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Solving a system of fractional Volterra integro-differential equations using cubic Hermit spline functions

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Abstract

In this article, we solve systems of fractional Volterra integro-differential equations in the sense of the Caputo fractional derivative, using cubic Hermite spline functions. We first construct the operational matrix for the fractional derivative of the cubic Hermite spline functions. Then, using this matrix and key properties of these functions, we transform systems of fractional Volterra integro-differential equations into a system of algebraic equations, which can be solved numerically to obtain approximate solutions. Numerous examples show that the results obtained by this method align closely with the results presented by some previous works.

Keywords. Fractional Volterra integro-differential equations, Cubic Hermite spline functions, Caputo derivative, Operational matrix.
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1. Introduction

Fractional differential equations (FDEs) and fractional Volterra integro-differential equations (FVIDEs) play important role in various scientific fields, including physics, chemistry, biology, mechanics, fluid flow, image processing, engineering, and more. Numerous authors have investigated analytical results concerning the existence and uniqueness of solutions to FVIDEs (see [3, 15, 16] for examples). Consequently, solving these equations has attracted the attention of many researchers. Since most FVIDEs systems lack exact analytical solutions, approximate and numerical methods have become the preferred approach. Some of these numerical approximations are mentioned as follows.

Abdelkawy et al. [2] successfully solved nonlinear variable-order fractional Fredholm integro-differential equations using a collocation method based on fractional-order Legendre functions. In [6], an operational matrix method based on triangular functions was utilized to solve a system of fractional Integro-differential equations. The authors in [22] solved a system of linear fractional integro-differential equations using least squares method and shifted Chebyshev polynomials, while Shahmorad et al. [34] employed a geometric approach for solving nonlinear fractional integro-differential equations. Haydari et al. [16] utilized a wavelet collocation method for solving systems of nonlinear singular fractional Volterra integro-differential equations based on Chebyshev polynomials, and in [36], two ideas based on discretization of the fractional differential operator and integral form of the FDEs were used to develop higher order numerical techniques for solving FDEs. In addition, optimal homotopya symptotic method was employed to solve a system of fractional order Volterra Integro-differentiale quations in [5]. Furthermore, Khader et al. [20] solved system of linear and nonlinear fractional integro-differential equations of Volterra type using the Chebyshev pseudo-spectral method. For further research works on this problem, we recommend interested readers to refer to

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[1, 4, 7, 9, 10, 18, 26, 30–32, 35–37]. A comprehensive history of fractional differential operators and their applications can be found in [11, 14, 19, 21, 24, 27–29], with [19] and [29] being two excellent references on FDEs.

In this paper, we would like to solve a fractional initial value problem (FIVP) in the form

$$\begin{cases} {}_*D_0^{\alpha_i} y_i(x) = f_i\big(x, y_1(x), ..., y_n(x)\big) + \int_0^x k_i\big(x, t, y_1(t), ..., y_n(t)\big)dt, \ 0 < \alpha_i < 1, \ 0 < x \le 1, \\ y_i(0) = b_i, \ i = 0, 1, ..., n. \end{cases} \tag{1.1}$$

where $*D_0^{\alpha_i}$ denotes the Caputo fractional derivative of order α_i, b_i are real constants, and f_i, k_i are continuous functions.

A common strategy, to solve fractional equations, the solution is first written as a linear combination of basis functions. Then, by using a truncated series of basis functions and applying operational matrices, the problem is reduced to a system of linear or nonlinear algebraic equations that can be easily solved.

In this article, we first construct the operational matrix for fractional derivative of cubic Hermit spline functions (CHSFs). Next, we express the unknown functions as a linear combination of CHSFs with unknown coefficients. Finally, using the operational matrix for derivative and the collocation method, we reduce the problem (1.1) to a system of algebraic equations, which can be solved to find the unknown coefficients.

This paper is organized into five sections. In section 2 we give some preliminaries and definitions needed for our work. In section 3, the numerical method is presented. A convergence analysis of the method is discussed in section 4, and we show that if the unknown function lies in $H^4(\Omega)$, where Ω is the domain of the problem, then the order of convergence will be $O(2^{-4J})$. Some numerical examples are presented to show the efficiency and validity of the method in section 5. We finish the paper with a conclusion and suggest a future work.

2. Preliminaries and notations

2.1. **The fractional derivative in the Caputo sense and some other definitions.** In this section, we recall the basic concepts and definitions of fractional calculus that are most frequently used in the following sections.

Fractional derivatives and integrals of order $\alpha > 0$ have various definitions. The two most important ones are the Riemann-Liouville and Caputo definitions [19, 29], which we describe below.

Definition 2.1. [19] Let α be a positive real number. The Riemann-Liouville fractional integral operator of order α, denoted by J_a^α, is defined on the interval $[a, b]$ and acts on functions in $L^1[a, b]$ as

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{n-1} f(t)dt, \ a \le x \le b.$$

Specifically, when $\alpha = 0$, we define $J_a^0 f(x) = f(x)$.

Definition 2.2. [19] Let $\alpha \ge 0$ and $m = \lceil \alpha \rceil$. Then the Caputo fractional derivative (CFD) of order α is defined as

$${}_*D_a^\alpha f(x) = J_a^{m-\alpha} D^m f(x),$$

when $D^m f \in L^1[a, b]$. Using Definition 2.1, it is then written in the following form

$${}_*D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt.$$

In special cases, for the Caputo derivative when $\beta \geqslant 0$, we have

$${}_*D_a^\alpha C = 0, \quad (C \text{ is constant}),$$

$${}_*D_a^\alpha \ x^\beta = \begin{cases} 0, & for \beta = \{0, 1, 2, ..., m-1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \ x^{\beta-\alpha}, & for \ \beta \in \mathbb{N} \ and \ \beta \geqslant m \quad or \quad \beta \notin \mathbb{N} \ and \ \beta > m-1. \end{cases}$$

It is worth noting that the ceiling function $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α and the floor function $\lfloor \alpha \rfloor$ is the largest integer less than or equal to α.

Recall that the operator ${}_*D_a^\alpha$ is linear, i.e.,

$${}_*D_a^\alpha \Big(Af(x) + Bg(x) \Big) = A \ {}_*D_a^\alpha f(x) + B \ {}_*D_a^\alpha g(x),$$

where A and B are arbitrary constants.

Definition 2.3. For an interval $x \in [a,b]$, let $p \geqslant 1$. The set of functions

$$\left\{ f | f : [a,b] \longrightarrow \mathbb{C} : \int_a^b |f(x)|^p < \infty \right\},$$

forms a normed vector space over the field $\mathbb{C}$, denoted by $L^p[a,b]$.

