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Numerical idea to solve three-dimensional nonlinear Volterra integral equations with 3D-Legendre polynomials

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Abstract

In this paper, a three-dimensional Legendre polynomial (3D-LPs) is used for solving the nonlinear three-dimensional Volterra integral equations (VIEs). Converting the main problem to a nonlinear algebraic system using 3D-LPs, which can be generalized to equations in higher dimensions, then the nonlinear system will be solved. Some results concerning the error analysis have been achieved. Several examples are included to demonstrate the validity and applicability of the method. Moreover, we prove a theorem and a corollary about a sufficient condition for the minimum of mean square error under the Legendre coefficients and the uniqueness of the solution of the nonlinear VIEs. In addition, illustrative examples are included to demonstrate the validity and applicability of the presented method.

Keywords. Nonlinear Volterra integral equations, Three-dimensional Legendre polynomials, Nonlinear algebraic systems.
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1. Introduction

Nonlinear integral equations are used in the modeling of many phenomena in engineering, chemistry, physics, finance, and other disciplines, see [40, 48, 49] and the references therein. Note that most nonlinear integral equations do not have solutions in a closed form. So it is important to propose new methods for finding numerical solutions of these nonlinear equations [2]. Recently, several numerical methods have been developed to solve nonlinear integral equations (NIEs) such as two-dimensional models [13], time collocation and time discretization method [7], two-dimensional Chebyshev polynomials [45], the collocation method [14], Haar wavelets method [24], the Harmonic wavelet method [9], the Alpert Multi-wavelets method [47], the Legendre multi-wavelets [23], the Bernstein polynomials [37], semi-orthogonal B-spline wavelet collocation method [46], the sinc-collocation method [51], the homotopy perturbation method [4], the Adomian decomposition method [15, 29], homotopy analysis method [11, 12], improved $tan(\phi/2)$-expansion method [20, 21, 36], and many others. The Chebyshev collocation method is used to find an approximate solution for nonlinear integral equations by Yang [50]. Maleknejad and Kajani used a combination of Legendre and Block-Pulse function on the interval $[0, 1]$, to solve the linear integral equation of the second kind [27]. The Chebyshev and Legendre polynomials methods to obtain the approximate solution of mixed Volterra-Fredholm singular integral equations were been investigated by Mohamed and Taher [41]. Nemati and Ordokhani investigated two-dimensional nonlinear Volterra integral equations using Legendre orthogonal polynomials [44]. Mashayekhi et al. solved the nonlinear mixed Volterra-Fredholm integral equations based on hybrid functions approximation [38]. Bakhshi et al. used the differential transform method to solve three-dimensional non-linear Volterra equation [3]. Manafian and Bolghar utilized the 3D-block-pulse functions to solve nonlinear the three-dimensional Volterra integral-differential equations [30]. By Legendre polynomials, the VIEs of the second kind have been solved by Liu [26]. Also, Maleknejad and Sohrabi [28] solved nonlinear Volterra-Fredholm-Hammerstein integral equations in terms of Legendre polynomials. Moreover, in [39] the

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nonlinear mixed Volterra-Fredholm integral equations are investigated with the help of the modified 3D block pulse functions. The singularly perturbed Volterra integro-differential equations with delay were considered based on the properties of the exact solution, a hybrid difference scheme with appropriate quadrature rules on a Shishkin-type mesh [16]. The study of nonlinear models and its application has been the subject of much research in the last several decades [5, 6]. Solutions of nonlinear models are accurate solutions of integrable equations in models of nonlinear systems of partial differential equations (NLSPDEs). Therefore, researchers have developed several approaches for constructing closed-form solutions to nonlinear differential equations, such as the generalized fifth-order KdV-like equation [31], variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [32], Biswas-Milovic equation for Kerr law nonlinearity [33], the Gerdjikov-Ivanov model [34], the Kundu-Eckhaus equation [35], the nonlinear the fifth-order integrable equations [22], the fractional generalized CBS-BK equation [52], the (3+1)-D Burger system by bilinear analysis [17], non-local integral conditions by the reduced differential transform method [42], and the combined KdV-mKdV equations with solitary wave solutions [1].

One of the examples of NIEs is three-dimensional nonlinear VIEs, which are obtained from modeling in engineering applications.

The idea of nonlinear Volterra integral equations has been successfully used in several scientific domains in recent years to mimic a variety of real-world occurrences, such as using spectral methods in fluid dynamics and application of nonlinear VIS in this area [8], an investigation of a two-dimensional nonlinear Volterra integral equations using Legendre polynomials [43], the shifted Legendre direct technique for variational problems [10], a direct scheme using the shifted Legendre series expansion for near optimum control [18] and analytical and numerical methods for Volterra equations [25]. In this paper, we investigate the general three-dimensional nonlinear VIEs [18, 25] in the following form

$$u(x,y,z)+\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}E(x,y,z,\eta,\xi,\rho,u(\eta,\xi,\rho))d\rho d\eta d\xi=\lambda(x,y,z), \tag{1.1}$$

so that $E(x,y,z,\eta,\xi,\rho,u(\eta,\xi,\rho)) = \Lambda(x,y,z,\eta,\xi,\rho)(u(\eta,\xi,\rho))^p$, $u(x,y,z)$ is an unknown function, $\lambda(x,y,z)$ is a continuous function defined on $[-1,1]^3$, $E(x,y,z,\eta,\xi,\rho,u(\eta,\xi,\rho))$ on $[-1,1]^6\times R$, and $\Lambda(x,y,z,\eta,\xi,\rho)$ on $[-1,1]^6$.

