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Exact solutions of the nonlinear heat conduction equation using an analytical approach

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Abstract

This paper presents new solutions to the nonlinear heat equation using the Exp-function method. The method employs an exponential form to construct diverse solution models, including one-soliton, two-soliton, hyperbolic, and trigonometric soliton solutions. These solutions are crucial for modeling wave phenomena in studying the stress of water surfaces. By utilizing exponential structures, the complexity of the equation is reduced, and computational efficiency is enhanced. This approach offers a robust framework for solving higher-order nonlinear partial differential equations and explains for the behavior of solitons in complex systems.

Keywords. The Exp-function approach, The nonlinear heat equation, Solitary and soliton solutions.
2010 Mathematics Subject Classification. 02.60.Lj, 02.70.Wz, 02.90.+p.

1. Introduction

A proficient and extremely successful mathematical instrument for obtaining precise traveling wave solutions to nonlinear evolution equations (NLEEs) that arise in science, engineering, and mathematical physics is the improved Exp-function technique. The nonlinear heat equation uses the MSE approach to obtain precise solutions involving parameters to NLEEs [5, 6, 16]. When the parameters obtain their particular values, the solitary wave solutions can be obtained from the accurate traveling wave solutions [3, 15, 22]. This study extends the nonlinear heat equation by incorporating available variables of the recently introduced time and space derivatives. We thoroughly analyze the equation using the analytical approach and create various structures, including exponential, trigonometric, and hyperbolic functions [18]. Solitons and periodic solutions for the fifth–order KdV equation using the EFM have been investigated by Chun [7]. The Exp-function method, along with Hirota's and tanh-coth methods, have been applied for solving solitary wave solutions of the generalized shallow water wave equation by Wazwaz [33]. Wu et al. [35] have applied the Exp-function method and its application to nonlinear equations. Analyze some recent papers with applications of the EFM and discuss the main deficiencies of this method as represented in [20]. Various techniques have been employed to address the problem, each yielding only specific solutions. The primary objective of this document is to derive the chemical equation using the improved Exp-function method and to present novel wave solutions. The approaches that have been well recognized in recent works are the homotopy analysis method [1, 10], the variational iteration method [17], the Exp-function method [11], the Cauchy problem for matrix factorizations of the Helmholtz equation [19], nonlinear eigenvalue problems [13], the homotopy perturbation method [9], Hirota's bilinear operator [32], the boundary value problem for nonlinear first-order differential problems [29], the F-expansion method [2], the Jacobi elliptic function method [8], the tanh-function method [12], and so on. Nonlinear wave solutions of equations are of great help to understanding nonlinear physical phenomena and analyzing the wave mechanism. High-dimensional nonlinear partial differential equations (NLPDEs) pose greater complexity compared to the low-dimensional NLPDEs. They typically manifest richer and more complex nonlinear behavior and have great significance and wide application in the field of mathematical physics. The exact or numerical solutions to these models, which explain phenomena across various fields, have become an important area of research, leading to the development of numerous methods by

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scientists working in the field. Here, we investigate the analytical behavior of the nonlinear heat model in two forms (x,t) and (x,y,t).

In the following, the nonlinear heat conduction equation [34] is mentioned in (x,t) as

$$u_t - a(u^3)_{xx} - u + u^3 = 0, \tag{1.1}$$

and it is considered in (x,y,t) as

$$u_t - a(u^3)_{xx} - a(u^3)_{yy} - u + u^3 = 0. \tag{1.2}$$

The tanh approach was used to solve this problem and derive new solutions [34]. We present solutions for trigonometric and elliptic functions and the hyperbolic ones obtained by the Exp-function method. This method allows the simultaneous derivation of three different sorts of solutions. The nonlinear heat partial differential equation can be solved using the improved Exp-function approach to find new, more general closed-form wave solutions. This approach is among the potent strategies that have emerged recently to establish more precise wave solutions to nonlinear partial differential equations. We have obtained several new precise solutions, defined in terms of hyperbolic and trigonometric functions. These solutions include soliton and periodic wave solutions with arbitrary parameters [23, 25–28]. This method is particularly useful for obtaining exact soliton solutions, and its application to fractional models has provided deeper insights into the parametric effects of wave behavior in neural systems. By leveraging the expansion method, this study aims to explore new wave profiles of the chemical equation and analyze how different parameters, such as orders and nonlinearity, influence the traveling behavior of solitons. In this work, we also focus on the impact of these parameters on the stability and coherence of the resulting wave profiles, which could have significant applications for understanding neural signaling in both healthy and diseased states. The study provides potential applications of the obtained soliton solutions in various fields, including neuroscience, engineering, and applied mathematics. To fully appreciate its physical significance, soliton solutions are highly valuable [14, 21, 24, 31].

The paper is managed as follows: In section 2, the Exp-function method is presented. In section 3, the results of the mentioned equation are obtained. In section 4, we investigate the nonlinear heat equation in (x,y,t) form. Moreover, the conclusion and advantages are pointed out in section 5.

2. The Exp–function method

Take the following nonlinear partial differential equation as

$$F_1(\Phi, \Phi_t, \Phi_x, \Phi_{xx}, \Phi_{yy}, \Phi_{tt}, \Phi_{tx}, \Phi_{ty}, ...) = 0, \tag{2.1}$$

with

$$\Phi(x,y,t) = u(\Gamma), \quad \Gamma = x + y - ct, \tag{2.2}$$

where c is constant; therefore, Eq. (2.1) transforms to

$$F_2(\Phi, -c\Phi', \Phi', \Phi'', \Phi'', ...) = 0. \tag{2.3}$$

The Exp–function method [18] is introduced with the following function:

$$\Phi(\Gamma) = \frac{\sum_{n=-a_1}^{a_2} A_n \exp(n\Gamma)}{\sum_{m=-b_1}^{b_2} B_m \exp(m\Gamma)}. \tag{2.4}$$

We introduce a new variable or a series expansion to transform the original nonlinear differential equation into a simpler form. Equate the highest-order nonlinear term with the highest-order derivative term to establish the solution's structure. But the positive integers a_2 and b_2 can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in Eq. (2.3). We substitute the assumed solution into the transformed equation. Then, we determine the coefficients in the assumed solution by solving the resulting system of algebraic equations. Substituting the determined coefficients back into the assumed solution, we acquire the exact solution of the unique equation.