Definition 2.4. For each $f \in L^p[a,b]$, the norms of the function f are defined as follows:

$$||f||_p = \big(\int_a^b |f(x)|^p dx \big)^{\frac{1}{p}},$$

$$||f||_\infty = \big(ess\ sup_{x\in[a,b]} |f(x)| \big).$$

Definition 2.5. Let Ω be an open subset of $\mathbb{R}$. Then the space $H^s(\Omega)$ defined by

$$H^s(\Omega) = \{ f \in L^2(\Omega);\ f^{(\alpha)} \in L^2(\Omega), \forall \alpha,\ 0 \le \alpha \le s \}$$

is called the Sobolev space of order s [23]. The norm for this space is defined by

$$\|f\|_{s,\Omega} = \left(\sum_{\alpha=0}^{s} \|f^{(\alpha)}\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

For the non-integer value $s \in (0,1)$, the fractional Sobolev space is defined as [8]

$$W^{s,2}(a,b) = \left\{ f \in L^2(a,b) : \frac{f(x) - f(y)}{|x-y|^{\frac{1}{2}+s}} \in L^2([a,b] \times [a,b]) \right\},$$

with the corresponding norm

$$||f||_{W^{s,2}(a,b)} = \left[\int_a^b |f(x)|^2 dx + \int_a^b \int_a^b \frac{|f(x) - f(y)|^2}{|x-y|^{1+2s}} dx dy \right]^{\frac{1}{2}}.$$

If $s = n + \mu > 1$ where $n \in \mathbb{N}$ and $\mu \in (0,1)$, then the fractional Sobolev space is defined as

$$W^{s,2}(a,b) = \left\{ f \in H^n(a,b) : D^n f \in W^{\mu,2}(a,b) \right\}.$$

2.2. **Cubic Hermite spline functions on** $[0,1]$**.** A cubic Hermite spline (or cubic Hermite interpolator) is a spline function such that each piece is a third-degree polynomial specified in Hermite form. In other words, cubic Hermite spline functions are defined by [12, 13, 25]

$$\phi_1(x) = \begin{cases} (x+1)^2(-2x+1), & x \in [-1,0], \\ (1-x)^2(2x+1), & x \in [0,1], \\ 0, & o.w., \end{cases} \tag{2.1}$$

$$\phi_2(x) = \begin{cases} (x+1)^2 x, & x \in [-1,0], \\ (1-x)^2 x, & x \in [0,1], \\ 0, & o.w., \end{cases} \tag{2.2}$$

these functions satisfy the following interpolation properties.

$$\phi_1(k) = \phi_2'(k) = \delta_{0,k}, \quad \phi_1'(k) = \phi_2(k) = 0,\ (k \in \mathbb{Z}),$$

where $\delta_{0,k}$ is the Kronecker delta. The integer transformations of ϕ_1 and ϕ_2 form a basis for the space of C^1-continuous piecewise cubic functions on $\mathbb{R}$ that interpolate function values and their first derivatives at $k \in \mathbb{Z}$. The dilation and translation parameters are denoted by 2^j and k, respectively where $j, k \in \mathbb{Z}$. For $l = 1, 2$, we can express any function in this space as a linear combination of $\phi_l(2^j x - k)$.

Suppose

$$B_{j,k}(x) = supp\big[\phi_l^{j,k}(x)\big] = clos\big\{x : \phi_l^{j,k}(x) \neq 0\big\},$$

with simple calculations, it can be shown that

$$B_{j,k}(x) = \Big[\frac{k-1}{2^j}, \frac{k+1}{2^j}\Big], \text{ for } j,k \in \mathbb{Z}.$$

We define the index set as follows

$$S_j = \big\{B_{j,k} \cap (0,1) \neq \emptyset\big\}, \quad j,k \in \mathbb{Z}.$$

It is evident that for $j \in \mathbb{Z}$, $S_j = \big\{0,1,2,...,2^j\big\}$.

To define cubic Hermite functions on $[0,1]$, we set

$$\phi_l^{j,k}(x) = \phi_l^{j,k}(x).\chi_{[0,1]}(x), \qquad j \in \mathbb{Z},\ k \in S_j,\ l = 1,2.$$

2.3. **Function approximation.** Let $\Phi_j(.)$ be a $2(2^j+1)$-dimensional vector:

$$\Phi_j(.) = \Big[\phi_1^{j,0}(.),\, \phi_2^{j,0}(.), ..., \phi_1^{j,2^j}(.),\, \phi_2^{j,2^j}(.)\Big]^T, \qquad j \in \mathbb{Z}. \tag{2.3}$$

Due to the interpolatory nature of the functions ϕ_1 and ϕ_2, $j = J$ for a fixed, it is possible to approximate a function $f \in H^4[0,1]$ using cubic Hermite functions as

$$f(x) \simeq \sum_{\kappa=0}^{2^J} \Big(c_{1,\kappa}\ \phi_1^{J,\kappa}(x) + c_{2,\kappa}\ \phi_2^{J,\kappa}(x)\Big) = C^T\ \Phi_J(x), \tag{2.4}$$

where

$$c_{1,\kappa} = f(\frac{\kappa}{2^J}),\ \ c_{2,\kappa} = 2^{-J}\ f'(\frac{\kappa}{2^J}), \quad \kappa = 0,1,\dots,2^J,$$

and C is a N-dimensional vector defined as

$$C = \Big[c_{1,0},\, c_{2,0}, ..., c_{1,2^J},\, c_{2,2^J}\Big]^T,$$

where $N = 2(2^J+1)$.

2.4. **Fractional Derivative Operational Matrix.** The Caputo fractional derivative (CFD) of order $\alpha_i, (0 \le \alpha_i < 1, i = 1,2,...,n)$ for functions $\phi_r^{J,k}(.), k \in S_j (r = 1,2)$ can be approximated as

$$\begin{aligned} {}_*D_0^{\alpha_i}\phi_r^{J,\ell}(x) &= {}_*D_0^{\alpha_i}\phi_r(2^Jx-\ell) \\ &\simeq \textstyle\sum_{k=0}^{2^J}\Big\{{}_*D_0^{\alpha_i}\phi_r(k-\ell)\phi_1(2^Jx-k) + 2^{-J}D\big({}_*D_0^{\alpha_i}\phi_r(x)\big)\big|_{x=k-\ell}\ \phi_2(2^Jx-k)\Big\}, \end{aligned} \tag{2.5}$$

where D denotes the classical first derivative and $\ell = 0,1,\dots,2^J$. Using the relation (2.5) we can find the CFD of order α_i for the vector function Φ_J in form

$${}_*D_0^{\alpha_i}\Phi_J(x) \simeq \mathcal{D}_{\alpha_i}\ \Phi_J(x), i = 1,2,...,n, \tag{2.6}$$

where $\mathcal{D}_{\alpha_i}$ $(i = 1,2,...,n)$ are $N \times N$ operational matrices of fractional derivative. For the special case $J = 1$, $i = 2$ and $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, the matrices $\mathcal{D}_{\alpha_1}$ and $\mathcal{D}_{\alpha_2}$ are as follows:

