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Efficient numerical solution of singularly perturbed two-point boundary value problems using double exponential non-classical sinc-collocation method

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Abstract

In this paper, we present the application of the double exponential non-classical sinc method for solving a specific class of singularly perturbed two-point boundary value problems. This method is particularly effective for problems with singularities, infinite domains, or boundary layers. We discuss the convergence and error estimation of our approach. Using three illustrative examples, we demonstrate the superior performance of our method compared to existing approaches.

Keywords. Non-classical, Sinc collocation method, Singularly perturbed, Boundary value problems, Convergence.
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1. Introduction

In this paper, the singularly perturbed singular two-point boundary value problems are addressed as follows:

$$\varepsilon y''(x) + \frac{\lambda}{x} y'(x) + b(x)y(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

with boundary conditions:

$$y(0) = \alpha, \quad y(1) = \beta, \tag{1.2}$$

where $0 < \varepsilon << 1$ is a perturbation parameter, $b(x)$, $f(x)$ are continuous functions in $(0,1)$, $f(x) > 0$, and λ, α, β are finite constants.

Such problems typically arise in various fields of science and engineering. For example, in disciplines such as quantum mechanics, fluid mechanics, aerodynamics, geophysics, and chemical reactor theory [12], etc. The key characteristic of these problems is the presence of the parameter ε, which multiplies some or all higher-order terms. The presence of ε makes the solution difficult, since as we decrease the parameter ε, the instability of the problem increases. Most developed methods for solving singular perturbation problems include collocation with polynomials and tension splines [5], a class of simple exponential B-spline [15], boundary value technique [14], continuous genetic algorithm [1], a new efficient high-order four-step multiderivative method [18], spectral poly-sinc collocation method [3], and numerical method for singular boundary-value problems [7], etc. Unfortunately, these methods are not applicable for solving singularly perturbed singular problems. In [6, 10, 11], the authors applied the numerical integration method, the compression method based on a class of variable mesh spline, and the double exponential sinc method, respectively.

In 1974, Takahasi and Mori introduced double exponential (DE) transformations for the numerical calculation of one-dimensional integrals [8]. The success of this method in numerical integration led researchers to explore its application in other numerical methods. In 1997, Sugihara applied this transformation in the sinc method and for n points obtained the convergence rate, which was much faster than the convergence rate $O(e^{-\frac{kn}{logn}})$ in a state that was

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used by typical mapping [19]. This transformation, when combined with the sinc method, has been increasingly used to solve various problems, as in [4].

In this paper, we use a double exponential non-classical sinc-collocation method for solving problem (1.1) with boundary conditions (1.2). The concept of using non-classical weight functions was initially employed by Shizgal to solve the Boltzmann equation and related problems [16], and Shizgal and Chen later utilized these bases to determine the eigenvalues and eigenfunctions of the Schrödinger equation [17], and Alipanah applied the non-classical pseudospectral method to solve singular boundary value problems in physiology [9].

The non-classical sinc-collocation method is known as a more efficient method for dealing with singularity problems because the singularity in this kind of problem occurs at the endpoints of the interval. Still, the sinc function does not need to be smooth at the endpoints [9]. In addition, another thing that distinguishes this method is that most of the sinc grid points are gathered near the endpoints of the interval, which helps control the singularity.

The paper is organized as follows: In section 2, we discuss some basic information about sinc function. In section 3, we explain non-classical sinc functions. In section 4, we discuss the error analysis of our method. Finally, we solve some examples to show the efficiency of our method.

2. Sinc functions

The sinc function is defined across the entire real line as

$$sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

With evenly spaced the transformed sinc functions nodes for $h > 0$ are defined as:

$$S(j,h)(x) = sinc\left(\frac{x - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \cdots.$$

The sinc functions are defined on, the infinite domain $D_s = \{w = u + iv : |v| < d \leq \pi/2\}$, whereas the Equation (1.1) is defined on the finite interval $(0, 1)$. To transform the finite interval $(0, 1)$ to the infinite interval, we apply the following conformal map:

$$\Phi_{DE}(w) = ln\left[\frac{1}{\pi} ln\left(\frac{w}{1-w}\right) + \sqrt{1 + \left(\frac{1}{\pi} ln\left(\frac{w}{1-w}\right)\right)^2}\right],$$

which maps the eye-shaped region

$$D_E = \left\{ w = x + iy : \left| arg\left[\frac{1}{\pi} ln\left(\frac{w}{1-w}\right) + \sqrt{1 + \left(\frac{1}{\pi} ln\left(\frac{w}{1-w}\right)\right)^2}\right]\right| < d \leq \pi/2 \right\},$$

onto D_s.

Now, on the finite interval, we obtain the sinc functions as follows

$$S_j(w) = S(j,h)o\Phi(w) = sinc\left(\frac{\Phi(w) - jh}{h}\right), \tag{2.1}$$

for $w \in D_E$. The function

$$w = \Phi^{-1}(z) = \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh(z)\right) + \frac{1}{2},$$

is inverse of $z = \Phi(w)$.

In $(0, 1)$, the sinc grid points are determined as follows

$$x_k = \Phi^{-1}(kh) = \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh(kh)\right) + \frac{1}{2}, \quad k = 0, \pm 1, \pm 2, \cdots. \tag{2.2}$$