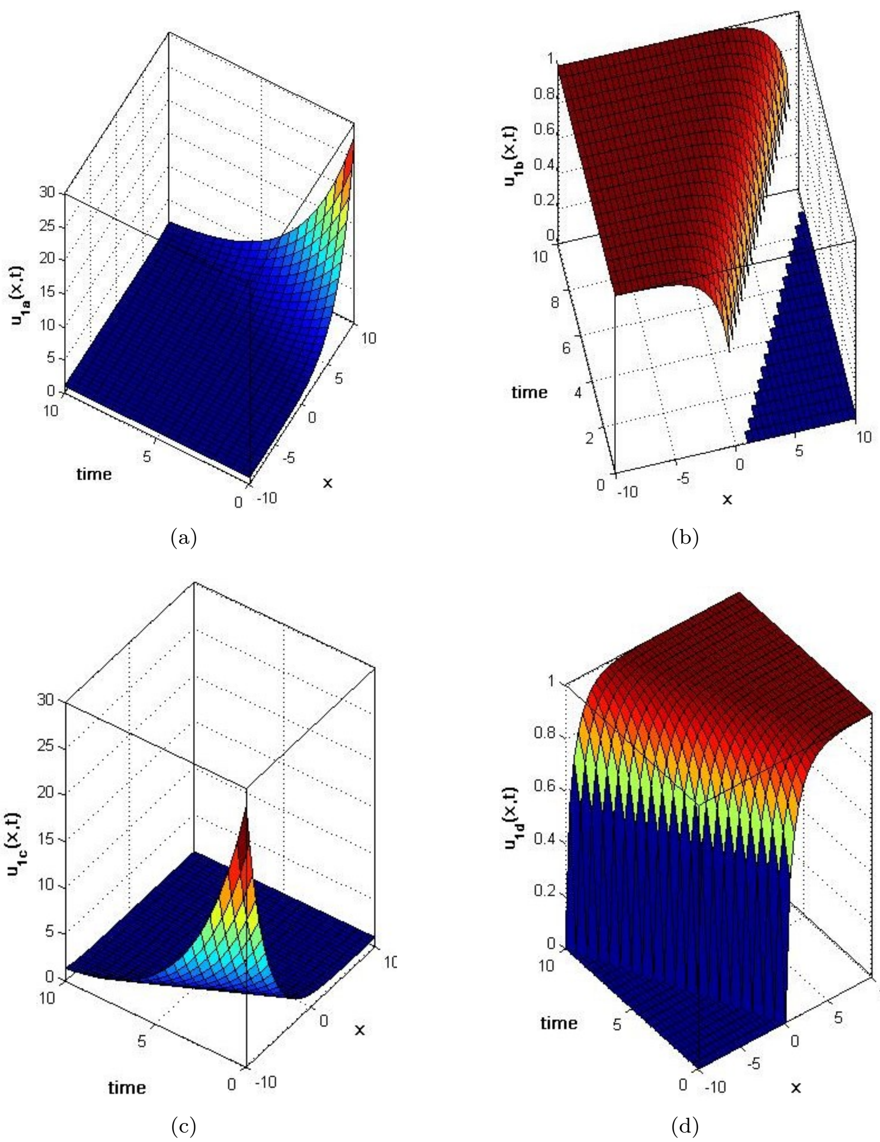

FIGURE 1. The solitary solution: (a)u_{1a}, (b) u_{1b}, (c)u_{1c}, and (d) u_{1d}, all positioned at a = 1 in Eqs. (3.18)-(3.21).

3. The heat equation to (x, t)

This section will construct the test function by giving a model consisting of one space and time, as below