In this paper, we examine an application of three-dimensional 3D-Legendre polynomials (3D-LPs) for solving three-dimensional nonlinear VIEs. In connection with this method, we introduce 3D-LPs for solving nonlinear VIEs, which have not been evaluated earlier and we propose a suitable algorithm for their evaluation. This study aimed to examine the regulatory role of three-dimensional Legendre polynomial via converting the main problem to a nonlinear algebraic system on the nonlinear three-dimensional Volterra integral equations in which can be generalized to equations in higher dimensions then the nonlinear system will be solved.

The organization of the paper is as follows. In section 2, 3D Legendre polynomials are considered. In section 3, we introduce the error and analysis to investigate the existence and uniqueness of the solution. Then convert the main problem to a nonlinear system, and their evaluation is the issue of section 4. Some simple numerical examples are provided in section 5 to illustrate the capabilities of the method concerning the error norms. Finally, the conclusions are provided in section 6.

2. Properties of 3D Legendre polynomials

2.1. **Basic Preliminaries.** The three-dimensional Legendre polynomials (LPs) are defined on Ω is given as [19]

$$\psi_{m,n,p}(x,y,z)=L_m(x)\,L_n(y)\,L_p(z)\,,\qquad m,n,p=0,1,..., \tag{2.1}$$

where L_m, L_n and L_p are the renowned Legendre polynomials, respectively of order m,n and p, which are defined on the interval $[-1,1]$ and can be concluded with the help of the following recursive procedure [19]:

$$L_0(x)=1,$$
$$L_1(x)=x,$$
$$\vdots$$

$$L_{m+1}(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x).$$

By disjointing the 3D-LPs from each other, the expected expression can be clearly written as

$$\psi_{i,j,l}(x,y,z)\psi_{i',j',l'}(x,y,z) = \begin{cases} \psi_{i,j,l}(x,y,z), & i=i', j=j', l=l', \\ 0, & \text{Otherwise}, \end{cases} \tag{2.2}$$

also the corresponding orthogonality with 3D-LPs can be seen in below as

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \psi_{i,j,l}(x,y,z)\psi_{m,n,p}(x,y,z)dzdydx = \begin{cases} \frac{8}{(2i+1)(2j+1)(2l+1)}, & i=m, j=n, l=p, \\ 0, & \text{Otherwise}. \end{cases} \tag{2.3}$$

Suppose that $L^2[-1,1]^3$, the inner product in this space is given as

$$\langle \lambda(x,y,z), \mu(x,y,z) \rangle = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \lambda(x,y,z)\mu(x,y,z)dzdydx,$$

and the norm can be written as

$$\|\lambda(x,y,z)\|_2 = \langle \lambda(x,y,z), \lambda(x,y,z) \rangle^{\frac{1}{2}} = \left(\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mid \lambda(x,y,z) \mid^2 dzdydx \right)^{\frac{1}{2}}.$$

Then, 3D Legendre polynomials can be written as

$$\psi(x,y,z) = [\psi_{0,0,0}(x,y,z), ..., \psi_{0,0,m}(x,y,z), ..., \psi_{0,m,m}(x,y,z), ..., \psi_{m,m,m}(x,y,z)]^T_{(m+1)^3\times 1},$$

and

$$\begin{aligned} \lambda(x,y,z) \simeq \lambda_m(x,y,z) &= \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m} \lambda_{i,j,l}\psi_{i,j,l}(x,y,z) \\ &= F^T\psi(x,y,z) = \psi^T(x,y,z)F, \end{aligned} \tag{2.4}$$

where F and $\psi(x,y,z)$ are $(m+1)(m+1)(m+1)$ vectors with the form

$$F = [\lambda_{0,0,0}(x,y,z), ..., \lambda_{0,0,m}(x,y,z), ..., \lambda_{0,m,m}(x,y,z), ..., \lambda_{m,m,m}(x,y,z)]^T_{(m+1)^3\times 1}, \tag{2.5}$$

$$\psi(x,y,z) = [\psi_{0,0,0}(x,y,z), ..., \psi_{0,0,m}(x,y,z), ..., \psi_{0,m,m}(x,y,z), ..., \psi_{m,m,m}(x,y,z)]^T_{(m+1)^3\times 1}, \tag{2.6}$$

Theorem 2.1. *[30] Suppose that $\lambda \in L^2([-1,1]^3)$, then*

$$\lambda_m(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m} \lambda_{i,j,l}\psi_{i,j,l}(x,y,z) \simeq \lambda(x,y,z), \tag{2.7}$$

and

$$\lambda_{i,j,l} = \frac{\langle \lambda(x,y,z), \psi_{i,j,l}(x,y,z) \rangle}{\|\psi_{i,j,l}(x,y,z)\|_2^2}. \tag{2.8}$$

Besides, we have

$$\Sigma = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left(\lambda(x,y,z) - \lambda_m(x,y,z)\right)^2 dzdydx, \tag{2.9}$$

attains to minimum value.