Definition 2.1. The approximation of the function $y(x)$ using non-classical sinc functions over the interval $(0, 1)$ is defined as follows

$$y(x) \simeq \hat{y}(x) = \sum_{j=-m}^{n} \frac{U(x)}{U(x_j)} f(jh) sinc\left(\frac{\Phi(x) - jh}{h}\right), \tag{2.3}$$

where $U(x) > 0$, is the weight function. At the interpolation points $x_k = kh$ we have

$$\hat{y}(kh) = y(kh), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Definition 2.2. We denote $B(D_E)$ as the class of functions y that are analytic in D_E, and

$$\int_{\Psi(L+u)} |y(w)dw| \to 0, \quad u \to \pm\infty,$$

where $L = \{iv : |v| < d\}$ and on the boundary of D_E (denoted by ∂D_E)

$$N(f, D_E) \equiv \int_{\partial D_E} |y(w)dw| < \infty.$$

Theorem 2.3. *Assuming $\Phi' y \in B(D_E)$ and the weight function satisfies the condition $\frac{U(x)}{U(x_j)} < c_1$. Also suppose that constants β_1, β_2 are positive, and c such that*

$$|y(x)| \leq c \begin{cases} e^{-\beta_1 |\phi(x)|}, & x \in \Gamma_a, \\ e^{-\beta_2 |\phi(x)|}, & x \in \Gamma_b, \end{cases}$$

where

$$\Gamma_a = \{x \in \Gamma : \Phi(x) \in (-\infty, 0)\}, \tag{2.4}$$

and

$$\Gamma_b = \{x \in \Gamma : \Phi(x) \in [0, \infty)\}. \tag{2.5}$$

Then $\forall x \in \Gamma = \{w \in \mathbb{C} : w = \Psi(v), v \in \mathbb{R}\}$ we have, [9]

$$\left| y(x) - \sum_{j=-m}^{n} y(x_j) \frac{U(x)}{U(x_j)} sinc\left(\frac{\Phi(x) - jh}{h}\right) \right| \leq c_1 k_1 m^{1/2} e^{-\sqrt{\pi d_1 \alpha m}}. \tag{2.6}$$

In which the mesh-size h is chosen as:

$$h = \sqrt{\frac{\pi d_1}{\beta_1 m}}, \tag{2.7}$$

and

$$n = \left[\left| \frac{\beta_1}{\beta_2} m \right| \right]. \tag{2.8}$$

Lemma 2.4. *Suppose that Φ is a one-to-one conformal mapping from the simply connected domain D_E onto D_s, then*

$$\delta_{j,k}^{(0)} = [S(j,h) o \Phi(x)]\Big|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.9}$$

$$\delta_{j,k}^{(1)} = h \frac{\lambda}{d\Phi} [S(j,h) o \Phi(x)]\Big|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{2.10}$$

$$\delta_{j,k}^{(2)} = h^2 \frac{d^2}{d\Phi^2} [S(j,h) o \Phi(x)]\Big|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases} \tag{2.11}$$

3. The non-classical DE-sinc-collocation method

At the boundary conditions, the transformed sinc functions $S_k(x)$ are not differentiable. To address this difficulty, we use the non-classical transformed sinc functions [2], specified as follows:

$$\frac{U(x)}{U(x_j)}S_j(x). \tag{3.1}$$

Equation (3.1) must be vanished at the boundary point $x=0$, hence, the weight functions $x, x(1-x),\cdots$ are used. Consequently

$$\lim_{x\to 0}\frac{U(x)}{U(x_j)}S_j(x)=\lim_{x\to 0}\left(\frac{U(x)}{U(x_j)}S_j(x)\right)'=0.$$

Therefore, they are well-defined and differentiable at zero. To approximate the exact solution $y(x)$ by using the new bases introduced in (3.1), we define $u(x)$ as

$$u(x)=\sum_{i=-m}^{n}c_i\frac{U(x)}{U(x_i)}S_i(x), \tag{3.2}$$

where c_i are unknown constants to be determined. The approximate solution of the problem requires the addition of the following polynomial since at the boundary $x=0$, the non-classical sinc basis functions are zero. Let

$$V(x)=\alpha\,\psi_1(x)+\beta\,\psi_2(x)+c_{-m-1}\,\psi_3(x)+c_{n+1}\,\psi_4(x), \tag{3.3}$$

where $\psi_i(x)\quad i=1,\cdots,4$ are boundary basis polynomials which are obtained from Hermite interpolation at points 0 and 1 [9]. And the polynomials $\psi_i(x)$ are as follows:

$$\begin{aligned}
\psi_1(x)&=(2x+1)(1-x)^2,\\
\psi_2(x)&=x(1-x)^2,\\
\psi_3(x)&=x^2(3-2x),\\
\psi_4(x)&=x^2(x-1).
\end{aligned}$$

Now, we can define the new approximate solution as

$$y^*(x)=u(x)+V(x). \tag{3.4}$$

The new approximate solution satisfies the boundary conditions (1.2) as follows

$$y^*(0)=u(0)+V(0)=\alpha,\quad y^*(1)=u(1)+V(1)=\beta.$$

Substituting $y^*(x)$ into (1.1), multiplying both sides by $\frac{h^2}{\Phi'^2}$ and discretizing the result at the sinc grid points x_k, $k=-m,\cdots,n$, we have

$$\begin{aligned}
&h^2\sum_{j=-m}^{n}\left(\varepsilon\omega_{2,j}(x_k)+\frac{\lambda}{x_k\Phi'(x_k)}\omega_{1,j}(x_k)+\frac{b(x_k)}{\Phi'(x_k)^2}\omega_{0,j}(x_k)\right)c_j\\
&+\frac{h^2}{\Phi'(x_k)^2}\left(\varepsilon V''(x_k)+\frac{\lambda}{x_k}V'(x_k)+b(x_k)V(x_k)\right)=\frac{h^2}{\Phi'(x_k)^2}f\left(x_k\right),\quad k=-m-1,\cdots,n,
\end{aligned} \tag{3.5}$$

where

$$\omega_{m,j}(x)=\frac{1}{(\Phi'(x))^m}\frac{d^m}{dx^m}\left(\frac{U(x)}{U(x_j)}S_j(x)\right), \tag{3.6}$$

$$\omega_{0,j}(x)=\frac{U(x)}{U(x_j)}S_j(x). \tag{3.7}$$

Simplifying (3.5) we get the following system with the unknowns $c_j, j=-m-1,\cdots,n$,

