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Distinct solitary wave solutions for the (3+1)-dimensional integrable pKP–BKP equation using the modified extended direct algebraic technique

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Abstract

This study aims to derive solitons and other traveling wave solutions for the pKP-BKP equation, which integrates the potential Kadomtsev–Petviashvili (pKP) and B-type Kadomtsev–Petviashvili (BKP) equations in three spatial dimensions. This equation is used to describe long water waves in oceans, impoundments, and estuaries, as well as to predict tsunamis, analyze river, tidal, and irrigation flows, and simulate weather patterns. The modified extended direct algebraic approach is employed to obtain various types of exact solutions, including dark solitons, combo dark-singular solitons, singular solitons, hyperbolic solutions, singular periodic solutions, exponential solutions, rational solutions, and Jacobi elliptic solutions. The derived solutions are visualized using Mathematica software, with contour, 2D, and 3D graphical representations to illustrate their dynamic behavior.

Keywords. pKP–BKP equation, Dark soliton, Rational solution, Modified extended direct algebraic method.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Partial differential equations (PDEs) are essential tools for modeling complex physical and engineering phenomena, including wave propagation, energy transport, and system dynamics. Recent advancements in PDE-based modeling have contributed to various fields, such as seismic analysis for estimating compressional wave attenuation in carbonate reservoirs [6, 7], and adaptive control strategies for unmanned submarines [3]. Additionally, PDEs have been widely used in Internet of Things (IoT) and cloud computing optimization [8], field-Programmable Gate Array (FPGA)-based encryption systems [25], and photovoltaic energy modeling with improved Maximum Power Point Tracking (MPPT) algorithms [18]. In power generation, PDEs aid in optimizing maintenance and repair strategies for combined cycle power plants [5, 26]. These applications highlight the versatility of PDEs in solving real-world challenges and their importance in deriving exact solutions for nonlinear evolution equations.

Numerous physical applications, including atmospheric systems, optics, plasma physics, nonlinear fiber optics, and fluid dynamics, involve the study of solitons. Many researchers have introduced new solutions for higher-order integrable equations. For instance, Lakestani et al. [17] established novel soliton solutions for nonlinear fifth-order integrable equations. Manafian and Lakestani [23] explored the interaction among a lump, periodic waves, and kink solutions in the fractional generalized CBS-BK equation. Manafian [24] derived new exact multi-soliton solutions for a higher-order nonlinear equation. Ma [19] obtained soliton solutions for a higher-dimensional integrable system using the bilinear approach. El-Shamy et al. [9] investigated new solitons in optical media incorporating higher-order

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dispersive and nonlinear effects. Ahmed et al. [2] established soliton solutions for the generalized Kundu-Eckhaus equation with additional dispersion.

The potential Kadomtsev–Petviashvili (pKP) equation and the B-type Kadomtsev–Petviashvili (BKP) equation were combined to create the nonlinear pKP–BKP equation [1, 4, 10–12, 15, 16, 20–22, 29–37]. Adding more terms to the suggested model resulted in a new (3+1)-dimensional integrable equation with interesting properties. During its brief building period, this model underwent intensive investigation. In [7], certain terms were removed from the original model to provide two reduced equation representations. The integrability of these equations was verified in [35].

For the last time, in [20], a linear mixing of the prospective KP equation and the BKP equation was published using the Hirota bilinear approach [13], which was designed for generating soliton and lump solutions. The technique relies heavily on Hirota bilinear derivatives [14].

An interesting method for comprehending nonlinear wave dynamics in complex systems is to examine the prospective KP (Korteweg-de Vries-Poisson) equation using the B-KP (Bilinear Korteweg-de Vries-Poisson) equation. With its capacity to analyse wave interactions and stability, the B-KP equation offers a reliable tool that helps to clarify how nonlinearity and dispersion impact wave behaviour. This approach advances our understanding of nonlinear events and their management by improving prediction skills and providing creative answers to practical problems involving dynamic systems. The (3+ 1)-dimensional mixed pKP-BKP issue is examined in this paper as [34, 36].

In contrast to prior studies that primarily focused on breather wave solutions and Hirota's bilinear approach, this study systematically derives a more diverse range of exact solutions using the Modified Extended Direct Algebraic (MEDA) technique. By doing so, we provide new insights into the complex wave structures of the (3+1)-dimensional pKP–BKP equation, with potential applications in nonlinear optics, fluid mechanics, and plasma physics.

The governing equation for the (3+1)-dimensional pKP–BKP equation is given by:

$$\begin{aligned} &W_{\mathrm{xt}} + \mu_1 \left(15 W_x W_{\mathrm{xxx}} + 15 (W_x)^3 + W_{\mathrm{xxxxx}}\right)_x + \mu_2 \left(6 W_x W_{\mathrm{xx}} + W_{\mathrm{xxxx}}\right) + \mu_3 \left(W_{\mathrm{xxxy}} + 3\left(W_x W_y\right)_x\right) \\ &+ \mu_4 W_{\mathrm{xx}} + \mu_5 W_{\mathrm{xy}} + \mu_6 W_{\mathrm{xz}} - \frac{\mu_3^2}{5\mu_1} W_{\mathrm{yy}} = 0. \end{aligned} \tag{1.1}$$

where W represents the potential function of the independent variables $x, y, z,$ and the temporal variable t, and the coefficients (μ_i, $i = 1, 2, 3, 4, 5, 6$) are real constants. This equation describes the nonlinear wave dynamics in a (3+1)-dimensional framework and incorporates multiple physical effects, including nonlinearity, dispersion, and cross-interactions between spatial dimensions. Where, $W(x, y, z, t)$ represents the wave potential, dependent on three spatial coordinates (x, y, z) and time t, W_{xt} is a mixed derivative term captures the temporal evolution of wave propagation along the x-direction, $\mu_1 \left(15 W_x W_{xxx} + 15(W_x)^3 + W_{xxxxx}\right)_x$ is a nonlinear and dispersive term, $15 W_x W_{xxx}$ represents nonlinear wave interactions, $15(W_x)^3$ denotes the self-interaction effects of the wave profile, W_{xxxxx} is a higher-order dispersion term that influences wave stability and shape and the entire expression is differentiated concerning x, emphasizing its spatial evolution. Additional nonlinear-dispersive term $\mu_2(6 W_x W_{xx} + W_{xxxx})$ and $6 W_x W_{xx}$ describes the interplay between wave amplitude and curvature, W_{xxxx} is a fourth-order dispersion term that regulates the wave steepness and stability. Coupled cross-dimensional effects in $\mu_3 \left(W_{xxxy} + 3(W_x W_y)_x\right)$, and W_{xxxy} represents mixed spatial derivatives coupling the x- and y-directions and $3(W_x W_y)_x$ indicates nonlinear interaction effects between wave propagation in x and disturbances in y.

Our study's novelty is the way we applied the modified extended direct algebraic (MEDA) technique to the (3+1)-dimensional integrable pKP–BKP equation. This allowed us to derive different classes of exact solutions, such as Jacobi elliptic solutions, hyperbolic solutions, singular periodic solutions, dark solitons, combo dark-singular solitons, and singular solitons. These findings add significantly to the body of literature in the following areas: Diversity of solutions, in contrast to earlier research like [36] and [34], which mostly concentrated on breather wave solutions and Hirota's bilinear approach for reduced versions of the equation, our study methodically builds and categorises a wider range of wave solutions. In particular, the explicit derivation of Jacobi elliptic solutions is a substantial expansion that has not been covered in previous studies. Methodological advancements, our work uses the MEDA methodology, which enables a systematic balance between the nonlinear and highest-order derivative components, resulting in a richer set of precise solutions than [36] and [34], which used the simplified Hirota's method and other transformations.

This methodological enhancement not only provides new solutions but also improves the interpretability of wave structures in physical contexts. Graphical and analytical representation, in this study incorporates an extensive graphical analysis using Mathematica, illustrating the different solution types in 2D and 3D visualizations. This step enhances the understanding of wave dynamics, which is less emphasized in previous works [36] and [34]. Physical insights, by examining the (3+1)-dimensional pKP–BKP equation with extended solution classes, we offer new insights into nonlinear wave propagation in oceanography, plasma physics, and fluid mechanics. The ability to capture various solution behaviors, including periodic and singular structures, presents a more comprehensive picture of wave interactions in complex systems compared to prior studies. The structure of this research article is organized as follows: Section 1 introduction pKP–BKP equation. Section 2 presents the methodology of the modified extended direct algebraic method. Section 3 discusses the obtained results, while section 4 provides graphical representations, including 3D, 2D, and contour simulations. Section 5 offers a detailed analysis of the results and discussions. Section 6 concludes the study, summarizing key findings. Finally, section 7 outlines potential directions for future research.

Future research can explore stability properties and interactions of these solutions in complex environments. Furthermore, by linking the graphical illustrations with physical interpretations, our study enhances the understanding of how different wave structures manifest in real-world applications, such as energy transport in plasmas, nonlinear wave modulation in optical fibers, and fluid dynamics modeling. These findings provide valuable insights into controlling and predicting nonlinear wave behavior in practical scenarios.

2. Methodology

To demonstrate the fundamental principles of the modified extended direct algebraic (MEDA) approach [27, 28]. Consider a PDE with four independent variables, given by:

$$G\,(\,W,\; W_t,\; ,W_x\;,W_{xx}\;,W_{yy}\;,W_{zz}\;,W_{xt}\;,W_{xy}\;,W_{xz}...\,)=0. \tag{2.1}$$

As long as G is a polynomial of W and its partial derivatives for time t and space (x, y, z). Currently, the primary steps of the suggested technique are as follows:

Step I: Using the resulting transformation:

$$W\;(x,\;y,\;z,\;t)\;=\;P\,(\xi), \qquad \xi=\beta_1\,x+\beta_2\,y+\beta_3\,z-\;\rho\;t, \qquad \rho\neq 0, \tag{2.2}$$

where the wave numbers and soliton frequency are shown by $\beta_1,\;\beta_2,\;\beta_3$, and ρ.

Step II: So, Eq. (2.1) turns into:

$$G(P,\,P',\,P'',\,P''',\,P^{(4)},\,P^{(5)}\ldots)=0. \tag{2.3}$$

The key concept behind this technique is that the solution to Eq. (2.3) can be expressed as follows:

$$P(\xi)=\sum_{i=-N}^{N} r_i\;R^i(\xi). \tag{2.4}$$

The differential equation above has an explicit solution given by $P\,(\xi)$, provided that $r_N^2+r_{-N}^2\neq 0$, where r_i represents real-valued constants that must be considered.

$$R'(\xi)=\sqrt{j_0+j_1\;R(\xi)+j_2\;R^2(\xi)+j_3\;R^3(\xi)+j_4\;R^4(\xi)+j_6\;R^6(\xi)}, \tag{2.5}$$

the constant real values $j_n,\;(n=0,1,2,3,4,6)$ determine the probable outcomes of the applied technique.

Step III: Homogeneous balance condition:

In this step, I calculate the values of j_n to make sure that the modified equation's highest-order derivatives and nonlinear terms stay balanced. Using the homogeneous balance condition, we determine the largest exponent N in the expansion of $P(\xi)$.

Step V: Coefficient Matching:

To determine the constants j_n, we substitute the polynomial expansion of $P(\xi)$ into the transformed ordinary differential equation along with the Riccati equation. This substitution results in an algebraic equation where terms involving different powers of $R(\xi)$ appear. By setting the coefficients of these terms to zero, we derive a system of algebraic constraints on j_n. Solving these constraints provides explicit values for j_n, ensuring consistency with the governing equation and enabling the classification of various exact solutions.