Proof. For $i,j,l = 0,1,2,...,m$

$$\frac{\partial \Sigma}{\partial \lambda_{i,j,l}} = -2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left(\lambda(x,y,z) - \sum_{a=0}^{m}\sum_{b=0}^{m}\sum_{c=0}^{m} \lambda_{a,b,c}\phi_{a,b,c}(x,y,z) \right) \psi_{i,j,l}(x,y,z)dzdydx = 0, \tag{2.10}$$

with orthogonality of 3D-LPs we have

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\lambda_{i,j,l}\psi_{i,j,l}^2(x,y,z)dzdydx=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\lambda(x,y,z)\psi_{i,j,l}(x,y,z)dzdydx, \tag{2.11}$$

and

$$\begin{aligned}\lambda_{i,j,l}&=\frac{(2i+1)(2j+1)(2l+1)}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\lambda(x,y,z)\psi_{i,j,l}(x,y,z)dzdydx\\&=\frac{\langle\lambda(x,y,z),\psi_{i,j,l}(x,y,z)\rangle}{\langle\psi_{i,j,l}(x,y,z),\psi_{i,j,l}(x,y,z)\rangle}.\end{aligned} \tag{2.12}$$

In addition, for $i,j,l,i',j',l'=0,1,...,m$

$$\begin{aligned}\frac{\partial^2\Sigma}{\partial\lambda_{i,j,l}\partial\lambda_{i',j',l'}}&=2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\psi_{i,j,l}(x,y,z)\psi_{i',j',l'}(x,y,z)dzdydx\\&=\begin{cases}\frac{16}{(2i+1)(2j+1)(2l+1)}, & i=i',j=j',l=l',\\0, & \text{Otherwise},\end{cases}\end{aligned} \tag{2.13}$$

for $n,p,q=0,1,...,m$, we obtain

$$D_{n,p,q}=\det\begin{bmatrix}\frac{\partial^2\Sigma}{\partial\lambda_{0,0,0}\partial\lambda_{0,0,0}} & \frac{\partial^2\Sigma}{\partial\lambda_{0,0,0}\partial\lambda_{0,0,1}} & \cdots & \frac{\partial^2\Sigma}{\partial\lambda_{0,0,0}\partial\lambda_{n,p,q}}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial^2\Sigma}{\partial\lambda_{0,0,q}\partial\lambda_{0,0,0}} & \frac{\partial^2\Sigma}{\partial\lambda_{0,0,q}\partial\lambda_{0,0,1}} & \cdots & \frac{\partial^2\Sigma}{\partial f_{0,0,q}\partial\lambda_{n,p,q}}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial^2\Sigma}{\partial\lambda_{0,p,q}\partial\lambda_{0,0,0}} & \frac{\partial^2\Sigma}{\partial\lambda_{0,p,q}\partial\lambda_{0,0,1}} & \cdots & \frac{\partial^2\Sigma}{\partial\lambda_{0,p,q}\partial\lambda_{n,p,q}}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial^2\Sigma}{\partial\lambda_{n,p,q}\partial\lambda_{0,0,0}} & \frac{\partial^2\Sigma}{\partial\lambda_{n,p,q}\partial\lambda_{0,0,1}} & \cdots & \frac{\partial^2\Sigma}{\partial\lambda_{n,p,q}\partial\lambda_{n,p,q}}\end{bmatrix}_{(n+1)(p+1)(q+1)}. \tag{2.14}$$

This is just sufficient condition for the minimum of mean square error Σ under the 3D-LPs coefficients $\lambda_{i,j,l}$. Also, we have

$$D_{n,p,q}=\det\begin{bmatrix}\frac{16}{1} & & & & & & \mathbf{0}\\ & \ddots & & & & & \\ & & \frac{16}{2q+1} & & & & \\ & & & \ddots & & & \\ & & & & \frac{16}{2p+1} & & \\ & & & & & \ddots & \\ \mathbf{0} & & & & & & \frac{16}{(2n+1)(2p+1)(2q+1)}\end{bmatrix}_{(n+1)(p+1)(q+1)}>0. \tag{2.15}$$

□

3. The error and existence analysis

In this sections, we present that the expressed method in the previous sections.

3.1. **Existence and uniqueness of solution.** In this subsection, we investigate the existence and uniqueness of solution of the Equation (1.1). To this end, we first contemplate the equations of the form

$$u(x,y,z)+\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}E(x,y,z,\eta,\xi,\rho,u(\eta,\xi,\rho))d\rho d\xi d\eta=\lambda(x,y,z),\qquad (x,y,z)\in([-1,1])^3, \tag{3.1}$$

and we extend the Theorem (3.1) from [25], to three-dimensional case in the following theorem.

Theorem 3.1. *Assume that the λ and E be continuous for all $x,y,z,\eta,\xi,\rho\in[-1,1]$ and $-\infty<u<\infty$. Let also the kernel E satisfies the Lipschitz condition, i.e.*

$$|E(x,y,z,\eta,\xi,\rho,u(\eta,\xi,\rho))-E(x,y,z,\eta,\xi,\rho,v(\eta,\xi,\rho))|\leq L|u(\eta,\xi,\rho)-v(\eta,\xi,\rho)|,$$

for a nonnegative $0<L$, then the Equation (3.1) has a unique continuous solution.

Proof. For the proof, see [30, 49]. □

3.2. **Error estimate.**

Theorem 3.2. *Let $u(x,y,z)\in H^{\alpha}([-1,1]^3)$ (Sobolev space), and*

$$u_m(x,y,z)=\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}u_{i,j,l}\psi_{i,j,l}(x,y,z), \tag{3.2}$$

be the best approximation polynomial $u(x,y,z)$ in L^2-norm, then continuous for all x,y,z, $\eta,\xi,\rho\in[-1,1]$ and $-\infty<u<\infty$. Let also the kernel E satisfy the Lipschitz condition, i.e.

$$\|u(x,y,z)-u_m(x,y,z)\|_{L^2([-1,1]^3)}\leq C_0 m^{-\alpha}\parallel u(x,y,z)\parallel_{H^{\alpha}([-1,1]^3)}, \tag{3.3}$$

where C_0 is a positive constant which is dependent on the selected norm and independent of $u(x,y,z)$ and m.