$$\mathcal{D}_{\alpha_1}=\begin{pmatrix} 0 & 0 & \frac{-96}{77}\frac{2^{\frac{7}{4}}}{\Gamma(\frac{3}{4})} & \frac{16}{7}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{1}{77}\frac{-608\ 2^{\frac{1}{4}}+640}{\Gamma(\frac{3}{4})} & \frac{1}{77}\frac{-2640\ 2^{\frac{1}{4}}+3168}{\Gamma(\frac{3}{4})} \\ 0 & 0 & \frac{-4}{77}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{10}{21}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{1}{231}\frac{-736\ 2^{\frac{1}{4}}+872}{\Gamma(\frac{3}{4})} & \frac{1}{231}\frac{-3344\ 2^{\frac{1}{4}}+3982}{\Gamma(\frac{3}{4})} \\ 0 & 0 & \frac{96}{77}\frac{2^{\frac{7}{4}}}{\Gamma(\frac{3}{4})} & \frac{-16}{7}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{1}{77}\frac{5122\ 2^{\frac{1}{4}}-640}{\Gamma(\frac{3}{4})} & \frac{1}{77}\frac{2816\ 2^{\frac{1}{4}}+3168}{\Gamma(\frac{3}{4})} \\ 0 & 0 & \frac{32}{231}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{80}{21}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{1}{231}\frac{-1408\ 2^{\frac{1}{4}}+1664}{\Gamma(\frac{3}{4})} & \frac{1}{231}\frac{-4928\ 2^{\frac{1}{4}}+5984}{\Gamma(\frac{3}{4})} \\ 0 & 0 & 0 & 0 & \frac{96}{77}\frac{2^{\frac{7}{4}}}{\Gamma(\frac{3}{4})} & \frac{-16}{7}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} \\ 0 & 0 & 0 & 0 & \frac{32}{231}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{80}{21}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} \end{pmatrix},$$

$$\mathcal{D}_{\alpha_2}=\begin{pmatrix} 0 & 0 & \frac{-8\sqrt{2}}{5\sqrt{\pi}} & \frac{8\sqrt{2}}{\sqrt{\pi}} & \frac{96-72\sqrt{2}}{5\sqrt{\pi}} & \frac{80-56\sqrt{2}}{\sqrt{\pi}} \\ 0 & 0 & \frac{-2\sqrt{2}}{15\sqrt{\pi}} & \frac{2\sqrt{2}}{\sqrt{\pi}} & \frac{124-88\sqrt{2}}{15\sqrt{\pi}} & \frac{34-24\sqrt{2}}{\sqrt{\pi}} \\ 0 & 0 & \frac{8\sqrt{2}}{5\sqrt{\pi}} & \frac{-8\sqrt{2}}{\sqrt{\pi}} & \frac{-32(3-2\sqrt{2})}{15\sqrt{\pi}} & \frac{16(4\sqrt{2}-5)}{\sqrt{\pi}} \\ 0 & 0 & \frac{8\sqrt{2}}{15\sqrt{\pi}} & \frac{8\sqrt{2}}{\sqrt{\pi}} & \frac{32(5\sqrt{2}-7)}{15\sqrt{\pi}} & \frac{-16(30\sqrt{2}-45)}{15\sqrt{\pi}} \\ 0 & 0 & 0 & 0 & \frac{96}{77}\frac{2^{\frac{7}{4}}}{\Gamma(\frac{3}{4})} & \frac{-16}{7}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} \\ 0 & 0 & 0 & 0 & \frac{32}{231}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{80}{21}\frac{2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} \end{pmatrix}.$$

3. Methodology Description

In this section, we solve the FIVP (1.1) using cubic Hermite spline functions. For the unknown functions $y_i (i = 1, 2, ..., n)$ in (1.1), we use (2.4) to obtain the approximation:

$$y_i(x) \simeq y_{i_J}(x) = \sum_{k=0}^{2^J} \Big\{ Y^i_{1,k}\ \phi_1^{J,k}(x) + Y^i_{2,k}\ \phi_2^{J,k}(x) \Big\} = Y^{i^T}\Phi_J(x), \tag{3.1}$$

where $Y^i (i = 1, 2, .., n)$ are unknown vectors defined by:

$$Y^i = \Big[Y^i_{1,0}, Y^i_{2,0}, Y^i_{1,1}, Y^i_{2,1}, \dots, Y^i_{1,2^J}, Y^i_{2,2^J} \Big]^T.$$

Using the operational matrix from (2.6), we approximate ${}_*D_0^{\alpha_i} y_i(x)$ as

$${}_*D_0^{\alpha_i} y_i(x) \simeq {}_*D_0^{\alpha_i} Y^{i^T}\Phi_J(x) \simeq Y^{i^T}\mathcal{D}_{\alpha_i}\Phi_J(x). \tag{3.2}$$

Substituting (3.1) and (3.2) into Eq. (1.1), for $i = 1, 2, ..., n$ and $0 < \alpha_i < 1$ we get

$$Y^{i^T}\mathcal{D}_{\alpha_i}\Phi_J(x) \simeq f_i\big(x, Y^{1^T}\Phi_J(x), ..., Y^{n^T}\Phi_J(x)\big) + \int_0^x k_i\big(x, t, Y^{1^T}\Phi_J(t), \dots, Y^{n^T}\Phi_J(t)\big)dt, \quad 0 < x \le 1, \tag{3.3}$$

also by replacing (3.1) in the initial condition of problem (1.1), we obtain

$$Y^{i^T}\Phi_J(0) = b_i, \quad i = 1, 2, \dots, n. \tag{3.4}$$

By collocating Eq. (3.3) at the $N-1$ equally spaced nodes $x_j \in [0, 1]$, for $j = 1, 2, ..., N-1$, we have

$$Y^{i^T}\mathcal{D}_{\alpha_i}\Phi_J(x_j) \simeq f_i\big(x_j, Y^{1^T}\Phi_J(x_j), \dots, Y^{n^T}\Phi_J(x_j)\big) + \int_0^{x_j} k_i\big(x_j, t, Y^{i^T}\Phi_J(t), \dots, Y^{i^T}\Phi_J(t)\big)dt, \quad j = 1, 2, 3, \dots, N-1. \tag{3.5}$$

The integrals in Eq. (3.5) are evaluated using a suitable Newton-Cotes method, thus, we obtain:

$$\begin{aligned} Y^{i^T}\mathcal{D}_{\alpha_i}\Phi_J(x_j) \simeq\, & f_i\big(x_j, Y^{1^T}\Phi_J(x_j), \dots, Y^{n^T}\Phi_J(x_j)\big) \\ & + \sum_{\ell=0}^{m_j} w_\ell k_i\big(x_j, t, Y^{i^T}\Phi_J(t_\ell), \dots, Y^{i^T}\Phi_J(t_\ell)\big)dt, \quad j = 1, 2, 3, \dots, N-1, \end{aligned} \tag{3.6}$$

where $w_\ell, \ell = 0, 1, \dots, m_j$ are the Newton-Cotes quadrature weights. Combining (3.6) and (3.4) yields a system of $n \times N$ algebraic equations with $n \times N$ unknowns, which is solved for the unknown vectors Y^i. Thus, the solutions y_i to (1.1) are obtained.

