\tau\right)\right) d\left(\tau\right) \\ &+ \frac{\left(1-\alpha\right)}{M\left(\alpha\right)} g'\left(t\right) f_2\left(t, x\left(t\right)\right) B\left(t\right) + \frac{\alpha}{M\left(\alpha\right)} \int_0^t g'\left(\tau\right) f_2\left(\tau, x\left(\tau\right)\right) dB\left(\tau\right). \end{aligned} \tag{4.8}$$

Assuming $B(t)$ as a Brownian motion is differentiable, so we can write

$$x\left(t\right) - x\left(0\right) = \frac{\left(1-\alpha\right)}{M\left(\alpha\right)} g'\left(t\right) f_1\left(t, x\left(t\right)\right) + \frac{\alpha}{M\left(\alpha\right)} \int_0^t g'(\tau) f_1\left(\tau, x\left(\tau\right)\right) d\tau$$

$$+\frac{(1-\alpha)}{M(\alpha)}g'(t)f_2\left(t,x\left(t\right)\right)B\left(t\right)+\frac{\alpha}{M\left(\alpha\right)}\int_0^t g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)B'(\tau)d\tau. \tag{4.9}$$

By considering Equation (4.9) at two points $t_{\omega+1}=(\omega+1)\Delta t$ and $t_\omega=(\omega)\Delta t$, and by takes the difference of them we can get the following equation:

$$\begin{aligned}x(t_{\omega+1})-x(t_\omega)=&\frac{(1-\alpha)}{M(\alpha)}g'(t_{\omega+1})\left[f_1(t_{\omega+1},x(t_{\omega+1}))+f_2(t_\omega,x(t_{\omega+1}))B(t_{\omega+1})\right]\\&-\frac{(1-\alpha)}{M(\alpha)}g'(t_\omega)\left[f_1(t_\omega,x(t_\omega))+f_2(t_\omega,x(t_\omega))B(t_\omega)\right]\\&+\frac{\alpha}{M(\alpha)}\int_{t_\omega}^{t_{\omega+1}}g'(\tau)f_1(\tau,x(\tau))\,d\tau\\&+\frac{\alpha}{M(\alpha)}\int_{t_\omega}^{t_{\omega+1}}g'(\tau)f_2(\tau,x(\tau))B'(\tau)\,d\tau.\end{aligned}\tag{4.10}$$

Let us put some simplicity again for the Equation (4.10);

$$\begin{aligned}&g'\left(\tau\right)f_1\left(\tau,x\left(\tau\right)\right)=\emptyset_1\left(\tau,x\left(\tau\right)\right),\\&g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)B'\left(\tau\right)=\emptyset_2\left(\tau,x\left(\tau\right)\right).\end{aligned}\tag{4.11}$$

By consider the interpolation (4.4) and putting in Equation (4.10). And putting the interpolation polynomials of $g'\left(t\right)$ at this point, we have:

$$\begin{aligned}x\left(t_{\omega+1}\right)-x\left(t_\omega\right)=&\frac{(1-\alpha)}{M\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\left[f_1\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)+f_2\left(t_\omega,x\left(t_{\omega+1}\right)\right)B\left(t_{\omega+1}\right)\right]\\&-\frac{(1-\alpha)}{M\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\left[f_1\left(t_\omega,x\left(t_\omega\right)\right)+f_2\left(t_\omega,x\left(t_\omega\right)\right)B\left(t_\omega\right)\right]\\&+\frac{\alpha}{M\left(\alpha\right)}\int_{t_\omega}^{t_{\omega+1}}\emptyset_1\left(\tau,x\left(\tau\right)\right)d\tau+\frac{\alpha}{M\left(\alpha\right)}\int_{t_\omega}^{t_{\omega+1}}\emptyset_2\left(\tau,x\left(\tau\right)\right)d\tau.\end{aligned}\tag{4.12}$$

Then we replace the functions $\emptyset_1\left(\tau,x\left(\tau\right)\right)$ and $\emptyset_2\left(\tau,x\left(\tau\right)\right)$, by its Lagrange interpolation polynomial in Equation (4.12) and putting their values, so we have the following scheme:

$$\begin{aligned}x\left(t_{\omega+1}\right)-x\left(t_\omega\right)=&\frac{(1-\alpha)}{M\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\left\{f_1\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)+f_2\left(t_\omega,x\left(t_{\omega+1}\right)\right)B\left(t_{\omega+1}\right)\right\}\\&-\frac{(1-\alpha)}{M\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\left\{f_1\left(t_\omega,x\left(t_\omega\right)\right)+f_2\left(t_\omega,x\left(t_\omega\right)\right)B\left(t_\omega\right)\right\}\\&+\frac{\alpha}{M\left(\alpha\right)}\left\{\frac{3}{2}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}f_1\left(t_\omega,x\left(t_\omega\right)\right)\Delta t-\frac{1}{2}\frac{g\left(t_\omega\right)-g\left(t_{\omega-1}\right)}{\Delta t}f_1\left(t_{\omega-1},x\left(t_{\omega-1}\right)\right)\Delta t\right\}\\&+\frac{\alpha}{M\left(\alpha\right)}\left\{\frac{3}{2}\;\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\;f_2\left(t_\omega,x\left(t_\omega\right)\right)\;\frac{B\left(t_{\omega+1}\right)-B\left(t_\omega\right)}{\Delta t}\Delta t\right.\\&\left.-\frac{1}{2}\frac{g\left(t_\omega\right)-g\left(t_{\omega-1}\right)}{\Delta t}f_2\left(t_{\omega-1},x\left(t_{\omega-1}\right)\right)\;\frac{B\left(t_\omega\right)-B\left(t_{\omega-1}\right)}{\Delta t}\Delta t\right\}.\end{aligned}\tag{4.13}$$

If we arrange all operations then we have;

$$\begin{aligned}x\left(t_{\omega+1}\right)-x\left(t_\omega\right)=&\frac{(1-\alpha)}{M\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\left\{f_1\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)+f_2\left(t_\omega,x\left(t_{\omega+1}\right)\right)B\left(t_{\omega+1}\right)\right\}\\&-\frac{(1-\alpha)}{M\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\left\{f_1\left(t_\omega,x\left(t_\omega\right)\right)+f_2\left(t_\omega,x\left(t_\omega\right)\right)B\left(t_\omega\right)\right\}\\&+\frac{\alpha}{M\left(\alpha\right)}\left\{\frac{3}{2}\left(g\left(t_{\omega+1}\right)-g\left(t_\omega\right)\right)f_1\left(t_\omega,x\left(t_\omega\right)\right)-\frac{1}{2}\left(g\left(t_\omega\right)-g\left(t_{\omega-1}\right)\right)f_1\left(t_{\omega-1},x\left(t_{\omega-1}\right)\right)\right\}\\&+\frac{\alpha}{M\left(\alpha\right)}\left\{\frac{3}{2}\;\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\;f_2\left(t_\omega,x\left(t_\omega\right)\right)\left(B\left(t_{\omega+1}\right)-B\left(t_\omega\right)\right)\right.\\&\left.-\frac{1}{2}\frac{g\left(t_\omega\right)-g\left(t_{\omega-1}\right)}{\Delta t}f_2\left(t_{\omega-1},x\left(t_{\omega-1}\right)\right)\left(B\left(t_\omega\right)-B\left(t_{\omega-1}\right)\right)\right\}.\end{aligned}\tag{4.14}$$

Similarly, by applying the above operations on the functions $y(t)$, $v(t)$, and $z\left(t\right)$. we can obtain the same result.

4.3. **Numerical scheme for Atangana-Baleanu order stochastic equation with global derivative version.** First, we introduce the Atangana-Baleanu version of the Equation (2.7):

$$ {}_{0}^{AB}D_{g}^{a}x\left(t\right)=f_1\left(t,x\left(t\right)\right)+f_2\left(t,x\left(t\right)\right),\qquad x\left(t_0\right)=x_0. \tag{4.15}$$

By assuming that $g(\mathrm{t})$ is differentiable, we will write:

$$ {}_{0}^{AB}D_{g}^{a}x\left(t\right)=g'\left(t\right)f_1\left(t,x\left(t\right)\right)+g'\left(t\right)f_2\left(t,x\left(t\right)\right). \tag{4.16}$$

Using the Caputo-Fabrizio integral definition, the previous equation can be rewritten as follows:

$$\begin{aligned} x\left(t\right)-x\left(0\right)=&\frac{\left(1-\alpha\right)}{AB\left(\alpha\right)}g'\left(t\right)f_1\left(t,x\left(t\right)\right)+\frac{\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha\right)}\int_0^t g'\left(\tau\right)f_1\left(\tau,x\left(\tau\right)\right)(t-\tau)^{(\alpha-1)}d\left(\tau\right)\\ &+\frac{\left(1-\alpha\right)}{AB\left(\alpha\right)}g'\left(t\right)f_2\left(t,x\left(t\right)\right)B\left(t\right)+\frac{\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha\right)}\int_0^t g'\left(\tau\right)\left(t-\tau\right)^{(\alpha-1)}dB\left(\tau\right). \end{aligned} \tag{4.17}$$

Assuming $B(t)$ as a Brownian motion is differentiable, so we can write:

$$\begin{aligned} x\left(t\right)-x\left(0\right)=&\frac{\left(1-\alpha\right)}{AB\left(\alpha\right)}g'\left(t\right)f_1\left(t,x\left(t\right)\right)+\frac{\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha\right)}\int_0^t g'\left(\tau\right)f_1\left(\tau,x\left(\tau\right)\right)(t-\tau)^{(\alpha-1)}d\left(\tau\right)\\ &+\frac{\left(1-\alpha\right)}{AB\left(\alpha\right)}g'\left(t\right)f_2\left(t,x\left(t\right)\right)B\left(t\right)+\frac{\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha\right)}\int_0^t g'\left(\tau\right)\left(t-\tau\right)^{(\alpha-1)}B'\left(\tau\right)d\tau. \end{aligned} \tag{4.18}$$

We have at the point $t_{\omega+1}=(\omega+1)\Delta t$

$$\begin{aligned} x\left(t_{\omega+1}\right)-x\left(0\right)=&\frac{\left(1-\alpha\right)}{AB\left(\alpha\right)}g'\left(t_{\omega+1}\right)f_1\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)\\ &+\frac{\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha\right)}\int_0^{t_{\omega+1}} g'\left(\tau\right)f_1\left(\tau,x\left(\tau\right)\right)\left(t_{\omega+1}-\tau\right)^{(\alpha-1)}d\tau\\ &+\frac{\left(1-\alpha\right)}{AB\left(\alpha\right)}g'\left(t_{\omega+1}\right)f_2\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)B\left(t_{\omega+1}\right)\\ &+\frac{\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha\right)}\int_0^{t_{\omega+1}} g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)(t_{\omega+1}-\tau)^{(\alpha-1)}B'\left(\tau\right)d\tau. \end{aligned} \tag{4.19}$$

In the following, we put some simplicity in the Equation (4.19):

$$\begin{aligned} &g'\left(t_\omega\right)f_1\left(t_\omega,x\left(t_\omega\right)\right)=\emptyset_1\left(\tau,x\left(\tau\right)\right),\\ &g'\left(t_\omega\right)f_2\left(t_\omega,x\left(t_\omega\right)\right)B'(t)=\emptyset_2\left(\tau,x\left(\tau\right)\right). \end{aligned} \tag{4.20}$$