Proof. See ([8]). □

Remark 3.3. Suppose $\overline{u}_m$ is a function that are obtained by us. Also, to error estimate it we have

$$\begin{aligned}\|u(x,y,z)-\overline{u_m}(x,y,z)\|_2\leq\ &\|u(x,y,z)-u_m(x,y,z)\|_2\\ &\leq\|u_m(x,y,z)-\overline{u_m}(x,y,z)\|_2,\end{aligned} \tag{3.4}$$

then, we have

$$\begin{aligned}\|u(x,y,z)-\overline{u_m}(x,y,z)\|_2^2&=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}|u_m(x,y,z)-\overline{u_m}(x,y,z)|^2dzdydx\\ &=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}(u_{i,j,l}-\overline{u}_{i,j,l})\psi_{i,j,l}(x,y,z)\right|^2dzdydx,\end{aligned} \tag{3.5}$$

from the Holder inequality we have

$$\begin{aligned}&\leq\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}\mid u_{i,j,l}-\overline{u}_{i,j,l}\mid^2\right)\left(\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}\mid\psi_{i,j,l}(x,y,z)|^2\right)dzdydx\\ &=\left(\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}\mid u_{i,j,l}-\overline{u}_{i,j,l}\mid^2\right)\left(\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\mid\psi_{i,j,l}(x,y,z)|^2dzdydx\right)\\ &=\parallel U-\overline{U}\parallel_2^2\left(\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}\frac{8}{(2i+1)(2j+1)(2l+1)}\right),\end{aligned}$$

where

$$U=[u_{0,0,0},...,u_{0,0,m},...,u_{0,m,m},...,u_{m,m,m}]^T_{(m+1)^3\times 1},$$

therefore, by using Theorem (3.2) we conclude

$$\|u(x,y,z)-\overline{u_m}(x,y,z)\|_2 \le C_0 m^{-\alpha}\ \|\|_{H^{\alpha}([-1,1]^3)} + \sqrt{\beta}\ \| U-\overline{U}\ \|_2, \tag{3.6}$$

where $\beta=\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m}\frac{8}{(2i+1)(2j+1)(2l+1)}$.

4. Convert the main problem to a nonlinear system

In this section, we want to convert the main problem (1.1) to a nonlinear system using a method which can be generalized to equations in higher dimensions. Approximating functions $u(x,y,z), \lambda(x,y,z), (u(x,y,z))^p$ and $\Lambda(x,y,z,\eta,\xi,\rho)$ with respect to 3D-LPs by the way described in Section 2 we get

$$\begin{aligned} &u(x,y,z)\simeq \psi^T(x,y,z)U=U^T\psi(x,y,z),\\ &\lambda(x,y,z)\simeq \psi^T(x,y,z)F=F^T\psi(x,y,z),\\ &u^p(x,y,z)\simeq \psi^T(x,y,z)U_p,\\ &\Lambda(x,y,z,\eta,\xi,\rho)\simeq \psi^T(x,y,z)\Lambda\phi(\eta,\xi,\rho), \end{aligned} \tag{4.1}$$

where $\psi(x,y,z)$ is defined in (2.4), the vectors U, F, U_p, and matrix Λ are 3D-LPs coefficients of $u(x,y,z)$, $\lambda(x,y,z)$, $(u(x,y,z))^p$ and $\Lambda(x,y,z,\eta,\xi,\rho)$, respectively.

Lemma 4.1. *Take $(m+1)^3$-vectors U and U^p be 3D-LPs coefficients of $u(x,y,z)$ and $u^p(x,y,z)$, respectively. If*

$$U=[u_{0,0,0},...,u_{0,0,m},...,u_{0,m,m},...,u_{m,m,m}]^T_{(m+1)^3\times 1}, \tag{4.2}$$

then

$$U_p=[u^p_{0,0,0},...,u^p_{0,0,m},...,u^p_{0,m,m},...,u^p_{m,m,m}]^T, \tag{4.3}$$

where $p\ge 1$, is a positive integer.

Proof. For the proof, see [40]. □

We take

$$u^p(x,y,z)\simeq \psi^T(x,y,z)U_p=U_p^T\psi(x,y,z).$$

Also, the operational matrix of 3D-LPs defined over $[-1,1]^3$ as

$$\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)\psi^T(\eta,\xi,\rho)d\rho d\xi d\eta = \tag{4.4}$$

$$\begin{bmatrix} \int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)\psi_{0,0,0}(\eta,\xi,\rho)d\rho d\xi d\eta\\ \vdots\\ \int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)\psi_{0,0,m}(\eta,\xi,\rho)d\rho d\xi d\eta\\ \vdots\\ \int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)\psi_{0,m,m}(\eta,\xi,\rho)d\rho d\xi d\eta\\ \vdots\\ \int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)\psi_{m,m,m}(\eta,\xi,\rho)d\rho d\xi d\eta \end{bmatrix}^T_{1\times(m+1)^3} = \begin{bmatrix} g_{0,0,0}(x,y,z)\\ \vdots\\ g_{0,0,m}(x,y,z)\\ \vdots\\ g_{0,m,m}(x,y,z)\\ \vdots\\ g_{m,m,m}(x,y,z) \end{bmatrix}^T_{1\times(m+1)^3},$$