We recall that the matrix method solves the algebraic systems.

4. Convergence Analysis

Theorem 4.1. *[12, 13] Let $f \in H^4[0,1]$, and f_J be the projection of f on the space V_J. Then we have*

$$\inf \|f(x) - f_J(x)\|_{L^2([0,1])} \le 2^{-4J}\|f\|_{4,(0,1)} = O(2^{-4J}). \tag{4.1}$$

Definition 4.2. Let $F = [f_1, f_2, ..., f_n]^T \in C\big([0,1] \times \mathbb{R}^n, \mathbb{R}^n\big)$ and $K = [k_1, k_2, ..., k_n]^T \in C\big([0,1]^2 \times \mathbb{R}^n, \mathbb{R}^n\big)$. Assume these functions satisfy the following Lipschitz conditions for some real constants $L_p, L_q > 0$

$$\|F(x, z_1) - F(x, z_2)\|_2 \le L_p\|z_1 - z_2\|_2, \quad \|K(x, t, z_1) - K(x, t, z_2)\|_2 \le L_q\|z_1 - z_2\|_2, z_1, z_2 \in \mathbb{R}^n. \tag{4.2}$$

Let $Y = [y_1, y_2, ..., y_n]^T \in C^1(\mathbb{R}^n)$ be the solution of the nonlinear fractional Volterra integro-differential system (1.1). We rewrite this system in the following form:

$$\begin{cases} {}_*D_0^\alpha Y(x) = \mathsf{F}\big(x, Y(x)\big) + \int_0^x K(x, t, Y(t))dt, \quad 0 < \alpha < 1,\ 0 < x \le 1, \\ Y(0) = Y_0, \end{cases} \tag{4.3}$$

where $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]$ and $Y_0 = [b_1, b_2, ..., b_n]$.

Let Y and Y_J denote the exact and approximate solutions of the FIVP (4.3), respectively. Then we have

$$\begin{cases} {}_*D_0^\alpha Y_J(x) = \mathsf{F}(x, Y_J(x)) + \int_0^x K(x, t, Y_J(t))dt + r_J(x), \quad 0 < x \le 1, \\ Y_J(0) = Y_0, \end{cases} \tag{4.4}$$

where $r_J(x)$ is the residual term of the approximation. In the following theorem we find an upper bound for $r_J(x)$.

Theorem 4.3. *Let $Y(x) \in H^4[0,1]$ and $Y_J(x)$ denote the exact and approximate solutions of the FIVP (4.3), respectively. Assume ${}_*D_0^\alpha Y(x) \in H^4[0,1]$, with F and K Lipschitz continuous in their second and third arguments, respectively." Then we have*

$$\|r_J(x)\|_2 = \|{}_*D_0^\alpha Y_J(x) - \mathsf{F}\big(x, Y_J(x)\big) - \int_0^x K\big(x, t, Y_J(t)\big)dt\|_2 = O(2^{-4J}). \tag{4.5}$$

Proof. From the definition of finction $r_J(x)$, we can conclude that:

$$\begin{aligned} \|r_J(x)\|_2 &= \left\| {}_*D_0^\alpha Y_J(x) - \mathsf{F}\big(x, Y_J(x)\big) - \int_0^x K\big(x, t, Y_J(t)\big)dt \right\|_2 \\ &= \| \left\{ {}_*D_0^\alpha Y_J(x) - {}_*D_0^\alpha Y(x) + \mathsf{F}\big(x, Y(x)\big) + \int_0^x K\big(x, t, Y(t)\big)dt \right\} \\ &- \left\{ \mathsf{F}\big(x, Y_J(x)\big) + \int_0^x K\big(x, t, Y_J(t)\big)dt \right\} \|_2 \\ &\le \big\{ ||{}_*D_0^\alpha Y(x) - {}_*D_0^\alpha Y_J(x)||_2 + ||\mathsf{F}\big(x, Y(x)\big) - \mathsf{F}\big(x, Y_J(x)\big)||_2 \big\} \\ &+ \left\{ \int_0^x ||K\big(x, t, Y(t)\big) - \int_0^x K\big(x, t, Y_J(t)\big)||_2 dt \right\}. \end{aligned}$$

Applying (4.1) for the first relation and (4.2) for the last two relations, we have

$$\begin{aligned} &\|{}_*D_0^\alpha Y(x) - {}_*D_0^\alpha Y_J(x)\|_2 = O(2^{-4J}), \\ &\big\|\mathsf{F}\big(x, Y(x)\big) - \mathsf{F}\big(x, Y_J(x)\big)\big\|_2 \leq L_p \|Y(x) - Y_J(x)\|_2\,, \\ &\Big\|K\Big(x,t,Y(x)\Big) - K\Big(x,t,Y_J(x)\Big)\Big\|_2 \leq L_q \|Y(x) - Y_J(x)\|_2\,. \end{aligned} \tag{4.6}$$

Using (4.1) for the last two relations, we get

$$\big|\big|\mathsf{F}\big(x, Y(x)\big) - \mathsf{F}\big(x, Y_J(x)\big)\big|\big|_2 = O(2^{-4J}), \tag{4.7}$$

$$\big|\big|K\big(x,t,Y(x)\big) - K\big(x,t,Y_J(x)\big)\big|\big|_2 = O(2^{-4J}). \tag{4.8}$$

Combining (4.6), (4.7), and (4.8) completes the proof. □

Theorem 4.4. *Let $Y(x) \in H^4[0,1]$ and $Y_J(x)$ are the exact and approximate solution of FIVP (4.3), respectively, and the functions F, K are Lipschitz continuous functions with respect to second and third variable. Then, we have*

$$e_J(x) = \|Y(x) - Y_J(x)\|_2 = O(2^{-4J}). \tag{4.9}$$

Proof. Applying the Riemann-Liouville fractional integral on both sides of FDEs in (4.3), (4.4), and the related initial values, we get

$$Y(x) = Y(0) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}\mathsf{F}\big(z, Y(z)\big)dz + \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}\int_0^z K\big(z,t,Y(t)\big)dtdz, \tag{4.10}$$

and

$$\begin{aligned} Y_J(x) &= Y(0) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}\mathsf{F}\big(z, Y_J(z)\big)dz \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}\int_0^z K\big(z,t,Y_J(t)\big)dtdz - \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}e_J(z)dz. \end{aligned} \tag{4.11}$$