$$\sum_{j=-m}^{n}\left[\varepsilon\left(\frac{U(x_k)}{U(x_j)}\right)\delta_{jk}^{(2)}+h\left(\frac{U}{U(x_j)}\left(\frac{\lambda}{x_k\Phi'}+\frac{\varepsilon\Phi''}{\Phi'^2}\right)+\frac{U'}{U(x_j)}\left(\frac{2\varepsilon}{\Phi'}\right)\right)(x_k)\delta_{jk}^{(1)}\right.$$
$$\left.+h^2\left(\frac{U}{U(x_j)}\frac{b}{\Phi'^2}+\frac{U\prime}{U(x_j)}\frac{\lambda}{x_k\Phi'^2}+\frac{U''}{U(x_j)}\left(\frac{\varepsilon}{\Phi'^2}\right)\right)(x_k)\delta_{jk}^{(0)}\right]c_j$$
$$+\frac{h^2}{\Phi'(x_k)^2}\left(\varepsilon V''(x_k)+\frac{\lambda}{x_k}V'(x_k)+b(x_k)V(x_k)\right)$$
$$=\frac{h^2}{\Phi'(x_k)^2}f\left(x_k\right),\quad k=-m-1,\cdots,n, \tag{3.8}$$

where we used $c_{-m-1}=0$. Since our system is too big and full, we cannot use direct methods. For this reason, we are solving the linear system of Equations (3.8) using the iterative method, and the unknowns c_j, and $y^*(x)$ are obtained.

4. Error analysis

Suppose that the solution $y\in B(D_E)$ of Equations (1.1) and (1.2) are uniquely defined, and $\left\{\frac{\lambda}{x\phi'},\frac{b}{\phi'^2}\right\}\in B(D_E)$ [6]. Now, we can write the system of Equations (3.8) in matrix form. Assume $I^{(n)}, n=0,1,2$ is the matrix $I^{(n)}=[\delta_{kj}^{(n)}]$, where $\delta_{kj}^{(n)}$ is (k,j)th element. Obviously, from (1.2) we get that

$$V(x)=\alpha(2x+1)(1-x)^2+\beta x^2(3-2x)+y'(0)x(1-x)^2+y'(1)x^2(x-1). \tag{4.1}$$

Now the system (3.8) can be expressed as follows

$$\mathbf{A}\mathbf{c}=\mathbf{q}, \tag{4.2}$$

where the matrix $\mathbf{A}$ and vectors $\mathbf{c}$ and $\mathbf{q}$ are

$$\mathbf{c}=[0,c_{-m},\cdots,c_n,0],$$
$$\mathbf{A}=\left[-\varepsilon\left(D(U)\right)I^{(2)}+hD\left(U\left(\frac{\lambda}{x\Phi'}+\frac{2\varepsilon\Phi''}{\Phi'^2}\right)+\varepsilon U'\right)I^{(1)}-h^2D\left(U\frac{b}{\Phi'^2}+U'\frac{\lambda}{x\Phi'^2}+U''\frac{1}{\Phi'^2}\right)I^{(0)}\right]\mathbf{g},$$
$$\mathbf{g}=\left[\frac{1}{U(x_{-m-1})},\ldots,\frac{1}{U(x_{n+1})}\right]^T,$$
$$\mathbf{q}=-h^2D\left(\frac{1}{\phi'^2}\right)\mathbf{V},$$
$$\mathbf{V}=\left[\varepsilon V''(x_{-m-1})+\frac{\lambda}{x_{-m-1}}V'(x_{-m-1})+bV(x_{-m-1}),\ldots,\varepsilon V''(x_{n+1})+\frac{\lambda}{x_n}V'(x_{n+1})+bV(x_n)\right]^T,$$
$$\mathbf{f}=\left[f(x_{-m-1}),\ldots,f(x_{n+1})\right].$$

In order to determine a bound on the error $|y(x)-y^*(x)|$, we need to get a bound on $||\mathbf{A}\hat{\mathbf{y}}-\mathbf{q}||$ where

$$\hat{\mathbf{y}}=\left[\hat{y}_{-m-1},\ldots,\hat{y}_{n+1}\right]^T,$$

with

$$\hat{y}_i=(y-V)(x_i).$$

Lemma 4.1. *Suppose that $\phi' y\in B(D_E)$ and β_1, β_2 are two positive constants, and c is a constant, then we have*

$$\left|\hat{(y)}(x)\right|=\left|\left(y-V\right)(x)\right|\le c\begin{cases}e^{-\beta_1 e^{|\Phi(x)|}}, & x\in\Gamma_a,\\ e^{-\beta_2 e^{|\Phi(x)|}}, & x\in\Gamma_b,\end{cases} \tag{4.3}$$

where Γ_a *and* Γ_b *are selected as (2.4) and (2.5). Assume that* h *and* N *are the same as (2.7) and (2.8). let the weight function* U *be defined in a way that* $\max_{s\in\{0,1,2\}} \frac{U^{(s)}(x)}{U(x_j)} < c_1$, *where* $U^{(s)}$ *is the order derivative of sth. Then we have*

$$||\boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{q}|| \leq c_1 k_2 m^{\frac{1}{2}} e^{-\sqrt{\pi d\alpha m}}, \tag{4.4}$$

where $\boldsymbol{A}$, $\hat{\boldsymbol{x}}$ *and* $\boldsymbol{q}$ *were defined before.*

Proof. Suppose that the nuclei $G_n, n = 0, 1, 2$ defined as

$$G_n(x,z) = \frac{1}{2\pi i(\Phi')^n}\frac{\partial^n}{\partial x^n}\left(\frac{\sin\left(\frac{\pi\Phi(x)}{h}\right)U(x)}{(\Phi(w)-\Phi(x))U(z)}\right). \tag{4.5}$$

The series expansion for $\hat{y}(x)$ can be written as

$$y(x) - \sum_{j=-\infty}^{\infty}\omega_{0,j}(x)\hat{y}(x_j) = \int_{\partial D}\frac{G_0(x,z)}{\sin\left(\frac{\pi\Phi(z)}{h}\right)}\Phi'(z)\hat{y}(z)dz. \tag{4.6}$$