so that for each $n,p,q=0,1,...,m$ we have

$$\begin{aligned} g_{n,p,q}(x,y,z) &\simeq \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{l=0}^{m} g^{i,j,l}_{n,p,q}\psi_{i,j,l}(x,y,z)\\ &= G^T_{n,p,q}\psi(x,y,z)=\psi^T(x,y,z)G_{n,p,q}, \end{aligned} \tag{4.5}$$

where

$$G_{n,p,q}=[g^{0,0,0}_{n,p,q},...,g^{0,0,m}_{n,p,q},...,g^{0,m,m}_{n,p,q},...,g^{m,m,m}_{n,p,q}]^T_{(m+1)^3\times 1}. \tag{4.6}$$

Therefore, we get

$$\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)\psi^{T}(\eta,\xi,\rho)d\rho d\xi d\eta=\psi^{T}(x,y,z)G, \tag{4.7}$$

so that

$$G=[G_{0,0,0}\mid \dots \mid G_{0,0,m}\mid \dots \mid G_{0,m,m}\mid \dots \mid G_{m,m,m}]_{(m+1)^3\times(m+1)^3}. \tag{4.8}$$

Now, we can write

$$\begin{aligned}\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)[u(\eta,\xi,\rho)]^{p}drdtds &\simeq \left(\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\Lambda(x,y,z,\eta,\xi,\rho)\psi^{T}(\eta,\xi,\rho)\right)U_{p}d\rho d\xi d\eta\\ &=\psi^{T}(x,y,z)GU_{p}.\end{aligned} \tag{4.9}$$

In general one obtains

$$\psi^{T}(x,y,z)U+\psi^{T}(x,y,z)GU_{p}=\psi^{T}(x,y,z)F, \tag{4.10}$$

then we get

$$U+GU_{p}=F, \tag{4.11}$$

by solving the above nonlinear system (4.11), U obtain and finally u can be calculated.

5. Numerical Applications

In this section, the examples are presented, and the efficiency of the 3D Legender polynomials is implemented to two examples.

Example 5.1. We consider the first example for nonlinear VIE as

$$u(x,y,z)-\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}\frac{1}{4}(x^{3}+\eta)z\rho u(\eta,\xi,\rho)d\rho d\xi d\eta=\lambda(x,y,z), \tag{5.1}$$

which $(x,y,z)\in[-1,1]^3$ and

$$\begin{aligned}&k(x,y,z,\eta,\xi,\rho)=\frac{1}{4}(x^{3}+\eta)z\rho,\\ &\lambda(x,y,z)=y\sin(x)-\frac{1}{12}z(y^{2}-1)(z^{2}-1)\times(\cos(1)-\sin(1)-\sin(x)+x^{3}\cos(x)-x^{3}\cos(1)+x\cos(x)).\end{aligned} \tag{5.2}$$

The closed-form of solution is $u(x,y,z)=y\sin(x)$. The errors for 3D-LPs for this example are given in Table 1.

Remark 5.2. First, we used the 3D-Legendre method for solving the nonlinear 3D VIEs. By applying the presented method to the nonlinear 3D VIEs gives the approximate solution of $u(x,y,z)$. Figure 1 illustrates the capabilities of the method concerning the error norms.

Example 5.3. As the last example, we consider the nonlinear VIE as

$$u(x,y,z)+\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}k(x,y,z,\eta,\xi,\rho)[u(\eta,\xi,\rho)]^{2}d\rho d\xi d\eta=\lambda(x,y,z), \tag{5.3}$$

which $(x,y,z)\in[-1,1]^3$ and

$$\begin{aligned}&k(x,y,z,\eta,\xi,\rho)=-\frac{1}{5}(x+\eta)(y^{2}+\rho)z\xi,\\ &\lambda(x,y,z)=x^{2}yz-\frac{1}{7200}z(y^{4}-1)(z+1)(11x^{6}+6x-5)\times(4y^{2}z^{2}-4y^{2}z+4y^{2}+3z^{3}-3z^{2}+3z-3).\end{aligned} \tag{5.4}$$

The closed form of solution is $u(x,y,z)=x^{2}yz$. The error for 3D-Legendre for this example is given in Table 2.

Remark 5.4. First, we used the 3D-Legendre method for solving the nonlinear 3D VIEs. Applying the presented method to the nonlinear 3D VIEs gives the approximate solution of $u(x,y,z)$. Figure 2 illustrates the capabilities of the method concerning the error norms.

TABLE 1. Error for 3D-Legendre of Example 5.1.

$(x,y,z)=2^{-i}$	$m=2$	$m=3$	$m=4$	$m=5$
i=1	1.4182×10^{-4}	6.2077×10^{-4}	8.9742×10^{-4}	6.1234×10^{-5}
i=2	2.0239×10^{-4}	1.3032×10^{-3}	1.5835×10^{-3}	1.1719×10^{-4}
i=3	2.0519×10^{-4}	2.5406×10^{-3}	2.8992×10^{-3}	2.1251×10^{-4}
i=4	2.8127×10^{-3}	4.5342×10^{-3}	5.0207×10^{-3}	3.3166×10^{-4}
i=5	9.1822×10^{-3}	6.8279×10^{-3}	5.2703×10^{-3}	2.1339×10^{-4}

TABLE 2. Error for 3D-Legendre of Example 5.3.