Step VI: Computation and classification of solutions for Eq. (2.5):
The algebraic equations obtained from the coefficient matching step are solved to find explicit values for j_n. These values determine different classes of solutions, such as Jacobi elliptical solutions, solitons, (dark, combo dark-singular, singular), hyperbolic, periodic, exponential, and rational solutions. Graphical representations of the solutions are provided for various parameter values to illustrate their behavior. The constants j_n are not arbitrarily chosen but are derived through a systematic mathematical framework involving transformations, balance principles, and coefficient matching. This structured approach ensures that the obtained solutions satisfy the given PDE and align with the physical properties of wave dynamics in the (3+1)-dimensional pKP–BKP equation.

Case 1: When $j_0 = j_1 = j_3 = j_6 = 0$, then:

$$A_{1.1}(\xi) = \sqrt{-\frac{j_2}{j_4}}\,\mathrm{sech}\left[\xi\sqrt{j_2}\right],\ j_2 > 0 \text{ and } j_4 < 0,$$

$$A_{1.2}(\xi) = \sqrt{-\frac{j_2}{j_4}}\,\sec\left[\xi\sqrt{-j_2}\right],\ j_2 < 0 \text{ and } j_4 > 0.$$

Case 2: If $j_0 = \frac{j_2^2}{4j_4}$, $j_1 = j_3 = j_6 = 0$, then:

$$A_{2.1}(\xi) = \sqrt{-\frac{j_2}{2j_4}}\,\tanh\left[\xi\sqrt{-\frac{j_2}{2}}\right],\ j_2 < 0 \text{ and } j_4 > 0,$$

$$A_{2.2}(\xi) = \sqrt{\frac{j_2}{2j_4}}\,\tan\left[\xi\sqrt{\frac{j_2}{2}}\right],\ j_2 > 0 \text{ and } j_4 > 0.$$

Case 3: If $j_3 = j_4 = j_6 = 0$, then:

$$A_{3.1}(\xi) = \frac{j_1}{2j_2}\left[\sinh\left[2\xi\sqrt{j_2}\right] - 1\right],\ j_2 > 0 \text{ and } j_0 = 0,$$

$$A_{3.2}(\xi) = \frac{j_1}{2j_2}\left[\sin\left[2\,\xi\sqrt{-j_2}\right] - 1\right],\ j_2 < 0 \text{ and } j_0 = 0,$$

$$A_{3.3}(\xi) = \exp^{(\xi\sqrt{j_2})} - \frac{j_1}{2j_2},\ j_0 = \frac{j_1^2}{4j_2} \text{ and } j_2 > 0.$$

Case 4: If $j_0 = j_1 = j_2 = j_6 = 0$, then:

$$A_{4.1}(\xi) = \frac{4j_3}{j_3^2\,\xi^2 - 4j_4}.$$

Case 5: If $j_0 = j_1 = j_6 = 0$, then:

$$A_{5.1}(\xi) = -\frac{j_2}{j_3}\left[\tanh\left[\frac{\xi}{2}\sqrt{j_2}\right] + 1\right],\ j_2 > 0,\ \text{and } j_2 = \frac{4j_4}{j_3^2},$$

$$A_{5.2}(\xi) = -\frac{j_2}{j_3}\left[\coth\left[\frac{\xi}{2}\sqrt{j_2}\right] + 1\right],\ j_2 > 0,\ \text{and } j_2 = \frac{4j_4}{j_3^2},$$

$$A_{5.3}(\xi) = \frac{j_2\,\mathrm{sech}^2\left[\frac{\xi\sqrt{j_2}}{2}\right]}{2\sqrt{j_2 j_4}\,\tanh\left[\frac{\xi\sqrt{j_2}}{2}\right] - j_3},\ j_3^2 \neq 4j_2\,j_4,\ j_2 > 0, \text{ and } j_4 > 0,$$

$$A_{5.4}(\xi) = \frac{j_2\,\sec^2\left[\frac{\xi\sqrt{-j_2}}{2}\right]}{2\sqrt{-j_2 j_4}\,\tan\left[\frac{\xi\sqrt{-j_2}}{2}\right] + j_3},\ j_3^2 \neq 4j_2\,j_4,\ j_2 < 0, \text{ and } j_4 > 0.$$

Case 6: If $j_2 = j_4 = j_6 = 0$, then:

$$A_{6.1}(\xi) = \wp\left(\frac{\xi\sqrt{j_3}}{2}; -\frac{4j_1}{j_3}, -\frac{4j_0}{j_3}\right), \; j_3 > 0.$$

Case 7: If $j_1 = j_3 = 0$, then:

$$A_{7.1}(\xi) = \sqrt{\frac{2j_2 \mathrm{sech}^2\left(\xi\sqrt{j_2}\right)}{2\sqrt{j_4^2 - 4j_2\, j_6} - \left(\sqrt{j_4^2 - 4j_2 j_6} + j_4\right)\mathrm{sech}^2\left[\xi\sqrt{j_2}\right]}}, \; j_2 > 0,$$

$$A_{7.2}(\xi) = \sqrt{\frac{2j_2 \sec^2\left[\xi\sqrt{-j_2}\right]}{2\sqrt{j_4^2 - 4j_2 s_6} - \left(\sqrt{j_4^2 - 4j_2 j_6} - j_4\right)\sec^2\left[\xi\sqrt{-j_2}\right]}}, \; j_2 < 0,$$

$$A_{7.3}(\xi) = \sqrt{\frac{8j_2 \tanh^2\left[\xi\sqrt{-\frac{j_2}{3}}\right]}{3j_4\left(\tanh^2\left[\xi\sqrt{-\frac{j_2}{3}}\right] + 3\right)}}, \; j_2 < 0,$$

$$A_{7.4}(\xi) = \sqrt{\frac{8j_2 \tan^2\left[\xi\sqrt{\frac{j_2}{3}}\right]}{3j_4\left(3 - \tan^2\left[\xi\sqrt{\frac{j_2}{3}}\right]\right)}}, \; j_2 > 0.$$

Case 8: If $j_1 = j_3 = j_6 = 0$, then:

No.	j_0	j_2	j_4	$A(\xi)$
1	1	$-(\omega^2+1)$	ω^2	$\mathrm{sn}\,(\xi,\omega)$ or $\mathrm{cd}\,(\xi,\omega)$
2	$\omega^2 - 1$	$-(\omega^2 - 2)$	-1	$\mathrm{dn}\,(\xi,\omega)$
3	$-\,\omega^2$	$2\omega^2 - 1$	$1 - \omega^2$	$\mathrm{nc}\,(\xi,\omega)$
4	- 1	$2 - \omega^2$	$\omega^2 - 1$	$\mathrm{nd}\,(\xi,\omega)$
5	1	$2 - 4\omega^2$	1	$\mathrm{dn}(\xi,\omega)\;\mathrm{nc}(\xi,\omega)\;\mathrm{sn}(\xi,\omega)$
6	$\omega^4 - 2\omega^3 + \omega^2$	$-\frac{4}{\omega}$	$-\omega^2 + 6\omega - 1$	$\frac{j\;\mathrm{cn}(\xi,\omega)\;\mathrm{dn}(\xi,\omega)}{1+j\;\mathrm{sn}^2(\xi,\omega)}$
7	$\frac{1}{4}$	$\frac{\omega^2}{2} - 1$	$\frac{\omega^4}{4}$	$\frac{\mathrm{sn}(\xi,\omega)}{1+\mathrm{dn}(\xi,\omega)}$ or $\frac{\mathrm{cn}(\xi,\omega)}{\sqrt{1-j^2}\;+\;\mathrm{dn}(\xi,\omega)}$

Subsequently, by inserting the obtained constants r_i into Eq. (2.4) along with the general solutions of Eq. (2.5), several exact wave solutions to Eq. (1.1) can be obtained.

3. Analysis of (3+1)-dimensional pKP-BKP equation

The wave transformation in Eq. (2.2) reduces Eq. (1.1) to the following ordinary differential equation (ODE):

$$\begin{aligned}&\beta_1^6\mu_1 P^{(6)} + 15\beta_1^5\mu_1 P' P^{(4)} + \left(\beta_1^4\mu_2 + \beta_2\beta_1^3\mu_3\right)P^{(4)} + 15\beta_1^5\mu_1 P'' P^{(3)} + \left(6\beta_1^3\mu_2 + 6\beta_2\beta_1^2\mu_3\right)P'P'' \\ &+ 45\beta_1^4\mu_1\left(P'\right)^2 P'' + \left(\beta_1^2\mu_4 + \beta_2\beta_1\mu_5 + \beta_3\beta_1\mu_6 - \frac{\beta_2^2\mu_3^2}{5\mu_1} - \beta_1\rho\right)P'' = 0.\end{aligned} \tag{3.1}$$

Integrating Eq. (3.1) concerning ξ and setting the constant of integration to zero, yields the following ordinary differential equation (ODE):

$$\begin{aligned}&\mu_1\beta_1^6 P^{(5)} + \left(\mu_2\beta_1^4 + \mu_3\beta_2\beta_1^3\right)P^{(3)} + \left(\beta_1^2\mu_4 + \beta_2\beta_1\mu_5 + \beta_3\beta_1\mu_6 - \frac{\beta_2^2\mu_3^2}{5\mu_1} - \beta_1\rho + 15\beta_1^5\mu_1 P^{(3)}\right)P' \\ &+ \left(3\beta_1^3\mu_2 + 3\beta_2\beta_1^2\mu_3\right)(P')^2 + 15\beta_1^4\mu_1 (P')^3 = 0.\end{aligned} \tag{3.2}$$

Assuming $P' = Q$, then, Eq. (3.2) can be reduced to:

$$\begin{aligned}&\beta_1^6\, Q^{(4)} + \left(\beta_1^4\mu_2 + \beta_2\beta_1^3\mu_3\right)Q'' + \left(\beta_1^2\mu_4 + \beta_2\beta_1\mu_5 + \beta_3\beta_1\mu_6 - \frac{\beta_2^2\mu_3^2}{5\mu_1} - \beta_1\rho + 15\beta_1^5\mu_1 Q''\right)Q \\ &+ \left(3\beta_1^3\mu_2 + 3\beta_2\beta_1^2\mu_3\right)\, Q^2 + 15\beta_1^4\mu_1\, Q^3 = 0.\end{aligned} \tag{3.3}$$

Writing the general answer for Eq. (3.3) using the suggested method in Sect. 2 as follows under constraint $\beta_1\,\mu_1 \neq 0$:

$$Q = r_0 + r_1 R(\xi) + r_2 R^2(\xi) + r_{-1}\left(\frac{1}{R(\xi)}\right) + r_{-2}\left(\frac{1}{R^2(\xi)}\right), \tag{3.4}$$

where the constants $r_0, r_1, r_{-1}, r_2,$ and r_{-2} are those that will be computed so long as r_2 or $r_{-2} \neq 0$. By entering Eqs. (3.4) and (2.5) into Eq. (3.3), grouping coefficients of comparable powers, and setting them all to zero, one may generate a system of NLAEs. These can be solved using the Mathematica program to obtain the results shown below:

Case 1: $j_0 = j_1 = j_3 = j_6 = 0$.