So, the approximation error for the derivatives of $y(x)$ with non-classical sinc functions is equal to

$$\frac{d^n}{dx^n}y(x) - \sum_{j=-\infty}^{\infty}(\Phi'(x))^n\omega_{n,j}(x)\hat{y}(x_j) = \int_{\partial D}\frac{(\Phi'(x))^n G_n(x,z)}{\sin\left(\frac{\pi\Phi(z)}{h}\right)}\Phi'(z)\hat{y}(z)dz, \quad n = 0,1,2. \tag{4.7}$$

Let r_k denote the kth component of the residual vector $\mathbf{r} = \mathbf{A}\hat{\mathbf{y}} - \mathbf{q}$. Then, replacing c_j with $\hat{y}(x_j)$ in (3.5) and by subtracting from $\frac{h^2}{\Phi'^2}(Ly - f)$ we have

$$\begin{aligned}
r_k &= \left\{\mathbf{A}\hat{\mathbf{y}} - \mathbf{q} - \frac{h^2}{\Phi'(x)^2}(Ly - f)\right\}_k \\
&= h^2\sum_{j=-m}^{n}\left(\varepsilon\omega_{2,j}(x_k) + \frac{\lambda}{x_k\Phi'(x_k)}\omega_{1,j}(x_k) + \frac{b(x_k)}{\Phi'(x_k)^2}\omega_{0,j}(x_k)\right)\hat{y}(x_j) \\
&- \frac{h^2}{\Phi'(x_k)^2}f(x_k) + \frac{h^2}{\Phi'(x_k)^2}\left(\varepsilon V''(x_k) + \frac{\lambda}{x_k}V'(x_k) + b(x_k)V(x_k)\right) \\
&- \frac{h^2}{\phi'(x_k)^2}\left(\varepsilon y''(x_k) + \frac{\lambda}{x_k}y'(x_k) + b(x_k)y(x_k)\right) + \frac{h^2}{\phi'(x_k)^2}f(x_k) \\
&= r_k^{(1)} + r_k^{(2)} + r_k^{(3)},
\end{aligned}$$

where

$$\begin{aligned}
r_k^{(1)} &= -\frac{h^2}{\phi'(x_k)^2}\Big(\varepsilon y''(x_k) + \frac{\lambda}{x_k}y'(x_k) + b(x_k)y(x_k) - \varepsilon v''(x_k) - \frac{\lambda}{x_k}v'(x_k) - b(x_k)v(x_k)\Big) \\
&+ h^2\sum_{j=-m}^{n}\left(\varepsilon\omega_{2,j}(x_k) + \frac{\lambda}{x_k\Phi'(x_k)}\omega_{1,j}(x_k) + \frac{b(x_k)}{\Phi'(x_k)^2}\omega_{0,j}(x_k)\right)\hat{y}(x_j) \\
&- \frac{h^2}{\Phi'(x_k)^2}\left(\varepsilon\hat{y}''(x_k) + \frac{\lambda}{x_k}\hat{y}'(x_k) + b(x_k)\hat{y}(x_k)\right) \\
&= h^2\sum_{j=-m}^{n}\left(\varepsilon\omega_{2,j}(x_k) + \frac{\lambda}{x_k\Phi'(x_k)}\omega_{1,j}(x_k) + \frac{b(x_k)}{\Phi'(x_k)^2}\omega_{0,j}(x_k)\right)\hat{y}(x_j) \\
&- h^2\int_{\partial D_E}\left(\varepsilon G_2(x_k,z) + \frac{\lambda}{x_k\Phi'(x_k)}G_1(x_k,z) + \frac{b(x_k)}{\Phi'(x_k)^2}G_0(x_k,z)\right)\frac{\Phi'(z)\hat{y}(z)}{\sin\left(\frac{\pi\Phi(z)}{h}\right)}dz,
\end{aligned} \tag{4.8}$$

$$r_k^{(2)} = -h^2 \sum_{j=-\infty}^{-m-1} \left(\varepsilon \omega_{2,j}(x_k) + \frac{\lambda}{x_k \Phi'(x_k)} \omega_{1,j}(x_k) + \frac{b(x_k)}{\Phi'(x_k)^2} \omega_{0,j}(x_k) \right) \hat{y}(x_j)$$

$$-h^2 \sum_{j=n}^{\infty} \left(\varepsilon \omega_{2,j}(x_k) + \frac{\lambda}{x_k \Phi'(x_k)} \omega_{1,j}(x_k) + \frac{b(x_k)}{\Phi'(x_k)^2} \omega_{0,j}(x_k) \right) \hat{y}(x_j),$$

$$r_k^{(3)} = \frac{h^2}{\Phi'(x_k)^2} \left(f(x_k) - f(x_k) \right).$$

From (4.5) we can get that

$$G_0(x_k, z) = 0, \quad G_1(x_k, z) = \frac{(-1)^k U(x_k)}{2ih(\Phi(z) - kh)U(z)},$$

$$G_2(x_k, z) = \frac{(-1)^k}{2ih(\Phi(z) - kh)^2} \left(2 - (\Phi(z) - kh) \left(\frac{1}{\Phi'} \right)' (x_k) \right) \frac{U(x_k)}{U(z)} + \frac{(-1)^k U'(x_k)}{2ih(\Phi(z) - kh)U(z)} \left(\frac{2}{\Phi'} \right) (x_k).$$