$(x,y,z)=2^{-i}$	$m=2$	$m=3$	$m=4$	$m=5$
i=1	6.3895×10^{-5}	1.1892×10^{-4}	7.5366×10^{-5}	6.1234×10^{-5}
i=2	9.6872×10^{-5}	2.0219×10^{-4}	1.4401×10^{-4}	1.1718×10^{-4}
i=3	1.5005×10^{-4}	3.3771×10^{-4}	2.5953×10^{-4}	2.1251×10^{-4}
i=4	1.9369×10^{-4}	4.5417×10^{-4}	3.9461×10^{-4}	3.3166×10^{-4}
i=5	4.9335×10^{-5}	1.2582×10^{-4}	2.2263×10^{-4}	2.1339×10^{-4}

TABLE 3. Error for 3D-Legendre of Example 5.5.

$(x,y,z)=2^{-i}$	$m=2$	$m=3$	$m=4$	$m=5$
i=1	3.1331×10^{-3}	5.7993×10^{-3}	4.9624×10^{-3}	1.3552×10^{-5}
i=2	2.3543×10^{-3}	7.9789×10^{-3}	2.6482×10^{-3}	2.0083×10^{-4}
i=3	1.2799×10^{-3}	1.2948×10^{-2}	3.6194×10^{-4}	3.0621×10^{-4}
i=4	2.4581×10^{-3}	2.0006×10^{-2}	1.5702×10^{-3}	1.8660×10^{-4}
i=5	1.0537×10^{-2}	9.0936×10^{-3}	6.7727×10^{-2}	2.9802×10^{-4}

Example 5.5. As the last example, we consider the nonlinear VIE as

$$u(x,y,z)+\int_{-1}^{x}\int_{-1}^{y}\int_{-1}^{z}k(x,y,z,\eta,\xi,\rho)[u(\eta,\xi,\rho)]^3 d\rho d\xi d\eta=\lambda(x,y,z), \tag{5.5}$$

which $(x,y,z)\in[-1,1]^3$ and

$$k(x,y,z,\eta,\xi,\rho)=-\frac{1}{2}x^2 z\xi,$$
$$\lambda(x,y,z)=yz\exp(x)-\frac{1}{120\exp(3)}x^2 z(y^5+1)(z^4-1)(\exp(3)\exp(3x)-1)). \tag{5.6}$$

The closed form of solution is $u(x,y,z)=yz\exp(x)$. The error for 3D-Legendre for this example is given in Table 3.

Remark 5.6. First, we used the 3D-Legendre method for solving the nonlinear 3D VIEs. Applying the presented method to the nonlinear 3D VIEs gives the approximate solution of $u(x,y,z)$. Figure 3 illustrates the capabilities of the method concerning the error norms.