Subtrcting Eq. (4.11) from (4.10) we get

$$\begin{aligned} ||Y(x) - Y_J(x)||_2 = &\bigg\| \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}\Big(\mathsf{F}(z,Y(z)) - \mathsf{F}(z,Y_J(z))\Big)dz \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x\int_0^z (x-z)^{\alpha-1}\Big(K(z,t,Y(t)) - K(z,t,Y_J(t))\Big)dtdz \\ &- \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1} r_J(z)dz \bigg\|_2 \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^x |x-z|^{\alpha-1}dz\Big(||\mathsf{F}(x,Y(x)) - \mathsf{F}(x,Y_J(x))||_2 \\ &+ \int_0^z ||K(x,t,Y(t)) - K(x,t,Y_J(t))||_2 dt + ||r_J(x)||_2\Big) \\ &\leq \frac{1}{\Gamma(\alpha+1)}\Big(L_p||\mathsf{F}(x,Y(x)) - \mathsf{F}(x,Y_J(x))||_2 + L_q||K(x,t,Y(t)) - K(x,t,Y_J(t))||_2 + ||r_J(x)||_2\Big), \end{aligned}$$

and so

$$\begin{aligned} ||Y(x) - Y_J(x)||_2 \leq \frac{1}{\Gamma(\alpha+1)}\Big(&L_p||\mathsf{F}(x,Y(x)) - \mathsf{F}(x,Y_J(x))||_2 \\ &+ L_q||K(x,t,Y(t)) - K(x,t,Y_J(t))||_2 + ||r_J(x)||_2\Big). \end{aligned} \tag{4.12}$$

Applying the relations (4.7) and (4.8) in (4.12) implies

$$e_J(x) = ||Y(x) - Y_J(x)||_2 = O(2^{-4J}),$$

which completes the proof. □

TABLE 1. The Computational results for Example 5.1.

x_i	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{1ex} - y_{1app}\rvert$[5]	$\lvert y_{2ex} - y_{2app}\rvert$[5]
0	0	0	0	0
0.1	0	0	5.23×10^{-7}	1.40×10^{-7}
0.2	0	0	2.39×10^{-6}	6.23×10^{-7}
0.3	0	0	4.69×10^{-6}	9.53×10^{-7}
0.4	0	0	6.01×10^{-6}	7.15×10^{-8}
0.5	0	0	5.76×10^{-6}	4.25×10^{-6}
0.6	0	0	5.16×10^{-6}	1.38×10^{-5}
0.7	0	0	7.27×10^{-6}	3.13×10^{-5}
0.8	0	0	1.52×10^{-5}	5.01×10^{-5}
0.9	0	0	3.01×10^{-5}	9.75×10^{-5}
1	0	0	4.09×10^{-5}	1.44×10^{-4}

5. Numerical examples

In this section, we present the computational results from numerical experiments using the method described in the previous sections to support our theoretical analysis.

To determine the numerical convergence order using the relation (4.9) we assume $e_J(x) = O(2^{-pJ})$, and solve for p accordingly

$$p = \log_2 \frac{\|e_{J-1}(x)\|_{L^\infty([0,1])}}{\|e_J(x)\|_{L^\infty([0,1])}}. \tag{5.1}$$

We define the L_2 and L_∞ error norms as

$$L_2 error(y) = ||y - y_J||_2,$$

and

$$L_\infty error(y) = ||y - y_J||_\infty,$$

where y and y_J denote the exact and approximate solutions, respectively.

Example 5.1. [5] For our first example, consider the following system of fractional Volterra integro-differential equations

$$\begin{cases} {}_*D_0^{1/2} y_1(x) = \frac{2(\sqrt{x}+\frac{4x^{3/2}}{3})}{\sqrt{\pi}} - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + \int_0^x y_1(t)dt + \int_0^x y_2(t)dt, \\ {}_*D_0^{1/2} y_2(x) = \frac{2x^{3/2}(5+6x)}{15\sqrt{\pi}} - \frac{x^2}{2} + \frac{x^4}{4} + \int_0^x y_1(t)dt - \int_0^x y_2(t)dt, \\ y_1(0) = y_2(0) = 0, \quad 0 \le x \le 1. \end{cases}$$

The exact solutions of this problem are $y_1(x) = x + x^2$ and $y_2(x) = x^3 + x^2$.

In Table 1, we report the absolute errors y_1 and y_2 for $J = 1$ in this study and those obtained using the method proposed in [5]. The graph of absolute errors y_1 and y_2 for $J = 1$ are given in Figure 1.

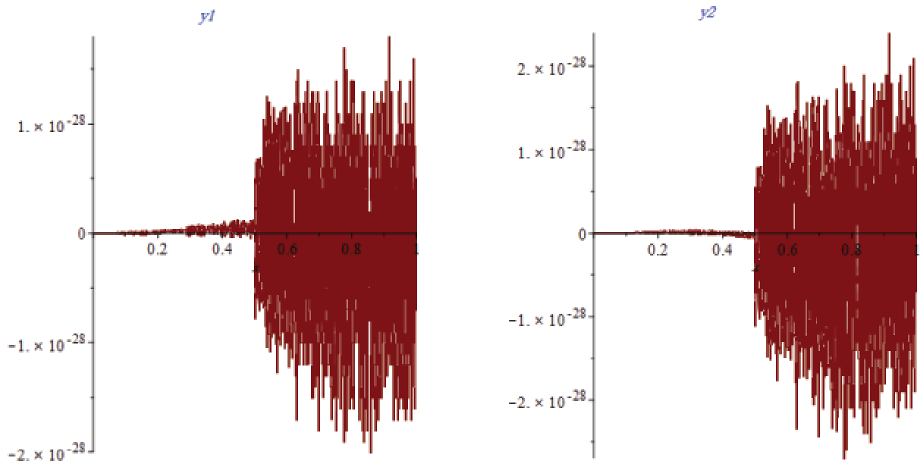

FIGURE 1. Plots of absolute errors $|y(x) - y_{1_J}(x)|$ and $|y(x) - y_{2_J}(x)|$ at $J = 1$, for Example 5.1.

Example 5.2. [5] In this example, for $\alpha \in (0,1)$ we consider the following system of fractional integro-differential equations

$$\begin{cases} {}_*D_0^{\alpha} y_1(x) = \frac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+2\alpha)} + \int_0^x (x-t)y_1(t)dt + \int_0^x (x-t)y_2(t)dt, \\ \\ {}_*D_0^{\alpha} y_2(x) = -\frac{2x^{2+3\alpha}}{2+9\alpha+9\alpha^2} - \frac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+2\alpha)} + \int_0^x (x-t)y_1(t)dt - \int_0^x (x-t)y_2(t)dt, \\ \\ y_1(0) = y_2(0) = 0, \quad 0 \le x \le 1, \quad 0 < \alpha < 1. \end{cases}$$

The exact solutions of this problem are $y_1(x) = x^{3\alpha}$ and $y_2(x) = -x^{3\alpha}$.