Now since $|I\Phi(z)| = d$ on ∂D_E, setting $u(z) = R\Phi(z)$ and using bounds on $\frac{1}{\Phi'}$ and its derivatives and lemmas assumptions on w we have,

$$|G_1(x_k, z)| \le \frac{c_1 c_1' h^{-1}}{\left((u(z) - kh)^2 + d^2\right)^{\frac{1}{2}}}, \quad |G_2(x_k, z)| \le \frac{c_1 c_2' h^{-1}}{\left((u(z) - kh)^2 + d^2\right)^{\frac{1}{2}}},$$

which results

$$h^2 \left| \varepsilon G_2(x_k, z) + \frac{\lambda}{x_k \phi'(x_k)} G_1(x_k, z) + \frac{b(x_k)}{\phi'(x_k)^2} G_0(x_k, z) \right| \le \frac{c_1 c_2 h}{\left((u(z) - kh)^2 + d_1^2\right)^{\frac{1}{2}}}, \tag{4.9}$$

where c_2 is a constant depending on h, d_1, ε, and the coefficients bonds of the differential equation and bounds on $\frac{1}{\Phi'}$, and its derivatives. Then, we obtain

$$||\mathbf{A}\hat{\mathbf{y}} - \mathbf{q}|| = \left(\sum_{k=-m}^{n} |r_k|^2 \right)^{\frac{1}{2}} \le \left(\sum_{k=-m}^{n} |r_k^{(1)}|^2 \right)^{\frac{1}{2}} + \left(\sum_{k=-m}^{n} |r_k^{(2)}|^2 \right)^{\frac{1}{2}} + \left(\sum_{k=-m}^{n} |r_k^{(3)}|^2 \right)^{\frac{1}{2}}. \tag{4.10}$$

On the right-hand side, for the first term, we have

$$\sum_{k=-m}^{n} |r_k^{(1)}|^2 \le \sum_{k=-m}^{n} \left| \int_{\partial D_E} \frac{c_1 c_2 h}{\left((u(z) - kh)^2 + d^2\right)^{\frac{1}{2}}} \frac{|\Phi'(z) \hat{y}(z)}{\left| \sin\left(\frac{\pi \Phi(z)}{h}\right) \right|} |dz| \right|^2 \le \frac{c_3 c_1^2}{\left(\sinh\left(\frac{\pi d_1}{h}\right)\right)^2}, \tag{4.11}$$

that is gotten by using (4.9), the bound $\sinh\left(\frac{\pi d_1}{h}\right) \le \sin\left(\frac{\pi \Phi(z)}{h}\right)$ on ∂D_E and the integrability of $|\hat{y}\phi'|$. As for the second term, using (4.3) and the supposition on the coefficients of the differential equation and bounds on $\frac{1}{\Phi'}$ and its derivatives we obtain

$$\begin{aligned} \sum_{k=-m}^{n} \left| r_k^{(2)} \right|^2 = \sum_{k=-m}^{n} \Bigg| \sum_{j<-m, j>n} \Bigg[& \varepsilon \left(\frac{U(x_k)}{U(x_j)} \right) \delta_{jk}^{(2)} \\ & + h \left(\frac{U}{U(x_j)} \left(\frac{\lambda}{x_k \Phi'} + \frac{\varepsilon \Phi''}{\Phi'^2} \right) + \frac{U'}{U(x_j)} 2\varepsilon \right) (x_k) \delta_{jk}^{(1)} \\ & + h^2 \left(\frac{U}{U(x_j)} \frac{b}{\Phi'^2} + \frac{U'}{U(x_j)} \frac{\lambda}{x_k \phi'^2} + \frac{U''}{U(x_j)} \left(\frac{\varepsilon}{\Phi'^2} \right) \right) (x_k) \delta_{jk}^{(0)} \Bigg] \hat{y}(x_j) \Bigg|^2 \end{aligned}$$

$$\leq c_1^2 c_3' \sum_{k=-m}^{n} \left(\sum_{j<-m, j>n} \gamma_{kj}^2 \sum_{j<-m, j>n} |\hat{y}(x_j)|^2 \right) \leq \frac{c_1^2 c_4}{h^2} e^{-2\beta_1 e^{mh}}, \tag{4.12}$$

where γ_{kj} is defined by

$$\gamma_{kj} = \max\left\{ |\delta_{kj}^{(0)}|, |\delta_{kj}^{(1)}|, |\delta_{kj}^{(2)}| \right\}.$$

For the function $f(x)$, we have

$$\sum_{k=-m}^{n} \left| r_k^{(3)} \right|^2 = 0. \tag{4.13}$$

Therefore, by using (4.10), (4.11), (4.12), and (4.13), we get that

$$||\mathbf{A}\hat{\mathbf{y}} - \mathbf{q}|| \leq c_1 k_2 m^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 m}}.$$

□

Theorem 4.2. *Assume that y and y^* are the exact and approximate solutions of Equation (1.1), respectively. And suppose that $\mathbf{c} = [0, c_{-m}, \cdots, c_n]$ is the solution of (4.2). If all assumptions of Lemma 4.1 are held, then*

$$\left| y(x) - y^*(x) \right| \leq c_1 \left(k_1 + \rho k_3 \right) e^{-\sqrt{\pi d_1 \beta_1 m}}.$$

Proof. Let the function η^* be defined by

$$\eta^*(x) = \sum_{j=-m}^{n} \frac{U(x)}{U(x_j)} \hat{y}(x_j) S_j(x).$$

From (3.4), we have

$$y(x) - y^*(x) = y(x) - y^*(x) + \eta^*(x) - \eta^*(x) = y(x) - u(x) - v(x) + \eta^*(x) - \eta^*(x),$$

now using $\hat{y}(x) = y(x) - V(x)$ we get

$$y(x) - y^*(x) = \left(\hat{y}(x) - \eta^*(x) \right) + \left(\eta^*(x) - u(x) \right).$$

Now, by using the triangle inequality on the above relation, we can obtain that

$$\left| y(x) - y^*(x) \right| \leq \left| \hat{y}(x) - \eta^*(x) \right| + \left| \eta^*(x) - u(x) \right|. \tag{4.14}$$

By Theorem 2.3, we get

$$\sup_{x \in \Gamma} \left| \hat{y}(x) - \eta^*(x) \right| \leq c_1 k_1 m^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 m}}. \tag{4.15}$$

Now, we can find a bound on the right-hand side second term of (4.14) as follows

$$\left| \eta^*(x) - u(x) \right| = \left| \sum_{j=-m}^{n} \frac{U(x)}{U(x_j)} \left(\hat{y}(x_j) - c_j \right) S_j(x) \right|,$$

then by applying the Cauchy-Schwartz inequality, we have

$$\left| \eta^*(x) - u(x) \right| \leq \left(\sum_{j=-m}^{n} \left| \hat{y}(x_j) - c_j \right|^2 \right)^{\frac{1}{2}} \left(\sum_{j=-m}^{n} \left| \frac{U(x)}{U(x_j)} S_j(x) \right|^2 \right)^{\frac{1}{2}} \leq c_1 \left(\sum_{j=-m}^{n} \left| \hat{y}(x_j) - c_j \right|^2 \right)^{\frac{1}{2}} = c_1 \left|\left| \hat{\mathbf{y}} - \mathbf{c} \right|\right|.$$