In this situation, we identify the following solution sets:

(1.1) $\mu_5 = \dfrac{5\beta_1\mu_1(\rho - \beta_3\mu_6) - 5\beta_1^2\mu_1\mu_4 + \beta_2^2\mu_3^2 - 80\beta_1^6 j_2^2\mu_1^2 - 20\beta_1^4 j_2\mu_1\mu_2 - 20\beta_2\beta_1^3 j_2\mu_1\mu_3}{5\beta_1\beta_2\mu_1}$,
$r_2 = -2j_4\beta_1,$ and $r_0 = r_1 = r_{-1} = r_{-2} = 0.$

(1.2) $\mu_5 = \dfrac{1}{5\beta_2\mu_1(4\beta_1 j_2 + 3r_0)^2}\Big(16\beta_1^2 j_2^2\left(\beta_1\mu_2^2 + \mu_1\left(-5\beta_1\mu_4 - 5\beta_3\mu_6 + 8\beta_1^3 j_2\mu_2 + 5\rho\right) + 16\beta_1^5 j_2^2\mu_1^2\right)$
$+r_0^2\left(9\beta_1\mu_2^2 + 15\mu_1\left(-3\beta_1\mu_4 - 3\beta_3\mu_6 + 40\beta_1^3 j_2\mu_2 + 3\rho\right) + 7680\beta_1^5 j_2^2\mu_1^2\right) + 24\beta_1 j_2 r_0\left(\beta_1\mu_2^2 + \mu_1\right.$
$\left.\left(-5\beta_1\mu_4 - 5\beta_3\mu_6 + 24\beta_1^3 j_2\mu_2 + 5\rho\right) + 120\beta_1^5 j_2^2\mu_1^2\right) + 180\beta_1^2\mu_1 r_0^3\left(35\beta_1^2 j_2\mu_1 + \mu_2\right) + 1575\beta_1^3\mu_1^2 r_0^4\Big)$,
$\mu_3 = \dfrac{\beta_1\left(-2\beta_1\mu_1\left(8\beta_1^2 j_2^2 + 30\beta_1 j_2 r_0 + 15r_0^2\right) - \mu_2\, 4\beta_1 j_2 + 3r_0\right)}{\beta_2(4\beta_1 j_2 + 3r_0)}$,
$r_2 = -2j_4\beta_1,\ r_1 = r_{-1} = 0,$ and $r_{-2} = 0.$

(1.3) $\mu_5 = \dfrac{\beta_1\mu_2^2 + 5\mu_1\left(-\beta_1\mu_4 - \beta_3\mu_6 + 8\beta_1^3 j_2\mu_2 + \rho\right) + 720\beta_1^5 j_2^2\mu_1^2}{5\beta_2\mu_1},\ \mu_3 = -\dfrac{\beta_1\left(20\beta_1^2 j_2\mu_1 + \mu_2\right)}{\beta_2}$,
$r_2 = -4\beta_1 j_4,$ and $r_0 = r_1 = r_{-1} = r_{-2} = 0.$

(1.1) The matching solutions of Eq. (1.1) from the previous set (1.1) are either a dark soliton solution or a singular periodic solution, which is elevated when $j_2 > 0$ or $j_2 < 0$.

$$W_{1.1,1}(x,y,z,t) = 2\beta_1\sqrt{j_2}\tanh\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{j_2}\right), \tag{3.5}$$

or

$$W_{1.1,2}(x,y,z,t) = -2\beta_1\sqrt{-j_2}\tan\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{-j_2}\right), \tag{3.6}$$

$$W_{1.1,3}(x,y,z,t) = 2\beta_1\sqrt{-j_2}\cot\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{-j_2}\right). \tag{3.7}$$

(1.2) The matching solutions of Eq. (1.1) from the previous set (1.2) are either a dark soliton solution or a singular periodic solution, which is elevated when $j_2 > 0$ or $j_2 < 0$.

$$W_{1.2,1}(x,y,z,t) = r_0\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right) + 2\beta_1\sqrt{j_2}\tanh\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\sqrt{j_2}\right), \tag{3.8}$$

or

$$W_{1.2,2}(x,y,z,t) = r_0\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right) - 2\beta_1\sqrt{-j_2}\tan\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\sqrt{-j_2}\right), \tag{3.9}$$

$$W_{1.2,3}(x,y,z,t) = r_0\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right) + 2\beta_1\sqrt{-j_2}\cot\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\sqrt{-j_2}\right). \tag{3.10}$$

(1.3) The matching solutions of Eq. (1.1) from the previous set (1.3) are either a dark soliton solution or a singular periodic solution, which is elevated when $j_2 > 0$ or $j_2 < 0$.

$$W_{1.3,1}(x,y,z,t) = 4\beta_1\sqrt{j_2}\tanh\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\sqrt{j_2}\right), \tag{3.11}$$

or

$$W_{1.3,2}(x,y,z,t) = 4\beta_1\sqrt{-j_2}\tan\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\sqrt{-j_2}\right), \tag{3.12}$$

$$W_{1.3,3}(x,y,z,t) = 4\beta_1\sqrt{-j_2}\cot\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\sqrt{-j_2}\right). \tag{3.13}$$

Case 2: $j_0 = \frac{j_2^2}{4j_4}$, and $j_1 = j_3 = j_6 = 0$.
In this situation, we identify the following solution sets:

$$(2.1)\ \mu_4 = \frac{5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + \rho\right) + \beta_2^2\mu_3^2 - 20\beta_1^6 j_2^2\mu_1^2 + 10\beta_1^4 j_2\mu_1\mu_2 + 10\beta_2\beta_1^3 j_2\mu_1\mu_3}{5\beta_1^2\mu_1},$$
$$r_2 = -2j_4\beta_1,\ r_0 = -j_2\beta_1,\ \text{and } r_1 = r_{-1} = r_{-2} = 0.$$

$$(2.2)\ \mu_4 = \frac{1}{5\beta_1^2\mu_1\left(\beta_1 j_2 + 3r_0\right)}\left(\beta_1 j_2\left(\beta_2^2\mu_3^2 - 5\beta_1\mu_1\left(\beta_2\mu_5 + \beta_3\mu_6 + 3\beta_1^5 j_2^2\mu_1 - \rho\right)\right) + 3r_0\left(\beta_2^2\mu_3^2\right.\right.$$
$$\left.\left. + 5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + 9\beta_1^5 j_2^2\mu_1 + \rho\right)\right) + 375\beta_1^5 j_2\mu_1^2 r_0^2 + 225\beta_1^4\mu_1^2 r_0^3\right),\ r_2 = -2\beta_1 j_4,$$
$$\mu_2 = \frac{-2\beta_1^2\mu_1\left(2\beta_1^2 j_2^2 + 15\beta_1 j_2 r_0 + 15r_0^2\right) - \beta_2\mu_3\left(\beta_1 j_2 + 3r_0\right)}{\beta_1\left(\beta_1 j_2 + 3r_0\right)},\ \text{and } r_1 = r_{-1} = r_{-2} = 0.$$

$$(2.3)\ \mu_4 = \frac{5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + \rho\right) + \beta_2^2\mu_3^2 - 20\beta_1^6 j_2^2\mu_1^2 + 10\beta_1^4 j_2\mu_1\mu_2 + 10\beta_2\beta_1^3 j_2\mu_1\mu_3}{5\beta_1^2\mu_1},$$
$$r_{-2} = -\frac{\beta_1 j_2^2}{2j_4},\ r_0 = -j_2\beta_1,\ \text{and } r_1 = r_{-1} = r_2 = 0.$$

$$(2.4)\ \mu_4 = \frac{5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + \rho\right) + \beta_2^2\mu_3^2 - 320\beta_1^6 j_2^2\mu_1^2 + 40\beta_1^4 j_2\mu_1\mu_2 + 40\beta_2\beta_1^3 j_2\mu_1\mu_3}{5\beta_1^2\mu_1},$$
$$r_0 = -2j_2\beta_1,\ r_2 = -2j_4\beta_1,\ r_{-2} = -\frac{\beta_1 j_2^2}{2j_4},\ \text{and } r_1 = r_{-1} = r_2 = 0.$$

$$(2.5)\ \mu_4 = -\frac{1}{5\beta_1^2\mu_1\left(2\beta_1 j_2 - 3r_0\right)}\left(-2\beta_1 j_2\left(\beta_2^2\mu_3^2 + 5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + 108\beta_1^5 j_2^2\mu_1 + \rho\right)\right)\right.$$
$$\left. + 3r_0\left(\beta_2^2\mu_3^2 - 5\beta_1\mu_1\left(\beta_2\mu_5 + \beta_3\mu_6 + 76\beta_1^5 j_2^2\mu_1 - \rho\right)\right) + 150\beta_1^5 j_2\mu_1^2 r_0^2 + 225\beta_1^4\mu_1^2 r_0^3\right),$$
$$\mu_2 = \frac{2\beta_1\mu_1\left(15r_0^2 - 28\beta_1^2 j_2^2\right)}{2\beta_1 j_2 - 3r_0} - \frac{\beta_2\mu_3}{\beta_1},\ r_{-2} = -\frac{\beta_1 j_2^2}{2j_4},\ r_2 = -2j_4\beta_1,\ r_1 = r_{-1} = 0.$$

(2.6) $\mu_4 = \dfrac{5\beta_1\left(-\beta_2\mu_5-\beta_3\mu_6+\rho\right)+\frac{\beta_2^2\ \mu_3^2}{\mu_1}+80\ \beta_1^6\ j_2^2\mu_1}{5\beta_1^2},\ \mu_2 = \dfrac{10\beta_1^3\ j_2\mu_1-\beta_2\mu_3}{\beta_1},$

$r_0 = -2j_2\beta_1,\ r_2 = -4j_4\beta_1,$ and $r_1 = r_{-1} = r_{-2} = 0.$

(2.7) $\mu_4 = \dfrac{\beta_2^2\mu_3^2+5\beta_1\mu_1\left(-\beta_2\mu_5-\beta_3\mu_6+256\beta_1^5 j_2^2\mu_1+\rho\right)}{5\beta_1^2\mu_1},\ \mu_2 = \dfrac{40\beta_1^3 j_2\mu_1-\beta_2\mu_3}{\beta_1},$

$r_0 = -4j_2\beta_1,\ r_2 = -4j_4\beta_1,\ r_{-2} = -\dfrac{\beta_1 j_2^2}{j_4},$ and $r_1 = r_{-1} = 0.$

(2.1) The corresponding solutions to Eq. (1.1) from the preceding set (2.1) are either a dark soliton solution or a singular periodic solution, which is elevated when $j_2 < 0$ or $j_2 > 0$.

$$W_{2.1,1}(x,y,z,t) = \beta_1\sqrt{-2j_2}\tanh\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{-\frac{j_2}{2}}\right), \tag{3.14}$$

or

$$W_{2.1,2}(x,y,z,t) = -\beta_1\sqrt{2j_2}\tan\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{\frac{j_2}{2}}\right). \tag{3.15}$$

(2.2) The corresponding solutions to Eq. (1.1) from the preceding set (2.2) are either a dark soliton solution or a singular periodic solution, which is elevated when $j_2 < 0$ or $j_2 > 0$.

$$\begin{aligned} W_{2.2,1}(x,y,z,t) = {} & \beta_1\left(j_2\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)+\sqrt{-2j_2}\tanh\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{-\frac{j_2}{2}}\right)\right) \\ & + r_0\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right), \end{aligned} \tag{3.16}$$

or

$$\begin{aligned} W_{2.1,2}(x,y,z,t) = {} & \beta_1\sqrt{2j_2}\tan\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{\frac{j_2}{2}}\right) - r_0\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right) \\ & - j_2\beta_1\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right). \end{aligned} \tag{3.17}$$