$$\Phi_t - s_1(\Phi^3)_{xx} - \Phi + \Psi^3 = 0, \tag{3.1}$$

by $\Gamma = \mu(\mathrm{x} - \mathrm{ct})$ transforms to the following

$$-c\mu\Phi' - s_1\mu^2(\Phi^3)'' - \Phi + \Phi^3 = 0, \tag{3.2}$$

and take

$$\Phi(x, t) = v^{-\frac{1}{2}}(x, t), \tag{3.3}$$

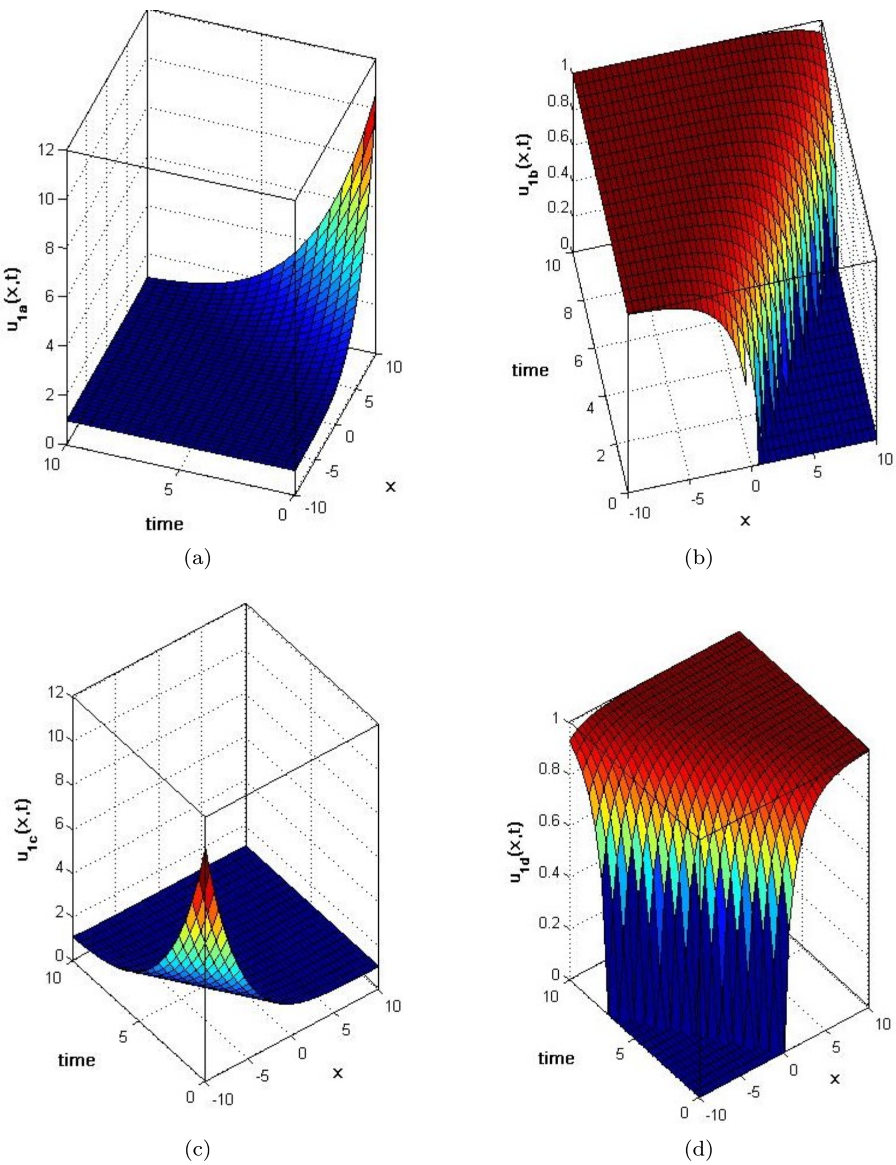

FIGURE 2. The solitary solution: (a)u_{1a}, and (b) u_{1b}, (c)u_{1c}, and (d) u_{1d}, all positioned at a = 2 in Eqs. (3.18)-(3.21).

into Eq. (3.2) to get

$$2c\mu v^2 v' + 6s_1\mu^2 vv'' - 15s_1\mu^2(v')^2 - v^3 + v^2 = 0. \tag{3.4}$$

To find the amounts of a_1 and b_1, we balance vv'' with v^2v' in Eq. (3.4) as

$$vv'' = \frac{a_{11}\exp((2a_1 + 3b_1)\Gamma) + ...}{a_{12}\exp(5b_1\Gamma) + ...}, \tag{3.5}$$

$$v^2v' = \frac{a_{13}\exp((3a_1+b_1)\Gamma)+...}{a_{14}\exp(4b_1\Gamma)+...} = \frac{a_{13}\exp((3a_1+2b_1)\Gamma)+...}{a_{14}\exp(5b_1\Gamma)+...}, \tag{3.6}$$

respectively. Balancing the highest order of the EFM in Eqs. (3.5) and (3.6), we obtain

$$2a_1+3b_1 = 3a_1+2b_1 \Rightarrow a_1 = b_1. \tag{3.7}$$

To find the amounts of a_2 and b_2, to vv'' and v^2v' in Eq. (3.2), we acquire

$$vv'' = \frac{...+a_{21}\exp(-(2a_2+3b_2)\Gamma)}{...+a_{22}\exp(-5b_2\Gamma)}, \tag{3.8}$$

$$v^2v' = \frac{...+a_{23}\exp(-(3a_2+b_2)\Gamma)}{...+a_{24}\exp(-4b_2\Gamma)} = \frac{...+a_{23}\exp(-(3a_2+2b_2)\Gamma)}{...+a_{24}\exp(-5b_2\Gamma)}. \tag{3.9}$$

Balancing the lowest order of the EFM in Eqs. (3.8) and (3.9), we get

$$-(2a_2+3b_2) = -(3a_2+2b_2) \Rightarrow a_2 = b_2. \tag{3.10}$$

Type I: $a_2 = a_1 = 1$ **and** $b_2 = b_1 = 1$.

We set $B_1 = 1$, $a_2 = a_1 = 1$, and $b_2 = b_1 = 1$. Then Eq. (2.4) transforms to

$$v(\Gamma) = \frac{A_1\exp(\Gamma)+A_0+A_{-1}\exp(-\Gamma)}{\exp(\Gamma)+B_0+B_{-1}\exp(-\Gamma)}. \tag{3.11}$$

Appending (3.11) into Eq. (3.4) and using the Maple software, one gets

$$\begin{aligned}&\frac{1}{\lambda}[G_4\exp(4\Gamma)+G_3\exp(3\Gamma)+G_2\exp(2\Gamma)+G_1\exp(\Gamma)+G_0+G_{-1}\exp(-\Gamma)\\&+G_{-2}\exp(-2\Gamma)+G_{-3}\exp(-3\Gamma)+G_{-4}\exp(-4\Gamma)] = 0,\end{aligned} \tag{3.12}$$

where

$$\lambda = \left[B_{-1}\exp(-\Gamma)+B_0+\exp(\Gamma)\right]^4, \tag{3.13}$$

and G_n are coefficients of $\exp(n\chi)$. Equating the coefficients of $\exp(n\Gamma)$ to be zero, we obtain the parameters $A_1, A_0, A_{-1}, B_0, B_{-1}$, μ, and c, as

$$\begin{cases} G_4 = 0,\ G_3 = 0,\ G_2 = 0,\ G_1 = 0,\\ G_0 = 0,\\ G_{-4} = 0,\ G_{-3} = 0,\ G_{-2} = 0,\ G_{-1} = 0.\end{cases} \tag{3.14}$$