Now, from (4.4) and (4.15), we have

$$||\hat{\mathbf{y}} - \mathbf{c}|| = \left|\left| \mathbf{A}^{-1} (\mathbf{A}\hat{\mathbf{y}} - \mathbf{q}) \right|\right| \leq \left|\left| \mathbf{A}^{-1} \right|\right| \, ||\mathbf{A}\hat{\mathbf{y}} - \mathbf{q}|| \leq c_1 \rho k_3 m^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 m}}, \tag{4.16}$$

where $\rho = ||\ \mathbf{A}^{-1}||$. Combining (4.15) with (4.16), we have

$$\left| y(x) - y^*(x) \right| \leq c_1 \left(k_1 + \rho k_3 \right) m^{\frac{1}{2}} e^{-\sqrt{\pi d_1 \beta_1 m}}.$$

□

According to Tables 6 and 12, the value of $||A^{-1}||$ is not very small. Therefore, we can conclude that our system is convergent.

5. Numerical Examples

In this section, we use the non-classical sinc-collocation method for three different examples [6]. In all examples, we assume that $\beta_1 = \beta_2 = \frac{1}{2}$ and $d_1 = \frac{\pi}{2}$.

For our method, the relative errors, can be given as

$$\max_{0<i<100} \frac{|y^*(x_i) - y(x_i)|}{|y(x_i)|},$$

and for obtaining the relative errors we used Maple 12 with digits equal to 40.

Example 5.1. Consider the following singular second-order boundary value problem:

$$-\varepsilon y'' + \frac{1}{x}y' + (1+x^2)y = (1-4\varepsilon)x^2 e^{x^2} + (3-2\varepsilon)e^{x^2}, \tag{5.1}$$

with the boundary conditions:

$$y(0) = 1 \quad \text{and} \quad y(1) = e^1, \tag{5.2}$$

with the exact solution $y(x) = e^{x^2}$. The relative errors of the presented method with different weights and $\varepsilon = 10^{-20}$ are revealed in Table 1 and for different weights and different ϵ in Table 3. Also, the condition numbers with different weights and $\varepsilon = 10^{-20}$ are revealed in Table 2, and for different weights and different ϵ in Table 4. We conclude that by increasing the number M, the condition numbers do not increase sharply.

Table 1. The relative errors for Example 5.1 with $\varepsilon = 10^{-20}$.

M	$U=1$	$U=x$	$U=x^2$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$	$U=xe^{-x^2}$
5	5.51E-5	1.01E-4	1.14E-4	4.01E-4	2.05E-4	7.91E-5	6.81E-5
10	3.57E-7	1.94E-7	3.29E-5	9.71E-7	2.94E-7	2.22E-6	2.00E-6
15	6.96E-9	8.66E-10	8.16E-7	2.48E-9	3.44E-9	4.82E-8	4.33E-8
20	3.84E-11	4.37E-11	2.68E-9	1.10E-10	1.02E-10	7.44E-10	8.57E-10

Table 2. The condition number for Example 5.1 with $\varepsilon = 10^{-20}$.

M	$U=1$	$U=x$	$U=x^2$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$	$U=xe^{-x^2}$
5	75.70960E5	14.90079E7	13.48664E8	59.45092E6	30.28740E7	33.10532E6	11.54977E7
10	26.35519E11	12.13343E14	93.45871E18	24.79420E15	11.70381E14	16.04732E14	10.18425E14
15	35.27975E16	13.00183E22	28.49029E28	54.77568E24	12.50379E22	17.54464E22	10.64140E22
20	19.80183E22	13.64159E30	28.57301E37	57.04349E33	13.11142E30	16.26341E30	10.96133E30

Table 3. The relative errors for Example 5.1 with $M = 20$.

ε	$U=1$	$U=x$	$U=xe^{x^2}$	$U=xe^{-x^2}$
10^{-5}	9.74E-10	4.43E-11	5.04E-9	8.75E-10
10^{-10}	1.09E-10	4.40E-11	1.86E-9	8.27E-10
10^{-15}	3.84E-11	4.37E-11	7.44E-10	8.57E-10
10^{-20}	3.84E-11	4.37E-11	7.44E-10	8.57E-10

TABLE 4. The condition number for Example 5.1 with $M = 20$.

ε	$U=1$	$U=x$	$U=xe^{x^2}$	$U=xe^{-x^2}$
10^{-1}	15.36292E12	32.95598E22	63.31165E22	94.51071E21
10^{-5}	28.39271E15	93.50583E22	25.18925E23	83.35720E22
10^{-10}	19.05625E20	93.33739E27	15.70809E28	78.02103E27
10^{-15}	19.82249E22	13.68908E30	16.32018E30	10.99950E30
10^{-20}	19.80183E22	13.64159E30	16.26341E30	10.96133E30

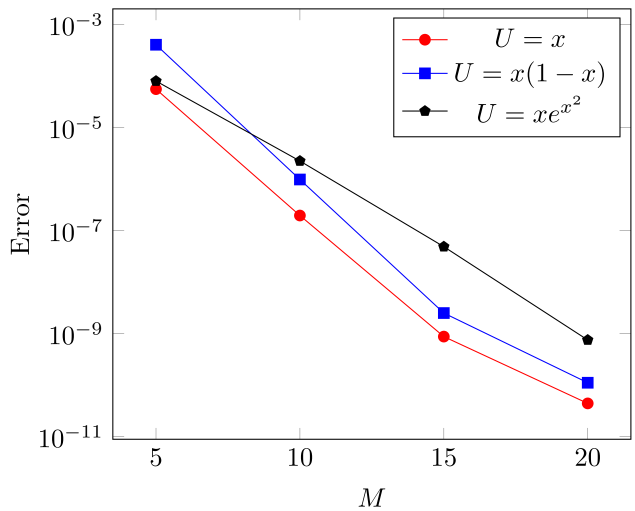

FIGURE 1. Relative error for Example 5.1 for $U = x, U = x(1-x)$ and $U = xe^{x^2}$.