(2.3) The corresponding solutions to Eq. (1.1) from the preceding set (2.3) are either a singular soliton solution or a singular periodic solution, which is elevated when $j_2 < 0$ or $j_2 > 0$.

$$W_{2.3,1}(x,y,z,t) = -\beta_1\sqrt{-2j_2}\coth\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{-\frac{j_2}{2}}\right), \tag{3.18}$$

or

$$W_{2.3,2}(x,y,z,t) = \beta_1\sqrt{2j_2}\cot\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{\frac{j_2}{2}}\right). \tag{3.19}$$

(2.4) The corresponding solutions to Eq. (1.1) from the preceding set (2.4) are either a singular soliton solution or a singular periodic solution, which is elevated when $j_2 < 0$ or $j_2 > 0$.

$$W_{2.4,1}(x,y,z,t) = \beta_1\sqrt{-8j_2}\coth\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{-2\,j_2}\right), \tag{3.20}$$

or

$$W_{2.4,2}(x,y,z,t) = \beta_1\sqrt{8j_2}\cot\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{2j_2}\right). \tag{3.21}$$

(2.5) The corresponding solutions to Eq. (1.1) from the preceding set (2.5) are either a singular soliton solution or a singular periodic solution, which is elevated when $j_2 < 0$ or $j_2 > 0$.

$$\begin{aligned} W_{2.5,1}(x,y,z,t) = {} & 2\beta_1\sqrt{-2j_2}\coth\left(\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\sqrt{-2j_2}\right) + r_0\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right) \\ & + 2j_2\,\beta_1\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right), \end{aligned} \tag{3.22}$$

or

$$W_{2.5,2}(x,y,z,t) = 2\beta_1\sqrt{2j_2}\cot\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{2j_2}\right). \tag{3.23}$$

(2.6) The corresponding solutions to Eq. (1.1) from the preceding set (2.6) are either a dark soliton solution or a singular periodic solution, which is elevated when $j_2 < 0$ or $j_2 > 0$.

$$W_{2.6,1}(x,y,z,t) = \beta_1\sqrt{-8j_2}\tanh\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{-\frac{j_2}{2}}\right), \tag{3.24}$$

or

$$W_{2.6,2}(x,y,z,t) = -\beta_1\sqrt{8j_2}\tan\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{\frac{j_2}{2}}\right). \tag{3.25}$$

(2.7) The corresponding solutions to Eq. (1.1) from the preceding set (2.7) are either a singular soliton solution or a singular periodic solution, which is elevated when $j_2 < 0$ or $j_2 > 0$.

$$W_{2.7,1}(x,y,z,t) = \beta_1\sqrt{-32\,j_2}\coth\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{-2j_2}\right), \tag{3.26}$$

or

$$W_{2.7,2}(x,y,z,t) = \beta_1\sqrt{32\,j_2}\cot\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{2\,j_2}\right). \tag{3.27}$$

Case 3: $j_3 = j_4 = j_6 = 0$.
In this situation, we identify the following solution sets:

(3.1) $\mu_4 = \dfrac{5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + \rho\right) + \beta_2^2\mu_3^2 - 80\beta_1^6 j_2^2\mu_1^2 - 20\beta_1^4 j_2\mu_1\mu_2 - 20\beta_2\beta_1^3 j_2\mu_1\mu_3}{5\beta_1^2\mu_1}$, $r_{-2} = -2j_0\beta_1$,
$r_0 = r_1 = r_{-1} = r_2 = 0$, and $j_1 = 0$.

(3.2) $\mu_4 = \dfrac{\beta_2^2\,\mu_3^2 + 5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + 64\beta_1^5 j_2^2\mu_1 + \rho\right)}{5\beta_1^2\mu_1}$, $\mu_2 = -\dfrac{\beta_2\mu_3 + 20\beta_1^3 j_2\mu_1}{\beta_1}$, $r_{-2} = -4j_0\beta_1$,
$r_0 = r_1 = r_{-1} = r_2 = 0$, and $j_1 = 0$.

(3.3) $\mu_4 = \dfrac{5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + \rho\right) + \beta_2^2\mu_3^2 - 5\beta_1^6 j_2^2\mu_1^2 - 5\beta_1^4 j_2\mu_1\mu_2 - 5\beta_2\beta_1^3 j_2\mu_1\mu_3}{5\beta_1^2\mu_1}$, $r_{-2} = -\dfrac{\beta_1 j_1^2}{2j_2}$,
$r_{-1} = -j_1\beta_1$, $r_0 = r_1 = r_2 = 0$, and $j_0 = \dfrac{j_1^2}{4j_2}$.

Therefore, the answers to Eq. (1.1) are as follows:
(3.1) The corresponding solutions to Eq. (1.1) from the preceding set (3.1) are either a singular soliton solution or a singular periodic solution, which is elevated when $j_2 > 0$ or $j_2 < 0$.

$$W_{3.1,1}(x,y,z,t) = 2\beta_1\sqrt{j_2}\coth\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{j_2}\right), \tag{3.28}$$

or

$$W_{3.1,2}(x,y,z,t) = 2\beta_1\sqrt{-j_2}\cot\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{-j_2}\right). \tag{3.29}$$

(3.2) The corresponding solutions to Eq. (1.1) from the preceding set (3.2) are either a singular soliton solution or a singular periodic solution, which is elevated when $j_2 > 0$ or $j_2 < 0$.

$$W_{3.2,1}(x,y,z,t) = 4\beta_1\sqrt{j_2}\coth\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{j_2}\right), \tag{3.30}$$

or

$$W_{3.2,2}(x,y,z,t) = 4\beta_1\sqrt{-j_2}\cot\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{-j_2}\right). \tag{3.31}$$

(3.3) The matching solutions of Eq. (1.1) from the previous set (3.3) is an exponential solution, which is elevated when $j_2 > 0$ and $j_1 \neq 2j_2 e^{\sqrt{j_2}(\beta_1 x+\beta_2 y+\beta_3 z-\rho t)}$.

$$W_{3.3}(x,y,z,t) = -\frac{2j_1\beta_1\sqrt{j_2}}{j_1 - 2j_2 e^{(\beta_1 x+\beta_2 y+\beta_3 z-\rho t)\sqrt{j_2}}}. \tag{3.32}$$

Case 4: $j_0 = j_1 = j_2 = j_6 = 0$.
In this situation, we identify the following solution set:

$$\mu_4 = \frac{5\beta_1\mu_1(-\beta_2\mu_5 - \beta_3\mu_6 + \rho) + \beta_2^2\mu_3^2}{5\beta_1^2\mu_1},\ \mu_2 = -\frac{\beta_2\mu_3}{\beta_1},\ r_1 = -j_3\beta_1,\ r_2 = -2\beta_1 j_4,$$
$$r_0 = r_{-1} = r_{-2} = 0.$$

The matching answers of Eq. (1.1) from the set above is the rational solution, which is elevated when $j_4 \neq \frac{1}{4}{j^2}_3(\beta_1 x+\beta_2 y+\beta_3 z-\rho t)^2$.

$$W_{4.1}(x,y,z,t) = \frac{4\beta_1 j_3^2(\beta_1 x+\beta_2 y+\beta_3 z-\rho t)}{(j_3(\beta_1 x+\beta_2 y+\beta_3 z-\rho t))^2 - 4j_4}. \tag{3.33}$$

Case 5: $j_0 = j_1 = j_6 = 0$.
In this situation, we identify the following solution set:

$$\mu_6 = \frac{\beta_2^2\mu_3^2 + 5\beta_1\mu_1(-\beta_1\mu_4 - \beta_2\mu_5 + 4\beta_1^5 j_2^2\mu_1 + \rho)}{5\beta_1\beta_3\mu_1},\ \mu_2 = \frac{-\beta_2\mu_3 - 5\beta_1^3 j_2\mu_1}{\beta_1},\ r_1 = -j_3\beta_1,\ r_2 = -2j_4\beta_1,$$
$$r_0 = r_{-1} = r_{-2} = 0.$$

(5.1) The matching solutions of Eq. (1.1) from the set above are either a dark soliton solution or a singular soliton solution, both of which are elevated in Eqs. (3.34) or (3.35) with $j_2 > 0$ and $j_3^2 = 4\,j_2\,j_4$.

$$W_{5.1,1}(x,y,z,t) = \beta_1\sqrt{j_2}\tanh\left(\frac{1}{2}(\beta_1 x+\beta_2 y+\beta_3 z-\rho t)\sqrt{j_2}\right), \tag{3.34}$$

or

$$W_{5.1,2}(x,y,z,t) = \beta_1\sqrt{j_2}\coth\left(\frac{1}{2}(\beta_1 x+\beta_2 y+\beta_3 z-\rho t)\sqrt{j_2}\right), \tag{3.35}$$

(5.2) The matching solutions of Eq. (1.1) from the set above are either a hyperbolic solution when $j_2 > 0$, or a periodic solution when $j_2 < 0$, both of which are elevated in Eqs. (3.36) or (3.37) with $j_3^2 \neq 4\,j_2\,j_4$.

$$W_{5.1,3}(\xi) = \frac{2\beta_1\sqrt{j_2}\left(2j_3\sqrt{j_2 j_4} - (j_3^2 - 4j_2 j_4)\sinh(\xi\sqrt{j_2}) - 4j_2 j_4\tanh\left(\frac{1}{2}\xi\sqrt{j_2}\right)\right)}{4j_2 j_4(\cosh(\xi\sqrt{j_2}) - 1) - j_3^2(\cosh(\xi\sqrt{j_2}) + 1)}, \tag{3.36}$$

or

$$W_{5.1,4}(\xi) = -\frac{2\beta_1 j_2\left(2j_3\sqrt{-j_2 j_4} - (j_3^2 - 4j_2 j_4)\sin(\xi\sqrt{-j_2}) - 4j_2 j_4\tan\left(\frac{1}{2}\xi\sqrt{-j_2}\right)\right)}{\sqrt{-j_2}\left((j_3^2 - 4j_2 j_4)\cos(\xi\sqrt{-j_2}) + j_3^2 + 4j_2 j_4\right)}, \tag{3.37}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.