Putting simplifications (4.20) and then considering the interpolation:

$$ \P_i\left(\tau\right)=\emptyset_i\left(t_\omega,x_\omega\right)\frac{\tau-t_{\omega-1}}{\Delta t}-\emptyset_i\left(t_{\omega-1},x_{\omega-1}\right)\frac{\tau-t_\omega}{\Delta t}. \tag{4.21}$$

We consider the interpolation (4.21) and putting in Equation (4.19), and divide the integration interval into equally spaced points as $t_\theta=\theta\Delta t,\theta=0,\ldots\omega.$ and implement the interpolation on each of these points. Then we interpolate the derivative of $g(t)$, so we have:

$$\begin{aligned} x_{\omega+1}=x_0+&\frac{\left(1-\alpha\right)}{AB\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_\omega\right)}{\Delta t}\left\{f_1\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)+f_2\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)B(t_{\omega+1})\right\}\\ +&\frac{\alpha(\Delta t)^\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+2\right)}\sum_{\theta=0}^{\omega}\emptyset_1\left(t_\theta,x_\theta\right)\left[(\omega-\theta+1)^{\alpha+1}(\omega-\theta+2+\alpha)-(\omega-\theta)^\alpha(\omega-\theta+2+2\alpha)\right]\\ -&\frac{\alpha(\Delta t)^\alpha}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+2\right)}\sum_{\theta=0}^{\omega}\emptyset_1\left(t_{\theta-1},x_{\theta-1}\right)\left[(\omega-\theta+1)^\alpha-(\omega-\theta)^\alpha\left(\omega-\theta+1+\alpha\right)\right] \end{aligned}$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{AB(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=0}^{\omega}\emptyset_2(t_\theta,x_\theta)\left[-(\omega-\theta)^{\alpha}(\omega-\theta+1+\alpha)-(\omega-\theta)^{\alpha}(\omega-\theta+2+2\alpha)\right]$$
$$-\frac{\alpha(\Delta t)^{\alpha}}{AB(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=0}^{\omega}\emptyset_2(t_{\theta-1},x_{\theta-1})\left[(\omega-\theta+1)^{\alpha}-(\omega-\theta)^{\alpha}(\omega-\theta+1+\alpha)\right], \tag{4.22}$$

then we replace the functions $\emptyset_1(t_\theta,x_\theta)$and $\emptyset_2(t_\theta,x_\theta)$,by its Lagrange interpolation polynomial in Equation (4.22) and putting their values:

$$\begin{aligned}x_{\omega+1}&=x_0+\frac{(1-\alpha)}{AB(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\left\{f_1(t_{\omega+1},x(t_{\omega+1}))+f_2(t_{\omega+1},x(t_{\omega+1}))B(t_{\omega+1})\right\}\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=1}^{\omega}f_1(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))\left[(\omega-\theta+1)^{\alpha}(\omega-\theta+2+\alpha)-(\omega-\theta)^{\alpha}(\omega-\theta+2+2\alpha)\right]\\
&-\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=1}^{\omega}f_1(t_{\theta-1},x_{\theta-1})(g(t_\theta)-g(t_{\theta-1}))\left[(\omega-\theta+1)^{\alpha+1}-(\omega-\theta)^{\alpha}(\omega-\theta+1+\alpha)\right]\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{B(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=1}^{\omega}f_2(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))(B(t_{\theta+1})-B(t_\theta))\Big[(\omega-\theta+1)^{\alpha}(\omega-\theta+2+\alpha)\\
&-(\omega-\theta)^{\alpha}(\omega-\theta+2+2\alpha)\Big]\\
&-\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=1}^{\omega}f_2(t_{\theta-1},x_{\theta-1})(g(t_\theta)-g(t_{\theta-1}))(B(t_\theta)-B(t_{\theta-1}))\Big[(\omega-\theta+1)^{\alpha+1}\\
&-(\omega-\theta)^{\alpha}(\omega-\theta+1+\alpha)\Big].\end{aligned} \tag{4.23}$$

Similarly, by applying the above operations on the functions $y(t)$, $v(t)$, and $z(t)$. we can obtain the same result.

4.4. Numerical scheme for stochastic equations with global derivative by considering Newton interpolation polynomial.

In this subsection, we present the numerical solution to the above problems while considering the new interpolation and the Caputo fractional derivative which is converted to Volterra type using the Caputo type of integral.

We consider the interpolation:

$$\begin{aligned}P_i(\tau)&=\emptyset_i(t_{\omega-2},x_{\omega-2})+\frac{\emptyset_i(t_{\omega-1},x_{\omega-1})-\emptyset_i(t_{\omega-2},x_{\omega-2})}{\Delta t}\times(\tau-t_{\omega-2})\\
&+\frac{\emptyset_i(t_\omega,x_\omega)-2\emptyset_i(t_{\omega-1},x_{\omega-1})+\emptyset_i(t_{\omega-2},x_{\omega-2})}{2(\Delta t)^2}\times(\tau-t_{\omega-2})(\tau-t_{\omega-1}),\end{aligned} \tag{4.24}$$

Replacing the function $\emptyset_1(t_\omega,x_\omega)$, $\emptyset_2(t_\omega,x_\omega)$ by its Newton interpolation polynomial in the Equation (4.1), then we have:

$$\begin{aligned}x(t_{\omega+1})-x(t_\omega)&=\left\{\frac{5}{12}\Delta t\ \emptyset_1(t_{\omega-2},x_{\omega-2})-\frac{4}{3}\ \Delta t\ \emptyset_1(t_{\omega-1},x_{\omega-1})+\frac{5}{12}\Delta t\ \emptyset_1(t_\omega,x_\omega)\right\}\\
&+\left\{\frac{5}{12}\Delta t\ \emptyset_2(t_{\omega-2},x_{\omega-2})-\frac{4}{3}\ \Delta t\ \emptyset_2(t_{\omega-1},x_{\omega-1})+\frac{5}{12}\Delta t\ \emptyset_2(t_\omega,x_\omega)\right\}.\end{aligned} \tag{4.25}$$

Putting for simplicity $g'(t_\omega)f_1(t_\omega,x_\omega)=\emptyset_1(t_\omega,x_\omega)\,,g'(t_\omega)f_2(t_\omega,x_\omega)B'(t_\omega)=\emptyset_2(t_\omega,x_\omega)$. Then, replacing their values in Equation (4.24). And putting the interpolation polynomials of $g'(t)$ at these points. We also, considering

$$f(t_{\omega-1},x_{\omega-1})=f(t_{\omega-1},x_\omega-\Delta tf(t_\omega,x_\omega)),$$

and

$$f(t_{\omega-2},x_{\omega-2})=f(t_{\omega-2},x_\omega-\Delta tf_1(t_\omega,x_\omega)-\Delta tf(t_{\omega-1},x_\omega-\Delta tf(t_\omega,x_\omega))).$$

So, we have:

$$\begin{aligned}x\left(t_{\omega+1}\right)-x\left(t_{\omega}\right)=&\Bigg\{\left(g\left(t_{\omega-1}\right)-g\left(t_{\omega-2}\right)\right)\frac{5}{12}f_1\left(t_{\omega-2},x_{\omega}-\Delta tf_1\left(t_{\omega},x_{\omega}\right)-\Delta tf_1\left(t_{\omega-1},x_{\omega}-\Delta tf_1\left(t_{\omega},x_{\omega}\right)\right)\right)\\&-\frac{4}{3}\left(g\left(t_{\omega}\right)-g\left(t_{\omega-1}\right)\right)f_1\left(t_{\omega-1},x_{\omega}-\Delta tf_1\left(t_{\omega},x_{\omega}\right)\right)+\frac{23}{12}\left(g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)\right)f_1\left(t_{\omega},x_{\omega}\right)\Bigg\}\\&+\Bigg\{\frac{5}{12}\left(g\left(t_{\omega-1}\right)-g\left(t_{\omega-2}\right)\right)f_2\left(t_{\omega-2},x_{\omega}-\Delta tf_2\left(t_{\omega},x_{\omega}\right)-\Delta tf_2\left(t_{\omega-1},x_{\omega}-\Delta tf_2\left(t_{\omega},x_{\omega}\right)\right)\right)\\&-\frac{4}{3}\left(g\left(t_{\omega}\right)-g\left(t_{\omega-1}\right)\right)f_2\left(t_{\omega-1},x_{\omega}-\Delta tf_2\left(t_{\omega},x_{\omega}\right)\right)+\frac{23}{12}\left(g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)\right)f_2\left(t_{\omega},x_{\omega}\right)\Bigg\}.\end{aligned}\tag{4.26}$$

4.5. **Numerical scheme for Caputo-Fabrizio order stochastic equation with global derivative version.** First, we introduce the equation with the Caputo-Fabrizio derivative version.

$${}_0^{CF}D_g^a x\left(t\right)=f_1\left(t,x\left(t\right)\right)+f_2\left(t,x\left(t\right)\right),\qquad x\left(t_0\right)=x_0.\tag{4.27}$$

If g is differentiable, then we have

$${}_0^{CF}D_g^a x\left(t\right)=g'\left(t\right)f_1\left(t,x\left(t\right)\right)+g'\left(t\right)f_2\left(t,x\left(t\right)\right).\tag{4.28}$$

Using the Caputo-Fabrizio integral definition, the Equation (4.28) can be rewritten as follows;

$$\begin{aligned}x\left(t\right)-x\left(0\right)=&\frac{\left(1-\alpha\right)}{M\left(\alpha\right)}g'\left(t\right)f_1\left(t,x\left(t\right)\right)\\&+\frac{\alpha}{M\left(\alpha\right)}\int_0^t g'\left(\tau\right)f_1\left(\tau,x\left(\tau\right)\right)d\left(\tau\right)+\frac{\left(1-\alpha\right)}{M\left(\alpha\right)}g'\left(t\right)f_2\left(t,x\left(t\right)\right)B\left(t\right)\\&+\frac{\alpha}{M\left(\alpha\right)}\int_0^t g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)dB\left(\tau\right).\end{aligned}\tag{4.29}$$

Assuming B(t) as a Brownian motion is differentiable. Then by considering Equation (4.29) at two points $t_{\omega+1}=(\omega+1)\Delta t$ and $t_{\omega}=(\omega)\Delta t$, and by taking the difference between them we can get the following equation:

$$\begin{aligned}x\left(t_{\omega+1}\right)-x\left(t_{\omega}\right)=&\frac{\left(1-\alpha\right)}{M\left(\alpha\right)}g'\left(t_{\omega+1}\right)\left[f_1\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)+f_2\left(t_{\omega},x\left(t_{\omega+1}\right)\right)B\left(t_{\omega+1}\right)\right]\\&-\frac{\left(1-\alpha\right)}{M\left(\alpha\right)}g'\left(t_{\omega}\right)\left[f_1\left(t_{\omega},x\left(t_{\omega}\right)\right)+f_2\left(t_{\omega},x\left(t_{\omega}\right)\right)B\left(t_{\omega}\right)\right]\\&+\frac{\alpha}{M\left(\alpha\right)}\int_{t_{\omega}}^{t_{\omega+1}}g'\left(\tau\right)f_1\left(\tau,x\left(\tau\right)\right)d\tau+\frac{\alpha}{M\left(\alpha\right)}\int_{t_{\omega}}^{t_{\omega+1}}g'\left(\tau\right)f_2\left(\tau,x\left(\tau\right)\right)B'\left(\tau\right)d\tau.\end{aligned}\tag{4.30}$$

Now, we put some simplicity again for the equation above;

$$\begin{aligned}&g'\left(t_{\omega}\right)f_1\left(t_{\omega},x\left(t_{\omega}\right)\right)=\emptyset_1\left(\tau,x\left(\tau\right)\right),\\&g'\left(t_{\omega}\right)f_2\left(t_{\omega},x\left(t_{\omega}\right)\right)B'\left(\tau\right)=\emptyset_2\left(\tau,x\left(\tau\right)\right).\end{aligned}\tag{4.31}$$