Solving the above sets and by utilizing the Maple software, we achieve the following sets of non-trivial solutions:

Type 1-1:

$$A_{-1} = 0,\ A_1 = B_{-1} = 0,\ B_0 = A_0,\ A_0 = A_0,\ \mu = \pm\frac{2}{3\sqrt{s_1}},\ c = \pm\sqrt{s_1}, \tag{3.15}$$

$$v_1(x,t) = \frac{A_0}{A_0 + exp\left[\pm\frac{2}{3\sqrt{s_1}}\left(x \mp \sqrt{s_1}t\right)\right]}. \tag{3.16}$$

Recalling that $\Phi = v^{-\frac{1}{2}}$ and using Eq. (3.16), we have

$$\Phi_1(x,t) = \left\{\frac{A_0}{A_0 + exp\left[\pm\frac{2}{3\sqrt{s_1}}\left(x \mp \sqrt{s_1}t\right)\right]}\right\}^{-\frac{1}{2}}. \tag{3.17}$$

If we choose $A_0 = 1$ and $A_0 = -1$, then the solution Eq. (3.17) gives (cf. Eqs. (54)–(57) in [34])

$$\Phi_{1a}(x,t) = \left\{\frac{1}{2} - \frac{1}{2}tanh\left[\frac{1}{3\sqrt{s_1}}\left(x - \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{3.18}$$

$$\Phi_{1b}(x,t) = \left\{\frac{1}{2} - \frac{1}{2}coth\left[\frac{1}{3\sqrt{s_1}}\left(x - \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{3.19}$$

and

$$\Phi_{1c}(x,t) = \left\{\frac{1}{2} + \frac{1}{2}tanh\left[\frac{1}{3\sqrt{s_1}}\left(x + \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{3.20}$$

$$\Phi_{1d}(x,t) = \left\{\frac{1}{2} + \frac{1}{2}coth\left[\frac{1}{3\sqrt{s_1}}\left(x + \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{3.21}$$

where obtained solutions by EFM are like obtained solutions by the tanh method in [34]. Also, by considering the above cases, we can find the periodic form of solutions as below:

$$\Phi_{1e}(x,t) = \left\{\frac{1}{2} + \frac{i}{2}tan\left[\frac{i}{3\sqrt{s_1}}\left(x - \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{3.22}$$

$$\Phi_{1f}(x,t) = \left\{\frac{1}{2} - \frac{i}{2}cot\left[\frac{i}{3\sqrt{s_1}}\left(x - \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{3.23}$$

and

$$\Phi_{1g}(x,t) = \left\{\frac{1}{2} - \frac{i}{2}tan\left[\frac{i}{3\sqrt{s_1}}\left(x + \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{3.24}$$

$$\Phi_{1h}(x,t) = \left\{\frac{1}{2} + \frac{i}{2}cot\left[\frac{1}{3\sqrt{s_1}}\left(x + \sqrt{s_1}t\right)\right]\right\}^{-\frac{1}{2}}, \quad s_1 > 0. \tag{3.25}$$

Case II: $b_1 = a_1 = 2$ **and** $b_2 = a_2 = 1$.

Here, we set $B_2 = 1$, $b_1 = a_1 = 2$, and $b_2 = a_2 = 1$. Then Eq. (2.4) transforms to

$$v(\Gamma) = \frac{A_2\exp(2\Gamma) + A_1\exp(\Gamma) + A_0 + A_{-1}\exp(-\Gamma)}{\exp(2\Gamma) + B_1\exp(\Gamma) + B_0 + B_{-1}\exp(-\Gamma)}. \tag{3.26}$$

Putting (3.26) into Eq. (3.2), we acquire

$$\begin{aligned}&\frac{1}{\lambda}[G_8\exp(8\Gamma) + G_7\exp(7\Gamma) + G_6\exp(6\Gamma) + G_5\exp(5\Gamma) + G_4\exp(4\Gamma)\\ &\quad + G_3\exp(3\Gamma) + G_2\exp(2\Gamma) + G_1\exp(\Gamma) + G_0 + G_{-1}\exp(-\Gamma) + G_{-2}\exp(-2\Gamma)\\ &\quad + G_{-3}\exp(-3\Gamma) + G_{-3}\exp(-3\Gamma) + G_{-4}\exp(-4\Gamma)] = 0,\end{aligned} \tag{3.27}$$

where

$$\lambda = [B_{-1}\exp(-\Gamma) + B_0 + B_1\exp(\Gamma) + \exp(2\Gamma)]^4, \tag{3.28}$$

and G_n are coefficients of $\exp(n\Gamma)$. Equating the coefficients of $\exp(n\Gamma)$ to be zero, we acquire the parameters $A_1, A_0, A_{-1}, B_1, B_0, B_{-1}$, μ, and c, as follows:

$$\begin{cases} G_8 = 0,\ G_7 = 0,\ G_6 = 0,\ G_5 = 0,\ G_4 = 0,\ G_3 = 0,\ G_2 = 0,\ G_1 = 0,\\ G_0 = 0,\\ G_{-4} = 0,\ G_{-3} = 0,\ G_{-2} = 0,\ G_{-1} = 0.\end{cases} \tag{3.29}$$