TABLE 5. Comparison between the relative error of our method and the method given in [6] for the Example 5.1.

x_k	Our Method	Method in [6]
0.01	1.40E-12	2.96E-4
0.02	3.30E-12	2.95E-4
0.03	3.09E-11	2.95E-4
0.1	1.91E-12	2.90E-4
0.2	2.83E-11	2.75E-4
0.3	2.24E-11	2.71E-4
0.96	9.48E-12	1.08E-4
0.97	9.46E-11	1.06E-4
0.98	8.84E-11	9.47E-5
0.99	1.55E-11	6.52E-5

Example 5.2. Consider the following singular second order boundary value problem:

$$-\varepsilon y'' + \frac{1}{x}y' = \left(-\varepsilon x + \frac{1}{x}\right)\sinh(x) + (1-2\varepsilon)\cosh(x), \tag{5.3}$$

with the boundary conditions:

$$y(0) = 0, \quad \text{and} \quad y(1) = \sinh(1), \tag{5.4}$$

Table 6. The $||A^{-1}||$ for Example 5.1 with $\varepsilon = 10^{-20}$.

M	$U=1$	$U=x$	$U=x^2$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$	$U=xe^{-x^2}$
5	27.42803E5	48.60486E5	34.54896E4	27.10031E5	10.15360E6	86.97636E4	43.16130E5
10	62.12743E10	18.13806E10	15.88346E10	58.40592E11	18.14930E10	18.14895E10	18.31541E10
15	66.98635E15	65.32627E15	30.83594E14	44.54835E18	65.31995E15	65.67539E15	65.09693E15
20	32.58815E21	20.40141E21	18.35840E18	13.99125E25	20.41143E21	17.97981E21	20.08085E21

the exact solution to the problem is $y(x) = x\sinh(x)$. The relative errors of our method with $\varepsilon = 10^{-20}$ and different weights are revealed in Table 7, and for different weights and different ϵ in Table 9. Also, the condition numbers with different weights and $\varepsilon = 10^{-20}$ are revealed in Table 8, and for different weights and different ϵ in Table 10. We conclude that by increasing the number M, the condition numbers do not increase sharply.

Table 7. The relative errors for Example 5.2 with $\varepsilon = 10^{-20}$.

M	$U=1$	$U=x$	$U=x^2$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$	$U=xe^{-x^2}$
5	1.27E-4	6.16E-5	4.63E-4	4.36E-3	4.61E-5	4.43E-4	1.32E-4
10	3.39E-4	5.54E-7	2.23E-4	1.19E-5	6.98E-6	1.06E-4	4.53E-5
15	6.93E-7	3.33E-8	3.28E-6	9.66E-8	9.33E-8	1.84E-6	6.80E-7
20	1.90E-9	1.57E-10	7.86E-9	2.37E-11	5.57E-11	6.80E-9	3.64E-9

Table 8. The condition number for Example 5.2 with $\varepsilon = 10^{-20}$.

M	$U=1$	$U=x$	$U=x^2$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$	$U=xe^{-x^2}$
5	21.46933E5	61.12036E5	52.68340E6	17.48515E5	44.61449E5	34.38492E5	24.31101E5
10	37.41427E10	80.07385E10	95.57743E14	19.10890E11	80.06047E10	81.13816E10	80.68151E10
15	40.99599E15	30.12971E16	71.36209E21	10.10127E17	30.11672E16	30.51115E16	29.99220E16
20	52.37252E20	94.61388E21	68.90019E28	38.68657E22	94.64009E21	83.54584E21	93.09428E21

Table 9. The relative errors for Example 5.2 with $M = 20$.

ε	$U=1$	$U=x$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$	
10^{-5}	4.89E-7	1.68E-10	8.70E-10	9.15E-11	1.32E-6	
10^{-10}	2.36e-9	1.60e-10	2.11E-10	6.30E-11	3.95E-8	
10^{-15}	1.89E-9	1.57E-10	2.36E-11	5.57E-11	6.80E-9	
10^{-20}	1.90E-9	1.57E-10	2.37E-11	5.57E-11	6.80E-9	

Table 10. The condition number for Example 5.2 with $M = 20$.

ε	$U=1$	$U=x$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$
10^{-1}	38.39574E20	73.18752E26	17.30204E14	54.40996E26	12.57436E27
10^{-5}	25.32654E20	22.34117E24	50.07911E18	18.75975E24	12.18753E25
10^{-10}	33.78915E20	41.72363E20	77.67836E22	38.04005E20	11.26861E21
10^{-15}	52.39697E20	94.72969E21	38.68692E22	94.75593E21	83.64889E21
10^{-20}	52.37252E20	94.61388E21	38.68657E22	94.64009E21	83.54584E21

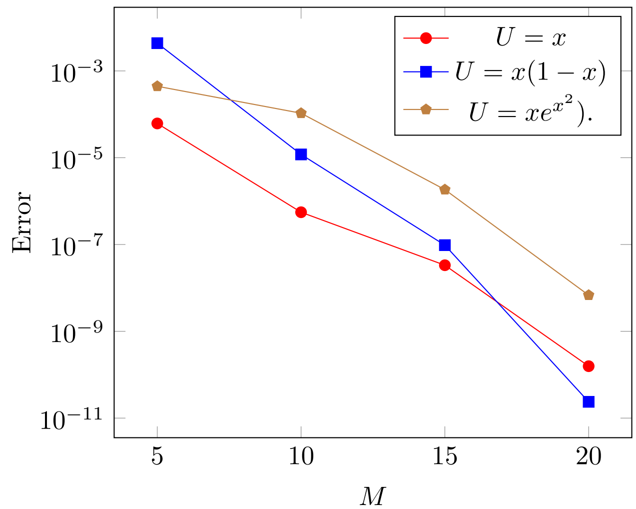

FIGURE 2. Relative error for Example 5.2 for $U = x, U = x(1-x)$ and $U = xe^{x^2}$.