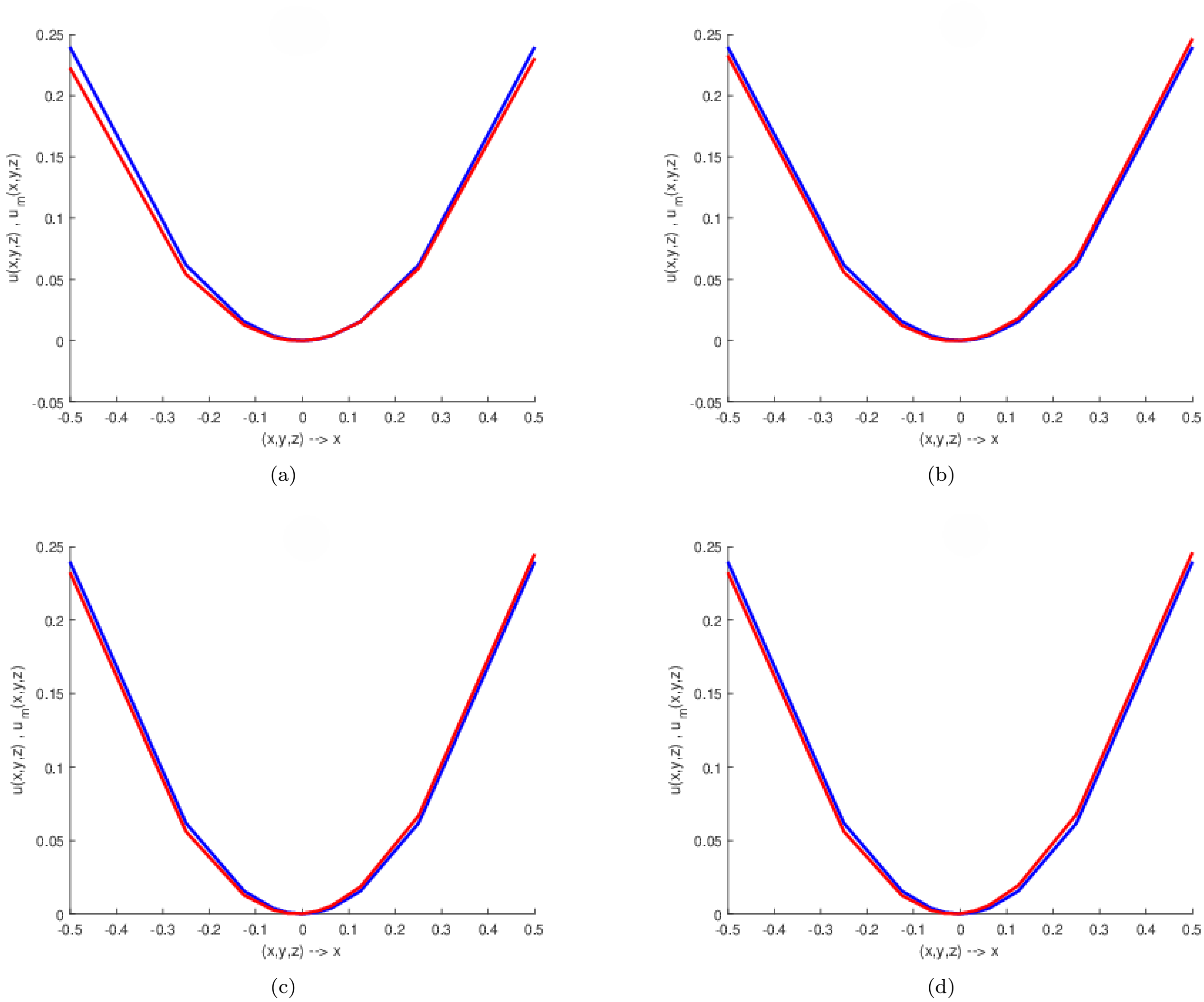

FIGURE 1. Graph of error function for Example 5.1 given (a) $m = 2$, (b) $m = 3$, (c) $m = 4$ and (d) $m = 5$, where the exact solution $u(x, y, z)$ with (blue color) and the numerical solution $u_m(x, y, z)$ with (red color).

6. Conclusion

In this article, we offered three-dimensional Legendre polynomials to find the approximate solution of the nonlinear VIEs. For illustration purposes, we solved some examples. The results of the examples showed that this method enjoys high accuracy. In the presented method, the unknown function is approximated using Legendre polynomials and the integral equation is converted to a system of algebraic equations. Our aim was based on the 3D-LPs and their operational matrix of integration, together with a set of suitable collocation nodes. The approximation of the solution together with enforcing the collocation nodes is used to lessen the computation of this problem to some algebraic equations. Also, since this is a simple method, it can be utilized by researchers in nonlinear sciences and engineering. We observed that these solutions possess good results using the mentioned method are presented in Figures 1–3 with good approximations, and also we could show the considerable errors in Tables 1–3 and has not