Solving the above system, we obtain the following results:

Type 1-2:

$$B_1 = A_1 = 0 = B_{-1} = B_0 = A_0 = A_2 = 0,\ A_{-1} = A_{-1},\ \mu = \pm\frac{2}{9\sqrt{s_1}}, \tag{3.30}$$

$$c = \mp 3\sqrt{s_1}, \qquad \mathbf{v}_1(x,t) = A_{-1}exp\left[\mp\frac{2}{3\sqrt{s_1}}\left(x \pm 3\sqrt{s_1}t\right)\right].$$

Noting that $\Phi = v^{-\frac{1}{2}}$ and using Eq. (3.30), we have

$$\Phi_1(\mathrm{x,t}) = \frac{1}{\sqrt{\mathrm{A}_{-1}}}\ \exp\left[\pm\frac{1}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} \pm 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right]. \tag{3.31}$$

If we choose $\mathrm{A}_{-1} = 1$, then the solution Eq. (3.31) gives

$$\Phi_{1\mathrm{a}}(\mathrm{x,t}) = \cosh\left[\frac{1}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} + 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right] + \sinh\left[\frac{1}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} + 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right], \quad s_1 > 0, \tag{3.32}$$

or

$$\Phi_{1\mathrm{b}}(\mathrm{x,t}) = \cosh\left[\frac{1}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} - 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right] - \sinh\left[\frac{1}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} - 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right], \quad s_1 > 0. \tag{3.33}$$

Also, by considering the above cases, we can find the periodic form of solutions as below:

$$\Phi_{1\mathrm{c}}(\mathrm{x,t}) = \cos\left[\frac{\mathrm{i}}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} + 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right] \imath\sin\left[\frac{\mathrm{i}}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} + 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right], \quad s_1 > 0, \tag{3.34}$$

or

$$\Phi_{1\mathrm{d}}(\mathrm{x,t}) = \cos\left[\frac{\mathrm{i}}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} - 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right] + \mathrm{i}\sin\left[\frac{\mathrm{i}}{3\sqrt{\mathrm{s}_1}}\left(\mathrm{x} - 3\sqrt{\mathrm{s}_1}\mathrm{t}\right)\right], \quad s_1 > 0. \tag{3.35}$$

4. The heat equation to (x, y, t)

Take the nonlinear heat equation [34] as

$$\Phi_\mathrm{t} - \mathrm{s}_1(\Phi^3)_{\mathrm{xx}} - \mathrm{s}_1(\Phi^3)_{\mathrm{yy}} - \Phi + \Phi^3 = 0, \tag{4.1}$$

using $\Gamma = \mu(\mathrm{x} + \mathrm{y} - \mathrm{ct})$, the above equation is transformed to

$$-\mathrm{c}\mu\Phi' - 2\mathrm{s}_1\mu^2(\Phi^3)'' - \Phi + \Phi^3 = 0, \tag{4.2}$$

we substitute

$$\Phi(\mathrm{x,y,t}) = \mathrm{v}^{-\frac{1}{2}}(\mathrm{x,y,t}), \tag{4.3}$$

into Eq. (4.2) to get

$$2\mathrm{c}\mu\mathrm{v}^2\mathrm{v}' + 12\mathrm{s}_1\mu^2\mathrm{v}\mathrm{v}'' - 30\mathrm{s}_1\mu^2(\mathrm{v}')^2 - \mathrm{v}^3 + \mathrm{v}^2 = 0. \tag{4.4}$$

To find the amounts of a_1 and b_1, we balance $\mathrm{v}\mathrm{v}''$ with $\mathrm{v}^2\mathrm{v}'$ in Eq. (4.4), as

$$\mathrm{v}\mathrm{v}'' = \frac{\mathrm{a}_{11}\exp((2\mathrm{a}_1 + 3\mathrm{b}_1)\Gamma) + ...}{\mathrm{a}_{12}\exp(5\mathrm{b}_1\Gamma) + ...}, \tag{4.5}$$

$$\mathrm{v}^2\mathrm{v}' = \frac{\mathrm{a}_{13}\exp((3\mathrm{a}_1 + \mathrm{b}_1)\Gamma) + ...}{\mathrm{a}_{14}\exp(4\mathrm{b}_1\Gamma) + ...} = \frac{\mathrm{a}_{13}\exp((3\mathrm{a}_1 + 2\mathrm{b}_1)\Gamma) + ...}{\mathrm{a}_{14}\exp(5\mathrm{b}_1\Gamma) + ...}. \tag{4.6}$$

Balancing Eqs. (4.5) and (4.6), we find

$$2\mathrm{a}_1 + 3\mathrm{b}_1 = 3\mathrm{a}_1 + 2\mathrm{b}_1 \Rightarrow a_1 = b_1. \tag{4.7}$$