Case 6: $j_1 = j_3 = j_6 = 0$.
In this situation, we identify the following solution sets:

(6.1) $$\mu_4 = \frac{5\beta_1(-\beta_2\mu_5 - \beta_3\mu_6 + \rho) + \frac{\beta_2^2\mu_3^2}{\mu_1} - 60\beta_1^6 j_0 j_4\mu_1}{5\beta_1^2},\ \mu_2 = -\frac{\beta_2\mu_3 + 4\beta_1^3 j_2\mu_1}{\beta_1},\ r_2 = -2j_4\,\beta_1,$$
$$r_0 = r_1 = r_{-1} = r_{-2} = 0.$$

(6.2) $$\mu_4 = \frac{5\beta_1(-\beta_2\mu_5 - \beta_3\mu_6 + \rho) + \frac{\beta_2^2\mu_3^2}{\mu_1} - 60\beta_1^6 j_0 j_4\mu_1}{5\beta_1^2},\ \mu_2 = -\frac{\beta_2\mu_3 + 4\beta_1^3 j_2\mu_1}{\beta_1},\ r_{-2} = -2j_0\beta_1,$$
$$r_0 = r_1 = r_{-1} = r_2 = 0.$$

(6.3) $\mu_4 = \dfrac{\beta_2^2\mu_3^2 + 5\beta_1\mu_1\left(-\beta_2\mu_5 - \beta_3\mu_6 + 48\beta_1\left(2j_2^2 + j_0j_4\right) + \rho\right)}{5\beta_1^2\mu_1}$, $r_{-2} = -2j_0\beta_1,\ r_2 = -2j_4\,\beta_1,$

$\mu_2 = -\dfrac{\beta_2\mu_3 + 28\beta_1^3 j_2\mu_1}{\beta_1}$, and $r_0 = r_1 = r_{-1} = 0.$

(6.4) $\mu_4 = \dfrac{-\beta_2\mu_5 - \beta_3\mu_6 - 12\beta_1^5 j_0j_4\mu_1 + \rho}{\beta_1},\ \mu_3 = 0,\ \mu_2 = -4j_2\mu_1\beta_1^2,\ r_2 = -2j_4\,\beta_1,\ r_0 = r_1 = 0,$

$r_{-1} = r_{-2} = 0.$

(6.5) $\mu_4 = \dfrac{-\beta_2\mu_5 - \beta_3\mu_6 - 12\beta_1^5 j_0j_4\mu_1 + \rho}{\beta_1},\ \mu_3 = 0,\ \mu_2 = -4j_2\mu_1\beta_1^2,\ r_{-2} = -2j_0\,\beta_1,\ r_0 = r_1 = 0,$

$r_{-1} = r_2 = 0.$

(6.6) $\mu_4 = \dfrac{-\beta_2\mu_5 - \beta_3\mu_6 + 96\beta_1^5 j_2^2\mu_1 + 48\beta_1^5 j_0j_4\mu_1 + \rho}{\beta_1},\ \mu_3 = 0,\ \mu_2 = -28j_2\mu_1\beta_1^2,\ r_{-2} = -2j_0\,\beta_1,$

$r_2 = -2j_4\,\beta_1,\ r_0 = r_1 = r_{-1} = 0.$

The collection of response set (6.1) demonstrates that Eq. (1.1) has exact solutions, which are given by:

Case (6.1,1): If $j_0 = 1,\ j_2 = -1-\omega^2,\ j_4 = \omega^2,$ and $0 < \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) are as follows:

$$W_{6.1,1}(x,y,z,t) = -2\beta_1\omega((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \text{JacobiEpsilon}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t)). \tag{3.38}$$

or

$$W_{6.1,2}(\xi) = -\frac{2\beta_1\,\omega\left(\omega\,\text{cn}(\xi)\,\text{sn}(\xi) + (\xi - \text{JacobiEpsilon}(\xi))\,\text{dn}(\xi)\right)}{\text{dn}(\xi)}, \tag{3.39}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 1$, then Eq. (3.38) reduces to the dark soliton solution:

$$W_{6.1,1.1}(x,y,z,t) = -2\beta_1\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \tanh\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\right). \tag{3.40}$$

Case (6.1,2): If $j_0 = \omega^2 - 1,\ j_2 = 2-\omega^2,\ j_4 = -1,$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.1,3}(x,y,z,t) = 2\beta_1\,\text{JacobiEpsilon}\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right). \tag{3.41}$$

If $\omega = 1$, then Eq. (3.41) reduces to the dark soliton solution:

$$W_{6.1,3.1}(x,y,z,t) = 2\beta_1 - \tanh\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right). \tag{3.42}$$

Case (6.1,3): If $j_0 = -\omega^2,\ j_2 = 2\omega^2 - 1,\ j_4 = 1-\omega^2,$ and $0 \le \omega < 1$, thus, The Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.1,4}(\xi) = \frac{2\beta_1\,(\omega+1)(\text{cn}(\xi)\,(\text{JacobiEpsilon}(\xi) + \xi\,(\omega-1)) - \text{dn}(\xi)\,\text{sn}(\xi))}{\text{cn}(\xi)}, \tag{3.43}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$, then Eq. (3.43) reduces to the singular periodic solution:

$$W_{6.1,4.1}(x,y,z,t) = -2\beta_1\,\tan\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right). \tag{3.44}$$

Case (6.1,4): If $j_0 = -1,\ j_2 = 2-\omega^2,\ j_4 = \omega^2 - 1,$ and $0 \le \omega < 1$, thus, The Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.1,5}(\xi) = \frac{2\beta_1\,(\omega+1)\,(\text{dn}(\xi)\,\text{JacobiEpsilon}(\xi) - \omega\,\text{cn}(\xi)\,\text{sn}(\xi))}{\text{dn}(\xi)}, \tag{3.45}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.

`Case (6.1,5):` If $j_0 = 1,\ j_2 = 2 - 4\omega^2,\ j_4 = 1,$ and $0 \leq \omega \leq 1$, thus, The Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.1,6}(\xi) = -2\beta_1 \left(\frac{\mathrm{dn}(\xi)\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)} - 2\mathrm{JacobiEpsilon}(\xi) + \xi \right), \tag{3.46}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.

If $\omega = 0,$ or $\omega = 1$, then Eq. (3.46) reduce to either a singular periodic solution or a dark soliton solution:

$$W_{6.1,6.1}(x, y, z, t) = 2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \tan(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right), \tag{3.47}$$

or,

$$W_{6.1,6.2}(x, y, z, t) = -2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right). \tag{3.48}$$

`Case (6.1,6):` If $j_0 = \frac{1}{4},\ j_2 = \frac{(\omega^2-2)}{2},\ j_4 = \frac{\omega^4}{4},$ and $0 < \omega \leq 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.1,7}(\xi) = \beta_1 \omega^2 \left((\mathrm{cs}(\xi)(\mathrm{dn}(\xi) - 1) + \mathrm{JacobiEpsilon}(\xi) + \frac{1}{2}\xi(\omega - 2) \right), \tag{3.49}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.

If $\omega = 1$, then Eq. (3.49) reduces to a dark soliton solution:

$$W_{6.1,7.1}(x, y, z, t) = -\beta_1 \left(\frac{1}{2}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \tanh\left(\frac{1}{2}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right) \right). \tag{3.50}$$

The collection of response set `(6.2)` demonstrates that Eq. (1.1) has exact solutions, which are given by:

`Case (6.2,1):` If $j_0 = 1,\ j_2 = -1 - \omega^2,\ j_4 = \omega^2,$ and $0 \leq \omega \leq 1$, thus, the Jacobi elliptic solution to Eq. (1.1) are as follows:

$$W_{6.2,1.1}(\xi) = -2\beta_1 \left(\xi - \mathrm{JacobiEpsilon}(\xi) - \frac{\mathrm{cn}(\xi)\,\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)} \right), \tag{3.51}$$

or,

$$W_{6.2,1.2}(\xi) = -2\beta_1 \left(\xi - \mathrm{JacobiEpsilon}(\xi) + \frac{\mathrm{dn}(\xi)\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)} \right), \tag{3.52}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.

If $\omega = 0$, or $\omega = 1$, then Eq. (3.51) reduces to either a singular periodic solution or a singular soliton solution. Additionally, if $\omega = 0$, then Eq. (3.52) reduces to a singular periodic solution:

$$W_{6.2,1.1}(x, y, z, t) = 2\beta_1 \cot(\beta_1 x + \beta_2 y + \beta_3 z - \rho t), \tag{3.53}$$

or,

$$W_{6.2,1.2}(x, y, z, t) = -2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \coth(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right). \tag{3.54}$$

$$W_{6.2,2.1}(x, y, z, t) = -2\beta_1 \tan(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.55}$$

`Case (6.2,2):` If $j_0 = \omega^2 - 1,\ j_2 = 2 - \omega^2,\ j_4 = -1,$ and $0 \leq \omega < 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.2,3}(x, y, z, t) = \frac{2\beta_1 (\omega + 1) \left(\mathrm{dn}(\xi)\,\mathrm{JacobiEpsilon}(\xi) - \omega\,\mathrm{cn}(\xi)\,\mathrm{sn}(\xi) \right)}{\mathrm{dn}(\xi)}, \tag{3.56}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.

`Case (6.2,3):` If $j_0 = -\omega^2,\ j_2 = 2\omega^2 - 1,\ j_4 = 1 - \omega^2,$ and $0 \leq \omega < 1$, thus, The Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.2,4}(\xi) = 2\beta_1 \omega \left(\mathrm{JacobiEpsilon}(\xi) + \xi(\omega - 1) \right), \tag{3.57}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$, then Eq. (3.57) reduces to a dark soliton solution:

$$W_{6.2,4.1}(x,y,z,t) = -2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \quad (3.58)$$

Case (6.2,4): If $j_0 = -1,\ j_2 = 2 - \omega^2,\ j_4 = \omega^2 - 1,$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.2,5}(x,y,z,t) = 2\beta_1 \,\text{JacobiEpsilon}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \quad (3.59)$$

If $\omega = 1$, then Eq. (3.59) reduces to a dark soliton solution:

$$W_{6.2,5.1}(x,y,z,t) = 2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \quad (3.60)$$

Case (6.2,5): If $j_0 = 1,\ j_2 = 2 - 4\omega^2,\ j_4 = 1,$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.2,6}(\xi) = -\frac{2\beta_1 \left(\left(-2\omega \text{cn}^2(\xi) + 2\omega - 1\right)\text{cs}(\xi) + \text{dn}(\xi)(\xi - 2\text{JacobiEpsilon}(\xi))\right)}{\text{dn}(\xi)}, \quad (3.61)$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.61) reduces to either a singular periodic solution or a singular soliton solution:

$$W_{6.2,6.1}(x,y,z,t) = 2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) + \cot(\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\right), \quad (3.62)$$

or

$$W_{6.2,6.2}(x,y,z,t) = -2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \coth(\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\right). \quad (3.63)$$

Case (6.2,6): If $j_0 = \frac{1}{4},\ j_2 = \frac{(\omega^2-2)}{2},\ j_4 = \frac{\omega^4}{4},$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.2,7}(\xi) = \frac{\beta_1}{2}\left(2\text{cs}(\xi)\left(1 + \text{dn}(\xi)\right) + 2\text{JacobiEpsilon}(\xi) + \xi(\omega - 2)\right), \quad (3.64)$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.64) reduces to either a singular periodic or a combo dark-singular solution:

$$W_{6.2,7.1}(x,y,z,t) = 2\beta_1 \cot(\beta_1 x + \beta_2 y + \beta_3 z - \rho t), \quad (3.65)$$

or

$$W_{6.2,7.2}(\xi) = \beta_1 \left(1 - \frac{\xi}{2} + \tanh(\xi) + \csc(\xi)\right). \quad (3.66)$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.

Combo dark-singular solitons provide a broader class of exact solutions, extending our understanding of nonlinear wave phenomena. Their unique structural features make them particularly useful in analyzing extreme wave events and energy localization in complex media.