We replace the functions $\emptyset_1\left(\tau,x\left(\tau\right)\right)$ and $\emptyset_2\left(\tau,x\left(\tau\right)\right)$, by its Newton interpolation polynomial in Equation (4.30) and putting their values. And by considering

$$g'\left(t_{\omega}\right)=g\left(t_{\omega}\right)-g\left(t_{\omega-1}\right),\quad B'\left(t_{\omega}\right)=B\left(t_{\omega-1}\right)-B\left(t_{\omega-2}\right),\quad f\left(t_{\omega-1},x_{\omega-1}\right)=f\left(t_{\omega-1},x_{\omega}-\Delta tf\left(t_{\omega},x_{\omega}\right)\right),$$

and

$$f\left(t_{\omega-2},x_{\omega-2}\right)=f\left(t_{\omega-2},x_{\omega}-\Delta tf_1\left(t_{\omega},x_{\omega}\right)-\Delta tf\left(t_{\omega-1},x_{\omega}-\Delta tf\left(t_{\omega},x_{\omega}\right)\right)\right).$$

So, we have the following scheme:

$$x\left(t_{\omega+1}\right)-x\left(t_{\omega}\right)=\frac{\left(1-\alpha\right)}{M\left(\alpha\right)}\frac{g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)}{\Delta t}\left[f_1\left(t_{\omega+1},x\left(t_{\omega+1}\right)\right)+f_2\left(t_{\omega},x\left(t_{\omega+1}\right)\right)B\left(t_{\omega+1}\right)\right]$$

$$
\begin{aligned}
&-\frac{(1-\alpha)}{M(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\left[f_1(t_\omega,x(t_\omega))+f_2(t_\omega,x(t_\omega))B(t_\omega)\right]\\
&+\frac{\alpha}{M(\alpha)}\Bigg\{(g(t_{\omega-1})-g(t_{\omega-2}))\frac{5}{12}f_1(t_{\omega-2},x_\omega-\Delta t f_1(t_\omega,x_\omega)-\Delta t f_1(t_{\omega-1},x_\omega-\Delta t f_1(t_\omega,x_\omega)))\\
&-\frac{4}{3}\,(g(t_\omega)-g(t_{\omega-1}))f_1(t_{\omega-1},x_\omega-\Delta t f_1(t_\omega,x_\omega))+\frac{23}{12}(g(t_{\omega+1})-g(t_\omega))f_1(t_\omega,x_\omega)\Bigg\}\\
&+\frac{\alpha}{M(\alpha)}\Bigg\{\frac{5}{12}(g(t_{\omega-1})-g(t_{\omega-2}))f_2(t_{\omega-2},x_\omega-\Delta t f_2(t_\omega,x_\omega)-\Delta t f_2(t_{\omega-1},x_\omega-\Delta t f_2(t_\omega,x_\omega)))\\
&\frac{B(t_{\omega-1})-B(t_{\omega-2})}{\Delta t}-\frac{4}{3}\,(g(t_\omega)-g(t_{\omega-1}))f_2(t_{\omega-1},x_\omega-\Delta t f_2(t_\omega,x_\omega))\\
&\frac{B(t_\omega)-B(t_{\omega-1})}{\Delta t}+\frac{23}{12}(g(t_{\omega+1})-g(t_\omega))f_2(t_\omega,x_\omega)\frac{B(t_{\omega+1})-B(t_\omega)}{\Delta t}\Bigg\},
\end{aligned}
\tag{4.32}
$$

Similarly, by performing the above operations on the functions $y(t)$, $v(t)$, and $z(t)$. we can obtain the same result.

$$
\begin{aligned}
y(t_{\omega+1})-y(t_\omega)=&\frac{(1-\alpha)}{M(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\left[g_1(t_{\omega+1},x(t_{\omega+1}))+g_2(t_\omega,x(t_{\omega+1}))B(t_{\omega+1})\right]\\
&-\frac{(1-\alpha)}{M(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\left[g_1(t_\omega,x(t_\omega))+g_2(t_\omega,x(t_\omega))B(t_\omega)\right]\\
&+\frac{\alpha}{M(\alpha)}\Bigg\{(g(t_{\omega-1})-g(t_{\omega-2}))\frac{5}{12}g_1(t_{\omega-2},x_\omega-\Delta t g_1(t_\omega,x_\omega)-\Delta t g_1(t_{\omega-1},x_\omega-\Delta t g_1(t_\omega,x_\omega)))\\
&-\frac{4}{3}\,(g(t_\omega)-g(t_{\omega-1}))g_1(t_{\omega-1},x_\omega-\Delta t g_1(t_\omega,x_\omega))+\frac{23}{12}(g(t_{\omega+1})-g(t_\omega))g_1(t_\omega,x_\omega)\Bigg\}\\
&+\frac{\alpha}{M(\alpha)}\Bigg\{\frac{5}{12}(g(t_{\omega-1})-g(t_{\omega-2}))g_2(t_{\omega-2},x_\omega-\Delta t g_2(t_\omega,x_\omega)-\Delta t g_2(t_{\omega-1},x_\omega-\Delta t g_2(t_\omega,x_\omega)))\\
&\frac{B(t_{\omega-1})-B(t_{\omega-2})}{\Delta t}\;-\frac{4}{3}\,(g(t_\omega)-g(t_{\omega-1}))g_2(t_{\omega-1},x_\omega-\Delta t g_2(t_\omega,x_\omega))\frac{B(t_\omega)-B(t_{\omega-1})}{\Delta t}\\
&+\frac{23}{12}(g(t_{\omega+1})-g(t_\omega))g_2(t_\omega,x_\omega)\frac{B(t_{\omega+1})-B(t_\omega)}{\Delta t}\Bigg\},
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
v(t_{\omega+1})-v(t_\omega)=&\frac{(1-\alpha)}{M(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\left[h_1(t_{\omega+1},x(t_{\omega+1}))+h_2(t_\omega,x(t_{\omega+1}))B(t_{\omega+1})\right]\\
&-\frac{(1-\alpha)}{M(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\left[h_1(t_\omega,x(t_\omega))+h_2(t_\omega,x(t_\omega))B(t_\omega)\right]\\
&+\frac{\alpha}{M(\alpha)}\Bigg\{(g(t_{\omega-1})-g(t_{\omega-2}))\frac{5}{12}h_1(t_{\omega-2},x_\omega-\Delta t h_1(t_\omega,x_\omega)-\Delta t h_1(t_{\omega-1},x_\omega-\Delta t h_1(t_\omega,x_\omega)))\\
&-\frac{4}{3}\,(g(t_\omega)-g(t_{\omega-1}))h_1(t_{\omega-1},x_\omega-\Delta t h_1(t_\omega,x_\omega))+\frac{23}{12}(g(t_{\omega+1})-g(t_\omega))h_1(t_\omega,x_\omega)\Bigg\}\\
&+\frac{\alpha}{M(\alpha)}\Bigg\{\frac{5}{12}(g(t_{\omega-1})-g(t_{\omega-2}))h_2(t_{\omega-2},x_\omega-\Delta t h_2(t_\omega,x_\omega)-\Delta t h_2(t_{\omega-1},x_\omega-\Delta t h_2(t_\omega,x_\omega)))\\
&\frac{B(t_{\omega-1})-B(t_{\omega-2})}{\Delta t}\;-\frac{4}{3}\,(g(t_\omega)-g(t_{\omega-1}))h_2(t_{\omega-1},x_\omega-\Delta t h_2(t_\omega,x_\omega))\frac{B(t_\omega)-B(t_{\omega-1})}{\Delta t}\\
&+\frac{23}{12}(g(t_{\omega+1})-g(t_\omega))h_2(t_\omega,x_\omega)\frac{B(t_{\omega+1})-B(t_\omega)}{\Delta t}\Bigg\},
\end{aligned}
\tag{4.34}
$$

$$
\begin{aligned}
z\left(t_{\omega+1}\right)-z\left(t_{\omega}\right)= & \frac{(1-\alpha)}{M(\alpha)} \frac{g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)}{\Delta t}\left[I_1\left(t_{\omega+1}, x\left(t_{\omega+1}\right)\right)+I_2\left(t_{\omega}, x\left(t_{\omega+1}\right)\right) B\left(t_{\omega+1}\right)\right] \\
& -\frac{(1-\alpha)}{M(\alpha)} \frac{g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)}{\Delta t}\left[I_1\left(t_{\omega}, x\left(t_{\omega}\right)\right)+I_2\left(t_{\omega}, x\left(t_{\omega}\right)\right) B\left(t_{\omega}\right)\right] \\
& +\frac{\alpha}{M(\alpha)}\left\{\left(g\left(t_{\omega-1}\right)-g\left(t_{\omega-2}\right)\right) \frac{5}{12} I_1\left(t_{\omega-2}, x_{\omega}-\Delta t I_1\left(t_{\omega}, x_{\omega}\right)-\Delta t I_1\left(t_{\omega-1}, x_{\omega}-\Delta t I_1\left(t_{\omega}, x_{\omega}\right)\right)\right)\right. \\
& \left.-\frac{4}{3}\left(g\left(t_{\omega}\right)-g\left(t_{\omega-1}\right)\right) I_1\left(t_{\omega-1}, x_{\omega}-\Delta t I_1\left(t_{\omega}, x_{\omega}\right)\right)+\frac{23}{12}\left(g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)\right) I_1\left(t_{\omega}, x_{\omega}\right)\right\} \\
& +\frac{\alpha}{M(\alpha)}\left\{\frac{5}{12}\left(g\left(t_{\omega-1}\right)-g\left(t_{\omega-2}\right)\right) I_2\left(t_{\omega-2}, x_{\omega}-\Delta t I_2\left(t_{\omega}, x_{\omega}\right)-\Delta t I_2\left(t_{\omega-1}, x_{\omega}-\Delta t I_2\left(t_{\omega}, x_{\omega}\right)\right)\right)\right. \\
& \frac{B\left(t_{\omega-1}\right)-B\left(t_{\omega-2}\right)}{\Delta t}-\frac{4}{3}\left(g\left(t_{\omega}\right)-g\left(t_{\omega-1}\right)\right) I_2\left(t_{\omega-1}, x_{\omega}-\Delta t I_2\left(t_{\omega}, x_{\omega}\right)\right) \frac{B\left(t_{\omega}\right)-B\left(t_{\omega-1}\right)}{\Delta t} \\
& \left.+\frac{23}{12}\left(g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)\right) I_2\left(t_{\omega}, x_{\omega}\right) \frac{B\left(t_{\omega+1}\right)-B\left(t_{\omega}\right)}{\Delta t}\right\} .
\end{aligned} \tag{4.35}
$$

4.6. **Numerical scheme for Atangana-Baleanu order stochastic equation with global derivative version.** First, we introduce the Atangana-Baleanu version of the equation

$$
{}_{0}^{AB}D_{g}^{a} x(t)=f_1(t, x(t))+f_2(t, x(t)), \qquad x\left(t_0\right)=x_0. \tag{4.36}
$$

Let us assume g is differentiable, then we will write

$$
{}_{0}^{AB}D_{g}^{a} x(t)=g'(t) f_1(t, x(t))+g'(t) f_2(t, x(t)). \tag{4.37}
$$