TABLE 11. Comparison between the relative error of our method and the method given in [6] for the Example 5.2.

x_k	Our Method	Method in [6]
0.01	9.00E-11	1.98E-2
0.02	7.00E-11	5.14E-1
0.03	2.82E-10	2.39E-1
0.1	1.21E-10	3.96E-2
0.2	8.89E-11	2.49E-2
0.3	1.26E-10	2.28E-2
0.96	1.00E-10	2.83E-3
0.97	9.62E-11	2.60E-3
0.98	1.37E-10	2.17E-3
0.99	1.22E-10	1.43E-3

TABLE 12. The $||A^{-1}||$ for Example 5.2 with $\varepsilon = 10^{-20}$.

M	$U=1$	$U=x$	$U=x^2$	$U=x(1-x)$	$U=sin(x)$	$U=xe^{x^2}$	$U=xe^{-x^2}$
5	19.55331E3	31.86158E3	61.60151E2	93.62503E2	23.32610E3	17.16164E3	12.81550E5
10	76.91374E5	72.35732E5	63.21751E5	17.34948E6	72.40360E5	72.32547E5	73.11254E5
15	20.60600E8	57.97620E8	27.86678E7	19.47865E9	57.97443E8	58.29597E8	57.79063E8
20	63.52512E10	40.12068E11	15.22061E9	16.42567E12	40.14178E11	35.26830E11	39.50941E11

Example 5.3. Consider the following singular second-order boundary value problem:

$$\varepsilon y'' + \frac{1}{x} y' = -y, \tag{5.5}$$

with the boundary conditions:

$$y(0) = 1 \quad \text{and} \quad y(1) = e^{\frac{-1}{2}}. \tag{5.6}$$

The approximate solution of Example 5.3 with two different weights $U(x) = x, sin(x)$ and $\varepsilon = 10^{-20}$ with $M = 5$, $M = 10$, $M = 15$ and $M = 20$ are revealed in Tables 13 and 14, respectively.

TABLE 13. Numerical results for Example 5.3 with $U(x) = x$ and $\epsilon = 10^{-20}$.

x_k	$M = 5$	$M = 10$	$M = 15$	$M = 20$
0.1	0.9950140400	0.9950124927	0.9950124790	0.9950124791
0.2	0.9802036770	0.9801986535	0.9801986733	0.9801986733
0.3	0.9560187299	0.9559974946	0.9559974817	0.9559974818
0.4	0.9231194121	0.9231163547	0.9231163465	0.9231163463
0.5	0.8824847888	0.8824968872	0.8824969024	0.8824969025
0.6	0.8352744667	0.8352702128	0.8352702113	0.8352702114
0.7	0.7827269175	0.7827045426	0.7827045383	0.7827045382
0.8	0.7261504962	0.7261490381	0.7261490369	0.7261490370
0.9	0.6669754063	0.6669768062	0.6669768110	0.6669768108

TABLE 14. Numerical results for Example 5.3 with $U(x) = sin(x)$ and $\epsilon = 10^{-20}$.

x_k	$M = 5$	$M = 10$	$M = 15$	$M = 20$
0.1	0.9950105895	0.9950124873	0.9950124791	0.9950124791
0.2	0.9801930723	0.9801986611	0.9801986733	0.9801986733
0.3	0.9559743743	0.9559974902	0.9559974818	0.9559974818
0.4	0.9231130735	0.9231163516	0.9231163464	0.9231163463
0.5	0.8825103053	0.8824968912	0.8824969025	0.8824969025
0.6	0.8352654399	0.8352702136	0.8352702114	0.8352702114
0.7	0.7826795534	0.7827045426	0.7827045382	0.7827045382
0.8	0.7261474709	0.7261490343	0.7261490370	0.7261490370
0.9	0.6669784533	0.6669768108	0.6669768108	0.6669768108

Example 5.4. Consider the following singular second-order boundary value problem:

$$-\varepsilon y'' + \frac{1}{x}y' + (1+x^2)y = (1-4\varepsilon)x^2e^{x^2} + (3-2\varepsilon)e^{x^2}, \tag{5.7}$$

with the boundary conditions:

$$y(0) = 1 \quad \text{and} \quad y(1) = e^1, \tag{5.8}$$

with the exact solution $y(x) = e^{x^2}$. The relative errors of the presented method with different weights and $\varepsilon = 10^{-12}$ with digits equal to 10^{-16} are revealed in Table 15.

TABLE 15. The relative errors for Example 5.4 with $\varepsilon = 10^{-12}$.

M	$U = 1$	$U = x$	$U = x^2$	$U = x(1-x)$	$U = sin(x)$	$U = xe^{x^2}$	$U = xe^{-x^2}$
5	5.51E-5	1.01E-4	1.14E-4	4.01E-4	2.05E-4	7.91E-5	6.81E-4
10	3.57E-7	1.94E-7	3.29E-5	9.71E-7	2.94E-7	2.22E-6	2.00E-6
15	6.96E-8	8.66E-10	8.19E-7	5.20E-7	3.44E-9	4.82E-8	4.33E-8
20	5.65E-10	1.77E-11	1.00E-8	2.75E-8	2.52E-11	1.81E-9	3.45E-10

6. Conclusion

In this article, we used the double exponential non-classical sinc-collocation method to address a class of singularly perturbed singular two-point boundary value problems. We have successfully obtained numerical solutions for three distinct examples and presented our method's results using six different weight configurations for each instance. Our study demonstrates that the double exponential non-classical sinc-collocation method is an effective approach for addressing singularly perturbed boundary value problems. Numerical experiments confirm its superior accuracy and efficiency compared to existing methods.

Conflicts of Interest

The authors declare that there are no conflicts of interest and grants regarding the publication of this paper.

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