The collection of response set (6.3) demonstrates that Eq. (1.1) has exact solutions, which are given by:
Case (6.3,1): If $j_0 = 1,\ j_2 = -1 - \omega^2,\ j_4 = \omega^2,$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) are as follows:

$$W_{6.3,1}(\xi) = -2\beta_1 \left(-\frac{\text{cn}(\xi)\text{dn}(\xi)}{\text{sn}(\xi)} - (\omega + 1)(\text{JacobiEpsilon}(\xi) - \xi)\right), \quad (3.67)$$

or

$$W_{6.3,2}(\xi) = -\frac{2\beta_1 \left(\omega^2 \text{cn}(\xi)\,\text{sn}(\xi) + (1 + \omega)\,\text{dn}(\xi)\,(\xi - \text{JacobiEpsilon}(\xi)) + \text{dn}^2(\xi)\,\text{sc}(\xi)\right)}{\text{dn}(\xi)}. \quad (3.68)$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.67) reduces to either a singular periodic solution or a singular soliton solution. Additionally, if $\omega = 0$, then Eq. (3.68) reduces to a singular periodic solution:

$$W_{6.3,1.1}(x,y,z,t) = 2\beta_1 \cot(\beta_1 x + \beta_2 y + \beta_3 z - \rho t), \tag{3.69}$$

or,

$$W_{6.3,1.2}(x,y,z,t) = -4\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \coth\left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\sqrt{2} \right) \right). \tag{3.70}$$

$$W_{6.3,2.1}(x,y,z,t) = -4\beta_1 \tan(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.71}$$

`Case (6.3,2):` If $j_0 = \omega^2 - 1,\ j_2 = 2 - \omega^2,\ j_4 = -1$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.3,3}(\xi) = 2\beta_1 \left((\omega + 2)\,\mathrm{JacobiEpsilon}(\xi) - \frac{\omega(\omega+1)\,\mathrm{cn}(\xi)\,\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)} \right), \tag{3.72}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 1$, then Eq. (3.72) reduces to a dark soliton solution:

$$W_{6.3,3.1}(x,y,z,t) = 2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.73}$$

`Case (6.3,3):` If $j_0 = -\omega^2,\ j_2 = 2\omega^2 - 1,\ j_4 = 1 - \omega^2$, and $0 \le \omega \le 1$, thus, The Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.3,4}(\xi) = \beta_1 \left((2\omega + 1)\left(\mathrm{JacobiEpsilon}(\xi) + \xi(\omega - 1)\right) - \frac{(\omega+1)\,\mathrm{dn}(\xi)\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)} \right), \tag{3.74}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.74) reduces to either a singular periodic solution or a dark soliton solution:

$$W_{6.3,4.1}(x,y,z,t) = -2\beta_1 \tan(\beta_1 x + \beta_2 y + \beta_3 z - \rho t), \tag{3.75}$$

or

$$W_{6.3,4.2}(x,y,z,t) = 2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.76}$$

`Case (6.3,4):` If $j_0 = -1,\ j_2 = 2 - \omega^2,\ j_4 = \omega^2 - 1$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.3,5}(\xi) = 2\beta_1 \left((\omega + 2)\,\mathrm{JacobiEpsilon}(\xi) - \frac{\omega(\omega+1)\,\mathrm{cn}(\xi)\,\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)} \right), \tag{3.77}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 1$, then Eq. (3.77) reduces to a dark soliton solution:

$$W_{6.3,5.1}(x,y,z,t) = 2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.78}$$

`Case (6.3,5):` If $j_0 = 1,\ j_2 = 2 - 4\omega^2,\ j_4 = 1$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.3,6}(\xi) = 2\beta_1 \left(\frac{(2\omega\,\mathrm{cn}^2(\xi) - 2\omega + 1)\,\mathrm{cs}(\xi)}{\mathrm{dn}(\xi)} - \mathrm{dn}(\xi)\,\mathrm{sc}(\xi) + 4\,\mathrm{JacobiEpsilon}(\xi) - 2\xi \right), \tag{3.79}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.79) reduces to either a singular periodic or a singular soliton solution:

$$W_{6.3,6.1}(x,y,z,t) = 4\beta_1 (\beta_1 x + \beta_2 y + \beta_3 z - \rho t) + \cot(2(\beta_1 x + \beta_2 y + \beta_3 z - \rho t)), \tag{3.80}$$

or

$$W_{6.3,6.2}(x,y,z,t) = -4\beta_1 ((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \coth(2(\beta_1 x + \beta_2 y + \beta_3 z - \rho t))). \tag{3.81}$$

`Case (6.3,6):` If $j_0 = \frac{1}{4},\ j_2 = \frac{(\omega^2-2)}{2},\ j_4 = \frac{\omega^4}{4},$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.3,7}(\xi) = \frac{\beta_1}{2}\left(2\mathrm{cs}(\xi)\left(1-\omega^2+\left(1+\omega^2\right)\mathrm{dn}(\xi)+\left(12\,\mathrm{JacobiEpsilon}(\xi)+\xi\left(\omega-2\right)\right)\right)\right), \tag{3.82}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.82) reduces to either a singular periodic solution or a singular soliton solution:

$$W_{6.3,7.1}(x,y,z,t) = 2\beta_1 \cot\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right), \tag{3.83}$$

or

$$W_{6.3,7.2}(x,y,z,t) = -\beta_1\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right) - 2\coth\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\right). \tag{3.84}$$

The collection of response set (6.4) demonstrates that Eq. (1.1) has exact solutions, which are given by:

`Case (6.4,1):` If $j_0 = 1,\ j_2 = -1-\omega^2,\ j_4 = \omega^2,$ and $0 < \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) are as follows:

$$W_{6.4,1}(x,y,z,t) = -2\beta_1\omega((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \mathrm{JacobiEpsilon}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t)), \tag{3.85}$$

or

$$W_{6.4,2}(\xi) = -\frac{2\omega\,\beta_1\left(\omega\,\mathrm{cn}(\xi)\,\mathrm{sn}(\xi)+\mathrm{dn}(\xi)\left(\xi\,-\mathrm{JacobiEpsilon}(\xi)\right)\right)}{\mathrm{dn}(\xi)}, \tag{3.86}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 1$, then, the Eq. (3.85) reduces to a dark soliton solution:

$$W_{6.4,1.1}(x,y,z,t) = -2\beta_1\left(\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right) - \tanh\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right)\right). \tag{3.87}$$

`Case (6.4,2):` If $j_0 = \omega^2 - 1,\ j_2 = 2 - \omega^2,\ j_4 = -1,$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.4,3}(x,y,z,t) = 2\beta_1\,\mathrm{JacobiEpsilon}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.88}$$

If $\omega = 1$, then Eq. (3.88) reduces to a dark soliton solition:

$$W_{6.4,3.1}(x,y,z,t) = 2\beta_1\,\tanh\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right). \tag{3.89}$$

`Case (6.4,3):` If $j_0 = -\omega^2,\ j_2 = 2\omega^2 - 1,\ j_4 = 1 - \omega^2,$ and $0 \le \omega < 1$, thus, The Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.4,4}(\xi) = \frac{2\beta_1\left(\omega+1\right)\left(\mathrm{cn}(\xi)\,\mathrm{JacobiEpsilon}(\xi)+\xi\left(\omega-1\right)\right)-\mathrm{dn}(\xi)\,\mathrm{sn}(\xi))}{\mathrm{cn}(\xi)}, \tag{3.90}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$, then Eq. (3.90) reduces to a singular periodic solution:

$$W_{6.4,4.1}(x,y,z,t) = -2\beta_1\,\tan\left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t\right). \tag{3.91}$$

`Case (6.4,4):` If $j_0 = -1,\ j_2 = 2 - \omega^2,\ j_4 = \omega^2 - 1,$ and $0 \le \omega < 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.4,5}(\xi) = \frac{2\beta_1\left(\omega+1\right)\left(\mathrm{dn}(\xi)\,\mathrm{JacobiEpsilon}(\xi)-\omega\,\mathrm{cn}(\xi)\,\mathrm{sn}(\xi)\right)}{\mathrm{dn}(\xi)}, \tag{3.92}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
`Case (6.4,5):` If $j_0 = 1,\ j_2 = 2 - 4\omega^2,\ j_4 = 1,$ and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.4,6}(\xi) = -2\beta_1\left(\frac{\mathrm{dn}(\xi)\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)} - 2\,\mathrm{JacobiEpsilon}(\xi) + \xi\right), \tag{3.93}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then, the Eq. (3.92) reduces to either a singular periodic solution or a dark soliton solution:

$$W_{6.4,6.1}(x,y,z,t) = 2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \tan(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right), \tag{3.94}$$

or

$$W_{6.4,6.2}(x,y,z,t) = -2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right). \tag{3.95}$$

`Case (6.4,6):` If $j_0 = \frac{1}{4},\ j_2 = \frac{(\omega^2-2)}{2},\ j_4 = \frac{\omega^4}{4}$, and $0 < \omega \leq 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.4,7}(\xi) = \frac{\beta_1 \omega^2}{2} \left(2\text{cs}(\xi)\left(\text{dn}(\xi) - 1\right) + 2\,\text{JacobiEpsilon}(\xi) + \xi\,(\omega - 2) \right), \tag{3.96}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 1$, then Eq. (3.96) reduces to a dark soliton solution:

$$W_{6.4,7.1}(x,y,z,t) = \frac{-\beta_1}{2} \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - 2\tanh\left(\frac{1}{2}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right) \right). \tag{3.97}$$

The collection of response set (6.5) demonstrates that Eq. (1.1) has exact solutions, which are given by:
`Case (6.5,1):` If $j_0 = 1,\ j_2 = -1 - \omega^2,\ j_4 = \omega^2$, and $0 < \omega \leq 1$, thus, the Jacobi elliptic solution to Eq. (1.1) are as follows:

$$W_{6.5,1}(\xi) = 2\beta_1 \left(\frac{\text{cn}(\xi)\ \text{dn}(\xi)}{\text{sn}(\xi)} + \text{JacobiEpsilon}(\xi) - \xi \right), \tag{3.98}$$

or

$$W_{6.5,2}(\xi) = -2\beta_1 \left(\frac{\text{dn}(\xi)\ \text{sn}(\xi)}{\text{cn}(\xi)} - \text{JacobiEpsilon}(\xi) + \xi \right), \tag{3.99}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$, or $\omega = 1$, then Eq. (3.98) reduces to either a singular periodic solution or a singular soliton solution. Additionally, if $\omega = 0$, then Eq. (3.99) reduces to the singular periodic solution:

$$W_{6.5,1.1}(x,y,z,t) = 2\beta_1\ \cot(\beta_1 x + \beta_2 y + \beta_3 z - \rho t), \tag{3.100}$$

or

$$W_{6.5,1.2}(x,y,z,t) = -2\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \coth(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) \right). \tag{3.101}$$

$$W_{6.5,2.1}(x,y,z,t) = -2\beta_1 \tan(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.102}$$

`Case (6.5,2):` If $j_0 = \omega^2 - 1,\ j_2 = 2 - \omega^2,\ j_4 = -1$, and $0 \leq \omega < 1$, thus, the Jacobi elliptic solution for Eq. (1.1) is as follows:

$$W_{6.5,3}(\xi) = \frac{2\beta_1(\omega + 1)(\text{dn}(\xi)\,\text{JacobiEpsilon}(\xi) - \omega\,\text{cn}(\xi)\,\text{sn}(\xi))}{\text{dn}(\xi)}, \tag{3.103}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
`Case (6.5,3):` If $j_0 = -\omega^2,\ j_2 = 2\omega^2 - 1,\ j_4 = 1 - \omega^2$, and $0 < \omega \leq 1$, thus, the Jacobi elliptic solution for Eq. (1.1) is as follows:

$$W_{6.5,4}(x,y,z,t) = 2\beta_1 \omega\,\text{JacobiEpsilon}(\beta_1 x + \beta_2 y + \beta_3 z - \rho t) + (\beta_1 x + \beta_2 y + \beta_3 z - \rho)\,(\omega - 1)). \tag{3.104}$$

If $\omega = 1$, then Eq. (3.104) reduces to a dark soliton solution:

$$W_{6.5,4.1}(x,y,z,t) = -2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.105}$$