Using the Caputo-Fabrizio integral definition, the Equation (4.37) can be rewritten as follows:

$$
\begin{aligned}
x(t)-x(0)= & \frac{(1-\alpha)}{AB(\alpha)} g'(t) f_1(t, x(t))+\frac{\alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t g'(\tau) f_1(\tau, x(\tau))(t-\tau)^{(\alpha-1)} d(\tau) \\
& +\frac{(1-\alpha)}{AB(\alpha)} g'(t) f_2(t, x(t)) B(t)+\frac{\alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t g'(\tau)(t-\tau)^{(\alpha-1)} dB(\tau).
\end{aligned} \tag{4.38}
$$

Assuming B(t) as a Brownian motion is differentiable, so we can write:

$$
\begin{aligned}
x(t)-x(0)= & \frac{(1-\alpha)}{AB(\alpha)} g'(t) f_1(t, x(t))+\frac{\alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t g'(\tau) f_1(\tau, x(\tau))(t-\tau)^{(\alpha-1)} d(\tau) \\
& +\frac{(1-\alpha)}{AB(\alpha)} g'(t) f_2(t, x(t)) B(t)+\frac{\alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t g'(\tau)(t-\tau)^{(\alpha-1)} B'(\tau) d\tau.
\end{aligned} \tag{4.39}
$$

By considering Equation (4.39) at point $t_{\omega+1}=(\omega+1)\Delta t$. And putting the interpolation polynomials of $g'(t)$ at this point. We have:

$$
\begin{aligned}
x\left(t_{\omega+1}\right)= & x(0)+\frac{(1-\alpha)}{AB(\alpha)} \frac{g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)}{\Delta t}\left\{f_1\left(t_{\omega+1}, x\left(t_{\omega+1}\right)\right)+f_2\left(t_{\omega+1}, x\left(t_{\omega+1}\right)\right) B(t_{\omega+1})\right\} \\
& +\frac{\alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^{t_{\omega+1}} \emptyset_1(\tau, x(\tau))\left(t_{\omega+1}-\tau\right)^{(\alpha-1)} d\tau \\
& +\frac{\alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^{t_{\omega+1}} \emptyset_2(\tau, x(\tau))\left(t_{\omega+1}-\tau\right)^{(\alpha-1)} d\tau.
\end{aligned} \tag{4.40}
$$

Then, by dividing the integration interval into equally spaced points as $t_\theta=\theta\Delta t, \theta=0,\ldots,\omega$. and implement the Newton polynomial interpolation on each of these points, and replacing $\emptyset_1(t, x(t))$, $\emptyset_2(t, x(t))$ with their values, we get:

$$
x\left(t_{\omega+1}\right)=x(0)+\frac{(1-\alpha)}{AB(\alpha)} \frac{g\left(t_{\omega+1}\right)-g\left(t_{\omega}\right)}{\Delta t}\left\{f_1\left(t_{\omega+1}, x\left(t_{\omega+1}\right)\right)+f_2\left(t_{\omega+1}, x\left(t_{\omega+1}\right)\right) B\left(t_{\omega+1}\right)\right\}
$$

$$
\begin{aligned}
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+1\right)}\sum_{\theta=2}^{\omega}\left\{f_1(t_{\theta-2},x_{\theta-2})\right\}\times\left\{(\omega-\theta+1)^{\alpha}-(\omega-\theta)\right\}\times\left(g\left(t_{\theta-1}\right)-g(t_{\theta-2})\right)\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+2\right)}\sum_{\theta=2}^{\omega}\left\{f_1(t_{\theta-1},x_{\theta-1})\left(g\left(t_{\theta}\right)-g\left(t_{\theta-1}\right)\right)-f_1(t_{\theta-2},x_{\theta-2})\times\left\{(\omega-\theta+1)^{\alpha}(\omega-\theta+3+2\alpha)\right.\right.\\
&\left.\left.-(\omega-\theta)^{\alpha}(\omega-\theta+3+3\alpha)\right\}\times\left(g\left(t_{\theta-1}\right)-g(t_{\theta-2})\right)\right)\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{2AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+3\right)}\sum_{\theta=2}^{\omega}f_1\left(t_{\theta},x_{\theta}\right)\left(g\left(t_{\theta+1}\right)-g(t_{\theta})\right)-2f_1(t_{\theta-1},x_{\theta-1})\left(g\left(t_{\theta}\right)-g\left(t_{\theta-1}\right)\right)+f_1(t_{\theta-2},x_{\theta-2})\\
&\times\left\{(\omega-\theta+1)^{\alpha}\{2(\omega-\theta)^2+(3\alpha+10)\left(\omega-\theta\right)+2\alpha^2+9\alpha+12\}\right.\\
&\left.-(\omega-\theta)^{\alpha}\{2\left(\omega-\theta\right)^2+(5\alpha+10)\left(\omega-\theta\right)+6\alpha^2+18\alpha+12\}\right\}\times\left(g\left(t_{\theta-1}\right)-g(t_{\theta-2})\right)\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+1\right)}\sum_{\theta=2}^{\omega}f_2(t_{\theta-2},x_{\theta-2})\times\left\{(\omega-\theta+1)^{\alpha}-(\omega-\theta)\right\}\times\left(g\left(t_{\theta-1}\right)-g(t_{\theta-2})\right)\times\left(B\left(t_{\theta-1}\right)-B(t_{\theta-2})\right)\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+2\right)}\sum_{\theta=2}^{\omega}\left\{f_2(t_{\theta-1},x_{\theta-1})\left(g\left(t_{\theta}\right)-g\left(t_{\theta-1}\right)\right)\times\left(B\left(t_{\theta}\right)-B\left(t_{\theta-1}\right)\right)\right\}-\left\{f_2(t_{\theta-2},x_{\theta-2})\right.\\
&\left.\times\left\{(\omega-\theta+1)^{\alpha}(\omega-\theta+3+2\alpha)-(\omega-\theta)^{\alpha}(\omega-\theta+3+3\alpha)\right\}\times\left(g\left(t_{\theta-1}\right)-g(t_{\theta-2})\right)\times\left(B\left(t_{\theta-1}\right)-B\left(t_{\theta-2}\right)\right)\right\}\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{2AB\left(\alpha\right)\mathit{\Gamma}\left(\alpha+3\right)}\sum_{\theta=2}^{\omega}f_2\left(t_{\theta},x_{\theta}\right)\left(g\left(t_{\theta+1}\right)-g(t_{\theta})\right)\times\left(B\left(t_{\theta+1}\right)-B\left(t_{\theta}\right)\right)-2f_2(t_{\theta-1},x_{\theta-1})\left(g\left(t_{\theta}\right)-g\left(t_{\theta-1}\right)\right)\\
&\times\left(B\left(t_{\theta}\right)-B\left(t_{\theta-1}\right)\right)+f_2(t_{\theta-2},x_{\theta-2})\times\left\{(\omega-\theta+1)^{\alpha}\{2\left(\omega-\theta\right)^2+(3\alpha+10)\left(\omega-\theta\right)+2\alpha^2+9\alpha+12\right\}\\
&-(\omega-\theta)^{\alpha}\left\{2\left(\omega-\theta\right)^2+(5\alpha+10)\left(\omega-\theta\right)+6\alpha^2+18\alpha+12\right\}\}\times\left(g\left(t_{\theta-1}\right)-g\left(t_{\theta-2}\right)\right)\times\left(B\left(t_{\theta-1}\right)-B\left(t_{\theta-2}\right)\right). \quad (4.41)
\end{aligned}
$$

where

$$
\begin{aligned}
&f_i(t_{\theta-1},x_{\theta-1})=f_i\left(t_{\theta-1},x_{\theta}-\Delta t f_i\left(t_{\theta},x_{\theta}\right)\right),\\
&f_i\left(t_{\theta-2},x_{\theta-2}\right)=f_i(t_{\theta-2},x_{\theta}-\Delta t f_i(t_{\theta},x_{\theta})-\Delta t f_i\left(t_{\theta-1},x_{\theta}-f_1\left(t_{\theta},x_{\theta}\right)\right),
\end{aligned}
$$

we can write final scheme:

$$
\begin{aligned}
x(t_{\omega+1})=x(0)&+\frac{(1-\alpha)}{AB(\alpha)}\frac{g(t_{\omega+1})-g(t_{\omega})}{\Delta t}\left\{f_1(t_{\omega+1},x(t_{\omega+1}))+f_2(t_{\omega+1},x(t_{\omega+1}))B(t_{\omega+1})\right\}\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\left\{f_1(t_{\theta-2},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta})-\Delta t f_1(t_{\theta-1},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta})))\right\}\\
&\times\left[(\omega-\theta+1)^{\alpha}-(\omega-\theta)^{\alpha}\right]\times\left[g(t_{\theta-1})-g(t_{\theta-2})\right]\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\{f_1(t_{\theta-1},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta}))\left(g(t_{\theta})-g(t_{\theta-1})\right)\\
&-f_1(t_{\theta-2},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta})-\Delta t f_1(t_{\theta-1},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta})))\}\\
&\times\left\{(\omega-\theta+1)^{\alpha}(\omega-\theta+3+2\alpha)-(\omega-\theta)^{\alpha}(\omega-\theta+3+3\alpha)\right\}\times\left[g(t_{\theta-1})-g(t_{\theta-2})\right]\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\{f_1(t_{\theta},x_{\theta})(g(t_{\theta+1})-g(t_{\theta}))\\
&-2f_1(t_{\theta-1},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta}))(g(t_{\theta})-g(t_{\theta-1}))\\
&+f_1(t_{\theta-2},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta})-\Delta t f_1(t_{\theta-1},x_{\theta}-\Delta t f_1(t_{\theta},x_{\theta})))\}\\
&\times\left\{(\omega-\theta+1)^{\alpha}\left[2(\omega-\theta)^2+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\right]\right.\\
&\left.-(\omega-\theta)^{\alpha}\left[2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\right]\right\}\times\left[g(t_{\theta-1})-g(t_{\theta-2})\right]\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\left\{f_2(t_{\theta-2},x_{\theta}-\Delta t f_2(t_{\theta},x_{\theta})-\Delta t f_2(t_{\theta-1},x_{\theta}-\Delta t f_2(t_{\theta},x_{\theta})))\right\}\\
&\times\left[(\omega-\theta+1)^{\alpha}-(\omega-\theta)^{\alpha}\right]\times\left[g(t_{\theta-1})-g(t_{\theta-2})\right]\times\left[B(t_{\theta-1})-B(t_{\theta-2})\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\{f_2(t_{\theta-1},x_\theta-\Delta t f_2(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))(B(t_\theta)-B(t_{\theta-1}))\\
&-f_2(t_{\theta-2},x_\theta-\Delta t f_2(t_\theta,x_\theta)-\Delta t f_2(t_{\theta-1},x_\theta-\Delta t f_2(t_\theta,x_\theta)))\}\\
&\times\{(\omega-\theta+1)^\alpha(\omega-\theta+3+2\alpha)-(\omega-\theta)^\alpha(\omega-\theta+3+3\alpha)\}\times[g(t_{\theta-1})-g(t_{\theta-2})]\times[B(t_{\theta-1})-B(t_{\theta-2})]\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\{f_2(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))(B(t_{\theta+1})-B(t_\theta))\\
&-2f_2(t_{\theta-1},x_\theta-\Delta t f_2(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))\\
&+f_2(t_{\theta-2},x_\theta-\Delta t f_2(t_\theta,x_\theta)-\Delta t f_2(t_{\theta-1},x_\theta-\Delta t f_2(t_\theta,x_\theta)))\}\\
&\times\left\{(\omega-\theta+1)^\alpha\left[2(\omega-\theta)^2+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\right]\right.\\
&\left.-(\omega-\theta)^\alpha\left[2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\right]\right\}\times[g(t_{\theta-1})-g(t_{\theta-2})]\times[B(t_{\theta-1})-B(t_{\theta-2})].
\end{aligned}\tag{4.42}
$$