The L_∞ and L_2 errors are obtained in Table 2 for different values of J using the presented method. Also, Table 2 shows the numerical convergence order for different values of α and J, which confirm our theoretical results. In Table 3, we report the absolute errors y_1 and y_2 for $\alpha = 1$ at $J = 7$ in this study and those obtained using the method proposed in [5]. The graph of approximate solutions y_1 and y_2 for $J = 5$ and $\alpha = 0.25, 0.5, 0.75, 0.85$, and 0.95 are given in Figure 2.

Example 5.3. As another example, we consider the following system of fractional Volterra integro-differential equations for $\alpha, \beta \in (0,1)$

$$\begin{cases} {}_*D_0^{\alpha} y_1(x) = g_1(x) - \frac{x^{5\alpha+1}}{5\alpha+1} - \frac{x^{7\beta+1}}{7\beta+1} - \frac{x^3}{3} - \frac{x^2}{2} + \int_0^x \big(y_1(t) + y_2(t)\big)dt, \\ \\ {}_*D_0^{\beta} y_2(x) = g_2(x) - \frac{x^{5\alpha+1}}{5\alpha+1} + \frac{x^{7\beta+1}}{7\beta+1} + \frac{x^3}{3} - \frac{x^2}{2} - 2x + \int_0^x \big(y_1(t) - y_2(t)\big)dt, \\ \\ y_1(0) = 1, y_2(0) = -1, \quad 0 \le x \le 1, \end{cases}$$

in which

$$g_1(x) = \frac{1}{\Gamma(2-\alpha)} x^{1-\alpha} + \frac{\Gamma(5\alpha+1)}{\Gamma(4\alpha+1)} x^{4\alpha},$$
$$g_2(x) = \frac{2}{\Gamma(3-\beta)} x^{2-\beta} + \frac{\Gamma(7\beta+1)}{\Gamma(6\beta+1)} x^{6\beta}.$$

The exact solutions of this problem are $y_1(x) = x + x^{5\alpha} + 1$ and $y_2(x) = x^2 + x^{7\beta} - 1$.

TABLE 2. L_∞ and L_2 errors for $y_1(x)$ and $y_2(x)$ using presented method for Example 5.2.

	J	$L_\infty error(y_1)$	p	$L_\infty error(y_2)$	p	$L_2 error(y_1)$	$L_2 error(y_2)$
	1	1.26×10^{-1}	–	1.26×10^{-1}	–	6.10×10^{-2}	6.33×10^{-2}
	2	7.55×10^{-2}	0.73	7.55×10^{-2}	0.73	2.62×10^{-2}	2.65×10^{-2}
	3	4.49×10^{-2}	0.74	4.49×10^{-2}	0.74	1.10×10^{-2}	1.10×10^{-2}
$\alpha = 0.25$	4	2.67×10^{-2}	0.74	2.67×10^{-2}	0.74	4.63×10^{-3}	4.64×10^{-3}
	5	1.58×10^{-2}	0.74	1.58×10^{-2}	0.74	1.94×10^{-3}	1.95×10^{-3}
	6	9.43×10^{-3}	0.74	9.43×10^{-3}	0.74	8.19×10^{-4}	8.19×10^{-4}
	7	5.60×10^{-3}	0.74	5.60×10^{-3}	0.74	3.44×10^{-4}	3.44×10^{-4}
	1	9.28×10^{-3}	–	9.28×10^{-3}	–	4.06×10^{-3}	4.05×10^{-3}
	2	3.28×10^{-3}	1.49	3.28×10^{-3}	1.49	1.02×10^{-3}	1.01×10^{-3}
	3	1.16×10^{-3}	1.49	1.16×10^{-3}	1.49	2.60×10^{-4}	2.56×10^{-4}
$\alpha = 0.5$	4	4.11×10^{-4}	1.5	4.11×10^{-4}	1.50	6.61×10^{-5}	6.49×10^{-5}
	5	1.45×10^{-4}	1.5	1.45×10^{-4}	1.50	1.67×10^{-5}	1.64×10^{-5}
	6	5.14×10^{-5}	1.49	5.14×10^{-5}	1.50	4.25×10^{-6}	4.18×10^{-6}
	7	1.81×10^{-5}	1.5	1.81×10^{-5}	1.5	1.07×10^{-6}	1.06×10^{-6}
	1	1.55×10^{-3}	–	1.55×10^{-3}	–	7.16×10^{-4}	6.92×10^{-4}
	2	3.26×10^{-4}	2.24	3.26×10^{-4}	2.24	1.12×10^{-4}	1.07×10^{-4}
	3	6.88×10^{-5}	2.24	6.88×10^{-5}	2.24	1.78×10^{-5}	1.68×10^{-5}
$\alpha = 0.75$	4	1.44×10^{-5}	2.25	1.44×10^{-5}	2.25	2.87×10^{-6}	2.70×10^{-6}
	5	3.04×10^{-6}	2.24	3.04×10^{-6}	2.24	4.71×10^{-7}	4.41×10^{-7}
	6	6.39×10^{-7}	2.24	6.39×10^{-7}	2.24	7.84×10^{-8}	7.30×10^{-8}
	7	1.34×10^{-7}	2.25	1.34×10^{-7}	2.25	1.32×10^{-8}	1.24×10^{-8}

TABLE 3. The Computational results for Example 5.2.

x_i	$\lvert y_{1ex} - y_{1app}\rvert$	$\lvert y_{2ex} - y_{2app}\rvert$	$\lvert y_{1ex} - y_{1app}\rvert$[5]	$\lvert y_{2ex} - y_{2app}\rvert$[5]
0	0	0	0	0
0.1	1.11×10^{-12}	1.11×10^{-12}	1.65×10^{-14}	4.50×10^{-14}
0.2	1.12×10^{-12}	1.15×10^{-12}	8.46×10^{-12}	2.84×10^{-12}
0.3	1.11×10^{-12}	1.42×10^{-12}	3.25×10^{-10}	3.11×10^{-11}
0.4	1.09×10^{-12}	2.89×10^{-12}	4.33×10^{-9}	1.61×10^{-10}
0.5	1.08×10^{-12}	7.98×10^{-12}	3.22×10^{-8}	5.36×10^{-10}
0.6	1.08×10^{-12}	2.17×10^{-11}	1.66×10^{-7}	1.27×10^{-9}
0.7	1.06×10^{-12}	5.32×10^{-11}	6.67×10^{-7}	2.24×10^{-9}
0.8	9.82×10^{-13}	1.17×10^{-10}	2.21×10^{-6}	2.77×10^{-9}
0.9	7.53×10^{-13}	2.36×10^{-10}	6.40×10^{-6}	2.03×10^{-9}
1	2.11×10^{-13}	4.44×10^{-10}	1.65×10^{-5}	1.99×10^{-9}

The L_∞ and L_2 errors are obtained in Table 4 for $\alpha = 0.8, \beta = 0.6$ and different values of J using the presented method. Also, Table 4 shows the numerical convergence order for different values of J, which confirm our theoretical results. The graphs of absolute errors functions $|y(x) - y_{1_J}(x)|$ and $|y(x) - y_{2_J}(x)|$ for $\alpha = 0.8, \beta = 0.6$ and $J = 7$ are given in Figure 3.