`Case (6.5,4):` If $j_0=-1,\ j_2=2-\omega^2,\ j_4=\omega^2-1,$ and $0\leq\omega\leq 1$, thus, the Jacobi elliptic solution for Eq. (1.1) is as follows:

$$W_{6.5,5}(x,y,z,t)=2\beta_1\,\text{JacobiEpsilon}(\beta_1 x+\beta_2 y+\beta_3 z-\rho t). \tag{3.106}$$

If $\omega=1$, then Eq. (3.106) reduces to a dark soliton solution:

$$W_{6.5,5.1}(x,y,z,t)=2\beta_1\tanh\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right). \tag{3.107}$$

`Case (6.5,6):` If $j_0=1,\ j_2=2-4\omega^2,\ j_4=1,$ and $0\leq\omega\leq 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.5,6}(\xi)=-\frac{2\beta_1\left(\left(-2\omega\,\text{cn}^2(\xi)+2\omega-1\right)\text{cs}(\xi)+\text{dn}(\xi)(\xi-2\,\text{JacobiEpsilon}(\xi))\right)}{\text{dn}(\xi)}, \tag{3.108}$$

where $\xi=\beta_1 x+\beta_2 y+\beta_3 z-\rho t$.
If $\omega=0$ or $\omega=1$, then Eq. (3.108) reduces to either a singular periodic solution or singular soliton solution:

$$W_{6.5,6.1}(x,y,z,t)=2\beta_1\left((\beta_1 x+\beta_2 y+\beta_3 z-\rho t)+\cot\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\right), \tag{3.109}$$

or

$$W_{6.5,6.2}(x,y,z,t)=-2\beta_1\left((\beta_1 x+\beta_2 y+\beta_3 z-\rho t)-\coth\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\right). \tag{3.110}$$

`Case (6.5,7):` If $j_0=\frac{1}{4},\ j_2=\frac{(\omega^2-2)}{2},\ j_4=\frac{\omega^4}{4},$ and $0\leq\omega\leq 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.5,7}(\xi)=\frac{\beta_1}{2}\left(2\,\text{cs}(\xi)\left(1+\text{dn}(\xi)\right)+2\,\text{JacobiEpsilon}(\xi)+\xi\left(\omega-2\right)\right), \tag{3.111}$$

where $\xi=\beta_1 x+\beta_2 y+\beta_3 z-\rho t$.
If $\omega=0$ or $\omega=1$, then Eq. (3.111) reduces to either a singular periodic solution or a combo dark-singular soliton solution:

$$W_{6.5,7.1}(x,y,z,t)=2\beta_1\,\cot\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right), \tag{3.112}$$

or

$$W_{6.5,7.2}(\xi)=\beta_1\left(1-\frac{\xi}{2}+\tanh\left(\xi\right)+\text{csch}\left(\xi\right)\right). \tag{3.113}$$

where $\xi=\beta_1 x+\beta_2 y+\beta_3 z-\rho t$.
The collection of response set (6.6) demonstrates that Eq. (1.1) has exact solutions, which are given by:

`Case (6.6,1):` If $j_0=1,\ j_2=-1-\omega^2,\ j_4=\omega^2,$ and $0\leq\omega\leq 1$, thus, the Jacobi elliptic solution to Eq. (1.1) are as follows:

$$W_{6.6,1}(\xi)=2\beta_1\left(\omega+1\right)\left(-\frac{\text{cn}(\xi)\ \text{dn}(\xi)}{(\omega+1)\text{sn}(\xi)}-\text{JacobiEpsilon}(\xi)+\xi\right), \tag{3.114}$$

or

$$W_{6.6,2}(\xi)=-\frac{2\beta_1\left(\omega^2\text{cn}(\xi)\,\text{sn}(\xi)+(\omega+1)\,\text{dn}(\xi)\left(\xi-\text{JacobiEpsilon}(\xi)\right)+\text{dn}^2(\xi)\,\text{sc}(\xi)\right)}{\text{dn}(\xi)}, \tag{3.115}$$

where $\xi=\beta_1 x+\beta_2 y+\beta_3 z-\rho t$.
If $\omega=0$, or $\omega=1$, then Eq. (3.114) reduces to either a singular periodic solution or a singular soliton solution. Additionally, if $\omega=0$, then Eq. (3.115) reduces to a singular periodic solution:

$$W_{6.6,1.1}(x,y,z,t)=2\beta_1\,\cot\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right), \tag{3.116}$$

or

$$W_{6.6,1.2}(x,y,z,t)=-4\beta_1\left((\beta_1 x+\beta_2 y+\beta_3 z-\rho t)-\coth\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right)\right). \tag{3.117}$$

$$W_{6.6,2.1}(x,y,z,t)=-2\beta_1\tan\left(\beta_1 x+\beta_2 y+\beta_3 z-\rho t\right). \tag{3.118}$$

Case (6.6,2): If $j_0 = \omega^2 - 1,\ j_2 = 2 - \omega^2,\ j_4 = -1$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution for Eq. (1.1) is as follows:

$$W_{6.6,3}(\xi) = 2\beta_1 \left((\omega + 2)\,\mathrm{JacobiEpsilon}(\xi) - \frac{\omega\,(\omega + 1)\,\mathrm{cn}(\xi)\ \mathrm{sn}(\xi)}{\mathrm{dn}(\xi)} \right), \tag{3.119}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 1$, then Eq. (3.119) reduces to a dark soliton solution:

$$W_{6.6,3.1}(x, y, z, t) = 2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.120}$$

Case (6.6,3): If $j_0 = -\omega^2,\ j_2 = 2\omega^2 - 1,\ j_4 = 1 - \omega^2$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution for Eq. (1.1) is as follows:

$$W_{6.6,4}(\xi) = 2\beta_1 \left((2\omega + 1)\ \mathrm{JacobiEpsilon}(\xi) + \xi\,(\omega - 1)) - \frac{(\omega + 1)\,\mathrm{dn}(\xi)\ \mathrm{sn}(\xi)}{\mathrm{cn}(\xi)} \right), \tag{3.121}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$, or $\omega = 1$, then Eq. (3.121) reduces to either a singular periodic solution or a dark soliton solution:

$$W_{6.6,4.1}(x, y, z, t) = -2\beta_1 \tan(\beta_1 x + \beta_2 y + \beta_3 z - \rho t), \tag{3.122}$$

or

$$W_{6.6,4.2}(x, y, z, t) = 2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.123}$$

Case (6.6,4): If $j_0 = -1,\ j_2 = 2 - \omega^2,\ j_4 = \omega^2 - 1$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution for Eq. (1.1) is as follows:

$$W_{6.6,5}(\xi) = 2\beta_1 \left((\omega + 2)\,\mathrm{JacobiEpsilon}(\xi) - \frac{\omega\,(\omega + 1)\,\mathrm{cn}(\xi)\ \mathrm{sn}(\xi)}{\mathrm{dn}(\xi)} \right), \tag{3.124}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 1$, then Eq. (3.124) reduces to a dark soliton solution:

$$W_{6.5,5.1}(x, y, z, t) = 2\beta_1 \tanh(\beta_1 x + \beta_2 y + \beta_3 z - \rho t). \tag{3.125}$$

Case (6.6,5): If $j_0 = 1,\ j_2 = 2 - 4\omega^2,\ j_4 = 1$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.6,6}(\xi) = 2\beta_1 \left(-2\xi - \mathrm{dn}(\xi)\,\mathrm{sc}(\xi) + 4\mathrm{JacobiEpsilon}(\xi) + \frac{\left(2\omega\,\mathrm{cn}^2(\xi) - 2\omega + 1\right)\mathrm{cs}(\xi)}{\mathrm{dn}(\xi)} \right), \tag{3.126}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.126) reduces to either a singular periodic solution or a singular soliton solution:

$$W_{6.6,6.1}(x, y, z, t) = 4\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) + \cot\left(2\ (\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\right) \right), \tag{3.127}$$

or

$$W_{6.6,6.2}(x, y, z, t) = -4\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \coth\left(2\ (\beta_1 x + \beta_2 y + \beta_3 z - \rho t)\right) \right). \tag{3.128}$$

Case (6.6,6): If $j_0 = \frac{1}{4},\ j_2 = \frac{(\omega^2 - 2)}{2},\ j_4 = \frac{\omega^4}{4}$, and $0 \le \omega \le 1$, thus, the Jacobi elliptic solution to Eq. (1.1) is as follows:

$$W_{6.6,7}(\xi) = \frac{\beta_1}{2} \left(2\mathrm{cs}(\xi) \left(1 - \omega^2 + \left(\omega^2 + 1\right) \mathrm{dn}(\xi) \right) + \left(\omega^2 + 1\right) \left(2\mathrm{JacobiEpsilon}(\xi) + \xi\,(\omega - 2) \right) \right), \tag{3.129}$$

where $\xi = \beta_1 x + \beta_2 y + \beta_3 z - \rho t$.
If $\omega = 0$ or $\omega = 1$, then Eq. (3.129) reduces to either a singular periodic solution or singular soliton solution:

$$W_{6.7,6.1}(x, y, z, t) = 2\,\beta_1\ \cot(\beta_1 x + \beta_2 y + \beta_3 z - \rho t), \tag{3.130}$$

or

$$W_{6.7,6.2}(x, y, z, t) = -\beta_1 \left((\beta_1 x + \beta_2 y + \beta_3 z - \rho t) - \coth \left(2 \left(\beta_1 x + \beta_2 y + \beta_3 z - \rho t \right) \right) \right). \tag{3.131}$$

4. Illustrations of the solutions

To illustrate the physical characteristics of some extracted solutions, this section displays the 2D, 3D, and contour diagrams of a few chosen solutions. Figure 1 displays a dark soliton of Eq. (3.5) with $j_2 = 1,\ \beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 3.5,\ \rho = 0.5,\ y = 0,\ z = 0$. Figure 2 displays a singular periodic solution of Eq. (3.25) including $j_2 = 0.5,\ \beta_1 = 1.6,\ \beta_2 = 3.1,\ \beta_3 = 3.6,\ \rho = 0.05,\ y = 0,\ z = 0$. Figure 3 depicts a singular soliton solution of Eq. (3.35) including $j_2 = 2,\ \beta_1 = 1.8,\ \beta_2 = 3.6,\ \beta_3 = 3.3,\ \rho = 0.055,\ y = 0$, and $z = 0$. Figure 4 depicts a combo dark-singular solution of Eq. (3.66) is illustrated using 2D and 3D plots, showing the interplay between the dark and singular characteristics. with $\beta_1 = -0.98,\ \beta_2 = 3.6,\ \beta_3 = 2.3,\ \rho = 0.06,\ y = 0,\ z = 0$.