Similarly, by performing the above operations on the functions $y(t)$, $v(t)$, and $z(t)$. we can obtain the same result:

$$
\begin{aligned}
y(t_{\omega+1})&=y(0)+\frac{(1-\alpha)}{AB(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\{g_1(t_{\omega+1},x(t_{\omega+1}))+g_2(t_{\omega+1},x(t_{\omega+1}))B(t_{\omega+1})\}\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\,\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\{g_1(t_{\theta-2},x_\theta-\Delta t g_1(t_\theta,x_\theta)-\Delta t g_1(t_{\theta-1},x_\theta-g_1(t_\theta,x_\theta))\Delta t\}\\
&\times\{(\omega-\theta+1)^\alpha-(\omega-\theta)\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\{g_1(t_{\theta-1},x_\theta-\Delta t g_1(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))-g_1(t_{\theta-2},x_\theta-\Delta t g_1(t_\theta,x_\theta)\\
&-\Delta t g_1(t_{\theta-1},x_\theta-\Delta t g_1(t_\theta,x_\theta))\times\{(\omega-\theta+1)^\alpha(\omega-\theta+3+2\alpha)\\
&-(\omega-\theta)^\alpha(\omega-\theta+3+3\alpha)\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{2AB(\alpha)\,\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\{g_1(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))-2g_1(t_{\theta-1},x_\theta-\Delta t g_1(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))\\
&+g_1(t_{\theta-2},x_\theta-\Delta t g_1(t_\theta,x_\theta)-\Delta t g_1(t_{\theta-1},x_\theta-\Delta t g_1(t_\theta,x_\theta))\times\{(\omega-\theta+1)^\alpha\{2(\omega-\theta)^2\\
&+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\}\\
&-(\omega-\theta)^\alpha\{2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\}\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\,\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\{g_2(t_{\theta-2},x_\theta-\Delta t g_2(t_\theta,x_\theta)-\Delta t g_2(t_{\theta-1},x_\theta-g_2(t_\theta,x_\theta))\Delta t\}\\
&\times\{(\omega-\theta+1)^\alpha-(\omega-\theta)\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\times(B(t_{\theta-1})-B(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\,\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\{g_2(t_{\theta-1},x_\theta-\Delta t g_2(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))\times(B(t_\theta)-B(t_{\theta-1}))\\
&-g_2(t_{\theta-2},x_\theta-\Delta t g_2(t_\theta,x_\theta)-\Delta t g_2(t_{\theta-1},x_\theta-\Delta t g_2(t_\theta,x_\theta))\times\{(\omega-\theta+1)^\alpha(\omega-\theta+3+2\alpha)\\
&-(\omega-\theta)^\alpha(\omega-\theta+3+3\alpha)\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\times(B(t_{\theta-1})-B(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{2AB(\alpha)\,\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\{g_2(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))\times(B(t_{\theta+1})-B(t_\theta))-2g_2(t_{\theta-1},x_\theta-\Delta t g_2(t_\theta,x_\theta))\\
&(g(t_\theta)-g(t_{\theta-1}))\times(B(t_\theta)-B(t_{\theta-1}))+g_2(t_{\theta-2},x_\theta-\Delta t g_2(t_\theta,x_\theta)-\Delta t g_2(t_{\theta-1},x_\theta-\Delta t g_2(t_\theta,x_\theta))\\
&\times\{(\omega-\theta+1)^\alpha\{2(\omega-\theta)^2+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\}\\
&-(\omega-\theta)^\alpha\{2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\}\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\times(B(t_{\theta-1})-B(t_{\theta-2})).
\end{aligned}\tag{4.43}
$$

$$
v(t_{\omega+1})=v(0)+\frac{(1-\alpha)}{AB(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\{h_1(t_{\omega+1},x(t_{\omega+1}))+h_2(t_{\omega+1},x(t_{\omega+1}))B(t_{\omega+1})\}
$$

$$
\begin{aligned}
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\{h_1(t_{\theta-2},x_\theta-\Delta t h_1(t_\theta,x_\theta)-\Delta t h_1(t_{\theta-1},x_\theta-\Delta t h_1(t_\theta,x_\theta)))\}\\
&\times[(\omega-\theta+1)^\alpha-(\omega-\theta)^\alpha](g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\{h_1(t_{\theta-1},x_\theta-\Delta t h_1(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))\\
&-h_1(t_{\theta-2},x_\theta-\Delta t h_1(t_\theta,x_\theta)-\Delta t h_1(t_{\theta-1},x_\theta-\Delta t h_1(t_\theta,x_\theta)))\}\\
&\times\{(\omega-\theta+1)^\alpha(\omega-\theta+3+2\alpha)-(\omega-\theta)^\alpha(\omega-\theta+3+3\alpha)\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\{h_1(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))\\
&-2h_1(t_{\theta-1},x_\theta-\Delta t h_1(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))\\
&+h_1(t_{\theta-2},x_\theta-\Delta t h_1(t_\theta,x_\theta)-\Delta t h_1(t_{\theta-1},x_\theta-\Delta t h_1(t_\theta,x_\theta)))\}\\
&\times\left\{(\omega-\theta+1)^\alpha\left[2(\omega-\theta)^2+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\right]\right.\\
&\left.-(\omega-\theta)^\alpha\left[2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\right]\right\}\times(g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\{h_2(t_{\theta-2},x_\theta-\Delta t h_2(t_\theta,x_\theta)-\Delta t h_2(t_{\theta-1},x_\theta-\Delta t h_2(t_\theta,x_\theta)))\}\\
&\times[(\omega-\theta+1)^\alpha-(\omega-\theta)^\alpha](g(t_{\theta-1})-g(t_{\theta-2}))(B(t_{\theta-1})-B(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\{h_2(t_{\theta-1},x_\theta-\Delta t h_2(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))(B(t_\theta)-B(t_{\theta-1}))\\
&-h_2(t_{\theta-2},x_\theta-\Delta t h_2(t_\theta,x_\theta)-\Delta t h_2(t_{\theta-1},x_\theta-\Delta t h_2(t_\theta,x_\theta)))\}\\
&\times\{(\omega-\theta+1)^\alpha(\omega-\theta+3+2\alpha)-(\omega-\theta)^\alpha(\omega-\theta+3+3\alpha)\}\\
&\times(g(t_{\theta-1})-g(t_{\theta-2}))(B(t_{\theta-1})-B(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\{h_2(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))(B(t_{\theta+1})-B(t_\theta))\\
&-2h_2(t_{\theta-1},x_\theta-\Delta t h_2(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))(B(t_\theta)-B(t_{\theta-1}))\\
&+h_2(t_{\theta-2},x_\theta-\Delta t h_2(t_\theta,x_\theta)-\Delta t h_2(t_{\theta-1},x_\theta-\Delta t h_2(t_\theta,x_\theta)))\}\\
&\times\left\{(\omega-\theta+1)^\alpha\left[2(\omega-\theta)^2+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\right]\right.\\
&\left.-(\omega-\theta)^\alpha\left[2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\right]\right\}\times(g(t_{\theta-1})-g(t_{\theta-2}))(B(t_{\theta-1})-B(t_{\theta-2})).
\end{aligned}
\tag{4.44}
$$

$$
\begin{aligned}
z(t_{\omega+1})=z(0)&+\frac{(1-\alpha)}{AB(\alpha)}\frac{g(t_{\omega+1})-g(t_\omega)}{\Delta t}\{I_1(t_{\omega+1},x(t_{\omega+1}))+I_2(t_{\omega+1},x(t_{\omega+1}))B(t_{\omega+1})\}\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\Big\{I_1(t_{\theta-2},x_\theta-\Delta t I_1(t_\theta,x_\theta))-\Delta t I_1(t_{\theta-1},x_\theta-I_1(t_\theta,x_\theta))\Delta t\Big\}\\
&\times\{(\omega-\theta+1)^\alpha-(\omega-\theta)^\alpha\}(g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{AB(\alpha)\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\Big\{I_1(t_{\theta-1},x_\theta-\Delta t I_1(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))\\
&-I_1(t_{\theta-2},x_\theta-\Delta t I_1(t_\theta,x_\theta)-\Delta t I_1(t_{\theta-1},x_\theta-\Delta t I_1(t_\theta,x_\theta)))\\
&\times\left[(\omega-\theta+1)^\alpha(\omega-\theta+3+2\alpha)-(\omega-\theta)^\alpha(\omega-\theta+3+3\alpha)\right]\Big\}(g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-1}}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\Big\{I_1(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))-2I_1(t_{\theta-1},x_\theta-\Delta t I_1(t_\theta,x_\theta))\\
&\times(g(t_\theta)-g(t_{\theta-1}))+I_1(t_{\theta-2},x_\theta-\Delta t I_1(t_\theta,x_\theta)-\Delta t I_1(t_{\theta-1},x_\theta-\Delta t I_1(t_\theta,x_\theta)))
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[(\omega-\theta+1)^{\alpha}\big(2(\omega-\theta)^2+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\big)\\
&-(\omega-\theta)^{\alpha}\big(2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\big)\Big](g(t_{\theta-1})-g(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\Gamma(\alpha+1)}\sum_{\theta=2}^{\omega}\Big\{I_2(t_{\theta-2},x_\theta-\Delta t I_2(t_\theta,x_\theta))-\Delta t I_2(t_{\theta-1},x_\theta-I_2(t_\theta,x_\theta))\Delta t\Big\}\\
&\times\{(\omega-\theta+1)^{\alpha}-(\omega-\theta)^{\alpha}\}(g(t_{\theta-1})-g(t_{\theta-2}))(B(t_{\theta-1})-B(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{AB(\alpha)\Gamma(\alpha+2)}\sum_{\theta=2}^{\omega}\Big\{I_2(t_{\theta-1},x_\theta-\Delta t I_2(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))(B(t_\theta)-B(t_{\theta-1}))\\
&-I_2(t_{\theta-2},x_\theta-\Delta t I_2(t_\theta,x_\theta)-\Delta t I_2(t_{\theta-1},x_\theta-\Delta t I_2(t_\theta,x_\theta)))\\
&\times\big[(\omega-\theta+1)^{\alpha}(\omega-\theta+3+2\alpha)-(\omega-\theta)^{\alpha}(\omega-\theta+3+3\alpha)\big]\\
&\times(g(t_{\theta-1})-g(t_{\theta-2}))(B(t_{\theta-1})-B(t_{\theta-2}))\\
&+\frac{\alpha(\Delta t)^{\alpha-2}}{2AB(\alpha)\Gamma(\alpha+3)}\sum_{\theta=2}^{\omega}\Big\{I_2(t_\theta,x_\theta)(g(t_{\theta+1})-g(t_\theta))(B(t_{\theta+1})-B(t_\theta))\\
&-2I_2(t_{\theta-1},x_\theta-\Delta t I_2(t_\theta,x_\theta))(g(t_\theta)-g(t_{\theta-1}))(B(t_\theta)-B(t_{\theta-1}))\\
&+I_2(t_{\theta-2},x_\theta-\Delta t I_2(t_\theta,x_\theta)-\Delta t I_2(t_{\theta-1},x_\theta-\Delta t I_2(t_\theta,x_\theta)))\\
&\times\Big[(\omega-\theta+1)^{\alpha}\big(2(\omega-\theta)^2+(3\alpha+10)(\omega-\theta)+2\alpha^2+9\alpha+12\big)\\
&-(\omega-\theta)^{\alpha}\big(2(\omega-\theta)^2+(5\alpha+10)(\omega-\theta)+6\alpha^2+18\alpha+12\big)\Big]\\
&\times(g(t_{\theta-1})-g(t_{\theta-2}))(B(t_{\theta-1})-B(t_{\theta-2})).
\end{aligned}
\tag{4.45}
$$

5. Numerical Simulation

In this section, the numerical method for Caputo-Fabrizio order and Atangana-Baleanu order stochastic equation with the global derivative version, mentioned in the previous section are used to solve the model, and the following results are reported for different values of fractional orders and random numbers. Not that, we plot the graphs of numerical solutions using MATLAB R2019b, based on the parameters presented in Table 1. In Figures 2-4, the red diagram represents uninfected cancer cells (x),

Table 1. Parameter estimation for the model.