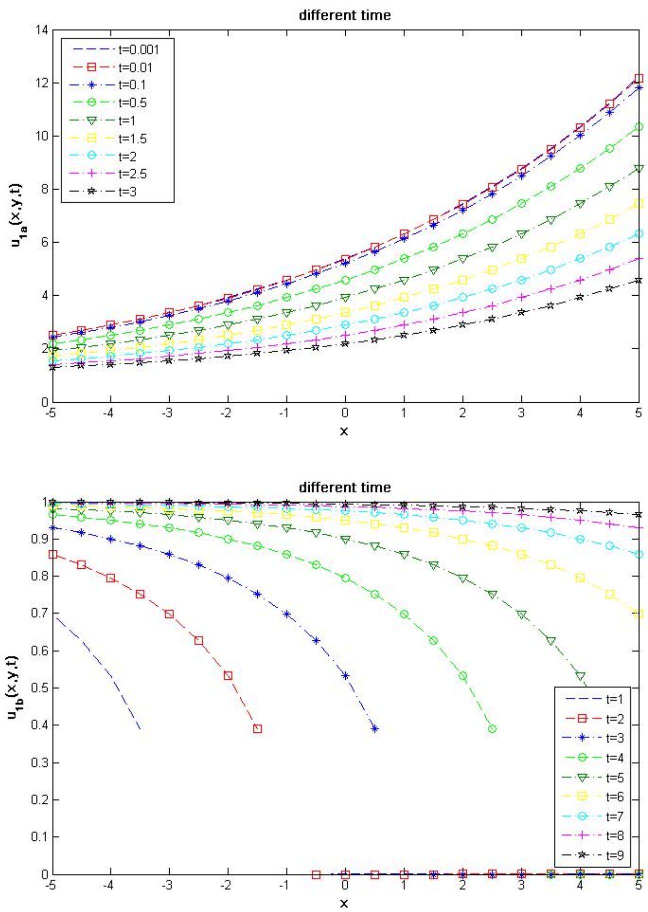

FIGURE 3. The evolution from $t = 0.001$ to $t = 3$ when $a = 2$ and $y = 10$ for u_{1a} and $t = 1$ to $t = 9$ when $a = 2$ and $y = 5$ for u_{1b} in Eqs. (4.18) and (4.19).

To find the amounts of a_2 and b_2, to vv'' and v^2v' in Eq. (4.2), we acquire

$$vv'' = \frac{... + a_{21}\exp(-(2a_2 + 3b_2)\Gamma)}{... + a_{22}\exp(-5b_2\Gamma)}, \tag{4.8}$$

$$v^2v' = \frac{... + a_{23}\exp(-(3a_2 + b_2)\Gamma)}{... + a_{24}\exp(-4b_2\Gamma)} = \frac{... + a_{23}\exp(-(3a_2 + 2b_2)\Gamma)}{... + a_{24}\exp(-5b_2\Gamma)}. \tag{4.9}$$

Balancing the lowest order of the EFM in Eqs. (4.8) and (4.9), we get

$$-(2a_2 + 3b_2) = -(3a_2 + 2b_2) \Rightarrow a_2 = b_2. \tag{4.10}$$

Case I: $a_1 = b_1 = 1$ **and** $a_2 = b_2 = 1$.

For simplicity, we set $B_1 = 1$, $a_1 = b_1 = 1$, and $a_2 = b_2 = 1$. Then Eq. (2.4) reduces to

$$v(\Gamma) = \frac{A_1\exp(\Gamma) + A_0 + A_{-1}\exp(-\Gamma)}{\exp(\Gamma) + B_0 + B_{-1}\exp(-\Gamma)}. \tag{4.11}$$

Appending (4.11) into Eq. (4.4) and using the Maple software, one gets

$$\frac{1}{\lambda}[G_4\exp(4\Gamma)+G_3\exp(3\Gamma)+G_2\exp(2\Gamma)+G_1\exp(\Gamma)+G_0+G_{-1}\exp(-\Gamma)$$
$$+G_{-2}\exp(-2\Gamma)+G_{-3}\exp(-3\Gamma)+G_{-4}\exp(-4\Gamma)]=0, \tag{4.12}$$

where

$$\lambda=\left[B_{-1}\exp(-\Gamma)+B_0+\exp(\Gamma)\right]^4, \tag{4.13}$$

and $\mathrm{G_n}$ are coefficients of $\exp(\mathrm{n}\Gamma)$. Equating the coefficients of $\exp(\mathrm{n}\chi)$ to be zero, we obtain the parameters $\mathrm{A_1, A_0, A_{-1}, B_0, B_{-1}}$, μ, and c, as

$$\begin{cases}\mathrm{G_4=0,\ G_3=0,\ G_2=0,\ G_1=0,}\\ \mathrm{G_0=0,}\\ \mathrm{G_{-4}=0,\ G_{-3}=0,\ G_{-2}=0,\ G_{-1}=0.}\end{cases} \tag{4.14}$$

Solving the above sets and by utilizing the Maple software, we achieve the following sets of non-trivial solutions:

Type 2-1:

$$A_{-1}=A_1=B_{-1}=0,\ B_0=a_0,\ A_0=A_0,\ \mu=\pm\frac{2}{3\sqrt{2s_1}},\ c=\pm\sqrt{2s_1}, \tag{4.15}$$

$$\mathrm{v_1(x,y,t)}=\frac{\mathrm{A_0}}{\mathrm{A_0}+\exp\left[\pm\frac{2}{3\sqrt{2s_1}}\left(\mathrm{x+y\mp\sqrt{2s_1}t}\right)\right]}. \tag{4.16}$$

Recalling that $\Phi=v^{-\frac{1}{2}}$ and using Eq. (4.16), we have

$$\Phi_1(\mathrm{x,y,t})=\left\{\frac{\mathrm{A_0}}{\mathrm{A_0}+\exp\left[\pm\frac{2}{3\sqrt{2s_1}}\left(\mathrm{x+y\mp\sqrt{2s_1}t}\right)\right]}\right\}^{-\frac{1}{2}}. \tag{4.17}$$