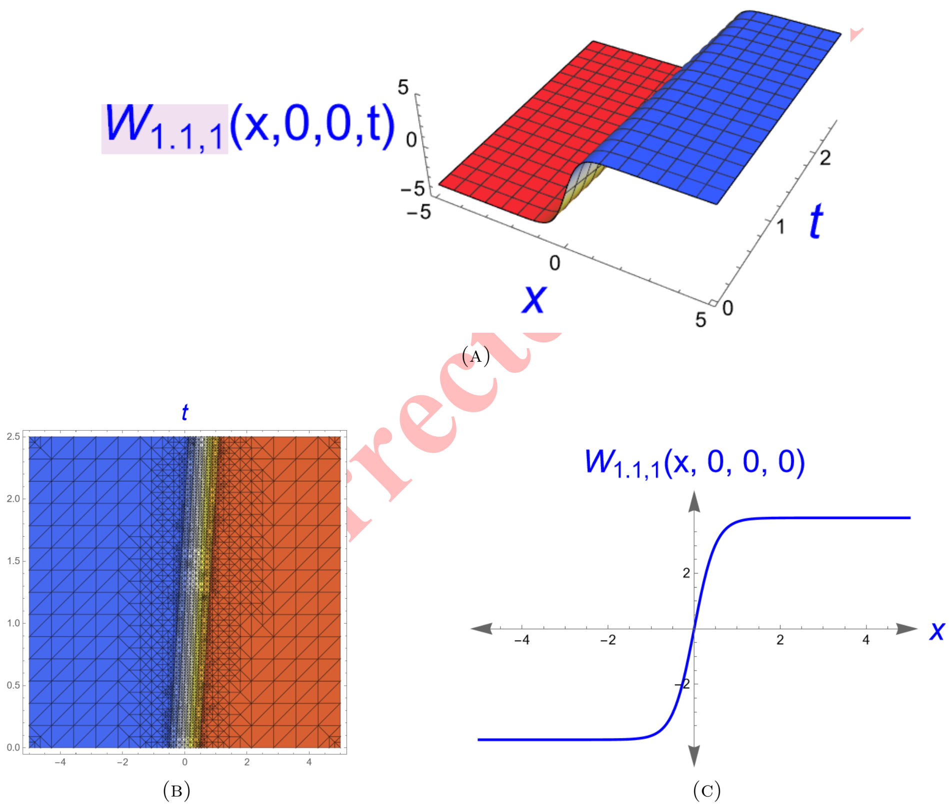
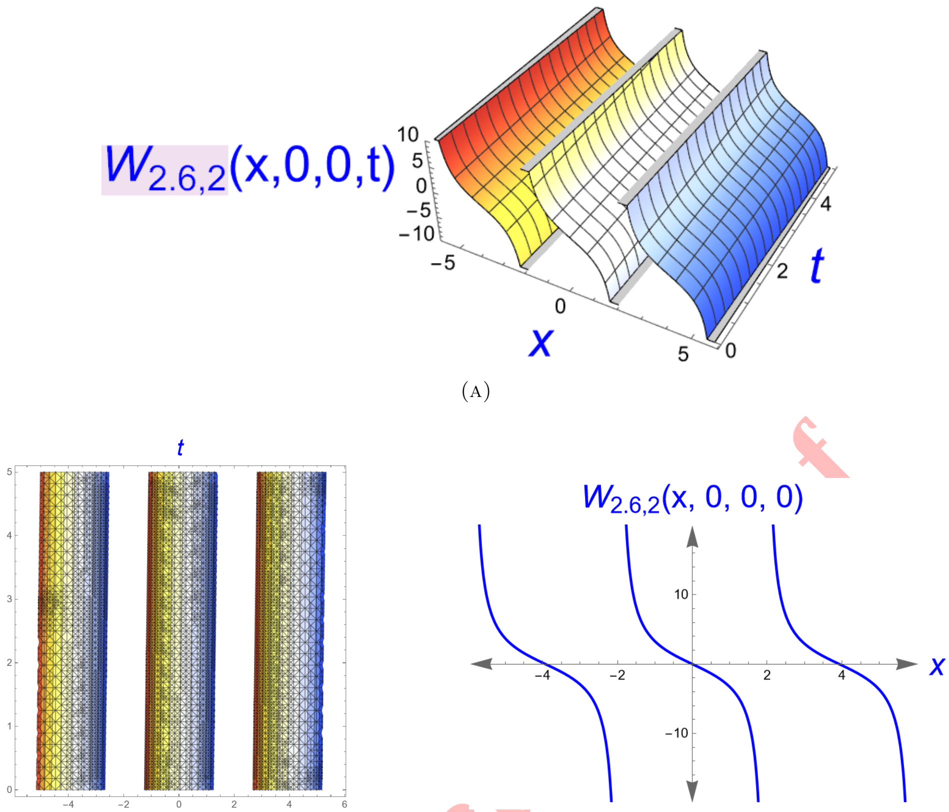


Figure 1. The dark soliton solution for Eq. (3.5).



(A) (B) (C)

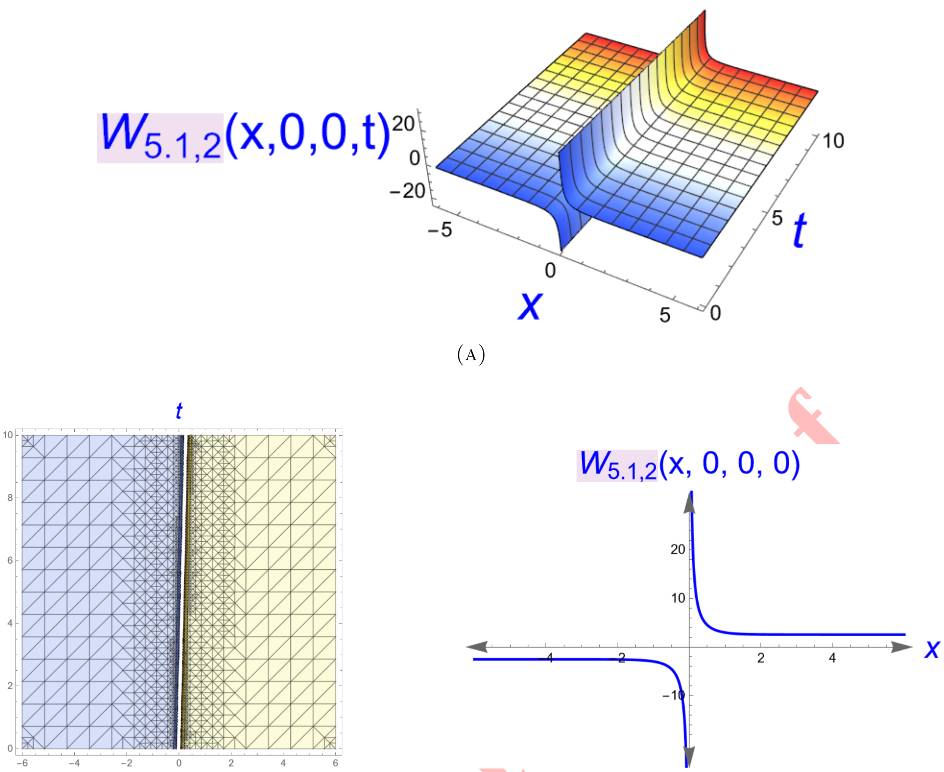
FIGURE 2. The singular periodic wave solution for Eq. (3.25).

5. Conclusion

In this study, we investigated the (3+1)-dimensional integrable pKP–BKP equation using the Modified Extended Direct Algebraic (MEDA) technique. The study successfully derived a diverse range of exact solutions, including dark solitons, singular solitons, hyperbolic solutions, singular periodic solutions, exponential solutions, rational solutions, and Jacobi elliptic solutions. These solutions were further visualized through 2D and 3D graphical representations to enhance the understanding of their dynamic behaviors.

Our approach provides a richer set of exact solutions compared to previous studies that primarily relied on Hirota's bilinear method and lump soliton solutions. These solutions, including previously unexplored Jacobi elliptic functions and combo dark-singular solitons, expand the understanding of multi-dimensional nonlinear wave propagation. The results of this study not only demonstrate the effectiveness of the MEDA method but also offer new perspectives for analyzing soliton interactions in applied physical sciences.

Moreover, the MEDA technique offers methodological advancements by enabling a systematic balance between nonlinear and highest-order derivative components, leading to a broader classification of wave structures. The results contribute to the theoretical analysis of nonlinear wave dynamics and hold potential applications in various physical systems, including oceanography, plasma physics, and fluid mechanics. Future research could explore the stability properties of these solutions and their interactions in real-world nonlinear systems.



(A) (B) (C)

Figure 3. The singular soliton solution for Eq. (3.35).

6. Future research directions

This study significantly contributes to the understanding of soliton solutions and nonlinear wave structures, but there are still many unexplored aspects that warrant further research. Based on our findings, several specific directions for future research can be explored to further advance the understanding of nonlinear wave dynamics and soliton interactions: Multi-soliton interactions, investigating the interactions between multiple combo dark-singular solitons can provide deeper insights into wave collision dynamics, energy exchange, and stability in nonlinear media. This could be extended to study higher-order soliton interactions in integrable and non-integrable systems. Modulation instability and perturbation analysis, examining how small perturbations affect the stability of the obtained soliton solutions is crucial for practical applications in fiber optics, fluid mechanics, and plasma waves. A detailed stability analysis can help in understanding the long-term behavior of solitons under realistic conditions. Higher-dimensional generalizations, extending the current (3+1)-dimensional pKP–BKP equation to (4+1) or higher-dimensional models may reveal additional soliton structures and more intricate wave behaviors. This can provide a broader perspective on wave propagation in multidimensional nonlinear systems. Experimental validation, comparing our theoretical predictions with experimental observations in nonlinear optics, plasma physics, and hydrodynamics is a crucial next step. This can confirm the physical relevance of our solutions and guide the development of real-world applications. Nonlocal and fractional extensions, investigating nonlocal solitons or fractional-order extensions of the pKP–BKP equation can provide new mathematical models that better describe nonlinear materials, quantum systems, and biological waves. Ultimately, this research enhances our understanding of nonlinear wave structures and serves as a

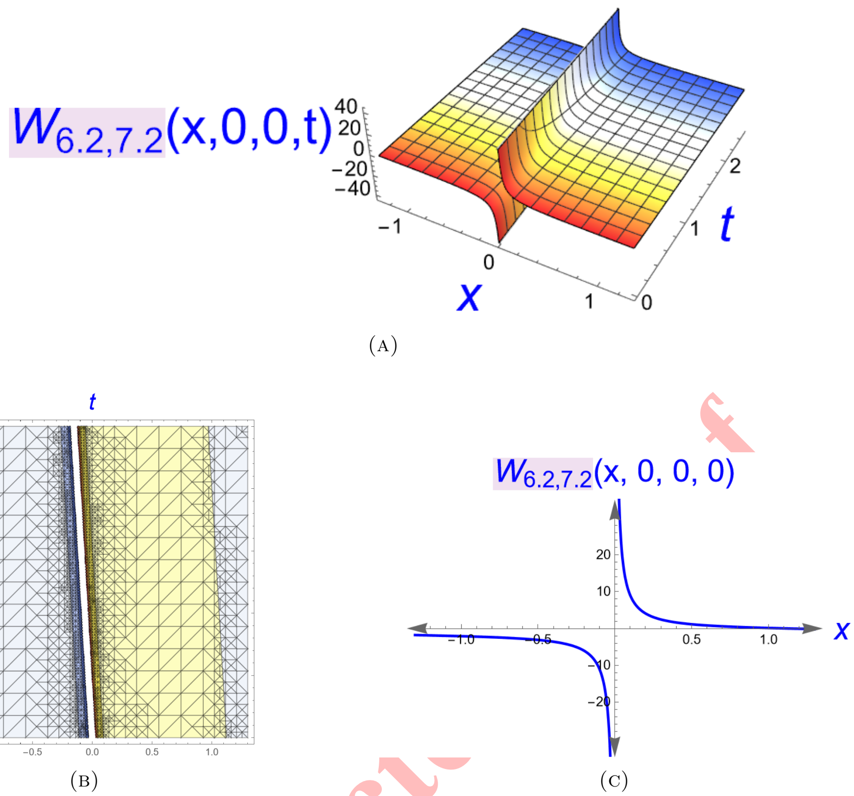


Figure 4. The combo dark-singular solution for Eq. (3.66).

foundation for future studies on soliton interactions, stability analysis, and practical applications in wave physics and engineering.

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