Parameter's	Values
r_1	0.5
a	5.1e-243
h_2	0.016e-243
d_1	0.5
c	5.048e-243
b	0.22e-63
m_1	0.6e-243
σ_i	$0 \leq i \leq 1$

the blue diagram represents infected cancer cells (y), the purple diagram represents virus-free cells (v), and the green diagram represents immune cells (z). Additionally, the figures on the left are approximated using Lagrange's polynomial, and the figures on the right are approximated using Newton's polynomial.

Figures 2-4 show the change process of each of the mentioned parameters in the disease-free state so that in the introduced random fraction model, the initial values

$$E_0=\left(\frac{r_1}{d_1},0,0,0\right) = (1,0,0,0),$$

are considered. Also, we can see the graphs in the disease-free state equations with different fractional orders are smooth. However, changing the orders of fractions from a value close to one to a value close to zero results in the instability of the graphs. Also, we can see the solution of the numerical scheme with the fractional derivative of Baleanu resulting from two

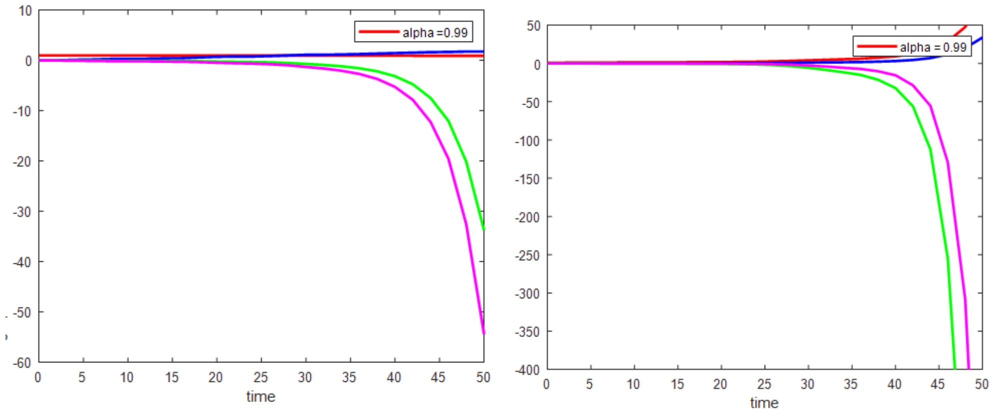

FIGURE 2. The simulations show the combined graphical behavior of subpopulations of cells in the stochastic fractional system (3.1) for $\alpha = 0.99$ and parameter for E_0. Left graph describes simulations approximated using Lagrange's polynomial, while right graph describes simulations approximated using Newton's polynomial.

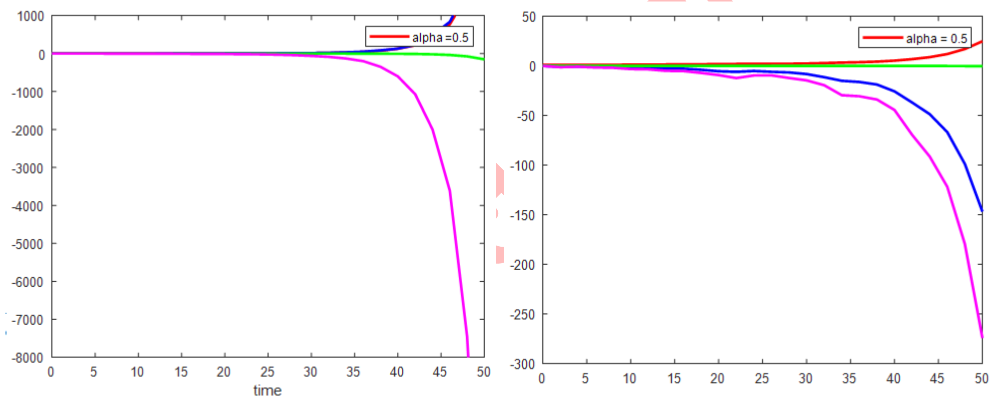

FIGURE 3. The simulations show the combined graphical behavior of subpopulations of cells in the stochastic fractional system (3.1) for $\alpha = 0.5$ and parameter for E_0. Left graph describes simulations approximated using Lagrange's polynomial, while right graph describes simulations approximated using Newton's polynomial.

different approximations using Lagrange's polynomial and Newton's polynomial is always stable. This means that with the introduction of a small error in the input variable, the error will not propagate, and the answers will be obtained almost accurately. In the following, to complicate the model and bring the model closer to the real conditions, we will consider the initial values as opposite to zero.

Note that in the figures below, the red diagram indicates uninfected cancer cells (x), the blue diagram indicates infected cancer cells (y), the purple diagram indicates virus-free cells (v), and the green diagram indicates immune cells (z). Also note that the figures on the right are approximated using Newton's polynomial, and the figures on the left are approximated using Lagrange's polynomial.

Figures 5-7 depict the stochastic fractional model in the disease state with initial values opposite to zero ($E_1 = (x_0 \neq 0, y_0 \neq 0, v_0 \neq 0, z_0 \neq 0) = (0.5, 0.3, 0.1, 0.1)$), based on the Atangana-Baleanu derivative. The fractional order changes from a value close to one to a value close to zero. The graphs demonstrate the effect of using Newton's polynomial in the approximation of the numerical solution in the numerical scheme. When compared to approximating the solution using Lagrange polynomials, the

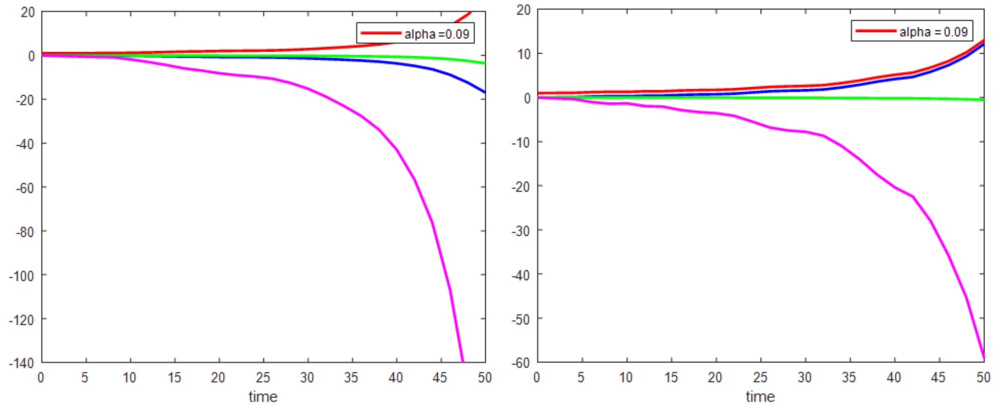

FIGURE 4. The simulations show the combined graphical behavior of subpopulations of cells in the stochastic fractional system (3.1) for α= 0.09 and parameter for E_0. Left graph describes simulations approximated using Lagrange's polynomial, while right graph describes simulations approximated using Newton's polynomial.

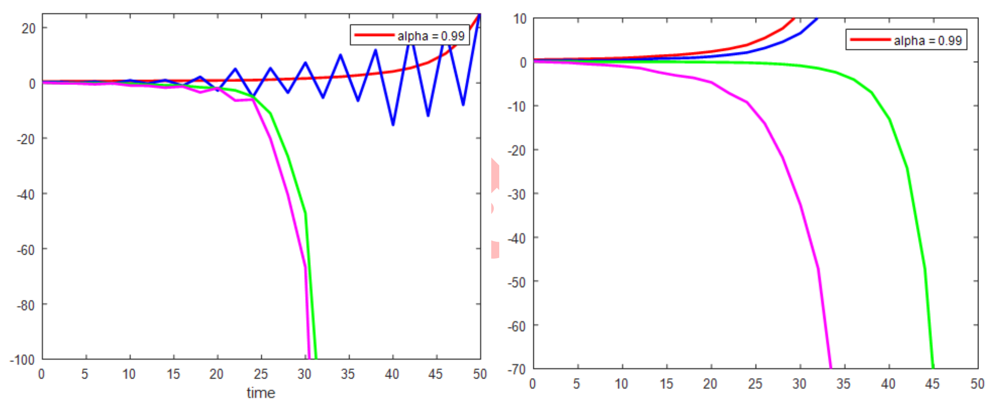

FIGURE 5. The simulations show the combined graphical behavior of subpopulations of cells in the stochastic fractional system (3.1) for α= 0.99 and the parameter for disease state. Left graph describes simulations approximated using Lagrange's polynomial, while right graph describes simulations approximated using Newton's polynomial.

graphs maintain smoothness. Consequently, the solutions also become stable. Also, the graphs show that the effect of Brownian motion becomes more prominent as the fractional order approaches zero.

In the figures below (Figures 8-11). Note that the figures on the right are approximated using Newton's polynomial, and the figures on the left are approximated using Lagrange's polynomial. Also, we consider the stochastic fractional model in the disease state with initial values opposite to zero.

Since tumor cells, unlike healthy cells, cannot produce enough interferon to fight viral infections, they are much more sensitive to the attack of viruses, and oncolytic viruses can multiply in cancer cells and infect those cells; as a result, this works by alerting the immune system to mount a defensive response against the tumor cells, which is effective throughout the body.

Therefore, according to the explanations provided, in the body of a cancer patient, if the treatment process using oncolytic viruses is carried out correctly, the cancer cells uninfected with the virus (variable X) should go through a reduction process, Figure 8. Over time, the number of cancer cells infected with the virus increases. The trend (variable of Y) becomes upward, Figure 9.

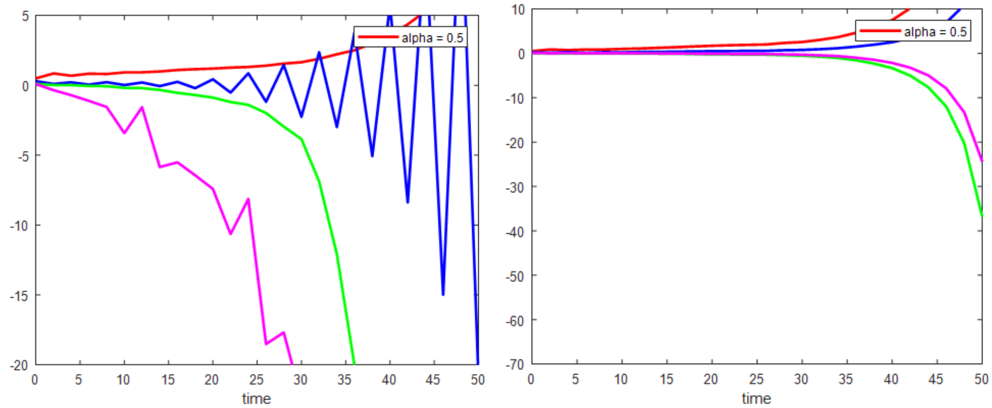

Figure 6. The simulations show the combined graphical behavior of subpopulations of cells in the stochastic fractional system (3.1) for α= 0.5 and the parameter for disease state. Left graph describes simulations approximated using Lagrange's polynomial, while right graph describes simulations approximated using Newton's polynomial.