If we choose $\mathrm{A_0}=1$ and $\mathrm{A_0}=-1$, then the solution Eq. (4.17) gives

$$\Phi_{1a}(\mathrm{x,y,t})=\left\{\frac{1}{2}-\frac{1}{2}\tanh\left[\frac{1}{3\sqrt{2s_1}}\left(\mathrm{x+y-\sqrt{2s_1}t}\right)\right]\right\}^{-\frac{1}{2}},\quad s_1>0, \tag{4.18}$$

$$\Phi_{1b}(\mathrm{x,y,t})=\left\{\frac{1}{2}-\frac{1}{2}\coth\left[\frac{1}{3\sqrt{2s_1}}\left(\mathrm{x+y-\sqrt{2s_1}t}\right)\right]\right\}^{-\frac{1}{2}},\quad s_1>0, \tag{4.19}$$

and

$$\Phi_{1c}(\mathrm{x,y,t})=\left\{\frac{1}{2}+\frac{1}{2}\tanh\left[\frac{1}{3\sqrt{2s_1}}\left(\mathrm{x+y+\sqrt{2s_1}t}\right)\right]\right\}^{-\frac{1}{2}},\quad s_1>0, \tag{4.20}$$

$$\Phi_{1d}(\mathrm{x,y,t})=\left\{\frac{1}{2}+\frac{1}{2}\coth\left[\frac{1}{3\sqrt{2s_1}}\left(\mathrm{x+y+\sqrt{2s_1}t}\right)\right]\right\}^{-\frac{1}{2}},\quad s_1>0. \tag{4.21}$$

Also, by considering the above cases, we can find the periodic form of solutions as below:

$$\Phi_{1a}(\mathrm{x,y,t})=\left\{\frac{1}{2}+\frac{\mathrm{i}}{2}\tan\left[\frac{\mathrm{i}}{3\sqrt{2s_1}}\left(\mathrm{x+y-\sqrt{2s_1}t}\right)\right]\right\}^{-\frac{1}{2}},\quad s_1>0, \tag{4.22}$$

$$\Phi_{1b}(\mathrm{x,y,t})=\left\{\frac{1}{2}-\frac{\mathrm{i}}{2}\cot\left[\frac{\mathrm{i}}{3\sqrt{2s_1}}\left(\mathrm{x+y-\sqrt{2s_1}t}\right)\right]\right\}^{-\frac{1}{2}},\quad s_1>0, \tag{4.23}$$

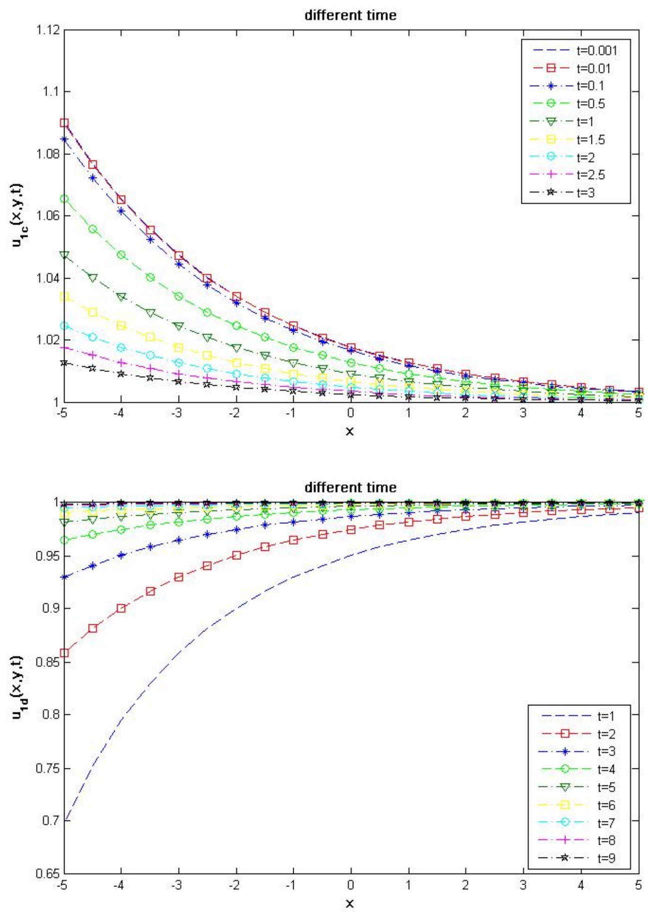

FIGURE 4. The evolution from $t = 0.001$ to $t = 3$ when $a = 2$ and $y = 10$ for u_{1c} and $t = 1$ to $t = 9$ when $a = 2$ and $y = 5$ for u_{1d} in Eqs. (4.20) and (4.21).

and

$$\Phi_{1c}(x, y, t) = \left\{ \frac{1}{2} - \frac{i}{2} \tan \left[\frac{i}{3\sqrt{2s_1}} \left(x + y + \sqrt{2s_1} t \right) \right] \right\}^{-\frac{1}{2}}, \quad s_1 > 0, \tag{4.24}$$

$$\Phi_{1d}(x, y, t) = \left\{ \frac{1}{2} + \frac{i}{2} \cot \left[\frac{i}{3\sqrt{2s_1}} \left(x + y + \sqrt{2s_1} t \right) \right] \right\}^{-\frac{1}{2}}, \quad s_1 > 0. \tag{4.25}$$

Case II: $a_1 = b_1 = 2$ **and** $a_2 = b_2 = 1$.