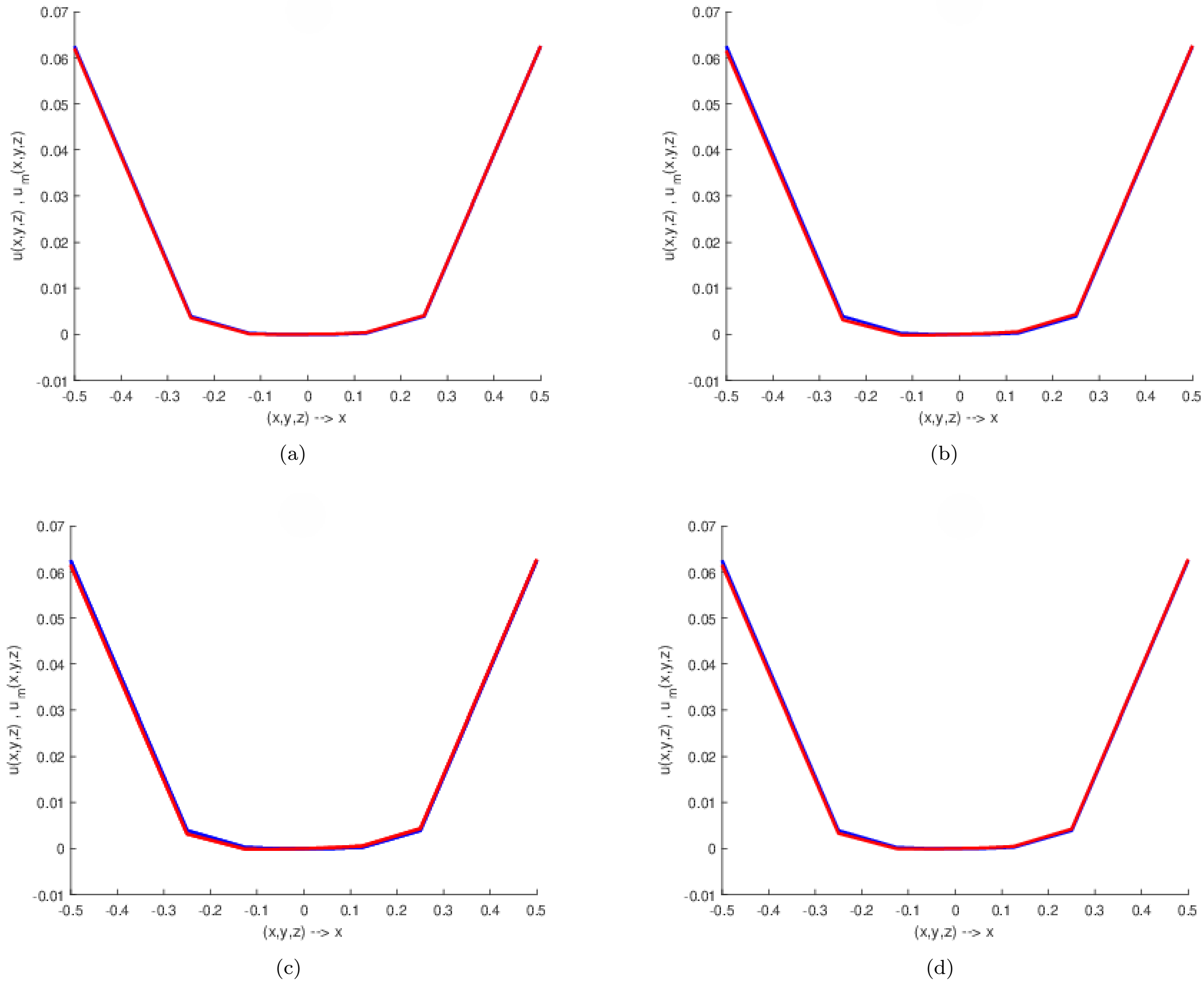

FIGURE 2. Graph of error function for Example 5.3 given (a) $m = 2$, (b) $m = 3$, (c) $m = 4$ and (d) $m = 5$, where the exact solution $u(x, y, z)$ with (blue color) and the numerical solution $u_m(x, y, z)$ with (red color).

been achieved previously for the considered model. These solutions have a wide range of physical significance and applications such as advanced scientific problems of biology, chemistry, physics, and engineering. Comparison between our proposed method with the exact solution shows that this method is effectively accurate and evidently, the error gets smaller as the calculation stages go ahead.

Conflict of Interest

The authors declare that they have no conflict of interest.

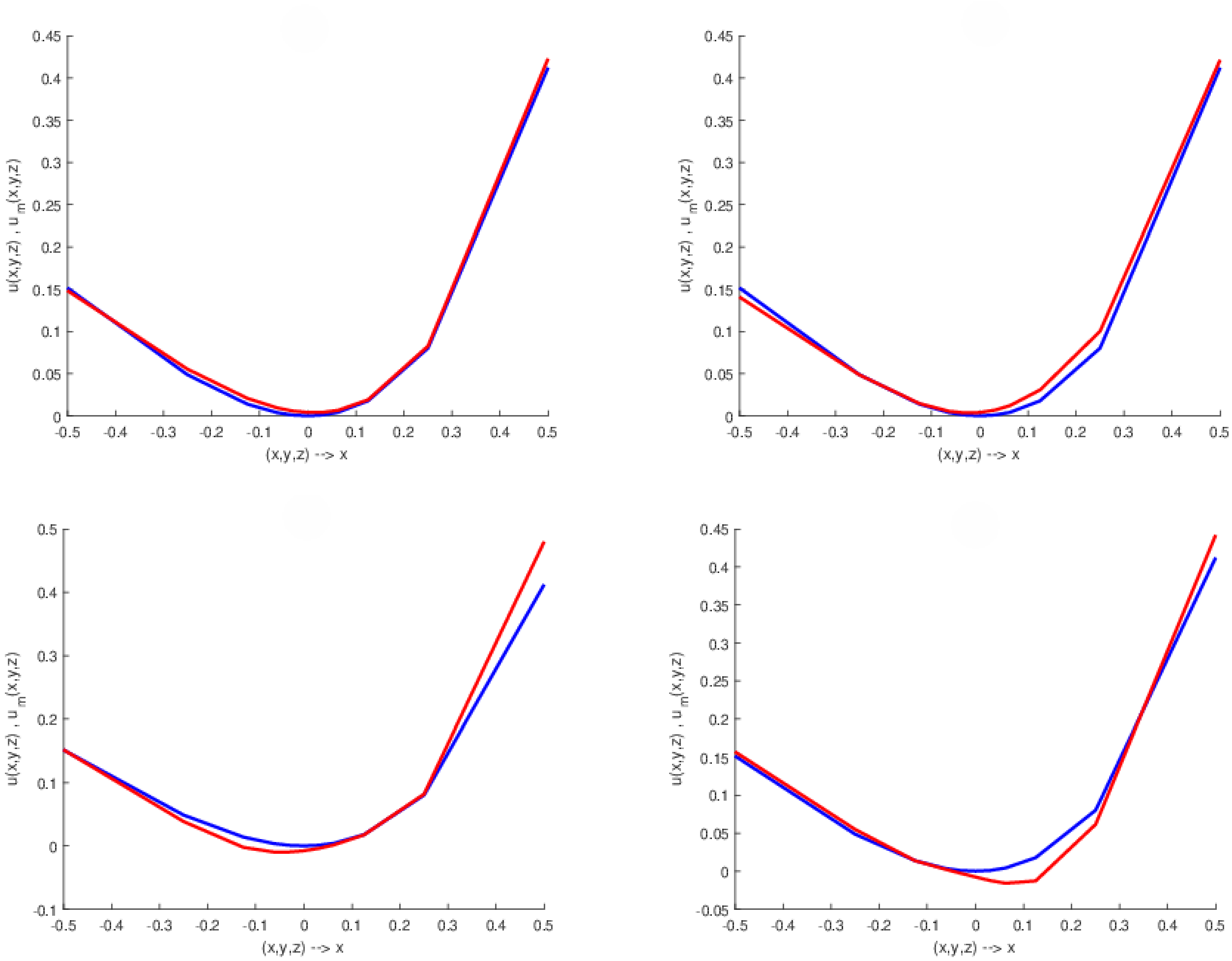

FIGURE 3. Graph of error function for Example 5.5 given (a) $m = 2$, (b) $m = 3$, (c) $m = 4$ and (d) $m = 5$, where the exact solution $u(x, y, z)$ with (blue color) and the numerical solution $u_m(x, y, z)$ with (red color).

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