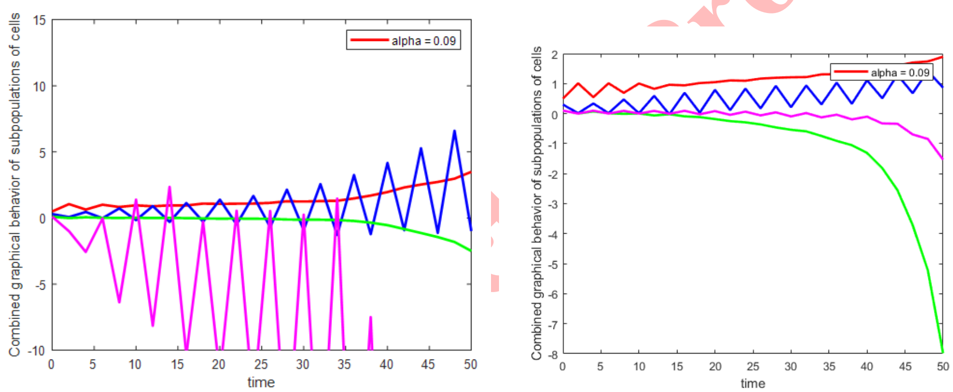

Figure 7. The simulations show the combined graphical behavior of subpopulations of cells in the stochastic fractional system (3.1) for α= 0.09 and the parameter for disease state. Left graph describes simulations approximated using Lagrange's polynomial, while right graph describes simulations approximated using Newton's polynomial.

It is also clear that when viruses enter the body, virus-free-cells (V variable) decrease, Figure 10. Due to the response of the body's immune system, the immune cells (variable Z) increase, Figure 11.

Of course, we don't always get these shapes, with the decreasing and increasing trends shown. As a part of the treatment process, we use a random process to infect tumor cells. We attach specific surface molecules that carry random genes, which code for therapeutic proteins. This helps the patient's immune system become aware of the tumor. But if this treatment works properly and we are lucky, we should achieve the same trends as indicated in the figures. In Figures 8-11, we have analyzed each of the model items that were shown in aggregate form in endemic mode and with non-zero initial values in Figures 5-7, separately. Also, to show more clearly the effect of Brownian motion, we have taken the fractional order as $\alpha = 0.99$ (close to zero). Then we will compare the simulation done using Lagrange's polynomial approximation and Newton's polynomial approximation for each of the items of the model have been discussed separately. We were able to reach the conclusion that the use of Newton's polynomial in the numerical method for approximating the solution leads to smoother graphs.

In the figures below (Figures 12-14), we compare the graphical behavior of non-infected cancer cells and infected cancer cells. This comparison considers different orders of fractional and randomness in the model (figures on the right) and the same model with the randomness removed (figures on the left). Also, the figures are approximated using Lagrange's polynomial. Note that,

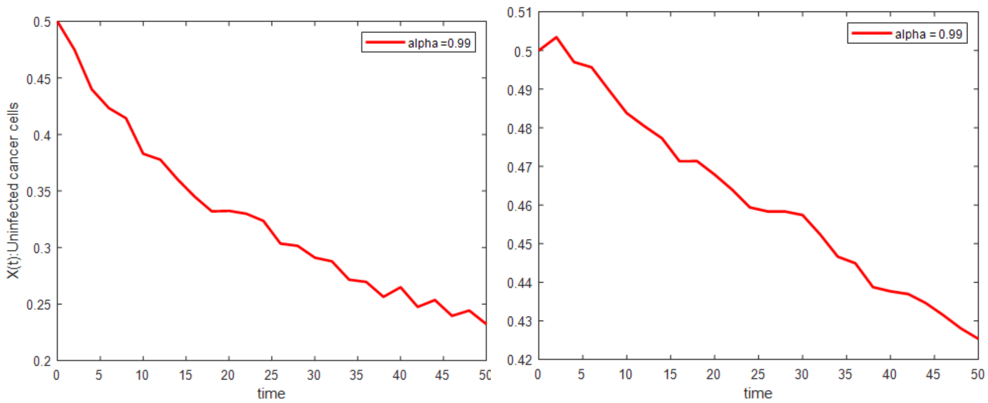

FIGURE 8. The comparison between the stochastic fractional system of $x(t)$, approximated using Lagrange's polynomial (left) and Newton's polynomial (right), for $\alpha = 0.99$ and parameter for disease state.

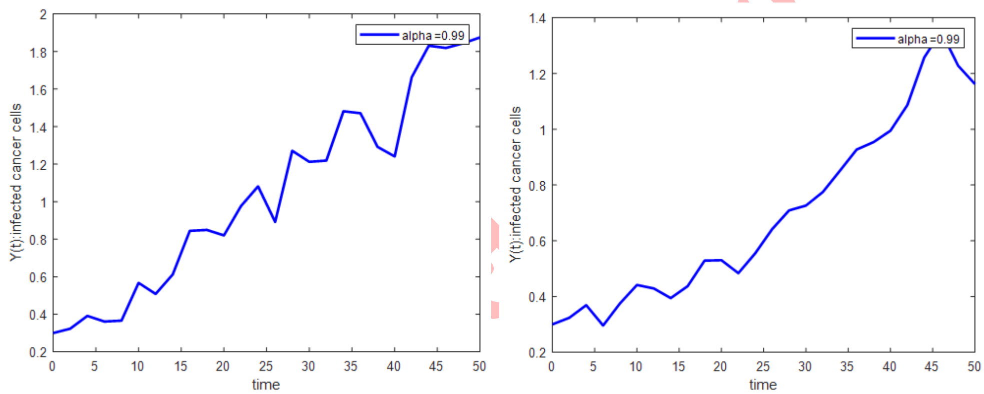

FIGURE 9. The comparison between the stochastic fractional system of $y(t)$, approximated using Lagrange's polynomial (left) and Newton's polynomial (right), for $\alpha = 0.99$ and parameter for disease state.

in the figures below, the red diagram represents uninfected cancer cells (x), while the blue represents infected cancer cells (y). We are considering the Stochastic fractional model in the disease state with initial values not equal to zero.

In Figures 12-19, we first compare the model in the mode without considering the memory effect in the model with two modes without random effect and with random effect (derived from the correct order of 1, and by removing and inserting Brownian motion) (Figures 12-13). Then, we consider the memory effect (derived from the fractional order) in different fractional orders from the order close to one to the order close to zero, for two cases without random effect and with random effect (Figures 14-19). Also, we approximate numerical solutions by using Lagrange's polynomial and Newton's polynomial. Therefore, according to the analysis of the graphs, we can conclude the graphs become unstable when the order of alpha is closer to zero. Also, by using Newton's polynomial we can approximate the model's solution in the random state providing smoother graphs. Therefore, it is confirmed once again that the numerical method using Newton's polynomial is more stable than Lagrange's polynomial.

6. CONCLUSION

In this research, we saw the use of advances in mathematical sciences as a practical tool for cancer treatment. This article examines cancer virus treatment as a new and low-risk method compared to previous treatment methods. During this research, we were able to convert the model of interaction between cancer cells and their response to the immune system in the form of

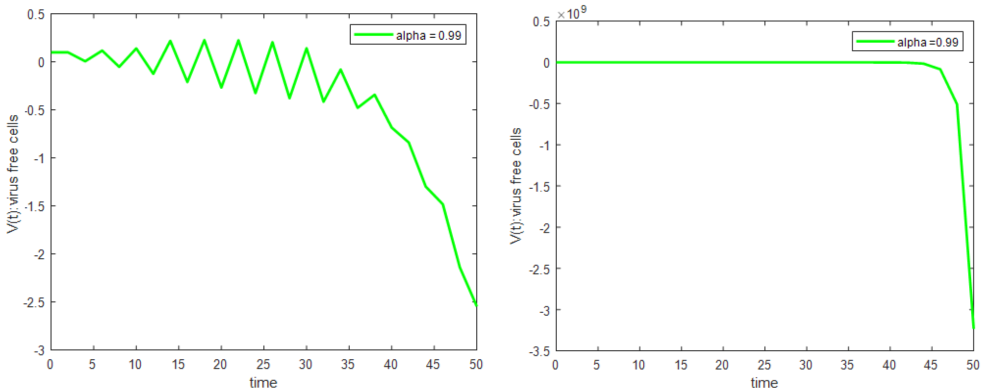

FIGURE 10. The comparison between the stochastic fractional system of $v(t)$, approximated using Lagrange's polynomial (left) and Newton's polynomial (right), for $\alpha = 0.99$ and parameter for disease state.

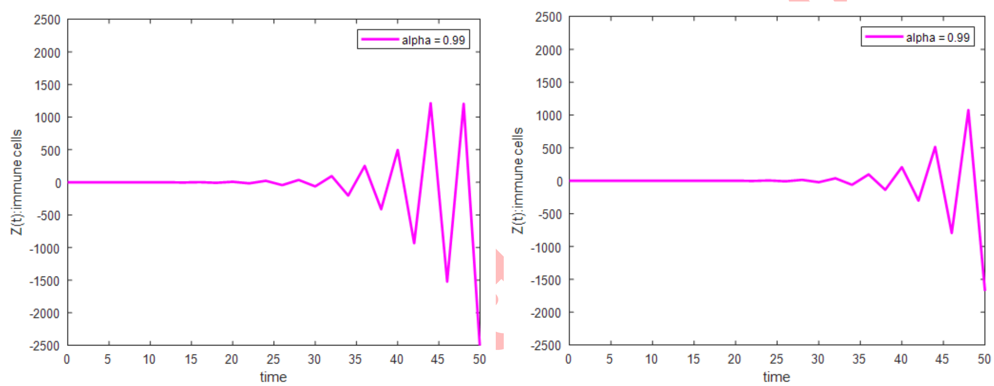

FIGURE 11. The comparison between the stochastic fractional system of $z(t)$, approximated using Lagrange's polynomial (left) and Newton's polynomial (right), for $\alpha = 0.99$ and parameter for disease state.

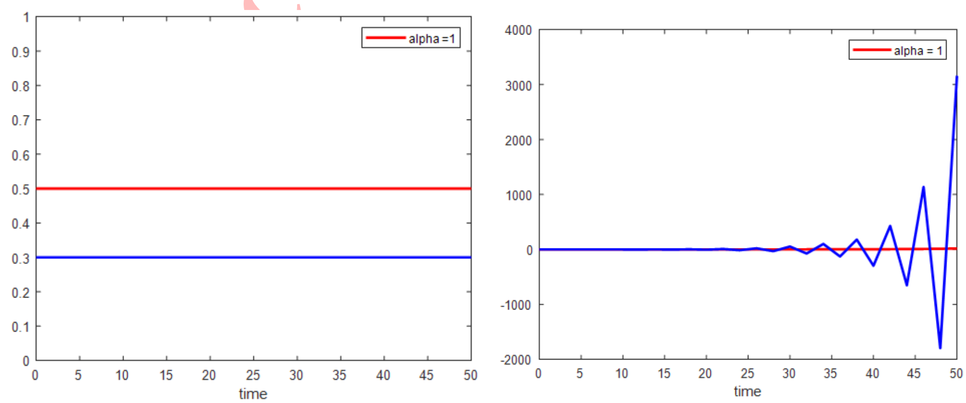

FIGURE 12. The comparison behavior of non-infected and infected cancer cells, between the ODE system and the stochastic differential system (SDE), for $\alpha = 1$ and parameter for disease state, that approximated using Lagrange's polynomial.