We set $A_2 = 1$, $a_1 = b_1 = 2$, and $a_2 = b_2 = 1$. Then, Eq. (2.4) gives the following

$$\Phi(\Gamma) = \frac{A_2 \exp(2\Gamma) + A_1 \exp(\Gamma) + A_0 + A_{-1} \exp(-\Gamma)}{\exp(2\Gamma) + b_1 \exp(\Gamma) + B_0 + B_{-1} \exp(-\Gamma)}. \tag{4.26}$$

Putting (4.26) into Eq. (4.2), we acquire

$$\frac{1}{\lambda}[G_8\exp(8\Gamma)+G_7\exp(7\Gamma)+G_6\exp(6\Gamma)+G_5\exp(5\Gamma)+G_4\exp(4\Gamma)$$
$$+G_3\exp(3\Gamma)+G_2\exp(2\Gamma)+G_1\exp(\Gamma)+G_0+G_{-1}\exp(-\Gamma)+G_{-2}\exp(-2\Gamma)$$
$$+G_{-3}\exp(-3\Gamma)+G_{-3}\exp(-3\Gamma)+G_{-4}\exp(-4\Gamma)]=0, \tag{4.27}$$

where

$$\lambda=[B_{-1}\exp(-\Gamma)+B_0+B_1\exp(\Gamma)+\exp(2\Gamma)]^4, \tag{4.28}$$

and $\mathrm{G_n}$ are coefficients of $\exp(\mathrm{n}\chi)$. Equating the coefficients of $\exp(\mathrm{n}\chi)$ to be zero, we acquire the parameters $\mathrm{A_1, A_0, A_{-1}, B_1, B_0, B_{-1}}$, μ, and c, as

$$\begin{cases} \mathrm{G_8=0,\ G_7=0,\ G_6=0,\ G_5=0,\ G_4=0,\ G_3=0,\ G_2=0,\ G_1=0,} \\ \mathrm{G_0=0,} \\ \mathrm{G_{-4}=0,\ G_{-3}=0,\ G_{-2}=0,\ G_{-1}=0.} \end{cases} \tag{4.29}$$

Solving the above system, we obtain the following results:

Type 2-2:

$$B_1=B_{-1}=A_1=A_0=A_2=B_0=0,\ A_{-1}=A_{-1},\ \mu=\pm\frac{2}{9\sqrt{2s_1}}, \tag{4.30}$$
$$c=\mp3\sqrt{2s_1},\qquad \mathbf{v}_1(x,y,t)=A_{-1}exp\left[\mp\frac{2}{3\sqrt{2s_1}}\left(x+y\pm3\sqrt{2s_1}t\right)\right].$$

Noting that $\Phi=v^{-\frac{1}{2}}$ and using Eq. (4.30), we have

$$\Phi_1(\mathrm{x,y,t})=\frac{1}{\sqrt{\mathrm{A_{-1}}}}\ \exp\left[\pm\frac{1}{3\sqrt{2\mathrm{s_1}}}\left(\mathrm{x+y}\pm3\sqrt{2\mathrm{s_1}}\mathrm{t}\right)\right]. \tag{4.31}$$

If we choose $\mathrm{A_{-1}}=1$, then the solution Eq. (4.31) gives

$$\Phi_{1a}(x,y,t)=cosh\left[\frac{1}{3\sqrt{2s_1}}\left(x+y+3\sqrt{2s_1}t\right)\right]+sinh\left[\frac{1}{3\sqrt{2s_1}}\left(x+y+3\sqrt{2s_1}t\right)\right],\quad s_1>0, \tag{4.32}$$

or

$$\Phi_{1b}(x,y,t)=cosh\left[\frac{1}{3\sqrt{2s_1}}\left(x+y-3\sqrt{2s_1}t\right)\right]-sinh\left[\frac{1}{3\sqrt{2s_1}}\left(x+y-3\sqrt{2s_1}t\right)\right],\quad s_1>0. \tag{4.33}$$

Also, by considering the above cases, we can find the periodic form of solutions as below:

$$\Phi_{1a}(x,y,t)=cos\left[\frac{i}{3\sqrt{2s_1}}\left(x+y+3\sqrt{2s_1}t\right)\right]-i\,sin\left[\frac{i}{3\sqrt{2s_1}}\left(x+y+3\sqrt{2s_1}t\right)\right],\quad s_1>0, \tag{4.34}$$

or

$$\Phi_{1b}(x,y,t)=cos\left[\frac{i}{3\sqrt{2s_1}}\left(x+y-3\sqrt{2s_1}t\right)\right]+i\,sin\left[\frac{i}{3\sqrt{2s_1}}\left(x+y-3\sqrt{2s_1}t\right)\right],\quad s_1>0. \tag{4.35}$$

While the Exp-function approach has proven effective in solving the heat nonlinear equation, it is not without limitations. One notable limitation is the reliance on specific parameter values for constructing exact solutions. The method may face challenges when generalized to other types of nonlinear equations or when the solution space involves more complex boundary conditions. The effectiveness of this method also hinges on the precise tuning of the related parameters, which can be time-consuming and require domain expertise. Furthermore, the method is currently limited to specific types of soliton solutions, and its application to other forms of nonlinear waves remains an area for future exploration. Despite these limitations, the Exp-function method opens up new avenues for solving nonlinear differential equations in a more structured and efficient way, especially when combined with other numerical techniques. For this purpose, at the different times, the surface graphics of the exact solutions are plotted in Figures (1)-(4). One can see that the exact solutions obtained by the EFM are quite accurate.

5. Conclusion

This study successfully demonstrates the application of the Exp-function approach to the nonlinear heat equation, providing various soliton solutions, including one-soliton, two-soliton, hyperbolic, and trigonometric solutions. Using rational expansion form in conjunction with exponential techniques simplifies the process of solving complex nonlinear partial differential equations, enhancing both solution accuracy and computational efficiency. Despite some limitations, this method represents a significant advancement in the study of nonlinear wave equations, offering a promising approach for future research on higher-order nonlinear systems. For the obtained soliton solutions, we have carried out theoretical and graphical analysis, showing that it holds a certain morphological oscillation in the process of time evolution, and analyzing the physical characteristics of velocity and direction. These results may help to understand the variety of the dynamic behavior of higher-dimensional nonlinear wave fields. Given the versatility of the nonlinear problems, future research could explore applications across fields such as engineering, physics, and applied mathematics, further extending the impact of this study beyond the nonlinear heat model.

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