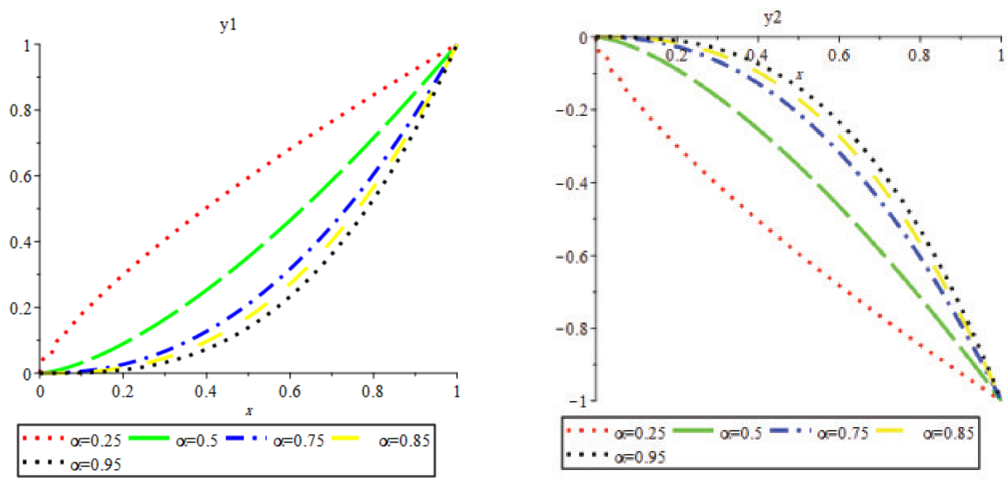

FIGURE 2. The plots of the approximate solutions $y_1(x)$ and $y_2(x)$ at $J = 5$, for Example 5.2.

TABLE 4. L_∞ and L_2 errors for $y_1(x)$ and $y_2(x)$ using presented method for Example 5.3.

J	$L_\infty error(y_1)$	p	$L_\infty error(y_2)$	p	$L_2 error(y_1)$	$L_2 error(y_2)$
1	2.88×10^{-3}	–	3.38×10^{-3}	–	1.71×10^{-3}	1.84×10^{-3}
2	1.68×10^{-4}	4.09	2.68×10^{-4}	9.49	3.04×10^{-5}	1.24×10^{-4}
3	1.59×10^{-5}	3.40	2.11×10^{-5}	3.66	6.66×10^{-6}	8.74×10^{-6}
4	1.69×10^{-6}	3.23	1.71×10^{-6}	3.62	6.82×10^{-7}	6.80×10^{-7}
5	1.78×10^{-7}	3.24	1.43×10^{-7}	3.57	7.83×10^{-8}	5.91×10^{-8}
6	1.92×10^{-8}	3.21	1.25×10^{-8}	3.51	9.03×10^{-9}	5.53×10^{-9}
7	2.67×10^{-9}	2.84	1.08×10^{-9}	3.53	1.15×10^{-9}	2.26×10^{-10}

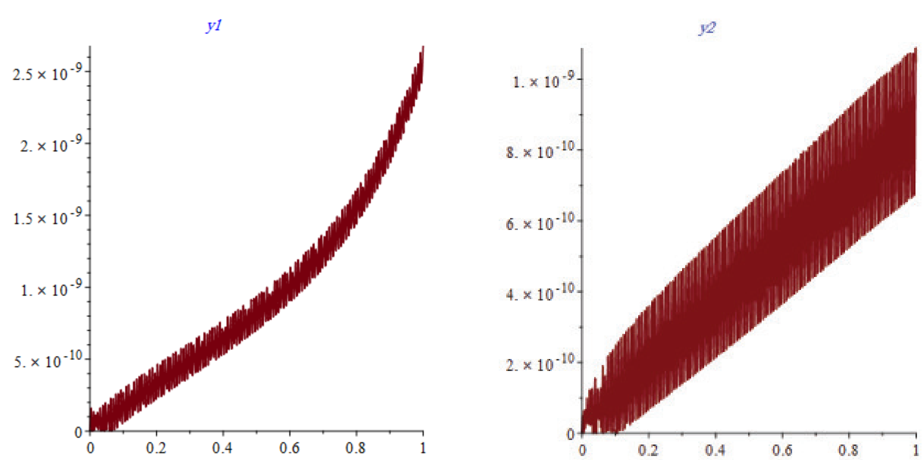

FIGURE 3. Plots of the absolute errors $|y(x) - y_{1_J}(x)|$ and $|y(x) - y_{2_J}(x)|$ at $\alpha = 0.8, \beta = 0.6$ and $J = 7$, for Example 5.3.

TABLE 5. L_∞ error for $\alpha = 0.6$ and $y_i(x)(i = 1, 2, 3)$, using presented method for Example 5.4.

J	$L_\infty error(y_1)$	p	$L_\infty error(y_2)$	p	$L_\infty error(y_3)$	p
1	1.38×10^{-3}	–	1.41×10^{-3}	–	8.08×10^{-3}	–
2	2.64×10^{-4}	2.39	1.17×10^{-4}	3.59	6.61×10^{-4}	3.61
3	5.01×10^{-5}	2.39	9.70×10^{-6}	3.59	5.14×10^{-5}	3.68
4	9.49×10^{-6}	2.39	8.00×10^{-7}	3.59	4.00×10^{-6}	3.68
5	1.80×10^{-6}	2.39	7.13×10^{-8}	3.48	3.19×10^{-7}	3.64
6	3.41×10^{-7}	2.39	5.97×10^{-9}	3.57	2.62×10^{-8}	3.60

TABLE 6. L_∞ error for $\alpha = 0.7$ and $y_i(x)(i = 1, 2, 3)$, using presented method for Example 5.4.