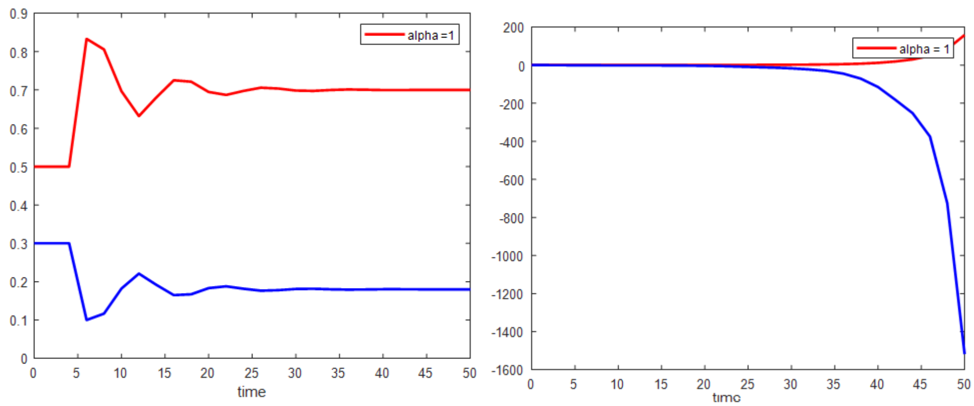

FIGURE 13. The comparison behavior of non-infected and infected cancer cells, between the ODE system and the stochastic differential system (SDE), for $\alpha = 1$ and parameter for disease state, that approximated using Newton's polynomial.

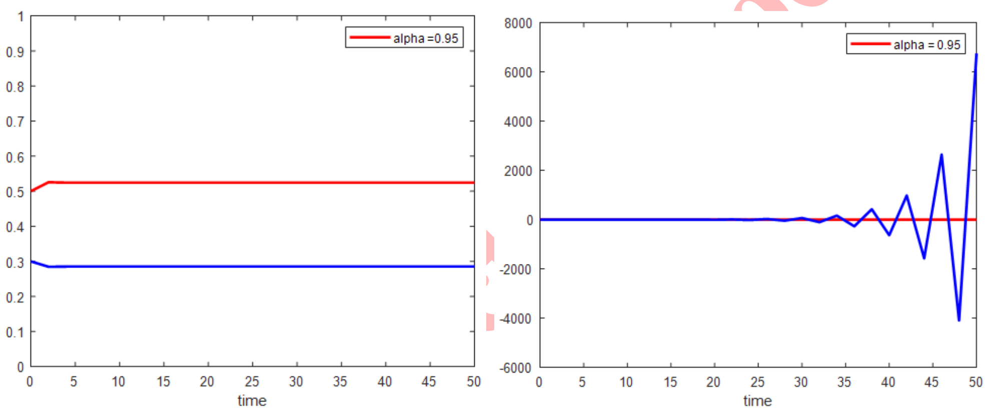

FIGURE 14. The comparison behavior of non-infected and infected cancer cells, between the fractional system (left) and the stochastic fractional system (right), for $\alpha = 0.95$ and parameter for disease state, that approximated using Lagrange's polynomial.

ordinary differential equations to the system of fractional and random differential equations by using Brownian motion, fractional operators Caputo-Fabrizio and Atangana-Baleanu. With this, we were able to apply the non-local effect and randomness of cancer cell growth in the model. Finally, using the numerical method, approximate the numerical solutions of the model. Also, in this numerical scheme, we rewrote the model as an integral version. Because the derivative is defined in the interval $(0, t]$. Therefore, in general, the derivative cannot be calculated at the point $t_0 = 0$. When the zero moment is considered as the origin, the process has not yet started, so no memory can be recorded. The initial conditions will be removed while using the integral we can maintain the initial conditions. Therefore, by remembering the initial condition by the integral, we can apply the non-local effect in the model. After carefully analyzing the numerical results, we have concluded that the numerical scheme is stable when using both the Newton polynomial and the Lagrange polynomial for the model with a disease-free state. However, for the model in the disease state, the numerical scheme is only stable when using Newton's polynomial to approximate the numerical solution.

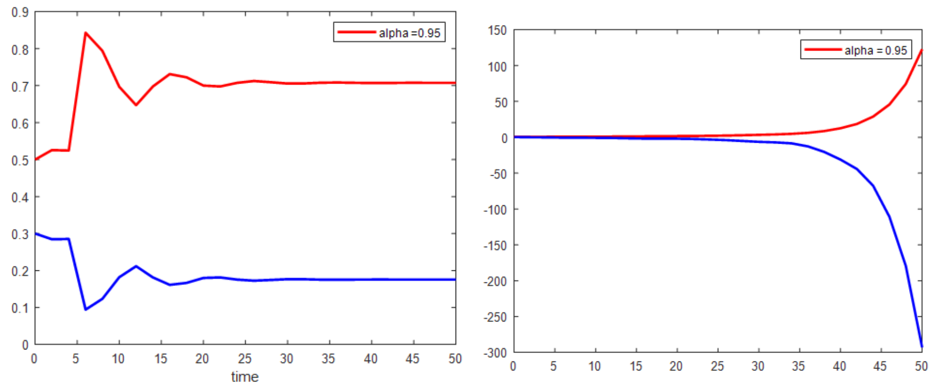

FIGURE 15. The comparison behavior of non-infected and infected cancer cells, between the fractional system (left) and the stochastic fractional system (right), for $\alpha = 0.95$ and parameter for disease state, that approximated using Newton's polynomial.

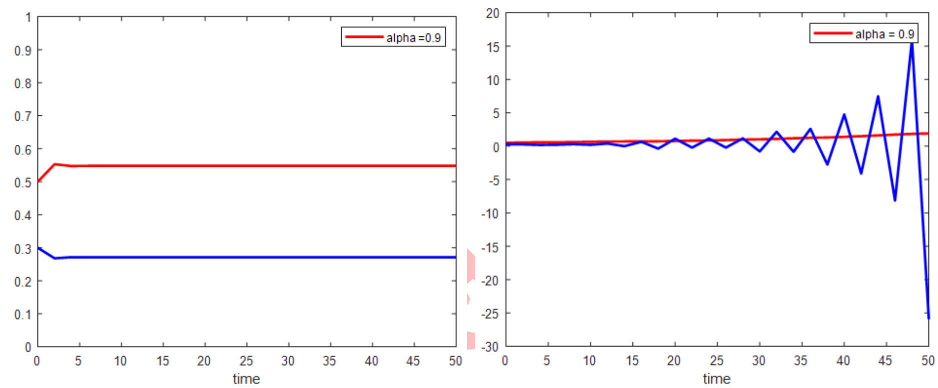

FIGURE 16. The comparison behavior of non-infected and infected cancer cells, between the fractional system (left) and the stochastic fractional system (right), for $\alpha = 0.9$ and parameter for disease state, that approximated using Lagrange's polynomial.

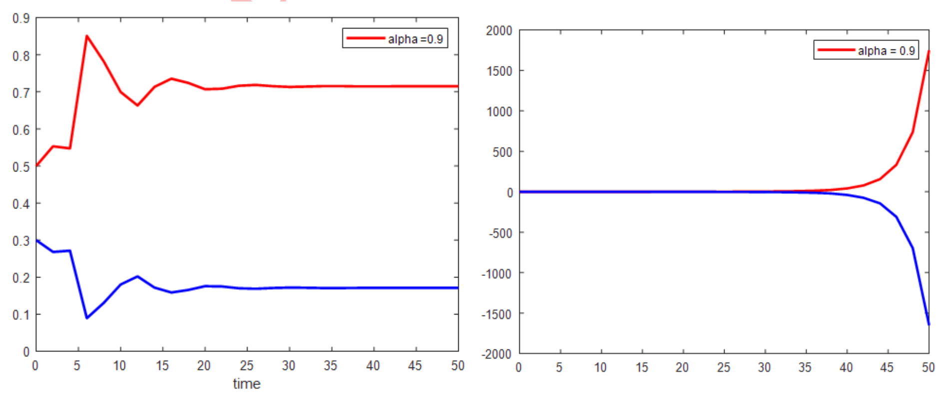

FIGURE 17. The comparison behavior of non-infected and infected cancer cells, between the fractional system (left) and the stochastic fractional system (right), for $\alpha = 0.9$ and parameter for disease state, that approximated using Newton's polynomial.

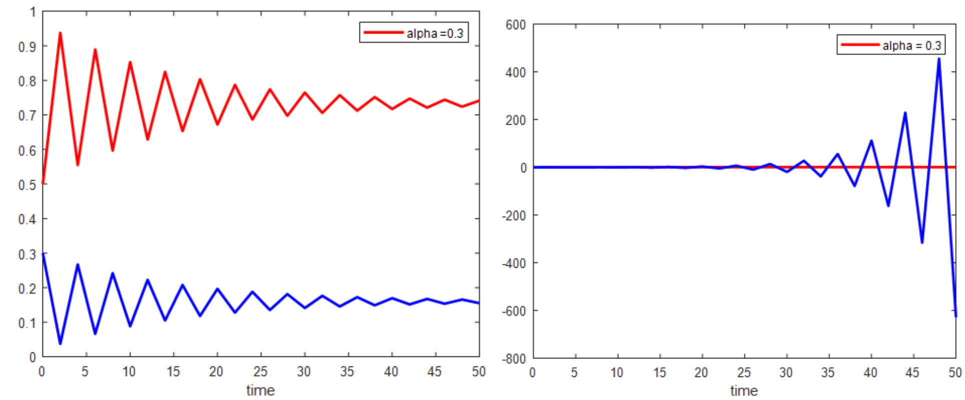

FIGURE 18. The comparison behavior of non-infected and infected cancer cells, between the fractional system (left) and the stochastic fractional system (right), for $\alpha = 0.3$ and parameter for disease state, that approximated using Lagrange's polynomial.

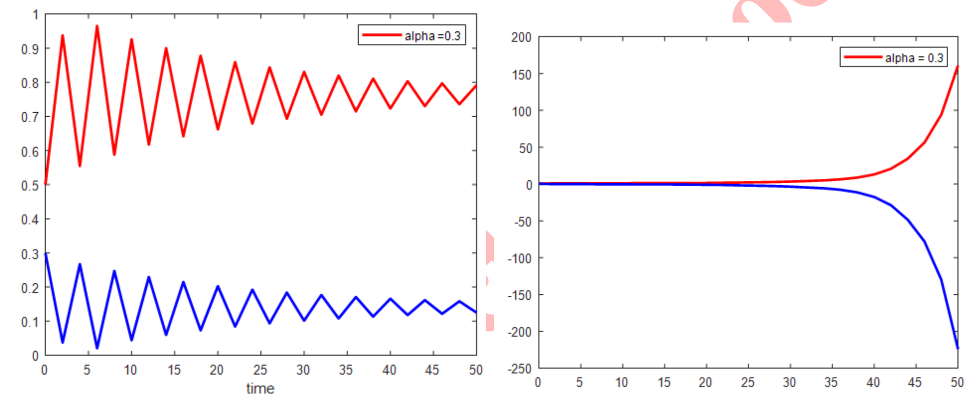

FIGURE 19. The comparison behavior of non-infected and infected cancer cells, between the fractional system (left) and the stochastic fractional system (right), for $\alpha = 0.3$ and parameter for disease state, that approximated using Newton's polynomial.

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