J	$L_\infty error(y_1)$	p	$L_\infty error(y_2)$	p	$L_\infty error(y_3)$	p
1	6.70×10^{-4}	–	3.43×10^{-3}	–	1.50×10^{-2}	–
2	9.71×10^{-5}	2.78	2.76×10^{-4}	3.63	1.31×10^{-3}	3.51
3	1.39×10^{-5}	2.80	2.24×10^{-5}	3.62	1.08×10^{-4}	3.59
4	2.00×10^{-6}	2.79	1.88×10^{-6}	3.57	8.80×10^{-6}	3.61
5	2.87×10^{-7}	2.79	1.63×10^{-7}	3.52	7.33×10^{-7}	3.58
6	4.14×10^{-8}	2.79	1.40×10^{-8}	3.54	2.08×10^{-7}	3.51

Example 5.4. As the last example, for $\alpha \in (0, 1)$ we consider the following linear initial value problem

$$\begin{cases} {}_*D_0^\alpha y_1(x) = g_1(x) - \int_0^x \big(y_(t) + y_2(t) + y_3(t)\big)dt, \\ \\ {}_*D_0^\alpha y_2(x) = g_2(x) - \int_0^x \big(y_1(t) - y_2(t) + y_3(t)\big)dt, \\ \\ {}_*D_0^\alpha y_3(x) = g_3(x) - \int_0^x \big(y_1(t) + y_2(t) - y_3(t)\big)dt, \\ \\ y_1(0) = y_2(0) = y_3(0) = 0, \quad 0 \le x \le 1, \quad 0 < \alpha < 1, \end{cases}$$

which

$$g_1(x) = \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}x^{3\alpha} + \frac{x^{4\alpha+2}}{16\alpha^2+12\alpha+2} + \frac{x^{6\alpha+2}}{36\alpha^2+18\alpha+2} + \frac{x^{8\alpha+2}}{64\alpha^2+24\alpha+2},$$
$$g_2(x) = \frac{\Gamma(6\alpha+1)}{\Gamma(5\alpha+1)}x^{5\alpha} + \frac{x^{4\alpha+2}}{16\alpha^2+12\alpha+2} - \frac{x^{6\alpha+2}}{36\alpha^2+18\alpha+2} + \frac{x^{8\alpha+2}}{64\alpha^2+24\alpha+2},$$
$$g_3(x) = \frac{\Gamma(8\alpha+1)}{\Gamma(7\alpha+1)}x^{7\alpha} + \frac{x^{4\alpha+2}}{16\alpha^2+12\alpha+2} + \frac{x^{6\alpha+2}}{36\alpha^2+18\alpha+2} - \frac{x^{8\alpha+2}}{64\alpha^2+24\alpha+2}.$$

The exact solutions of this problem are $y_1(x) = x^{4\alpha}$, $y_2(x) = x^{6\alpha}$ and $y_3(x) = x^{8\alpha}$.

The L_∞ errors y_i are obtained in Table 5–8 for different values of J using the presented method. Also, Table 5–8 show the numerical convergence order for different values of α and J, which confirm our theoretical results. The graph of approximate solutions y_1, y_2 and y_3 for $J = 5$ and $\alpha = 0.6, 0.7, 0.8$ and 0.9 are given in Figure 4.

TABLE 7. L_∞ error for $\alpha = 0.8$ and $y_i(x)(i = 1, 2, 3)$, using presented method for Example 5.4.

J	$L_\infty error(y_1)$	p	$L_\infty error(y_2)$	p	$L_\infty error(y_3)$	p
1	7.61×10^{-4}	–	6.74×10^{-3}	–	2.45×10^{-2}	–
2	8.26×10^{-5}	3.20	5.05×10^{-4}	3.73	2.10×10^{-3}	3.54
3	9.00×10^{-6}	3.19	4.29×10^{-5}	3.55	1.79×10^{-4}	3.55
4	9.80×10^{-7}	3.19	3.80×10^{-6}	3.49	1.53×10^{-5}	3.54
5	1.06×10^{-7}	3.20	3.52×10^{-7}	3.43	1.36×10^{-6}	3.49
6	1.16×10^{-8}	3.19	3.37×10^{-8}	3.38	1.26×10^{-7}	3.43

TABLE 8. L_∞ error for $\alpha = 0.9$ and $y_i(x)(i = 1, 2, 3)$, using presented method for Example 5.4.

J	$L_\infty error(y_1)$	p	$L_\infty error(y_2)$	p	$L_\infty error(y_3)$	p
1	2.33×10^{-3}	–	1.45×10^{-2}	–	4.46×10^{-2}	–
2	1.93×10^{-4}	3.59	1.05×10^{-3}	3.73	4.41×10^{-3}	3.33
3	1.59×10^{-5}	3.60	6.03×10^{-5}	4.12	3.12×10^{-4}	3.82
4	1.31×10^{-6}	3.60	5.24×10^{-6}	3.52	1.95×10^{-5}	4
5	1.08×10^{-7}	3.59	5.06×10^{-7}	3.37	1.78×10^{-6}	3.44
6	8.98×10^{-9}	3.60	5.15×10^{-8}	3.29	1.74×10^{-7}	3.53

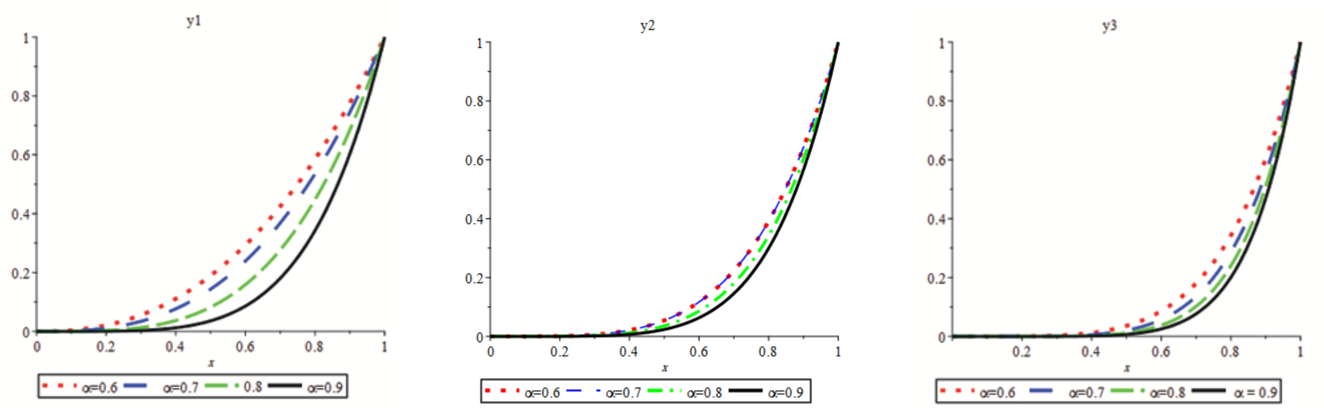

FIGURE 4. plots of the approximate solution $y_1(x), y_2(x)$ and $y_3(x)$ at $J = 6$, for Example 5.4.

6. CONCLUSION

In this paper, we utilize cubic Hermite spline functions to solve a nonlinear Duffing fractional differential equation with integral boundary conditions. We use the operational matrix of the Caputo-type fractional derivative and apply the collocation method to obtain the solution. We demonstrate that our method exhibits good agreement with the numerical order of convergence when the exact solution lies within $H^4[0, 1]$. However when the exact solution lies within the fractional Sobolev space $W^{s,2}(0, 1)$, where $1 \le s < 4$, we observe a reduction in convergence order to $O(2^{-sJ})$. Future work could focus on theoretically determining the convergence order of our method in fractional Sobolev spaces.

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